SHORTEST PATHS ALGORITHMS: THEORY AND EXPERIMENTAL EVALUATION

BORIS V. CHERKASSKY

CENTRAL INSTITUTE FOR ECONOMICS AND MATHEMATICS
KRASIKOVA ST. 32, 117418, MOSCOW, RUSSIA
CHER@CH.COMREL.MSK.SU

ANDREW V. GOLDBERG

COMPUTER SCIENCE DEPARTMENT, STANFORD UNIVERSITY
STANFORD, CA 94305, USA
GOLDBERG@CS.STANFORD.EDU

TOMASZ RADZIK

SORIE, CORNELL UNIVERSITY
ITHACA, NY 14853, USA
RADZIK@CS.CORNELL.EDU

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ABSTRACT. We conduct an extensive computational study of shortest paths algorithms, including some very recent algorithms. We also suggest new algorithms motivated by the experimental results and prove interesting theoretical results suggested by the experimental data. Our computational study is based on several natural problem classes which identify strengths and weaknesses of various algorithms. These problem classes and algorithm implementations form an environment for testing the performance of shortest paths algorithms. The interaction between the experimental evaluation of algorithm behavior and the theoretical analysis of algorithm performance plays an important role in our research.

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1. Introduction

The shortest paths problem is one of the most fundamental network optimization problems. This problem comes up in practice and arises as a subproblem in many network optimization algorithms. Algorithms for this problem have been studied for a long time. See *e.g.* [2, 5, 6, 7, 18, 19, 21]. However, advances in the theory of shortest paths algorithms are still being made. See *e.g.* [1, 9, 13]. A good description of the classical algorithms and their implementations appears in [10].

On a network with negative-length arcs, the best currently known time bound of $O(nm)$ is achieved by the Bellman-Ford-Moore algorithm [2, 7, 19]. (Here n and m denote the number of nodes and arcs in the network, respectively.) With the additional assumption that arc lengths are integers bounded below by $-N \le -2$, the $O(\sqrt{n}m \log N)$ bound [13] improves the Bellman-Ford-Moore bound unless N is very large. If the arc lengths are nonnegative, implementations of Dijkstra's algorithm achieve better bounds. An implementation of [8] runs in $O(m + n \log n)$ time. Under the assumption that arc lengths are integers in the interval $[0, \ldots, C]$, $C \ge 2$, the implementation of [1] runs in $O(m + n\sqrt{\log C})$ time.

As hardware becomes more powerful and more sophisticated algorithms need a shortest path subroutine, efficient shortest paths algorithms are of growing importance. This is the case for other network optimization problems as well, motivating broad computational investigation of available algorithms. In particular, a massive study of flow and matching algorithms was done for the First DIMACS Algorithm Implementation Challenge [15].

In this paper we study practical performance of several shortest paths algorithms, including established methods [2, 6, 7, 11, 18, 19, 20, 21], recently proposed algorithms [1, 14], and new algorithms. The development of the new algorithms was based on the experimental feedback. We give theoretical explanation of the observed behavior of the algorithms and prove complexity bounds on the new algorithms.

We also prove an interesting result suggested by the experimental data. This result, summarized in Theorem 11.1, shows that some algorithms, for example the Bellman-Ford-Moore algorithm, are *potential-invariant*, *i.e.,* behave in exactly the same way on two networks one of which is obtained from the other one by replacing the lengths by the reduced costs with respect of a potential function. This result has several interesting implications. Note, for instance, that any feasible shortest paths problem has an equivalent with nonnegative arc lengths. If the problem with nonnegative arc lengths is computationally simpler than the general problem, as is commonly believed, then the theorem suggests that a potential-invariant algorithm cannot be superior to all other algorithms on problems with nonnegative arc lengths.

An important part of our work is the development of several natural shortest paths problem generators and their use to create families of problems. Of special interest to us are the families that give insight into the relative algorithm performance, robustness, and dependence of the performance on the network structure and the arc cost distribution.

The collection of algorithms we test is larger than that of any previous study we are aware of, and the set of test problems is much richer. We show that the algorithm performance varies significantly more than previously believed and that some algorithms previously considered robust may fail dramatically. For example, we exhibit a family of problems that are hard for all established algorithms, although a recent algorithm of [14] solves these problems quickly (see Section 7).

Our work greatly improves the theoretical understanding of the shortest paths algorithm performance. In particular we identify several problem features that make problems hard or easy for the algorithms we study. The interaction between theoretical and experimental aspects of our work helps to produce more efficient codes and to identify important theoretical properties of the algorithms.

Although our research does not produce a single best code for the shortest paths problem, two codes we developed are very competitive in their domains, networks with nonnegative and mixed arc lengths, respectively. One of the codes is a new implementation of Dijkstra's algorithm using a double bucket data structure. Another code, which is a modification of a recent algorithm of Goldberg and Radzik [14], matches the $O(nm)$ bound of the Bellman-Ford-Moore algorithm and also achieves the optimal $O(m+n)$ time bound on acyclic networks.

Our codes, generators, and generator inputs form a testing environment for shortest paths algorithms. A new code can be compared against the existing ones to determine its relative performance. The environment can be augmented as interesting codes, problem generators, and problem families are developed. Our codes, generators, and generator inputs are available through a mail server.

The shortest paths environment can be used in several ways. Practitioners looking for an efficient code for an application can test our codes on their problems and select one that performs well. The number of codes which need to be compared can be narrowed down using the results of the current paper. Researchers evaluating a new shortest paths code can run the code on the problem families we suggest and compare its performance with the performance of our codes. The environment can also be used in teaching algorithms to demonstrate importance of proper algorithms and data structures.

This paper is organized as follows. Section 2 introduces definitions and notation. Section 3 reviews the labeling method for the shortest path problem. Section 4 describes the algorithms that we study and proves complexity bounds on the new algorithms. Section 5 describes our experimental setup and Sections 6 – 9 give the main experimental results. Section 10 gives additional experimental data for implementations of Dijkstra's algorithm. Section 11 discusses performance of individual algorithms and explains their experimental behavior. We make concluding remarks in Section 12.

```
procedure SCAN(v);
    for all (v, w) ∈ E do
        if d(v) + ℓ(v, w) < d(w) then
            d(w) ← d(v) + ℓ(v, w);
            S(w) ← labeled;
            π(w) ← v;
    S(v) ← scanned;
end.
```

FIGURE 1. *The* SCAN *operation.*

2. Definitions and Notation

The input to the single-source shortest paths problem is (G, s, ℓ), where $G = (V, E)$ is a directed graph, $\ell : E \to \mathbf{R}$ is a length function, and $s \in V$ is the source node. The goal is to find shortest paths from s to all other nodes of G or to find a negative length cycle in G. If G has a negative length cycle, we say that the problem is *infeasible*. We assume, without loss of generality, that all nodes are reachable from s in G. We denote $|V|$ by n, $|E|$ by m, and the biggest absolute value of an arc length by C.

