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Polygons and Chaos

Jay Kappraff
Department of Mathematics
New Jersey Institute of Technology
Kappraff@aol.com

Gary W. Adamson
P.O. Box 124571
San Diego, CA 92112-4571

Abstract

An infinite number of periodic trajectories are derived for the logistic equation of dynamic systems theory at a value of the parameter corresponding to the extreme point on the real axis of the Mandelbrot set. Beginning with the edge of a family of star n-gons as the seed, the trajectory of the logistic map cycles through a sequence of edges of other star n-gons. Each n-gon for n odd is shown to have its own characteristic cycle length. The logistic map is shown to be the first of two infinite families of maps, all exhibiting periodic trajectories, derived from two families of polynomials, the Chebyshev polynomials and another related to the Lucas sequence. These dynamics are shown to be closely related to properties of number.

1. Introduction

This paper describes a remarkable connection between the edges of star polygons and dynamical systems in the state of chaos. A sequence of dynamical maps are derived from the Chebyshev polynomials and another family of polynomials related to the Lucas series, and these maps exhibit periodic trajectories of all lengths with each polygon having its own characteristic cycle length. The first polynomial of the family is the logistic equation at a value of its parameter corresponding to the extreme left-hand point on the real axis of the Mandelbrot set. This leads to new connections between chaotic dynamics and both Euclidean geometry and the theory of numbers.

2. Star Polygons

Consider the polynomial equation $z^n = 1$ for z complex. The solutions are ,

$$z = \cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n} = e^{\frac{-2\pi k i}{n}} \quad \text{for } k = 0,1,2,3,\ldots,n\text{-}1 \qquad (1)$$

referred to as the n-th *roots of unity*. The points in a cartesian coordinate system,

$$(\cos\frac{2\pi k}{n}, -\sin\frac{2\pi k}{n}) \text{ for } k = 0,1,2,3,\ldots,n\text{-}1 \qquad (2)$$

lie at the vertices of a regular n-gon with unit radius that we shall refer to as a *cyclotomic n-gon*. If the point (1,0) is distinguished, the other points satisfy the equation,

$$\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \ldots + x^3 + x^2 + x + 1 = 0 \qquad (3)$$

referred to as the nth *cyclotomic polynomial* when n is prime. We shall continue to use this terminology for n non-prime but odd.

Consider the transformation:

$$S: z \to e^{-i\theta} z \qquad (4)$$

This transformation has the effect of rotating z clockwise by θ degrees, i.e., $\arg z \to \arg z - \theta$. If z_0 is taken as the seed of this transformation, then , $z_0 \to z_1 \to z_2 \to \ldots \to z_k \to \ldots$ where the sequence of points, $\{z_k\}$ for $k = 0,1,2,\ldots$ is the *trajectory* of the transformation.

Let $\theta = \dfrac{2\pi m}{n}$ for m and n relatively prime integers, and take the seed value to be $z_0 = 1$, i.e., $k = 0$, then the trajectory forms a *modular* system with indices *mk(mod n)* and *principal values* 0,1,2,…,n-1. The vectors $e_k^{(m)} = z_{k+1}^{(m)} - z_k^{(m)}$ represent the *directed edges* from vertex z_k to z_{k+1} and the system of edges, $e_k^{(m)}$ for k = 0, 1,2,…,n-1 are the edges of a *regular star polygon* symbolized by $\{n/m\}$. The star polygons $\{n/1\}$ are denoted simply as $\{n\}$ and represent *regular n-gons*. For $0 < m \le \lfloor \frac{n-1}{2} \rfloor$ the edges in the cycle are *clockwise* and have positive lengths. For $\lfloor \frac{n-1}{2} \rfloor \le m < n$, the stars are *counterclockwise* or *retrograde* and have negative edge lengths. If the star n-gon $\{n/m\}$ has an edge cycle oriented clockwise then $\{n/m\text{-}m\}$ has retrograde edges of the same length, i.e., $e_k^{(n-m)} = -e_k^{(m)}$. The three positively oriented star 7-gons are shown in Figure 1.

