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On the Squarefree Problem II

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One of the well-known problems in the analytical theory of numbers is the squarefree problem. We say that the natural number n is squarefree if n is a product of different prime numbers. The problem is to find $h=h(x)$, as small as possible, such that there is a squarefree number in the interval $(x, x+h]$ for sufficiently large x.

First, E. Fogels [1] proved, that, one can take $h=x^{\theta+\varepsilon}$, where $\theta=2/5$ and ε-arbitrary is a positive number. An important step was made by K. F. Roth [2]. He proved, using elementary methods that we can take $h=x^{\theta+\varepsilon}$, $\theta=1/4$ and with the help of the method of the trigonometric sums he improved the value of the constant $\theta=3/13$. . (We say that some methods or some proofs are elementary if they do not use estimates of trigonometric sums.)

Using the method of exponent pairs H. E. Richert [3], R. A. Rankin [4], P. G. Schmidt [5], S. W. Graham and G. Kolesnik [6] improved the estimate for θ, $\theta=2/9=0.2222\ldots$, $\theta=0.221982\ldots$, $\theta=109556/494419$ $=0.221585\ldots$, $\theta=1057/4785=0.208896\ldots$, respectively.

M. Filaseta [7] gave an elementary proof of the Roth's result ($\theta=3/13$).

In this paper we give an elementary proof of the result of Richert ($\theta=2/9$).

Moreover, we give further improvement for the constant θ, $\theta=17/77$ $=0.2207792\ldots$

(This result is announced without proof by the author in [11].)

We prove:

Theorem 1. *There is a constant c_1 such that for sufficiently large x there is a squarefree number in the interval $(x, x+c_1 . x^{2/9}]$.*

(The proof of Theorem 1 is elementary.)

Theorem 2. *There is a constant c_2 such that for sufficiently large x there is a squarefree number in the interval $(x, x+c_2 . x^{17/77}]$.*

Proof of the Theorems. Let S is the number of the integer numbers in the interval $(x, x+h]$ which are not squarefree. Our purpose is to prove that $S<h-1$. Denote $h=x^{\theta}$.

It is evident that

$$S \leqq \sum_{p-\text{prime}} \left(\left[\frac{x+h}{p^2} \right] - \left[\frac{x}{p^2} \right] \right)$$

and $S \leqq S_1 + S_2$, where

$$S_1 = \sum_{p \leqq x^\theta \sqrt{\log x}} \left(\left[\frac{x+h}{p^2} \right] - \left[\frac{x}{p^2} \right] \right);$$

$$S_2 = \sum_{p > x^\theta \sqrt{\log x}} \left(\left[\frac{x+h}{p^2} \right] - \left[\frac{x}{p^2} \right] \right);$$

$$S_1 \leqq \sum_{p \leqq x^\theta \sqrt{\log x}} (h/p^2 + 1) \leqq x^\theta \,.\, \sum_{n=2}^{\infty} \frac{1}{n^2} + \pi(x^\theta \sqrt{\log x}) < 2/3 \,.\, x^\theta \quad \text{for}$$

sufficiently large x.

Let estimate (1)

$$S_2 \leqq S_2' = \sum_{x^\theta \sqrt{\log x} \leqq n \leqq x^{1/2}} \left(\left[\frac{x+h}{n^2} \right] - \left[\frac{x}{n^2} \right] \right).$$

$n > x^\theta$, hence $\left[\frac{x+h}{n^2} \right] - \left[\frac{x}{n^2} \right] = 0$ or 1, and

(2) $\left[\frac{x+h}{n^2} \right] - \left[\frac{x}{n^2} \right] = 1$ if and only if there exist $k \in \mathbf{Z}$ such that $kn^2 \in (x, x+h]$

and this is equivalent to $\left\{ \frac{x}{n^2} \right\} \in (1 - h/n^2, 1)$.