A *potential function* is a real-valued function on nodes. Given a potential function d, we define a *reduced cost function* $\ell_d : E \to \mathbf{R}$ by

$$\ell_d(v, w) = \ell(v, w) + d(v) - d(w).$$

We say that an arc a is *admissible* if $\ell_d(a) \leq 0$, and denote the set of admissible arcs by E_d. The *admissible graph* is defined by $G_d = (V, E_d)$. Note that if $d(v) < \infty$ and $d(w) = \infty$, the arc (v, w) is admissible. If $d(v) = d(w) = \infty$, we define $\ell_d(v, w) = \ell(v, w)$.

A *shortest paths tree* of G is a spanning tree rooted at s such that for any $v \in V$, the reversal of the v to s path in the tree is a shortest path from s to v.

3. The Labeling Method

In this section we briefly outline the general *labeling* method for solving the shortest paths problem. (See *e.g.* [4, 10, 23] for more detail.) Most shortest paths algorithms are based on this method.

For every node v, the method maintains its potential $d(v)$, parent $\pi(v)$, and status $S(v) \in \{$*unreached*, *labeled*, *scanned*$\}$. The potential of a node v is also called the *distance label* of v. Initially for every node v, $d(v) = \infty$, $\pi(v) =$ nil, and $S(v) =$ unreached. The method starts by setting $d(s) = 0$ and $S(s) =$ labeled, and applies the SCAN operation to labeled nodes until none exists, in which case the method terminates.

The SCAN operation applies to a labeled node v. The operation is described in Figure 1. Note that if v is labeled, then $d(v) < \infty$ and $d(v) + \ell(v, w)$ is finite. After a SCAN operation, some unreached and scanned nodes may become labeled.

nodes/arcs	BF	STACK
16385	**0.39**	**44.29**
49152	11.92	1986.90
65537	**3.53**	**544.34**
196608	23.19	4613.02

FIGURE 2. *Relative performance of FIFO and LIFO selection rules. A table entry gives the running time in seconds (bold) and the number of scans per node.*

The method terminates if and only if G does not have negative length cycles. If the method terminates, the parent pointers define a correct shortest paths tree and, for any $v \in V$, $d(v)$ is the shortest path distance from s to v. The labeling method can be easily modified so that if G has negative cycles, the method finds such a cycle and terminates.

4. Labeling Algorithms

Different strategies for selecting labeled nodes to be scanned next lead to different algorithms. In this section we discuss some of these strategies.

The importance of a good ordering of the SCAN operations is illustrated in Figure 2. Here we compare the FIFO ordering used in the BF code and the LIFO ordering used in the STACK code on two Grid-SSquare problems (see Section 6) of modest size. As is usually the case, STACK performance is extremely poor compared to BF, although the codes differ by only two statements. As we shall see later, BF performs much worse than some other codes on this problem. Note that STACK has an exponential worst-case running time (see *e.g.* [22]).

4.1. Bellman-Ford-Moore Algorithm. The Bellman-Ford-Moore algorithm, due to Bellman [2], Ford [7], and Moore [19], maintains the set of labeled nodes in a FIFO queue. The next node to be scanned is removed from the head of the queue; a node that becomes labeled is added to the tail of the queue. Our code BF implements this algorithm.

We define a *pass* over the queue inductively. Initialization, during which the source s is added to the queue, is pass 0. For $i > 0$, pass i consists of processing nodes which were added to the queue during pass $i - 1$.

Performance of the Bellman-Ford-Moore algorithm is as follows.

Theorem 4.1. *(i) Each pass takes $O(m)$ time. (ii) The number of passes is bounded by the depth of the shortest paths tree. (iii) The algorithm runs in $O(nm)$ time in the worst case.*

Although the $O(nm)$ worst case bound is the best bound known for shortest paths algorithms, in practice the Bellman-Ford-Moore algorithm is often slower than other methods. We introduce the following *parent-checking* heuristic that usually improves performance of the algorithm. Suppose we have just removed a node v from the queue and the parent of v, $u = \pi(v)$,

nodes/arcs	BF	BFP
16385	**0.39**	**0.21**
49152	11.92	5.05
65537	**3.53**	**1.96**
196608	23.19	9.66

FIGURE 3. *Relative performance of* BF *and* BFP *on Grid-SSquare problems.*

$[L, U]$	BF	BFP
$[1, 1]$	**1.46**	**1.50**
	1.00	1.00
$[0, 10000]$	**25.28**	**23.15**
	19.04	16.75

FIGURE 4. *Relative performance of* BF *and* BFP *on Rand-Len problems with 131072 nodes and 524288 arcs.*

is in the queue. Note that $d(v)$ was last updated when u was scanned and $d(v)$ was set to $d(u) + \ell(u, v)$, and after that $d(u)$ decreased (causing u to be again added to the queue). Thus $\ell_d(u, v) < 0$. Intuitively, it is wasteful to scan v at this point because we know that $d(v)$ will decrease. The BFP algorithm is a variant of BF that scans a node only if its parent is not in the queue. One can easily prove the bounds of Theorem 4.1 for this algorithm.

The parent-checking idea can be extended. For example, one can check all predecessors of a node in the current tree. This is computationally expensive, however. An alternative is to periodically clean up the queue, removing from it all nodes with labeled predecessors in the tree. This approach can be used to obtain an algorithm that is usually better than BFP. In Section 4.5 we describe an even better algorithm motivated by this idea.

Figure 3 shows the performance of BF and BFP on a problem from Grid-SSquare family (see Section 6). On this problem, BF does about twice as many scans as BFP and runs about twice as slowly. Figure 4 shows the performance on problems from Rand-Len family (see Section 8). In the first problem all arcs have unit length and both BF and BFP do one scan per node. Because of the additional work of checking the parents, BFP is slightly slower. In the second problem the range of arc lengths is relatively large. On this problem BFP does slightly fewer scans per node and runs slightly faster.