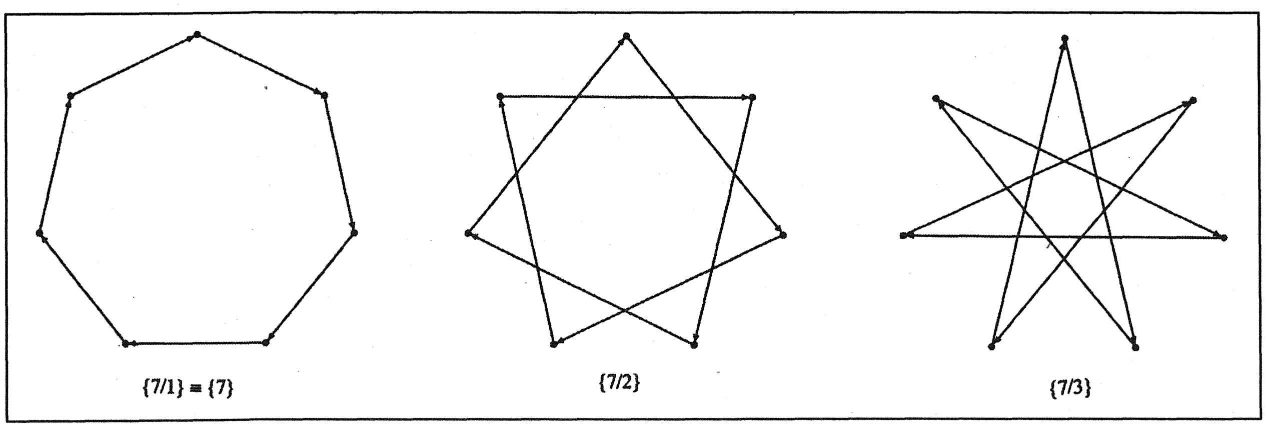

Figure 1: *The three positively oriented 7-gons.*

If n and m are not relatively prime, then $n/m = p/q$ in lowest terms and $\{n/m\}$ reduces to the star p-gon *{p/q}*. The case of the six positively oriented star polygons *{12/m}* for m = 1,2,…,6 are shown in Figure 2. Only m = 1 and 5 correspond to star 12-gons; {12/3} = {4} is a square while {12/4} = {3} an equilateral triangle, and {12/6} = {2} a digon (a degenerate case of a polygon).

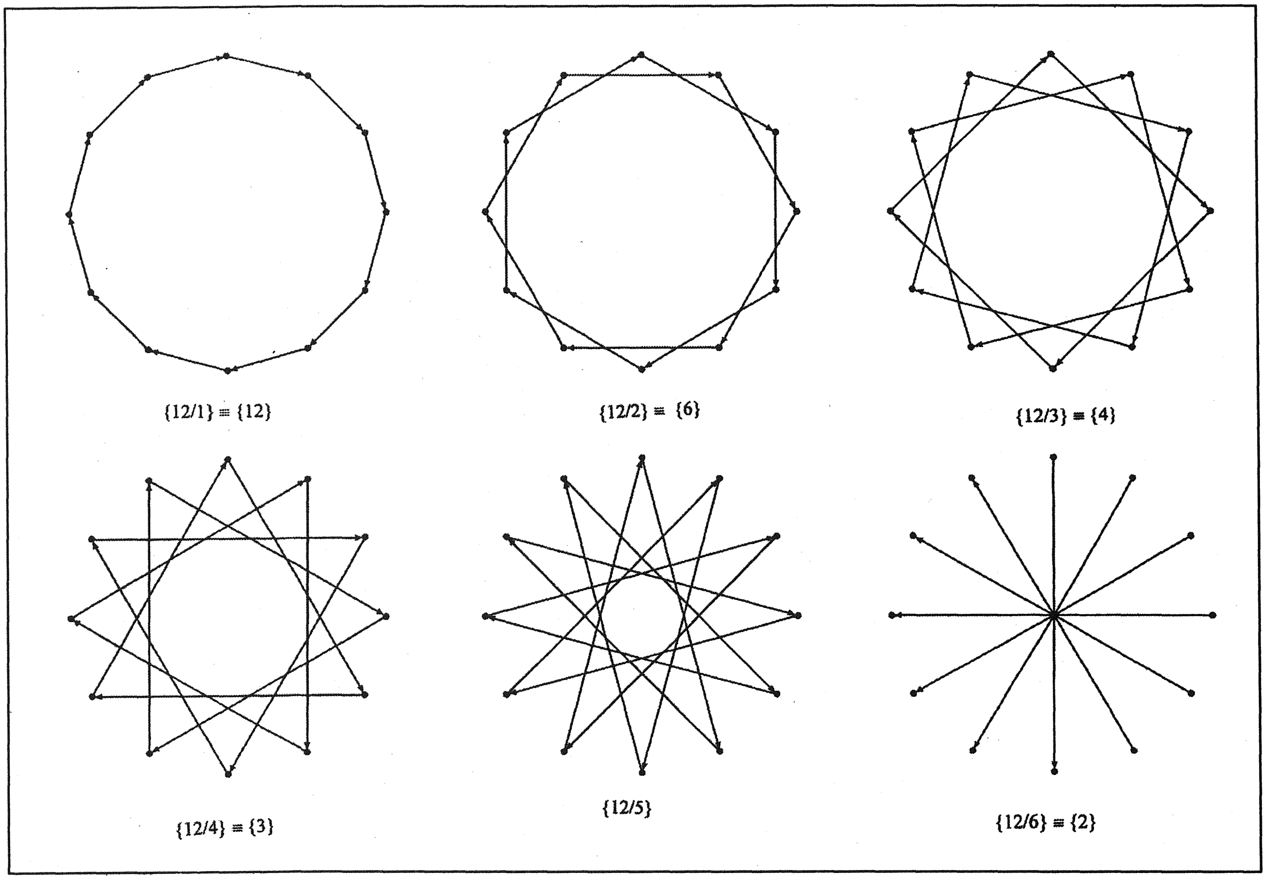

Figure 2: *The family of star 12-gons. Only {12} and {12/5} are star 12-gons. The star polygons {12/3} ≡ {4} is a 4-gon, {12/4} ≡ {3} is a 3-gon, and {12/6} ≡ {2} is a digon.*

The star polygons corresponding to n and all of its factors constitute a family of star n-gons. In general, n star polygons are related to any n-gon if the digon {2} and the polygon with a single vertex {1} are included. This follows from the property of numbers that,

$$\sum \phi(k) = n,$$

where the summation is over all of the factors k of n . $\phi\ (k)$ is the Euler phi-function which equals the number of integers relatively prime to and not greater than k. For example, the family of star 12-gons is represented by the factor tree,

$$1 \to 2 \to 4 \to 12 \leftarrow 6 \leftarrow 3$$

where $\phi(1) = 1,\ \phi(2) = 1,\ \phi(3) = 2,\ \phi(4) = 2,\ \phi(6) = 2,\ \phi(12) = 4$ with a sum of 12.