Define

(3) $$S(A, B) = \{ n \in (A, B] \cap \mathbf{N} \mid \exists k : kn^2 \in (x, x+h] \}.$$

Then $S_2 = |S(x^\theta \sqrt{\log x}, x^{1/2})|$.

Lemma 1. *Let a, b, c, d e are real numbers and $b>0$, $d>0$. Then:*

(i) *If* (4) $|S(x^\varphi, 2.x^\varphi)| \ll x^{a-b\cdot\varphi}$ *for every* $\varphi \in [u_1, v_1]$ *then* $|S(x^{u_1}, x^{v_1})| \ll x^{a-b\cdot u_1}$;

(ii) *If* (5) $|S(x^\varphi, 2.x^\varphi)| \ll x^{c+d\cdot\varphi}$ *for every* $\varphi \in [u_2, v_2]$ *then* $|S(x^{u_2}, x^{v_2})| \ll x^{c+d\cdot v_2}$;

(iii) *If* (6) $|S(x^\varphi, 2.x^\varphi)| \ll x^e$ *for every* $\varphi \in [u_3, v_3]$ *then* $|S(x^{u_1}, x^{v_1})| \ll x^e . \sqrt{\log x}$.

Proof of Lemma 1. We divide the interval (x^{u_i}, x^{v_i}) to shorter intervals of the type $[A, 2.A]$ and for every short interval we use the corresponding estimate (4), (5) or (6). □

Lemma 2 (Roth). *Let* $\varphi \in (0, 1)$. *Then* $|S(x^{\varphi}, 2.x^{\varphi})| \ll x^{\frac{1-\varphi}{3}}$.

Lemma 2 is proved first by Roth [2]. We give a new proof which contains new ideas useful for our later purposes.

Proof of Lemma 2. Let $u, u+l_1, u+l_2$ are three different elements of the set $S(x^{\varphi}, 2x^{\varphi}).(0<l_1<l_2)$.

Then, there exist integer numbers k_0, k_1, k_2 such that

$$k_i.(u+l_i)^2 \in (x, x+h], \quad i=0, 1, 2 \quad (l_0=0). \tag{6}$$

Consider the polynomials P_0, P_1, P_2 of three variables u, l_1, l_2 and depending of the parameters A_1, A_2, A_3.

$$P_0(u, \bar{l}, \bar{A}) = -2(A_1+A_3).u+(A_1+3A_2).l_1+(-3A_2+A_3).l_2;$$

$$P_1(u, \bar{l}, \bar{A}) = 2A_1.u+3A_1.l_1+3A_2.l_2;$$

$$P_2(u, \bar{l}, \bar{A}) = 2A_3.u-3A_2.l_1+3A_3.l_2.$$

Consider the identity

$$\frac{P_0}{u^2}+\frac{P_1}{(u+l_1)^2}+\frac{P_2}{(u+l_2)^2}=\frac{R}{u^2(u+l_1)^2(u+l_2)^2},$$

where

$$R(u, \bar{l}, \bar{A}) = u^2.(A_1l_1^3+9A_2l_1l_2.(l_1-l_2)+A_3l_2^3)$$

$$+ul_1l_2.((2A_1+6A_2)l_1+(-6A_2+2A_3)l_2)+l_1^2l_2^2.((A_1+3A_2).l_1+(-3A_2+A_3).l_2).$$

It is evident that if $u, l_1, l_2, l_3, A_1, A_2, A_3$ are integers then P_0, P_1, P_2 and R are also integer numbers.

We have $P_i \ll u$, $i=0, 1, 2$ and $R \ll l_2^3.u^2$.

From (6) follows

$$(k_0P_0+k_1P_1+k_2P_2=\frac{x.R}{u^2(u+l_1)^2(u+l_2)^2}+0\left(\frac{h}{u}\right). \tag{7}$$

Let assume that for every choice of the integer parameters A_1, A_2, A_3 $k_0P_0+k_1P_1+k_2P_2=0$.