In practice, BFP seems never to make more scans than BF and is never significantly slower. In the vast majority of cases, BFP is faster than BF and the two codes differ by only one "if" statement. We use the BFP code in our experiments below.

4.2. Dijkstra's Algorithm. Dijkstra's algorithm [6] selects a labeled node with the minimum potential as the next node to be scanned.

nodes/arcs	DIKH	DIKQ
16385	**0.20**	**5.15**
49152	1.00	1.00
32769	**0.50**	**22.11**
98304	1.00	1.00
65537	**1.29**	**122.22**
196608	1.00	1.00

FIGURE 5. *Relative performance of* DIKH *and* DIKQ *on Grid-SWide problem.*

Theorem 4.2. *If the length function is nonnegative, Dijkstra's algorithm scans each node exactly once.*

Remark. It is easy to show that if negative arc lengths are allowed, the number of scans may be exponential.

We first assume that arc lengths are nonnegative, and treat the other case at the end of this section. Also, when discussing below about R-heap and bucket-based implementations of Dijkstra's algorithm, we assume that the length function is integral.

The worst-case complexity of Dijkstra's algorithm on networks with nonnegative arc lengths depends on the way of finding the labeled node with the smallest distance label. A naive implementation that examines all labeled nodes to find the minimum runs in $O(n^2)$ time [6]. The implementation using k-ary heaps (see *e.g.* [4]) runs in $O(m \log n)$ time (for a constant k). The implementation using Fibonacci heaps [8] runs in $O(m+n \log n)$ time. The implementation using one-level R-heaps [1] runs in $O(m + (n \log C))$ time and the one using two-level R-heaps together with Fibonacci heaps, in $O(m + n\sqrt{\log C})$ time. We evaluated implementations that use k-ary heaps with k set to 3 (DIKH), Fibonacci heaps (DIKF, and one-level R-heaps (DIKR). (Note that the R-heap data structure is based on buckets and thus similar to bucket-based implementations discussed below.)

We also implemented the naive $O(n^2)$ algorithm (DIKQ). This implementation, however, performs poorly unless the average number of labeled nodes during the computation is small. For example, on two problems from Grid-SWide family (see Section 6), DIKQ is orders of magnitude slower than DIKH, which itself is relatively slow on this problem. (See Figure 5.) Because of the poor performance, we do not include DIKQ in our tests.

Another way to implement Dijkstra's algorithm is by using the bucket data structure, as proposed by Dial [5]. This implementation maintains an array of buckets, with the i-th bucket containing all nodes v with $d(v) = i$. When distance label of a node changes, the node is removed from a bucket corresponding to its old distance label (if the label was finite) and inserted into the bucket corresponding to the new one. The implementation maintains an index L. Initially, $L = 0$, and L has the property that all buckets $i < L$ are empty. The next node to be scanned is removed from bucket L or, if this bucket is empty, L is incremented. The following theorem follows easily from the observation that bucket deletions and insertions take linear time and at most nC buckets need to be examined by the algorithm.

Theorem 4.3. [5] *If the length function is nonnegative, Dial's algorithm runs in $O(m + nC)$ time.*

Although the algorithm, as stated, needs nC buckets, an observation that only $C + 1$ consecutive buckets can be occupied at any given time allows the use of $C + 1$ buckets. Our code DIKB implements Dial's algorithm. We maintain nodes in a bucket in the FIFO order. Our implementation places a limit of 300000 on the maximum arc length (which determines the number of buckets).

Next we introduce two simple ways to reduce the memory requirement of Dial's algorithm. In the *overflow bag* implementation, the number of buckets is set to $B < C+1$. At the i-th stage of the algorithm, the buckets contain nodes with distance labels in the range $[B_i, B_i + B - 1]$. The labeled nodes with distance label $B_i + B$ and above are maintained in a special set (the bag). Initially $i = 0$ and $B_i = 0$. When the value of L reaches $B_i + B$, the value of i is incremented and B_i is set to the minimum distance label of a node in the bag. Then the bag is scanned, nodes with distance labels in the range $[B_i, B_i + B - 1]$ are moved into appropriate buckets, and the next stage begins. The time-memory tradeoff of this implementation is as follows.

Theorem 4.4. *If the length function is nonnegative, the overflow bag implementation of Dijkstra's algorithm runs in $O(m + n((C/B) + B))$ time.*

Proof. Under this implementation, each node is scanned at most once, for the total of $O(m+n)$ time. The time for removing and inserting nodes from the buckets and the bag can be charged to the scanning of nodes. There are at most n passes through the buckets for a total of $O(nB)$ work. It remains to account for the work of examining nodes in the bag at the end of each stage of the algorithm. Note that if a node is added to the bag for the first time during stage i, then its distance label is at most $B_i + B + C$, so the node can be in the bag for $O(C/B)$ stages. Thus the work involved in examining the bag is $O(nC/B)$. ■

Choosing $B = \sqrt{C}$ yields an $O(m + n\sqrt{C})$ time bound. Our code DIKBM implements this algorithm. We set $B = \min(50000, C/3)$.

In the *approximate bucket* implementation, a bucket i contains nodes with distance labels in the range $[i\Delta, (i+1)\Delta - 1]$, where Δ is a fixed parameter. Nodes in the bucket are processed in the FIFO order. This implementation needs $\lceil C/\Delta \rceil + 1$ buckets. The time-memory tradeoff for this implementation is as follows.

Theorem 4.5. *If the length function is nonnegative, the approximate bucket implementation runs in $O(m\Delta + n(\Delta + C/\Delta))$ time.*

Proof. Each node can be scanned more than once since the buckets are approximate. However, a node cannot be scanned more than Δ times. Thus the total work involved in scanning nodes is $O(\Delta(m+n))$. The only work that cannot be charged to the scans is that of going through the buckets in search of a nonempty one. This work adds up to $O(n(C/\Delta))$. ■

Our code DIKBA implements this algorithm. We set $\Delta = \lceil C/2^{11} \rceil$.

The ideas of the above two algorithms can be combined to obtain the *double bucket* implementation of Dijkstra's algorithm. This implementation has two kinds of buckets, *high-level* and *low-level*. The number of low-level buckets is Δ. A high-level bucket i contains the set of nodes with distance labels in the range $[i\Delta, (i+1)\Delta - 1]$ except for the nonempty high-level bucket with the smallest index L. Nodes v with distance label in the range $[L\Delta, (L+1)\Delta - 1]$ are in the low-level bucket $d(v) - L\Delta$. After all low-level buckets are examined and the nodes in these buckets are scanned, L increases. If the corresponding high-level bucket is not empty, its nodes are moved to the corresponding low-level buckets and the next stage begins.