3. p-Cycles

Consider the (complex) transformation $z \to z^m$ restricted to the unit circle S^1 which can be denoted by $P_m : e^{i\theta} \to e^{im\theta}$. Observe that this map takes $\operatorname{Re} z = \cos\theta$ into $\operatorname{Re} z^m = \cos m\theta$. This is the defining property of the *Chebyshev polynomials* T_m, viz.,

$$T_m : \operatorname{Re} z \to \operatorname{Re} z^m. \tag{5a}$$

In dynamical system terms, there is a *semi-conjugacy* (see Appendix A) $h_T : S^1 \to [-1,1]$ defined by,

$$h_T(e^{i\theta}) = \cos\theta \tag{5b}$$

from the map $P_m : S^1 \to S^1$ to the map $T_m : [-1,1] \to [-1,1]$, $x \mapsto T_m(x)$. It should be noted that much of what we discuss in the sequel can be inferred from this semi-conjugacy.

Beginning with the seed $Re\ z_0$ where $z_0 = e^{i\theta_0}$,

$$Re\ z_0 \to Re\ z_1 \to Re\ z_2 \to \ldots \to Re\ z_k \to \ldots$$

where the sequence $\{Re\ z_k\}$ for $z_k = e^{im^k\theta_0}$ is the trajectory of T_m.

If $\theta_0 = \frac{2\pi}{n}$, then $z_k = e^{\frac{2\pi i m^k}{n}}$. Since $cos(-\theta) = cos\theta$, $Re\ z_p = Re\ z_0$ when p is the smallest positive integer such that ,

$$m^p \equiv \pm 1 (\operatorname{mod} n). \tag{6}$$

In this case the trajectory forms a cycle of length p, or what we shall refer to as a *p-cycle*. Since $(m^p)^2 = (m^2)^p$, Equation 6 is equivalent to

$$(m^2)^p \equiv 1 (\operatorname{mod} n) \tag{7}$$

If n is a *prime number*, then $(m^2)^{\frac{n-1}{2}} \equiv 1 (\operatorname{mod} n)$ so that p must be a factor of $\frac{n-1}{2}$ [1]. If n is odd but not prime, then $\frac{n-1}{2}$ is replaced by $\phi(\frac{n-1}{2})+1$ where $\phi(k)$ is the Euler phi-function. Table 1 shows the values of p for $m^2 = 4, 9, 16$ and 25 and $n = 7, 9, 11, 13, 15$, and 17.

Table 1. Exponents p such that $(m^2)^p \equiv 1(\bmod n)$

m^2	n = 7	n = 9	n = 11	n = 13	n=15	n=17
4	3	3	5	6	4	4
9	3	…	5	3	…	8
16	3	3	5	3	2	2
25	3	3	5	2	…	8

4. Polygons and Chaos

For k and n relatively prime integers and n odd, given a value of $k = 0,1,2,\ldots, \frac{n-1}{2}$, it can be shown that there exists a value of $j = 0,1,2,\ldots, \lfloor\frac{2n-1}{2}\rfloor$ such that,

$$2\cos\frac{2\pi k}{n} = 2\sin\frac{2\pi j}{2n} \tag{8}$$

for j and n relatively prime (e.g., j cannot be even). The values, $2\sin\frac{2\pi j}{2n}$, are the edge lengths of either of the star $2n$-gons $\{2n/j\}$ or $\{2n/2n\text{-}j\}$ depending on their signs [2]. Therefore we can view the p-cycles of the previous section corresponding to an n-cyclotomic polynomial as being a sequence of edges of star $2n$-gons.

This encourages us to consider the transformation of the circle of radius 2, S_2^1, defined by,

$$P_{m_2} : 2e^{i\theta} \rightarrow 2e^{im\theta}.$$

This map takes $2\operatorname{Re} z = 2\cos\theta$ into $2\operatorname{Re} z^m = 2\cos m\theta$, which is also the defining property of the *Lucas polynomials* [3],

$$L_m(2\cos\theta) = 2\cos m\theta. \tag{9a}$$

In other words, the maps $2z \rightarrow 2z^m$ restricted to S_2^1 are semi-conjugate (see Appendix A) to the Lucas polynomials $\{L_m\}$ for $m \geq 1$ via the map

$$h_L(2e^{i\theta}) = 2\cos\theta. \tag{9b}$$

This has some interesting dynamical consequences.

Letting $x = \cos\theta$, as the result of Equations 5b and 9b, $\frac{1}{2}L_m(2x) = T_m(x)$ and transformations 5a and 9a can be rewritten as,

$$x \mapsto T_m(x) \quad \text{and} \quad 2x \mapsto L_m(2x) \qquad \text{(10a and b)}$$

where the first few Chebyshev and Lucas polynomials are listed in Table 2,

Table 2. The Lucas and Chebyshev Polynomials

L_1	x	1	T_1	x	1
L_2	$x^2 - 2$	3	T_2	$2x^2 - 1$	3
L_3	$x^3 - 3x$	4	T_3	$4x^3 - 3x$	7
L_4	$x^4 - 4x^2 + 2$	7	T_4	$8x^4 - 8x^2 + 1$	17
L_5	$x^5 - 5x^3 + 5x$	11	T_5	$16x^5 - 20x^3 + 5x$	41
L_6	$x^6 - 6x^4 - 9x^2 - 2$	18	T_6	$32x^6 - 48x^4 + 18x^2 - 1$	99
L_7	$x^7 - 7x^5 + 14x^3 - 7x$	29	T_7	$64x^7 - 112x^5 + 56x^3 - 7x$	239
...	...	...		...	...