Then for the concrete choices

$$(A_1, A_2, A_3)=(1, 0, 0); \quad (A_1, A_2, A_3)=(0, 1, 0); \quad (A_1, A_2, A_3)=(0, 0, 1)$$

we obtain the identities

$$(8)\qquad \left|\begin{array}{l} k_0P_0^{(1)}+k_1P_1^{(1)}+k_2P_2^{(1)}=0\\ k_0P_0^{(2)}+k_1P_1^{(2)}+k_2P_2^{(2)}=0\\ k_0P_0^{(3)}+k_1P_1^{(3)}+k_2P_2^{(3)}=0.\end{array}\right.$$

We can consider (8) like a system of linear equations. The determinant is

$$\begin{vmatrix} P_0^{(1)} & P_1^{(1)} & P_2^{(1)}\\ P_0^{(2)} & P_1^{(2)} & P_2^{(2)}\\ P_0^{(3)} & P_1^{(3)} & P_2^{(3)}\end{vmatrix} = 36 . l_1 . l_2 . (l_2-l_1)\neq 0.$$

Then from (8) $k_0=k_1=k_2=0$ follows and this is a contradiction. So, our assumption is wrong and there exist (A_1, A_2, A_3), $A_i=0(1)$ such that $k_0P_0+k_1P_1+k_2P_2\neq 0$.

$$|k_0P_0+k_1P_1+k_2P_2|\geqq 1,\quad (k_i \text{ and } P_i \text{ are integers}).$$

Then

$$\left|\frac{x . R}{u^2(u+l_1)^2(u+l_2)^2}+0\left(\frac{h}{u}\right)\right|\geqq 1.$$

But $0\left(\frac{h}{u}\right)=0(1)$ and for sufficiently large x $|R|\gg x^{6\varphi-1}(u\in[x^\varphi, 2x^\varphi])$.

Then $u^2 . 1_2^3\gg|R|\gg x^{6\varphi}$, $l_2\gg x^{\frac{4\varphi-1}{3}}$.

We proved that if we take three different elements of $S(x^\varphi, 2x^\varphi): u$, $u+l_1$, $u+l_2(0<l_1<l_2)$, then $l_2\gg x^{\frac{4\varphi-1}{3}}$. Evidently this implies $|S(x^\varphi, 2x^\varphi)| \ll \frac{x^\varphi}{x^{\frac{4\varphi-1}{3}}} . 2\sim x^{\frac{1-\varphi}{3}}$. □

Lemma 3. *Let* $0<\theta<0.23$ *and* $\theta<\varphi\leqq 0.38$. *Then* $|S(x^\varphi, 2x^\varphi)|\ll x^{\frac{1+\varphi}{\sigma}}$.

This lemma and the method of the proof are the essential part of the paper.

Proof of Lemma 3. Let $f(z)=\frac{x}{z^2}$ and u, $u+l_1$, $u+l_2$ are three different elements of the set $S(x^\varphi, 2x^\varphi)$. $(0<l_1<l_2)$.

Let $a=\min(l_1, l_2-l_1)$ and $b=\max(l_1, l_2-l_1)$.

Consider the divided difference

$$(9)\qquad f[u,\ u+l_1,\ u+l_2]=\frac{f(u).(l_2-l_1)-f(u+l_1).l_2+f(u+l_2).l_1}{l_1.l_2.(l_2-l_1)}$$

$$=\frac{f''(\xi)}{2!}\sim x^{1-4\varphi},\quad \xi\in[u,\ u+l_2].$$

($f''(\xi)$ is between $\frac{3}{16}x^{1-4\varphi}$ and $3x^{1-4\varphi}$).

Denote (10) $A=f(u).(l_2-l_1)-f(u+l_1).l_2+f(u+l_2).l_1$

$$A=\frac{1}{2}.f''(\xi).l_1.l_2.(l_2-l_1)>0.$$

Let assume that $A<1$. Then $A=\{A\}$ $(0<A<1)$. But $\{f(u+l_i)\}\in(1-x^{\theta-2\varphi},1)$ for $i=0,\ 1,\ 2$ $(l_0=0)$ because $u+l_i\in S(x^\varphi,\ 2x^\varphi)$ (see (2)).