The number of high-level buckets needed by this implementation is $\lceil (C+1)/\Delta \rceil$. The running time of the implementation is as follows.

Theorem 4.6. *If the length function is nonnegative, the double bucket implementation runs in $O(m + n(\Delta + C/\Delta))$ time.*

Proof. Each node is scanned at most once. The number of high-level buckets that the algorithm processes is $O(nC/\Delta)$. The number of times a low-level bucket is examined is at most n. ■

For the best theoretical bound, the value of Δ should be $\Theta(\sqrt{C})$. Our code DIKBD implements this algorithm. We set Δ to (essentially) the biggest power of two that is less than $\sqrt{C}$.

The double bucket implementation can be generalized to the *k-level bucket* implementation in the following way. We consider only the case when the number of buckets at every level is the same and equal to $p = \lceil C^{1/k} \rceil$. The levels are numbered from 0 to $k-1$ and the buckets at each level are numbered from 0 to $p-1$. Consider level i. Associated with this level are *the base distance* B_i and the index of *the active bucket* a_i, Associated with bucket j, $0 \le j \le p-1$, at level i is the interval $[B_i + jp^i, B_i + (j+1)p^i - 1]$. If the distance label of a node v is in the interval associated with bucket j for some $a_i < j < p$, then v is in this bucket. If the distance label is in the interval associated with the active bucket and $i > 0$, then the node is at a lower level. If the distance label is greater than $B_i + p^{i+1} - 1$, then the node is at a higher level. For each level we maintain the total number of nodes at this level. Next we describe how to move a node into the appropriate bucket when its distance decreases and how to find a node with the smallest distance.

If the distance of a node decreases, we first try to relocate this node within the same level. If it drops into the active bucket, then we find the appropriate bucket at the lower level. We repeat this until we reach the lowest level or the first level such that the node does not drop into the active bucket. If bucket a_0 at the lowest level (level 0) is not empty, it contains all nodes with the smallest distance label. If this bucket is empty, we find the lowest nonempty level, then we find the first nonempty bucket at this level, make it the active bucket, and distribute the nodes from this bucket to lower levels.

The k-level bucket implementation requires $O(kC^{1/k})$ buckets and has the following time-memory tradeoff.

Theorem 4.7. *If the length function is nonnegative, the k-level bucket implementation runs in $O(m + n(k + C^{1/k}))$ time.*

Proof. Consider a node v whose distance label is decreased. If the level of the node does not change, the node can be moved into the new bucket in $O(1)$ time. If the node moves to a lower level, the appropriate level and bucket can be found in $O(i' - i'')$ time, where i' and i'' are the old level and the new level, respectively. Since there are $O(m)$ decreases of distance labels and each node may move only from a higher to a lower level, the total time spent on these operations is $O(m + nk)$.

To find a node with the smallest distance, we first find the lowest nonempty level in $O(k)$ time, using the information about the number of nodes at each level. The first nonempty bucket is found in $O(p)$ time. Then we have to distribute the nodes from this bucket to lower levels. The total time of this computation, throughout the whole algorithm, is $O(nk)$, since each time a node is inspected, it is moved to the lower level. We have to find the slammest distance node at most n times, so the total work involved is $O(n(k + p))$.

Thus the running time of the k-level implementation is $O(m + n(k + C^{1/k}))$. ∎

Setting $k = \lceil 2 \log C / \log\log C \rceil$ yields an $O(m + n \log C / \log\log C)$ time bound.

We conclude this section with a discussion of implementations of Dijkstra's algorithm when arc lengths can be negative. A "strict" implementation of the algorithm selects a labeled node with the smallest distance label at every step. This is what our code DIKH does.

An alternative is to maintain the value t of the biggest distance label of a node scanned so far, and to select a labeled node with the distance label of t or less if such a node exists and a labeled node with the smallest distance label otherwise. This strategy is more natural for bucket and R-heap implementations and we use it in the corresponding codes. If the nodes eligible for scanning are maintained in FIFO manner, one can show polynomial-time bounds for this variant of Dijkstra's algorithm on networks with arbitrary arc lengths.

4.3. Incremental Graph Algorithms. In this section we describe two algorithms. The first one was developed independently by Pape [21] and Levit [18]. The second algorithm was proposed by Pallottino [20]. He also introduced the incremental graph framework that unified these two algorithms. Our implementations of the above algorithms are called PAPE, and TWO_Q, respectively.

An algorithm in the *restricted scan* framework maintains a set W of nodes and scans only labeled nodes in W. The set W is monotone: once a node is added to W, the node remains in W. If there are labeled nodes but no labeled node is in W, some of the labeled nodes must be added to W. Nodes may also be added to W even if W already contains labeled nodes. Note that if the labeled nodes in W are processed in the FIFO order, then a simple modification of the analysis of the Bellman-Ford-Moore algorithm shows that in $O(nm)$ time, either the algorithm terminates or W grows. This leads to an $O(n^2m)$ time bound.

In Pape-Levit and Pallottino's algorithms define W as the set of nodes which have been scanned at least once; when no labeled node is in W, a labeled node is added to W. More precisely, these algorithms maintain the set of labeled nodes as two subsets, S_1 and S_2, the first containing labeled nodes which have been scanned at least once and the second containing those which have never been scanned ($S_1 \subseteq W$ and $S_2 \subseteq V - W$). The next node to be scanned is selected from S_1 unless S_1 is empty, in which case the node is selected from S_2 (*i.e.*, this node is added to W). We call S_1 the *high-priority set* and S_2 the *low-priority set*.

The Pape-Levit algorithm maintains S_1 as a LIFO stack and S_2 as a FIFO queue. (This algorithm is usually implemented using the *dequeue* data structure, which is a queue that allows insertions at either end. See *e.g.* [10, 20].) Initially the stack is empty and the queue contains s. The next node to be scanned is removed from the top of the stack if it is not empty and from the head of the queue otherwise. A node that becomes labeled is pushed to the top of the stack if the node has been scanned previously and added to the tail of the queue otherwise. The algorithm terminates when both the stack and the queue are empty. This algorithm has exponential worst-case time bound.