Notice that, ignoring signs, a coefficient S_R^C in row R and column C of Table 2 is given by the recursion relation, $S_j^{k+1} = S_{j-1}^{k-1} + S_j^k$ for the coefficients of the Lucas polynomials and $S_j^{k+1} = S_{j-1}^{k-1} + 2S_j^k$ for the coefficients of the Chebyshev polynomials. Also note that, ignoring signs, the sum of the coefficients of the Lucas polynomials form the Lucas sequence $\{l_k\}$: 1 3 7 11 18 29 ... which satisfies the recursion relation: $a_{n+2} = a_{n+1} + a_n$, i.e., it is a Fibonacci sequence beginning with 1,3. On the other hand the Chebyshev polynomials form the Chebyshev sequence (T-sequence) $\{t_k\}$: 1 3 7 17 41 99 239 ... which satisfies the recursion relation: $a_{n+2} = 2a_{n+1} + a_n$, i.e., a Pell sequence beginning with 1,3. The standard Fibonacci sequence (F-sequence) $\{f_k\}$ *is*: 1 2 3 5 8 13 21 ... and $l_k = f_{k-1} + f_{k+1}$ while the standard Pell sequence (P-sequence) $\{p_k\}$ *is* : 1 2 5 12 29 70 ... and $t_k = p_{k-1} + p_{k+1}$. Also,

$$\lim_{k\to\infty} \frac{l_{k+1}}{l_k} = \tau_1 \quad \text{and} \quad \lim_{k\to\infty} \frac{t_{k+1}}{t_k} = \tau_2$$

where $\tau_1 = \frac{1+\sqrt{5}}{2}$ and $\tau_2 = 1+\sqrt{2}$. Also $\tau_1 - \frac{1}{\tau_1} = 1$ and $\tau_2 - \frac{1}{\tau_2} = 2$. For this reason τ_1 and τ_2 are referred to as the golden and silver means respectively.

Consider the second Lucas polynomial L_2, $x^2 - 2$ and its iterative map,

$$x \mapsto x^2 - 2, \qquad \text{(11a)}$$

also known as the *logistic map*. It represents the extreme left-hand point on the real axis, i.e., $c = -2$, of the *Mandelbrot set* (see Figure 3) for the map,

$$x \mapsto x^2 + c, \qquad \text{(11b)}$$

where c is a complex number. The arrow notation in Equation 11 means that we choose a seed value x_0 and place it into the polynomial to get x_1. From x_1 we get x_2, etc. and in this way generate the sequence $x_0, x_1, x_2, \dots$, the *trajectory* of the map.

The map given by Equation 11a is a transformed version of the logistic map

$$x \mapsto \lambda x(1-x) \text{ for } \lambda = 4 \tag{12}$$

which has been studied in great detail [4], [5]. The fact that $c = -2$ in Equation 11b, means that this map is in a state of *chaos*. It can be shown that for values of $\lambda < 4$ *(or* $c > -2$) all points on the unit interval are "imprisoned" in the sense that their trajectories remain in the unit interval [1,0] for Equation 12 or for values of c corresponding to λ in Equation 11a, the trajectories remain on the interval $[-\lambda/2, \lambda/2]$. However, beginning at $\lambda = 4$ (or $c = -2$), orbits can escape; in fact the only imprisoned orbits lie on a *Cantor set* within the unit interval [0,1]. For any complex value of c, the boundary in the complex plane of the prisoner set is what is called the *Julia set*. Therefore, the Julia sets for real values $c \le -2$ are what we refer to as "Cantor dusts."

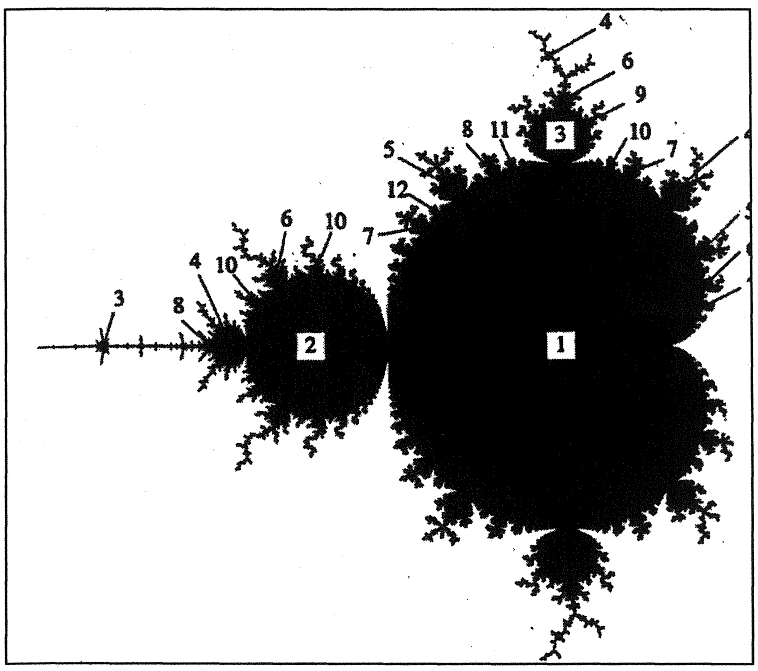

Figure 3: *The Mandelbrot set.*

The theory of dynamical system shows that as λ is increased to the *Feigenbaum limit* 3.569... the trajectories of the system go through period doubling bifurcations, i.e., cycles of length 2^n for $n = 1,2,3,\dots$ At the value 3.831... a trajectory with a cycle of length 3 appears, after which periods of every length are present according to the theorem of Sharkovskii. As a result of our analysis when λ is further increased to a value of 4, or alternatively c is decreased to $c = -2$, the cycles can be characterized as edges of star $2n$-gons for n odd in which each value of n has its own characteristic cycle length. Therefore, in a sense, the edges can be thought to dance about on the grains of a Cantor dust.