Hence $f(u+l_i)=N_i-\varepsilon_i$, $N_i\in\mathbf{Z}$, $\varepsilon_i\in(0,\ x^{\theta-2\varphi})$, $i=0,\ 1,\ 2$. From (9) follows that $\{A\}\leqq 2.l_2.x^{\theta-2\varphi}$. But $\{A\}=A\sim l_1.l_2.(l_2-l_1).x^{1-4\varphi}$ ((9), (10)) and then

$$l_1.l_2.(l_2-l_1).x^{1-4\varphi}\ll l_2.x^{\theta-2\varphi},\quad 1\leqq l_1.(l_2-l_1)\ll x^{\theta+2\varphi-1}.$$

But $\theta+2\varphi\leqq 0.23+2.0.38=0.99$ and for sufficiently large x our assumption leads to contradiction.

We proved that if x is sufficiently large $A\geqq 1$.

$$l_1.l_2.(l_2-l_1).x^{1-4\varphi}\gg 1$$

$$(11)\qquad a.b^2\gg x^{4\varphi-1}.$$

The next step is to take four elements of the set $S(x^\varphi,\ 2x^\varphi)-u,\ u+l_1,\ u+l_2,\ u+l_3$ $(0<l_1<l_2<l_3)$.

Let $a=\min(l_1,\ l_2-l_1,\ l_3-l_2)$;
$c=\max(l_1,\ l_2-l_1,\ l_3-l_2)$;
$b=l_3-a-c$.

We consider the case when $c=l_1$ or $c=l_3-l_2$. (In other words, the case when the maximal distance is not in the middle.)

Consider the divided difference

$$-f[u,\ u+l_1,\ u+l_2,\ u+l_3]=A/P=-\frac{f^{(3)}(\xi)}{3!}\sim x^{1-5\varphi},\quad \xi\in[u,\ u+l_3],$$

where

$$A=-f(u)(l_2-l_1)(l_3-l_2)(l_3-l_1)+f(u+l_1)(l_3-l_2).l_2.l_3$$
$$-f(u+l_2)(l_3-l_1).l_1.l_3+f(u+l_3)(l_2-l_1).l_1.l_2,$$

$$(12) \qquad P=l_1 . l_2 . l_3 . (l_2-l_1) . (l_3-l_2) . (l_3-l_1)>0,$$

$$A=P . -\frac{f^{(3)}(\xi)}{3!}>0.$$

Let assume that $A<1$. Then $A=\{A\}$ $(0<A<1)$ and $\{f(u+l_i)\}\in(1-x^{\theta-2\varphi}, 1)$ $i=0, 1, 2$, $(l_0=0)$ that is to say $f(u+l_i)=N_i-\varepsilon_i$, $N_i\in\mathbf{Z}$, $\varepsilon_i\in(0, x^{\theta-2\varphi})$, $i=0, 1, 2$.

From (12) we derive $\{A\}\leqq 12 . c^3 . x^{\theta-2\varphi}$.

But $\{A\}=A\sim P . x^{1-5\varphi}$ and $P\sim a . b^2 . c^3$. Hence $a . b^2 . c^3 . x^{1-5\varphi}\sim A=\{A\} \ll c^3 . . x^{\theta-2\varphi}$, $a . b^2\ll x^{\theta+3\varphi-1}$.

Since a and b are the distances between three different elements of $S(x^\varphi, 2x^\varphi)$ $(c=l_1$ or $c=l_3-l_2)$ from (11) there follows $a . b^2\gg x^{4\varphi-1}$.

We obtain $x^{4\varphi-1}\ll x^{\theta+3\varphi-1}$, $x^\varphi\ll x^\theta$. This is a contradiction if x is sufficiently large.