Theorem 4.8. [16, 22] *The Pape-Levit algorithm runs in $\Theta(n2^n)$ time in the worst case.*

Pallottino's algorithm maintains S_1 and S_2 using FIFO queues, Q_1 and Q_2. The next node to be scanned is removed from the head of Q_1 if the queue is not empty and from the head of Q_2 otherwise. A node that becomes labeled is added to the tail of Q_1 if it has been scanned previously and to the tail of Q_2 otherwise. The algorithm terminates when both queues are empty. As the above discussion of the restricted scan algorithms suggests, the worst-case running time of TWO_Q is polynomial.

Theorem 4.9. [20] *The two-queue algorithm runs in $O(n^2m)$ time in the worst case.*

Observe that in a restricted scan algorithm when there are no labeled nodes in W, then the current tree restricted to W is a shortest paths tree in the subgraph of the input graph induced by W and the reduced cost function is nonnegative on arcs connecting nodes in W. Both Pape-Levit and Pallottino's algorithms increase W only when there are no labeled nodes in W, in which case exactly one labeled node v is added to W. By the next time when there are no labeled nodes in W (*i.e.*, by the next time S_1 is empty) a shortest paths tree in the subgraph induced by $W \cup \{v\}$ is computed. Hence the "incremental graph algorithms" term.

4.4. The Threshold Algorithm. Glover et. al. [11] suggest the following method, which combines the ideas lying behind the Bellman-Ford-Moore, Dijkstra's, and incremental graph algorithms. (See also [12, 10].) The method partitions the set of labeled nodes into two subsets, NOW and NEXT, which are maintained as FIFO queues. At the beginning of each iteration of the algorithm, NOW is empty. The method also maintains a threshold parameter t which is set to a weighted average of the minimum and average distance labels of the nodes in NEXT. During an iteration, the algorithm transfers nodes v with $d(v) \leq t$ from NEXT to NOW and scans nodes in NOW. Nodes that become labeled during the iteration are added to NEXT. The algorithm terminates when NEXT is empty at the end of an iteration. Our code THRESH implements the threshold algorithm suggested in [11] with parameter values MINWT = 45 and WTCNG = 25.

The running time of THRESH is as follows.

Theorem 4.10. [12] *If the length function is nonnegative,* THRESH *runs in* $O(nm)$ *time.*

Note that the threshold parameter t is not necessarily monotone in our implementation. If t is updated only when at the beginning of an iteration the distance label of every node in NEXT is greater that t, then t becomes monotone. This version of the algorithm falls into the restricted scan framework discussed in the previous section and runs in $O(n^2m)$ time on networks with arbitrary arc lengths [10]. However, the version of THRESH that we implemented is that of [11], and we are not aware of any polynomial-time bound for this version in the arbitrary length case.

4.5. The Topological Ordering Algorithms. A generalization of the parent-checking idea discussed in Section 4.1 is as follows. Suppose both v and w are labeled and there is a path from v to w in the admissible graph containing a negative reduced cost arc. Then it is better to scan v before w, since we know that $d(w)$ is greater than the true distance from s to w. A recent algorithm of Goldberg and Radzik [14] is based on this idea. To simplify the algorithm description, we first assume that G has no cycles of length zero or less, and therefore for any d, the admissible graph G_d is acyclic.

The Goldberg-Radzik algorithm maintains the set of labeled nodes in two sets, A and B. Each labeled node is in exactly one set. Initially $A = \emptyset$ and $B = \{s\}$. At the beginning of each *pass*, the algorithm uses the set B to compute the set A of nodes to be scanned during the pass, and resets B to the empty set. A is a linearly ordered set. During the pass, elements are removed according to the ordering of A and scanned. The newly created labeled nodes are added to B. A pass ends when A becomes empty. The algorithm terminates when B is empty at the end of a pass.

The algorithm computes A from B as follows.

(1) For every $v \in B$ that has no outgoing arc with negative reduced cost, delete v from B and mark it as scanned.
(2) Let A be the set of nodes reachable from B in G_d. Mark all nodes in A as labeled.
(3) Apply topological sort to order A so that for every pair of nodes v and w in A such that $(v, w) \in G_d$, v precedes w and therefore v will be scanned before w.

The algorithm achieves the same bound as the Bellman-Ford-Moore algorithm.

Theorem 4.11. [14] *The Goldberg-Radzik algorithm runs in $O(nm)$ time.*

Now suppose G has cycles of zero or negative length. In this case G_d need not be acyclic. If, however, G_d has a negative length cycle, we can terminate the computation. If G_d has zero length cycles, we can contract such cycles and continue the computation. This can be easily done while maintaining the $O(nm)$ time bound. (See *e.g.* [13].)

Our code GOR is an implementation of the Goldberg-Radzik algorithm with one simplification. The implementation uses depth-first search to compute topological ordering of the admissible graph (see *e.g.* [4]). Instead of contracting zero length cycles, we simply ignore the back arcs discovered during the depth-first search. The resulting topological order is in the admissible graph minus the ignored arcs. This change does not affect the algorithm correctness or the above running time bound.

We also implement the following modification, GOR1, of GOR. Recall that we use depth-first search to compute the topological ordering. When an arc (v, w) is examined by the depth-first search, the arc is first scanned in the shortest-path sense, *i.e.*, if $d(v) + \ell(v, w) < d(w)$, $d(w)$ is set to $d(v) + \ell(v, w)$ and $\pi(w)$ is set to v. (Note that this changes the admissible graph.) The following theorem gives a theoretical justification for this change.

Theorem 4.12. GOR1 *runs in $O(nm)$ time. On an acyclic network,* GOR1 *terminates in one pass and therefore runs in $O(m + n)$ time.*

Proof. The proof of the first claim is similar to that of Theorem 4.11. To prove the second claim, we show that the first depth-first search topologically orders the nodes reachable from the source with respect to the *input* graph. Recall that at the beginning of the computation

optimization level	TEST 1 average running time			TEST 2 average running time		
	real	user	system	real	user	system
w/o optm.	1.2	1.2	0.0	11.1	10.8	0.1
-O1	1.05	1.0	0.0	9.3	9.15	0.1
-O2	1.0	0.95	0.0	9.1	8.9	0.1
-O3	0.9	0.9	0.0	8.3	8.05	0.1
-O4	0.9	0.9	0.0	8.2	8.0	0.1

FIGURE 6. *Average running times (in seconds) of the test programs.*

all nodes except for the source have infinite distance labels. Note also that an easy inductive argument shows that nodes processed by the depth-first search have finite distance labels. Thus when an arc (v, w) is first examined while processing v, $d(v)$ is finite and $d(w)$ infinite, so $d(w)$ will be updated and $\ell_d(v, w)$ will become zero. Thus (v, w) will become admissible and the search will start processing w. Therefore the depth-first search of the admissible graph will examine the nodes in exactly the same order as the depth-first search of the whole input graph, and the resulting order will be topological with respect to the input graph. The standard results for shortest paths in acyclic graphs imply that after the end of the first pass, the algorithm terminates. ∎

Remark. When counting the number of scans done by GOR and GOR1, we count both the shortest paths SCAN operations and processing of nodes done by the depth-first searchers.