4. Results

For any value of m the T_m and L_m transformations defined by Equation 10 result in p-cycles, some of which are listed in Table 1, of semi-edges and edges, respectively, of star $2n$-gons. We shall study the dynamics of the logistic map of Equation 11a corresponding to $m = 2$, in some detail giving the results of the p-cycles for cyclotomic n-gons where $n = 7, 9, 11, 13$, and 17.

For the cyclotomic 7-gon, values of $2\cos\frac{2\pi k}{7}$ for $k = 1,2,3$ are listed in Table 3 along with the values of j for which $2\sin\frac{\pi j}{14} = 2\cos\frac{2\pi k}{7}$ (ignoring signs). In other words, a given value of k corresponds to the edge of the {14/j} star 14-gon. Also listed in Table 3 is the actual edge corresponding to k taking into account its sign. For example, $2\cos\frac{2\pi k}{7} = -0.44509\ldots$ for $k = 2$ which corresponds to $-2\sin\frac{\pi j}{14}$ for $j = 1$. Therefore, this represents the edge of the {14/13} species of retrograde star 14-gon.

If $x_0 = 2\cos\frac{2\pi}{7}$ is taken to be the seed of Transformation 11a, then we find the remarkable result that, the iterates are the sequence of edge lengths of different species of star *14*-gons,

$$2\cos\frac{2\pi k}{7} \text{ for } k = 1,2,4,8,\ldots(\text{mod } 7), \qquad (13)$$

and since $8 \equiv 1$ (*mod* 7) the sequence repeats with the 3-cycle,

$$x_0 \to x_1 \to x_2 \to x_0, \text{ or}$$

$$2\cos\frac{2\pi}{7} = 1.2469\ldots \to 2\cos\frac{4\pi}{7} = -0.44509\ldots \to 2\cos\frac{8\pi}{7} = -1.80189\ldots$$

As a result of the fact that $\cos\frac{2\pi(n-k)}{n} = \cos\frac{2\pi k}{7}$, positive and negative values of the edges corresponding to k *(mod n)* are identical. As a result, $4 \equiv -3$ (*mod* 7) which corresponds in Sequence 13 to $k = 3$, and so the 3-cycle is represented by the k-values 1→2→3→1... in Table 3 and therefore by the sequences of edges j : 3→13→9→3... The *orders* of the edges in the cycles are also indicated in Table 3, beginning with the seed $i = 0$ and in Figure 4 by the boxed integers. These edges are shown in Figure 4 drawn from vertex number 0 of the 14-gon to the darkened vertex j. Notice the regular skip pattern of highlighted vertices : 4, 4, 6. We shall find this pattern to hold for all cycles. Also note that for each cycle, its mirror image, illustrated by open vertices, within the $2n$-gon is also a cycle, e.g., 11→1→5→11... is another 3-cycle for the cyclotomic 7-gon. Each cycle of a cyclotomic n-gon will have a corresponding mirror image cycle. Finally, Adamson has discovered that the product of the cycle values equals -1, e.g.,

$$(1.2469\ldots)(-0.44509\ldots)(1.80189\ldots) = -1,$$

and that the product of edges within any cycle will always equal ± 1 for $m = 2$, a result that can be proven from a dynamic/number theoretic description.

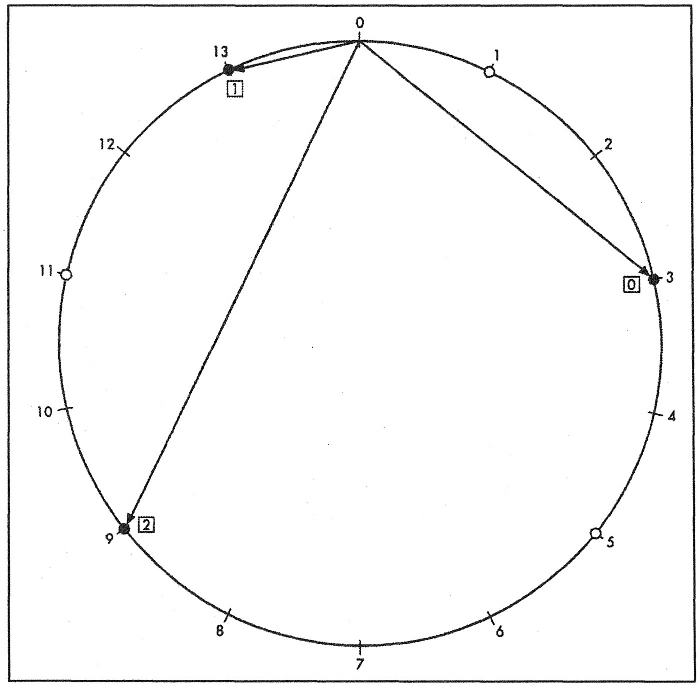

Figure 4: *The points k shown with closed circles within the following 14-gon represent the edge-lengths from point 0 to point k of the star 14-gon {14/k}. These are elements of the trajectories of the logistic equation corresponding to that 14-gon. The open circles represent the mirror image trajectory. The numbers in square boxes are the order of the points in the trajectory.*

Table 3 Cycles for the Logistic Equation Corresponding to Cyclotomic Polygons.