We proved that for sufficiently large x $A\geqq 1$. This implies $P . x^{1-5\varphi}\gg 1$, $a . b^2 . c^3\gg x^{5\varphi-1}$ and in particular

$$(13) \qquad c\gg x^{\frac{5\varphi-1}{6}} \quad (c\geqq a,\ c\geqq b).$$

Let $x^\varphi\leqq u_1<u_2<\ldots<u_{5m+s}\leqq 2x^\varphi$, $(s=0, 1, 2$ or $4)$ are all the elements of the set $S(x^\varphi, 2x^\varphi)$.

We divide them into groups containing five elements each one.

$$u_1,\ u_2,\ u_3,\ u_4,\ u_5;$$
$$u_6,\ u_7,\ u_8,\ u_9,\ u_{10};$$
$$\ldots$$
$$u_{5m-4}, \ldots, u_{5m}$$

and the last group $u_{5m+1}, \ldots, u_{5m+s}$.

Consider u_{5k+1}, u_{2k+2}, u_{5k+3}, u_{5k+4}, u_{5k+5} $(0\leqq k\leqq m-1)$.

Let $c=\max_{i=1,2,3,4}(u_{5k+i+1}-u_{5k+i})$. If $c=u_{5k+2}-u_{5k+1}$ we use (13) for the elements

$$u_{5k+1},\ u_{5k+2},\ u_{5k+3},\ u_{5k+4}$$

if $c=u_{5k+3}-u_{5k+2}$ for u_{5k+2}, u_{5k+3}, u_{5k+4}, u_{5k+5}, if $c=u_{5k+4}-u_{5k+3}$ for u_{5k+1}, u_{5k+2}, u_{5k+3}, u_{5k+4}, and if $c=u_{5k+5}-u_{5k+4}$ for u_{5k+2}, u_{5k+3}, u_{5k+4}, u_{5k+5}.

In all cases we obtain $c\gg x^{\frac{5\varphi-1}{6}}$ and $u_{5k+5}-u_{5k+1}\geqq c\gg . x^{\frac{5\varphi-1}{6}}$.

But $x^\varphi\geqq u_{5m}-u_1\geqq\sum_{i=0}^{m-1}(u_{5i+5}-u_{5i+1})\gg m . x^{\frac{5\varphi-1}{6}}$ and $m\ll x^{\frac{1+\varphi}{6}}$. $|S(x^\varphi, 2x^\varphi)| =5m+s\ll 5 . x^{\frac{1+\varphi}{6}}+4\ll x^{\frac{1+\varphi}{6}}$. □

Proof of Theorem 1. We choose $\theta=2/9$. From Lemma 1 and Lemma 3 there follows that

$$|S(x^{2/9}\sqrt{\log x},\ x^{1/3})| \ll x^{\frac{1+1/3}{6}} = x^{2/9},\ (1/3<0.38).$$

From Lemma 1 and Lemma 2 we obtain

$$|S(x^{1/3},\ x^{1/2})| \ll x^{\frac{1-1/3}{3}} = x^{2/9}.$$

$$S_2 = |S(x^{2/9}\sqrt{\log x},\ x^{1/2})| \ll x^{2/9}. \quad \blacksquare$$

Lemma 4. *Let* $0.32 \leqq \varphi \leqq 0.34$ *and* $0<\theta<0.23$. *Then* $|S(x^{\varphi},\ 2x^{\varphi})| \ll x^{\frac{11-6\varphi}{41}}$.
First we need to prove Lemma 5.

Lemma 5. *Let f is a real function defined in* $[a, b]$, $0<\delta<1$ *and* $N(\delta, f)$ *is the number of the integers in the interval* $(a, b]$ $(b-a>1)$ *such that* $\{f(n)\}\in(1-\delta, 1)$.
Then

$$|N(\delta, f)-\delta.(b-a)| \ll \sum_{k=1}^{N} \frac{|S_k|}{k} + \frac{b-a}{N} + O(1),$$

where

$$S_k = \sum_{a<x\leqq b} e^{2\pi i k f(x)}.$$

If $\delta=O\left(\frac{1}{N}\right)$ *then* $|N(\delta, f)-\delta.(b-a)| \ll \sum_{k=1}^{N} \frac{|S_k|}{N} + \frac{b-a}{N} + O(1)$. (A result very near to Lemma 5 is proved in the works of Stečkin but Lemma 5 is more suitable for our purposes.)