5. Experimental Setup

Our experiments were conducted on SUN Sparc-10 workstation model 41 with a 40MHZ processor running SUN Unix version 4.1.3. The workstation had 160 Meg. memory. Our codes were written in C and compiled with the SUN `cc` compiler version 1.0 using the `O4` optimization option.

We performed the machine calibration experiment designed by the organizers of the First DIMACS International Algorithm Implementation Challenge [15]. Figure 6 shows the average running times of the test programs compiled with different optimization levels.

Our implementations use the adjacency list representation of the input graph. We experimented with several folklore low-level representations of the graph and found that the one described in detail by Gallo and Pallottino [10] is the most efficient. Our implementations of the traditional algorithms (BF, PAPE, TWO_Q, THRESH) are also very similar to those described in [10]. We attempted to make our implementations of different algorithms uniform to make the running time comparisons more meaningful. We also tried to make the implementations efficient.

The codes compared in our main experiments are BFP, GOR, GOR1, DIKH, DIKBD, PAPE, TWO_Q, and THRESH. We do not include all the Dijkstra's algorithm implementations because they often perform very similarly. We chose DIKH because it is the most widely known version of Dijkstra's algorithm and DIKBD because it is the best overall implementation of Dijkstra's algorithm in our tests. We also compare DIKH, DIKBD, DIKR, DIKB, DIKBM, and DIKBA on a subset of the problems that shows strengths and weaknesses of these codes.

When tabulating results of our experiments, we give the running time in seconds (in bold) and the number of scan operations per node (below). The running time is the user CPU time and excludes the input and output times. To obtain a data point for a shortest paths code, we make five runs of the code on problems produced with the same generator parameters except for the pseudorandom generator seed. The data we tabulate is the average over the five runs.

We place a 20 minute limit on the user CPU time of each computation on a problem instance. This leaves over 15 minutes for the shortest paths computation (excluding input and output). Since all problems in our tests are solvable in well under a minute by the code that is fastest for this problem, the codes that exceed the limit on a problem are losing to the fastest code by over an order of magnitude.

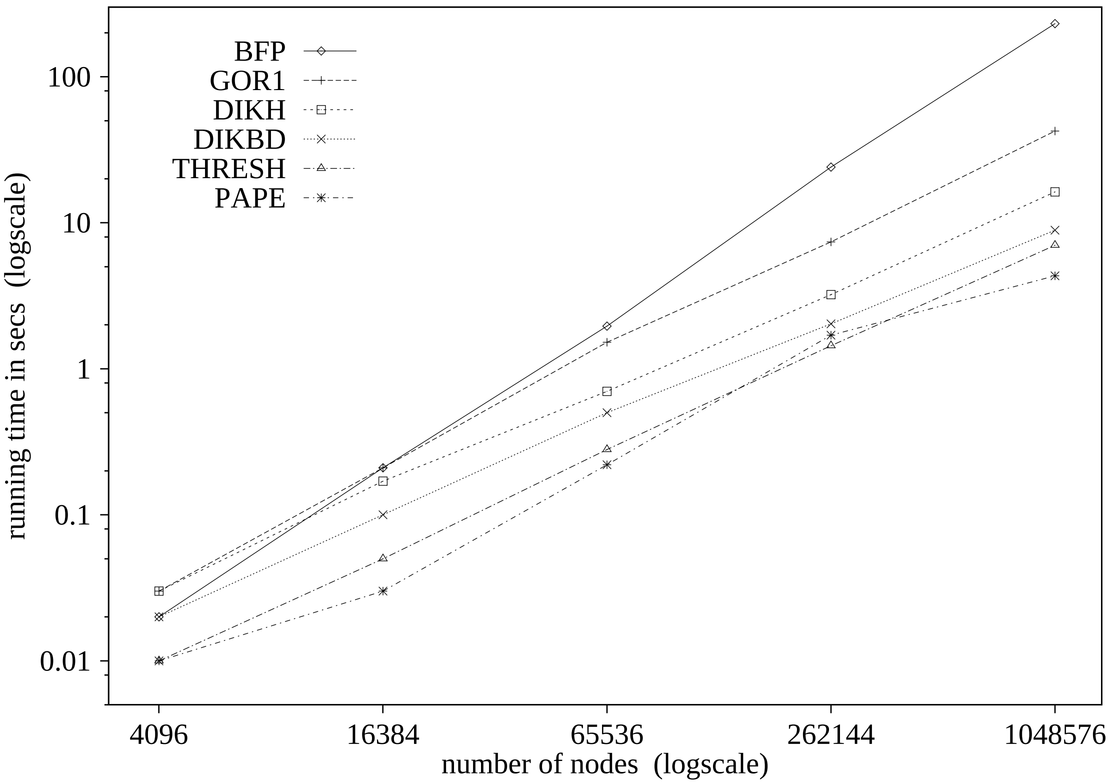
We also plot the data in addition to tabulating it. Our plots use regular or logarithmic scales, as appropriate for a particular problem family. To avoid crowding the plots, when two algorithms perform very similarly, we plot only one of them.

6. Simple SPGRID Problems

First we experiment with rectangular grid networks produced by our SPGRID generator. Nodes of these graphs correspond to points on the plane with integer coordinates $[x, y]$, $1 \leq x \leq X$, $1 \leq y \leq Y$. These points are connected "forward" by arcs of the form $([x, y], [x+1, y])$, $1 \leq x < X$, $1 \leq y \leq Y$, "up" by arcs of the form $([x, y], [x, y+1 (\mod Y)])$, $1 \leq x \leq X$, $1 \leq y \leq Y$, and "down" by arcs of the form $([x, y], [x, y-1 (\mod Y)])$, $1 \leq x \leq X$, $1 \leq y \leq Y$. Thus a *layer*, a set of nodes $[x, y]$ with x fixed and $1 \leq y \leq Y$, is a doubly connected cycle. There is also an additional source node connected to all nodes in the first layer, *i.e.*, the nodes with coordinates $[1, y]$, $1 \leq y \leq Y$. For the rectangular grid experiments, arc lengths are selected uniformly at random from the interval $[0, 10000]$.