	Cyclotomic *7*-gon				Cyclotomic *9*-gon			
K	$2\cos\frac{2\pi k}{7}$	$2\sin\frac{\pi j}{14}$	{14/j}	Order	$2\cos\frac{2\pi k}{9}$	$2\sin\frac{\pi j}{18}$	{14/j}	Order i
1	1.24696...	3	3	0	1.53208...	5	5	0
2	-0.44509...	1	13	1	0.34729...	1	1	1
3	-1.80189...	5	9	2	-0.500...	...	...	...
4					-1.87938...	7	11	2

	Cyclotomic *11*-gon				Cyclotomic *17*-gon			
K	$2\cos\frac{2\pi k}{11}$	$2\sin\frac{\pi j}{22}$	{14/j} j	Order I	$2\cos\frac{2\pi k}{17}$	$2\sin\frac{\pi j}{34}$	{34/j} j	Order i
1	1.68250...	7	7	0	1.8649...	13	13	0
2	0.83082...	3	3	1	1.4780...	9	9	1
3	-0.28462...	1	21	3	0.89147	5	5	0’

4 5 6 7 8	-1.30972... -1.91898...	5 9	17 13	2 4	0.1845... -0.5473... -1.205 -1.70043... -1.9659...	1 3 7 11 15	1 31 27 23 19	2 2' 1' 3' 3

Cyclotomic *13*-gon

K	$2\cos\frac{2\pi k}{13}$	$2\sin\frac{\pi j}{26}$	{26/j} j	Order I
1	1.77090...	9	9	0
2	1.13612...	5	5	1
3	0.24107...	1	1	4
4	-0.70920...	3	23	2
5	-1.49702...	7	19	3
6	-1.94188...	11	15	5

Next consider the cyclotomic 9-gon. Its values of $2\cos\frac{\pi k}{9}$ are listed in Table 3 along with the corresponding values of j for which $2\sin\frac{\pi j}{18}=2\cos\frac{2\pi k}{9}$. Using the value of k = 1 as the seed in the logistic Map 11a, the corresponding 3-cycle is derived, as before, from the sequence of k-values : 1, 2, 4, 8, ... (mod 9) where 8 ≡ -1 (mod 9) which corresponds to the edge k = 1 in Table 2. Therefore the 3-cycle is represented by k values : 1→2→4→1... Notice that k = 3 is not part of the cycle because 3 is not relatively prime to 9. The corresponding j value gives rise to the edge {18/3} = {6/1}, and this is the edge of a star 6-gon not an 18-gon. The sequence of edges is : 5→1→11→5...(not shown). This time the skip pattern is : 4,8,6 (the pattern would have been 4,4,4,6 if k=3 were to be included) and the product of the edges in the cycle equals to –1.

As a final example consider the cyclotomic 17-gon. Its k and j values are listed in Table 3 and its 4-cycle can be generated from the seed value corresponding to k = 1 as the sequence, once again, of k-values: 1, 2, 4, 8, 16,... (mod 17) where 16 ≡ -1 (mod 17) which corresponds in Table 2 to the edge k = 1. This leads to the sequence of star 34-gon edges corresponding to k-values: 1→2→4→8→1 ... whose edge lengths are found in Table 3, and to the cycle of edges: 13→9→1→19→13....

However, since 17 is a prime number there are eight distinct star 34-gons (not considering orientation) and we have accounted for only four of them where 4 is a factor of 8 according to the results of Section 1. The other four can be obtained by beginning with a seed value corresponding to k = 3 resulting in the sequence: 3, 6, 12, 24, 48, ... (mod 17) where 48 ≡ -3 (mod 17) which corresponds in Table 2 to the edge k = 3. But since 12 ≡ -5 (mod 17) corresponding to the edge k = 5 and 24 ≡ 7 (mod 17), this leads to the sequence of edges corresponding to the k values : 3->6->5->7->3... found in Table 3 and to the edge sequence : 5→27→31→23→5 ... This 4-cycle is distinguished from the other in Table 3 by order numbers denoted by primes ('). If the vertices corresponding to both of these edge cycles are highlighted (not shown), they lead to the skip pattern : 4, 4, 4, 6, 4, 4, 4, 4. Once again, the product of the edges equals -1 for the first 4-cycle and +1 for the second.

5. A Cycle Algorithm

These examples lead to a simple algorithm for any cyclotomic n-gon for n odd:

1. Compute $2\cos\frac{2\pi k}{n}$ for $k = 1,2,3,\ldots, \frac{n-1}{2}$.
2. The k-values for the cycle related to the cyclotomic n-gon is the sequence: 1,2,4,8,16,...(mod n), disregarding signs, i.e., 2^k (mod n) for $k = 0,1,2,3\ldots,p-1$. This sequence repeats when $2^k \equiv \pm 1$ (mod n) for $k = p$ in which case the cycle has length p and the values of i give the order of terms in the sequence beginning with a seed value of $k = 1$ corresponding to $i = 0$. Record the order numbers i for the sequence of iterates to the logistic Equation 3 as i: $0,1,2,\ldots,p-1$.
3. Once the k-values of the cycle are determined, compute the values of j such that $2\sin\frac{\pi j}{14} = 2\cos\frac{2\pi k}{7}$, ignoring signs. The values of j come from the sequence: $1,3,5,\ldots \lfloor\frac{2n-1}{2}\rfloor$ where j is relatively prime to $2n$.
4. If the sign of the cosine is positive, then the species of star polygon is $\{2n/j\}$; if the sign is negative then the star polygon is the retrograde version, $\{2n/2n-j\}$.
5. The process ends when all j values from step 3 are accounted for. If all of the sine values have not been used, then a second sequence of k-values gives another cycle of length p given by: 3^k (mod n) for $k = 0,1,2,3,\ldots p$ where $3^k \equiv \pm 1$ (mod n) for $k = p$ and steps 2-4 are repeated. Record the order numbers as 0',1',2',...,(p-1)'. If there are still additional unaccounted for j-values, then another p-cycle can be determined from the sequence 5^k (*mod n*), etc.
6. The period length is a divisor (or factor) of the number of integers $1,2,3,\ldots, \frac{n-1}{2}$ relatively prime to n.
7. Observe that the product of edges from a p-cycle satisfies the equation,

$$\prod_{k=2^j \, for j=0}^{j=e-1} \cos\frac{2\pi k}{n} = \pm 1$$

Try applying this algorithm to n =15. Note that among the integers k = 1,2,3,4,5,6,7 only 1,2,4, and 7 are relatively prime to 15, so that according step 6 of the Algorithm, the length of the cycle can be either 2 or 4. However, since $2^4 \equiv -1$ (mod 15), according to step 4 of the Algorithm the period is 4.

The results of applying this algorithm to the cyclotomic 11- and 13-gons are found in Table 3 and in Figures 5a and b. The cyclotomic 11-gon results in a 5-cycle of edges of the star 22-gon family while the cyclotomic 13-gon results in a 6-cycle within a 26-gon. The equal-tempered chromatic scale can be represented by a tone circle with 12 tones to the octave, or a 12-gon with each tone distant from the next by a semitone. Therefore a 24-sided polygon can be thought of as a tone circle in which each tone represents an interval of a quarter-tone. This means that the 5- and 6-cycles of the cyclotomic 11- and 13-gon, along with their symmetric opposites can be viewed as tonal subscales of almost quarter-tone intervals, one with tones slightly greater than quarter-tones and the other with tones slightly less.

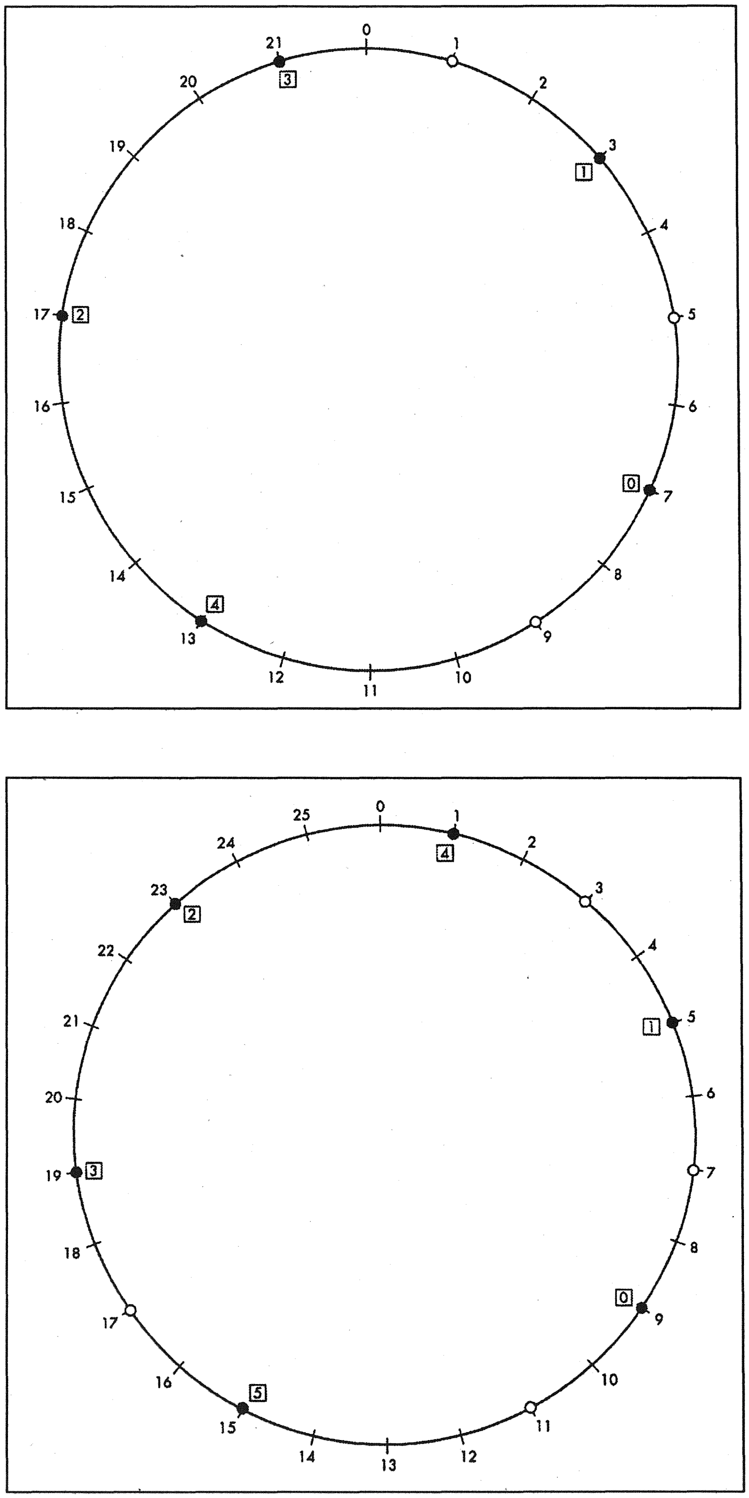

Figure 5: *Trajectories of the logistic equation corresponding to the edges of (a) a star 22-gon; (b) a star 26-gon (see figure caption of Figure 4).*