Proof of Lemma 5. Define

$$t(x)=\begin{cases} 0 & \text{if } 0\leqq\{x\}\leqq 1-\delta \\ 1 & \text{if } 1-\delta<\{x\}<1. \end{cases}$$

Let $T_N(x)$ and $t_N(x)$ are the trigonometrical polynomials of degree N of best onesided approximation for the function $t(x)$ in L_1 norm.

$$t_N(x) \leqq t(x) \leqq T_N(x).$$

Let

$$t_N(x) = a_0 + \sum_{k=-N}^{N} a_k . e^{2\pi i k x},$$

$$T_N(x) = A_0 + \sum_{k=-N}^{N} A_k . e^{2\pi i k x}.$$

From the well-known theorem about the onesided approximation (see T. Ganelius [8])

$$\|t-t_N\|_{L_1[0,1]}\leqq c.\frac{\text{Var}\ t}{N}\ll\frac{1}{N},$$

$$\|t-T_N\|_{L_1[0,1]}\leqq c.\frac{\text{Var}\ t}{N}\ll\frac{1}{N}$$

$$N(\delta,f)=\sum_{a<x<b} t(f(x))$$

are valid.

Hence

$$(b-a-1)a_0+\sum_{k=-N}^{N} a_kS_k\leqq N(\delta,f)\leqq(b-a+1)A_0+\sum_{k=-N}^{N} A_kS_k.$$

But

$$A_0=\int_0^1 T_N(x)dx=\int_0^1(T_N(x)-t(x))dx+\int_0^1 t(x)dx$$

$$=\int_0^1(T_N(x)-t(x))dx+\delta$$

$$|A_0-\delta|\leqq\|T_N-t\|_{L_1}\ll\frac{1}{N}$$

$$A_k=\int_0^1 T_N(x)e^{-2\pi ikx}dx=\int_0^1(T_N(x)-t(x))e^{-2\pi ikx}dx$$

$$+\int_0^1 t(x)e^{-2\pi ikx}dx.$$

But

$$\left|\int_0^1(T_N(x)-t(x))e^{-2\pi ikx}dx\right|\leqq\|T_N-t\|_{L_1}\ll\frac{1}{N}$$

$$\int_0^1 t(x)e^{-2\pi ikx}dx=\int_{1-\delta}^{1} e^{-2\pi ikx}dx=\frac{1-e^{-2\pi ik(1-\delta)}}{-2\pi ikx}\ll\frac{1}{k}.$$

It is evident that $\int_{1-\delta}^{1} e^{-2\pi ikx}dx\ll\delta\ll\frac{1}{N}$ when $\delta=0\left(\frac{1}{N}\right)$ and $|S_k|=|S_{-k}|$.

Hence

$$N(\delta,f)-(b-a)\delta\leqq c.\left(\frac{b-a}{N}+\sum_{k=1}^{N}\frac{|S_k|}{k}+1\right).$$

In the same way one can prove that

$$N(\delta, f)-(b-a)\delta \geqq c_1 \cdot \left(\frac{b-a}{N}+\sum_{k=1}^{N} \frac{|S_k|}{k}+1\right). \quad \square$$