6.1. Square Grids. Figure 7 presents results of experiments on Grid-SSquare family of square grids. For this family $X = Y$.

The best performance on this family is achieved by PAPE and TWO_Q. The performance of GOR, DIKBD, and THRESH is also good. These code loose to the best codes by less than a factor of 3. Somewhat slower is DIKH; it looses to the fastest codes by about a factor of four on the largest problem size.



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
4097	**0.02**	**0.02**	**0.03**	**0.03**	**0.02**	**0.01**	**0.01**	**0.01**
12288	2.74	2.26	4.82	1.00	1.00	1.25	1.25	1.05
16385	**0.21**	**0.08**	**0.21**	**0.17**	**0.10**	**0.03**	**0.04**	**0.05**
49152	5.05	2.29	5.25	1.00	1.00	1.26	1.26	1.05
65537	**1.96**	**0.37**	**1.52**	**0.70**	**0.50**	**0.22**	**0.24**	**0.28**
196608	9.66	2.28	7.41	1.00	1.00	1.27	1.27	1.13
262145	**24.07**	**2.02**	**7.40**	**3.22**	**2.03**	**1.70**	**1.53**	**1.44**
786432	19.68	2.29	8.11	1.00	1.00	1.27	1.27	1.16
1048577	**231.33**	**7.18**	**42.50**	**16.28**	**8.90**	**4.33**	**4.48**	**7.02**
3145728	41.78	2.30	11.25	1.00	1.00	1.27	1.27	1.19

FIGURE 7. *Grid-SSquare family data.*

The worst performance on this family is that of BFP. The second-worst code is GOR1. On the largest problem size, it is an order of magnitude slower than the fastest codes but an order of magnitude faster than the slowest code.

Remark. Our experiments show that the numbers of scans done by PAPE and TWO_Q on the same problem instances in the Grid-SSquare family are *exactly* the same. This is also the case for the Grid-SWide and Grid-SLong families of the next section. Closer examination of the distribution of the input graphs reveals that PAPE and TWO_Q on such problems are indeed very likely to perform the same number of scans.

When designing or implementing algorithms that use a shortest paths subroutine, it is often convenient to assume that all nodes of the network are reachable from the source. One way to assure this property is to introduce an artificial source and connect it to the original source by a zero length arc and to the other nodes of the graph by very long arcs. This is exactly how we obtain the Grid-SSquare-S family from the Grid-SSquare family.

Figure 8 shows the results of the Grid-SSquare-S experiment. Note that the Grid-SSquare-S graphs have about 1/3 more arcs than those in the previous experiment. Since the problem structure is similar, one would expect a slight increase in the running times on problems with the same number of nodes. However, the only code that meets this expectation is GOR1. Performance of all other codes suffers, but while PAPE and TWO_Q have a drastic change, other codes experience a relatively modest one.

The best codes in the first experiment are the worst by a wide margin in the second experiment. In particular, PAPE is the only code that ran over time limit on the second largest problem size. In the second experiment, TWO_Q performs much better than PAPE but much worse than the other codes.

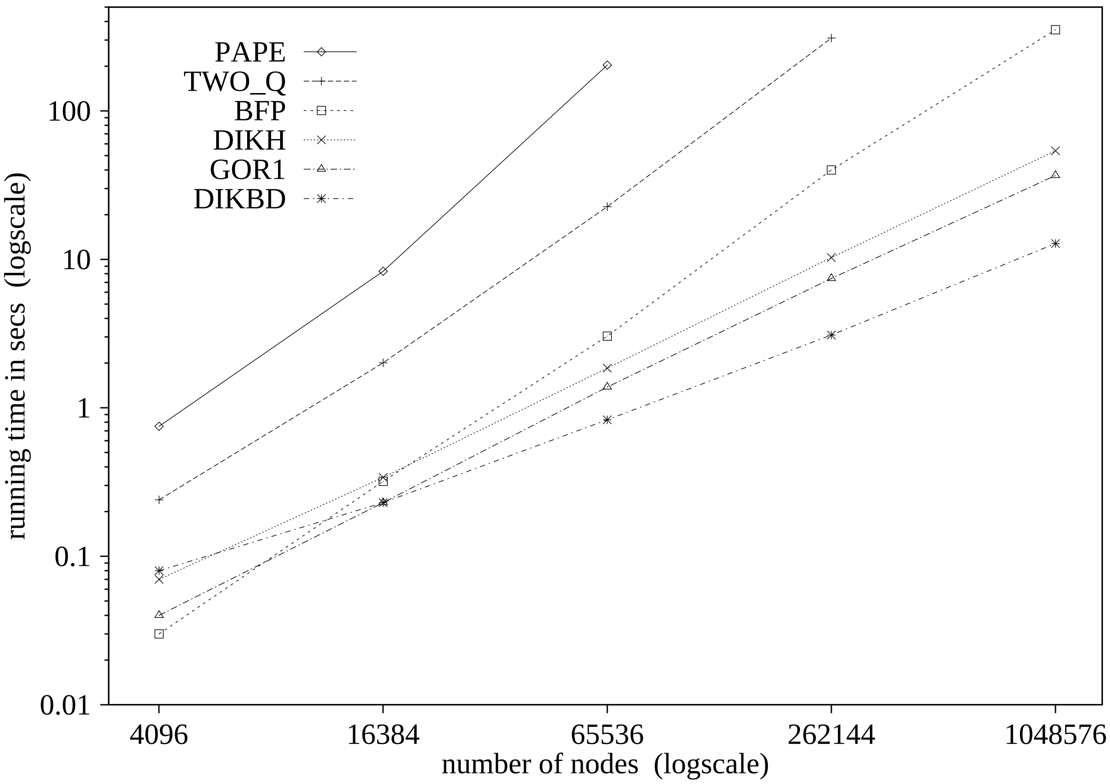
The performance of BFP decreases by roughly a factor of two, and the code remains uncompetitive with the best codes on this family.

The performance of DIKH decreases by a factor that slowly grows with the problem size. This factor is about 2 for the smaller problem sizes and over 3 for the largest size.