6. Other Lucas Polynomials

What we have discovered for the second Lucas polynomial, appears to hold true for all of the other Lucas polynomials from Table 2. For example, the third Lucas polynomial L_3 with alternating signs leads to the recursive map,

$$x \mapsto x^6 - 3x. \tag{13}$$

With $m = 3$, taking the seed $x_0 = 2\cos\frac{2\pi}{n}$ for odd n, results in the sequence $\{2\cos\frac{2\pi k}{n}\}$ where this time $k = 1,3,9,27,81,\ldots$ (mod n), i.e., 3^k (mod n) for $k = 0,1,2,3,\ldots,p-1$, a p-cycle as predicted by Table 1. The sequence is based on powers of 3 since we are using the 3rd Lucas polynomial. The cycle lengths can be determined by the Algorithm with the only change being that k-values of the cycle are now powers of 3 instead of 2. This generalizes to the m-th Lucas polynomial L_m with alternating signs in which case the iterates correspond to k-values that are powers of m. Presumably, these polynomial maps also represent dynamical systems in a state of chaos. However, the products of the edges in a cycle are not always equal ± 1.

7. Chaos and Number

There is an intimate relationship between chaos theory and number. We have shown that properties of number also lie at the basis of the polygon cycles. Let us once again consider the 2nd Lucas polynomial map. Expanding 1/7 in the base 4 (the square of 2), $1/7 = 0.021021021\ldots = 0.\overline{021}$, a repeating decimal with a 3-cycle. Likewise $1/11 = 0.\ 0.\overline{01131}$ expanded in base 4, a 5-cycle (see Appendix B). Our conjecture is that for n odd, $1/n$ expanded in base 4, has the identical cycle length as the cyclotomic n-gon analyzed in the previous section. Furthermore, the identical cycle lengths occur for $1/n$ in base 9, 16, 25 or any base m^2 as for the cycle lengths of cyclotomic n-gons corresponding to the m-th Lucas polynomial maps as shown in Table 2. The validity of this claim and other parts of this analysis were computer checked by Malcolm Lichtenstein [6].

8. Conclusion

Many things have come together in this study. We have shown the close relationship between chaos and number. The well known theorem of Sharkovskii predicts that once a period 3-cycle appears in a dynamical system, periods of all lengths occur. We have shown that at critical point of the Mandelbrot set where orbits of the logistic equation begin to escape, each of these periods can be characterized by a sequence of edge lengths of a family of star $2n$-gons, each n having a characteristic cycle length. Coxeter has shown star polygons to be related to the two-dimensional projections of higher-dimensional polyhedra or polytopes [7]. Geometry has shown itself once again to be the rich well-spring of mathematics. Rather than jettisoning these roots, the theory of dynamical systems and chaos has strongly embraced them.

Each of these star polygons can be looked at as a tone circle with the cycles represented by tones from the "octave." In particular, the 5-cycle from the 22-gon and 6-cycle of the 26-gon are promising candidates for new musical scales. After all, the chromatic scale was built from the circle of fifths related to the {12/5} star 12-gon.

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Appendix A

Given a pair of transformations T_1 on set A and T_2 on set B, the transformation h mapping A to B, such as the ones defined by Equations 5b and 9b, induce relationships between the transformations T_1 and T_2 via the following commuting diagram:

$$\begin{array}{ccc} A & \xrightarrow{T_1} & A \\ h\downarrow & & \downarrow h \\ B & \xrightarrow{T_2} & B \end{array}$$

In this relationship $h\,T_2 = T_1 h$ where h is not invertible. Such a transformation is known as a *semi-conjugacy*. When h is invertible an isomorphism is induced between the T_1 and T_2 characterized by $T_2 = h^{-1}T_1 h$, and h is called a *conjugacy*.

Appendix B

A decimal in base 10 can be written in any other base by the following procedure illustrated for converting $1/7 = 0.\overline{142857}$ in base 10 to the base 4.

1. Multiply the decimal in base 10 by 4 and record a 0 if the result is less than 1, otherwise record the integer part. For example, $0.\overline{142857}x4 = 0.57148...$ so record a 0 as the 1st in the first decimal place.
2. Multiply the result again by 4 to get 0.2857142… and record a 2 as the 2nd decimal place.
3. Multiply the decimal part of the preceding number by 4 to get : 1.1428568… and record a 1 as the 3rd decimal place.
4. Again multiply the decimal part of the preceding number by 4, but since the decimal part repeats we have the repeating decimal in base 4: $0.\overline{021}$

In general, consider the rational fraction $1/n = a_0$ where a_0 is the decimal expansion of $1/n$ in base 10. Its decimal expansion in base m is then : $0.b_1b_2b_3\ldots$ where b_n equals the integer part of : a_{n-1} x 4 (mod 1) for $n = 1,2,3,\ldots$