Proof of Lemma 4. We use the fact that if $u \in S(x^{\varphi}, 2x^{\varphi})$ then $\left\{\frac{x}{u^2}\right\} \in (1-x^{\theta-2\varphi}, 1)$ (see (2)). Now we use Lemma 5 with $f(u)=\frac{x}{u^2}$, $\delta=x^{\theta-2\varphi}$, $a=x^{\varphi}$, $b=2x^{\varphi}$. We obtain

$$|S(x^{\varphi}, 2x^{\varphi})| \ll x^{\theta-2\varphi}x^{\varphi}+\sum_{k=1}^{N}\frac{|S_k|}{N}+\frac{x^{\varphi}}{N}$$

(we will choose N later, but smaller than $x^{0.13}$) and

$$S_k=\sum_{u=x^{\varphi}}^{2x^{\varphi}} e^{2\pi i k x/u^2}.$$

First we consider the case $1 \leqq k < x^{3\varphi-1}$.

Lemma 6 (I. M. Vinogradov [9]). *Let $f \in C^2(a, b)$ and $f''(x) \sim 1/A$ in (a, b). Then*

$$\left|\sum_{a<x<b} e^{2\pi i f(x)}\right| \ll \frac{b-a}{\sqrt{A}}+\sqrt{A}.$$

In our case $f(u)=kx/u^2$, $f''(x) \sim kx/u^4 \sim kx^{1-4\varphi}$, $A=x^{4\varphi-1}/k > x^{\varphi}$ $(x^{3\varphi-1}>k)$. Hence

$$|S_k| \ll \frac{x^{\varphi}}{x^{\varphi/2}}+\frac{x^{2\varphi-1/2}}{\sqrt{k}}$$

$$\sum_{1 \leqq k \leqq x^{3\varphi-1}} \frac{|S_k|}{N} \ll x^{\varphi/2}\log x+\frac{x^{2\varphi-1/2}}{\sqrt{N}} \ll x^{0.18}.$$

So we consider k greater than $x^{3\varphi-1}$.

Now we reduce the sums S_k to a "shorter" sums.

Theorem 3. *Let $f \in C^3(a, b)$, $b-a>0$, $f''(x) \sim 1/A$, $f^{(3)}(x) \ll 1/(A.U)$ for every $x \in [a, b]$.*

Let x_n are the solutions of the equations $f'(x_n)=n$, $(x_n \in [a, b])$. Finally, let the functions

$$\frac{1}{f'(x+x_n)-n}-\frac{1}{\sqrt{2(f(x+x_n)-f(x_n)-nx).f''(x_n)}}$$

have finite number of parts of monotonity.

Then

$$\sum_{a<x<b} e^{2\pi i f(x)}=\frac{1+i}{\sqrt{2}}\sum_{f'(a)\leqq n\leqq f'(b)}\frac{e^{2\pi i}(f(x_n)-nx_n)}{\sqrt{f'(x_n)}}+0(\ln U)+0(\sqrt{A})+0(f'(b)-f'(a)).$$

(for proof see A. A. Karatsuba [10] p. 18-25).

In our case $f(u)=\frac{kx}{u^2}$, $A=\frac{x^{4\varphi-1}}{k}$, $U=x^{\varphi}$. All the conditions of Theorem 3 are satisfied : $U\gg A$, $A\gg 1$

$$\left(\frac{x^{4\varphi-1}}{k}\gg x^{4\cdot 0\cdot 32-1-0\cdot 13}=x^{0\cdot 15}\gg 1\right).$$

Denote $t=-k.x^{1-3\varphi}$, $u_n=-\left(\frac{2kx}{n}\right)^{1/3}$, $c_1=3.2^{-2/3}$, $c_2=3^{1/2}.2^{-1/3}$. Then

$$S_k=\frac{1+i}{\sqrt{2}}\cdot\sum_{n=2t-1}^{(t/4)+1}\frac{e^{2\pi i c_1\cdot(kx)^{1/3}\cdot n^{2/3}}}{c_2.(kx)^{-1/6}.n^{2/3}}+O(k.x^{1-3\varphi})+O\left(\frac{x^{2\varphi-1/2}}{\sqrt{k}}\right)+O(\log x).$$

But

$$\sum_{k=1}^{N}\frac{k.x^{3\varphi-1}}{N}\ll N.x^{1-3\varphi}\ll x^{1-3\cdot 0\cdot 32+0\cdot 13}=x^{0\cdot 17}$$

and

$$\sum_{k=1}^{N}\frac{x^{2\varphi-1/2}}{\sqrt{k}.N}\ll\frac{x^{2\varphi-1/2}}{\sqrt{N}}\ll x^{2\cdot 0\cdot 34-0\cdot 50}\ll x^{0\cdot 18}.$$

Now we make the Abel transformation and obtain

$$S_k\ll k^{-1/2}.x^{2\varphi-1/2}.\max_m|S(m)|, \tag{14}$$

where

$$S(m)=\sum_{n=2t-1}^{m} e^{2\pi i c_1(kx)^{1/3}\cdot n^{2/3}}$$

and $2t-1\leqq m\leqq(t/4)+1$.

Lemma 6 (Weil). *Let* $k\in\mathbf{N}$, $k\geqq 2$, $K=2^{k-1}$, $f\in C^k(a,b)$ *and* $0<\lambda_k\leqq f^{(k)}(\xi)$ $\leqq h.\lambda_k$ *for every* $\xi\in(a,b)$.
Then

$$\Big|\sum_{a<n<b} e^{2\pi i f(n)}\Big| \ll (b-a).\lambda_k^{\frac{1}{2K-2}}.h^{\frac{2}{K}}+(b-a)^{1-2/K}.\lambda_k^{-\frac{1}{2K-2}}$$

(for proof see Karatsuba [10] p. 26-29).

We apply Lemma 6 for $f(n)=c_1.(kx)^{1/3}.n^{2/3}$, $a=2t-1$, $b=m$ and $k=5$. Then $K=16$, $h=O(1)$, $\lambda_5=\dfrac{x^{13\varphi-4}}{k^4}$.

We obtain:

$$S(m)\ll k^{26/30}.x^{(26-77\varphi)/30}+k^{121/120}.x^{(121-367\varphi)/120}. \tag{15}$$

From (14)

$$|S_k|\ll k^{11/30}.x^{(11-17\varphi)/30}+k^{61/120}.x^{(61-127\varphi)/120}$$

$$\sum_{k=1}^{N}\frac{|S_k|}{N}+\frac{x^{\varphi}}{N}\ll N^{11/30}.x^{(11-17\varphi)/30}+N^{61/120}.x^{(61-127\varphi)/120}+\frac{x^{\varphi}}{N}. \tag{16}$$

We choose N to be the integer part of $x^{(47\varphi-11)/41}$. (This choice is correct because $(47.0.34-11)/41=0.12\ldots<0.13$.)

For this choice of $N.x^{59\varphi}\gg.N^{17}.x^{17}$ (this is equivalent to $\varphi\geqq 17/54$ $=0.31\ldots$) and $N^{11/30}.x^{(11-17\varphi)/30}\gg N^{61/120}.x^{(61-127\varphi)/120}$.

Finally we obtain

$$|S(x^{\varphi},2x^{\varphi})|\ll x^{\frac{11-6\varphi}{41}}. \ \square \tag{17}$$

Proof of Theorem 2. We choose $\theta=17/77=0.2207792\ldots$

From Lemma 1 and Lemma 2 there follows

$$|S(x^{0\cdot 34},x^{1/2})|\ll x^{\frac{1-0\cdot 34}{3}}=x^{0\cdot 22}.$$

From Lemma 1 and Lemma 3 we obtain

$$|S(x^{\theta}\sqrt{\log x},x^{25/77})|\ll x^{\frac{1+25/77}{6}}=x^{17/77}.$$

From Lemma 1 and Lemma 4 we have

$$|S(x^{25/77},x^{0\cdot 34})|\ll x^{\frac{11-6\cdot 25/77}{41}}=x^{17/77}\ (25/77=0.32\ldots<0.34).\ \blacksquare$$

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