The performance of GOR and THRESH decreases by about a factor of 3. For the smaller problem sizes, THRESH is the fastest code in this experiment, but it loses to DIKBD on larger problems. Slightly slower than THRESH is GOR.

For larger problems, DIKBD is the fastest code in this experiment. Its performance decreases only by a factor of about 1.5 on the largest problem size. On smaller problems the performance decreases by a factor of 4.

6.2. Wide and Long Grids. Next we examine how the performance depends on the shape of the grid. We study two problem families, Grid-SWide and Grid-SLong. The grids in the first family have $X = 16$, *i.e.,* the length of these grids is fixed and the width grows with the



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
4098	**0.03**	**0.04**	**0.04**	**0.07**	**0.08**	**0.75**	**0.24**	**0.03**
16385	4.78	4.51	4.73	1.00	1.00	153.32	38.14	2.24
16386	**0.32**	**0.24**	**0.23**	**0.34**	**0.23**	**8.31**	**2.01**	**0.16**
65537	9.19	4.57	5.19	1.00	1.00	326.03	71.31	2.29
65538	**3.04**	**1.17**	**1.38**	**1.85**	**0.83**	**203.40**	**22.62**	**0.96**
262145	17.43	4.59	6.48	1.00	1.00	1664.52	166.50	2.30
262146	**40.01**	**5.06**	**7.43**	**10.29**	**3.09**		**309.16**	**4.55**
1048577	34.12	4.62	7.53	1.00	1.00		489.18	2.27
1048578	**351.96**	**20.39**	**36.74**	**53.86**	**12.78**			**19.46**
4194305	70.97	4.62	9.46	1.00	1.00			2.26

FIGURE 8. *Grid-SSquare-S family data.*

problem size. The grids in the second family have $Y = 16$ and their length grows with the problem size.

The wide grids are easy for all algorithms, as one can see in Figure 9. The fastest codes for this problem family are PAPE and TWO_Q, and all other codes except DIKH are within a factor of 2 from the fastest codes. Even the slowest code, DIKH, loses by less than an order of magnitude.

The situation changes on long grids, as can be seen in Figure 10. The most affected code is BFP, which is very good on wide grids but very bad on long grids, where it is the slowest code by a wide margin. The performance of DIKH is also affected significantly; its performance improves, especially on big problems.

Other codes are less affected: their running times change by less than a factor of 4. The performance of GOR, DIKBD, PAPE, TWO_Q, and THRESH improves, while the performance of GOR1 degrades. The best codes for wide grids, PAPE and TWO_Q, remain the best for the long grids.

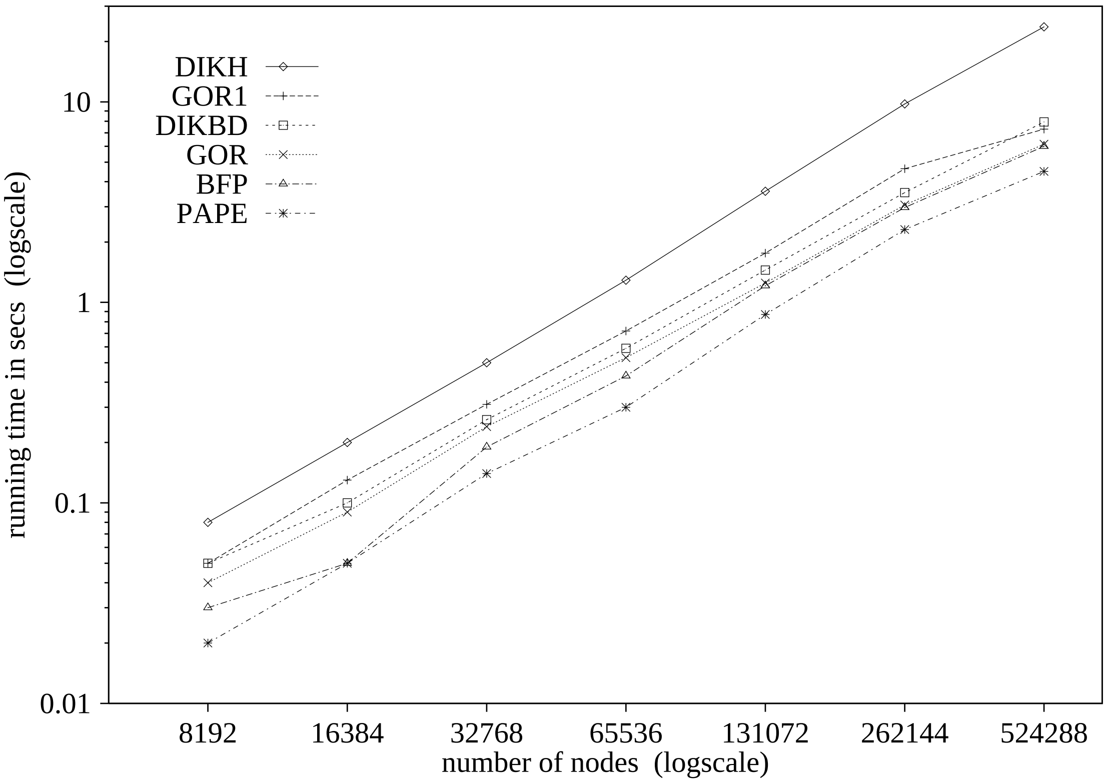
7. Harder SPGRID Problems

The SPGRID generator can also produce networks with structure that is very different from the simple grids described in the previous section. As in the case of simple grids, the networks considered in this section consist of layers and the source connected to the nodes of the first layer. Each layer is a simple cycle plus a collection of arcs connecting randomly selected pairs of nodes on the cycle. The length of the arcs inside a layer is small and nonnegative. There are arcs from one layer to the next one, as in simple grids, but in addition, there are generally arcs from lower to higher numbered layers. For the Grid-PHard problems the inter-layer arcs have nonnegative length, and for Grid-NHard problems, nonpositive length. The length of these arcs is selected uniformly at random from a wide range of integers. Additionally, in the Grid-PHard problems the length of an arc from layer x_1 to layer x_2 is multiplied by $(x_2 - x_1)^2$.

The Grid-PHard and Grid-NHard networks are significantly more complicated than simple grids. For example, these networks are non-planar. A more relevant difference is a complex layer structure of these networks, which has the property that a path between two nodes with many arcs is likely to have shorter length than a path with fewer arcs. This makes it difficult to direct the computation based on local information, so some algorithms may be forced to perform many re-scans.

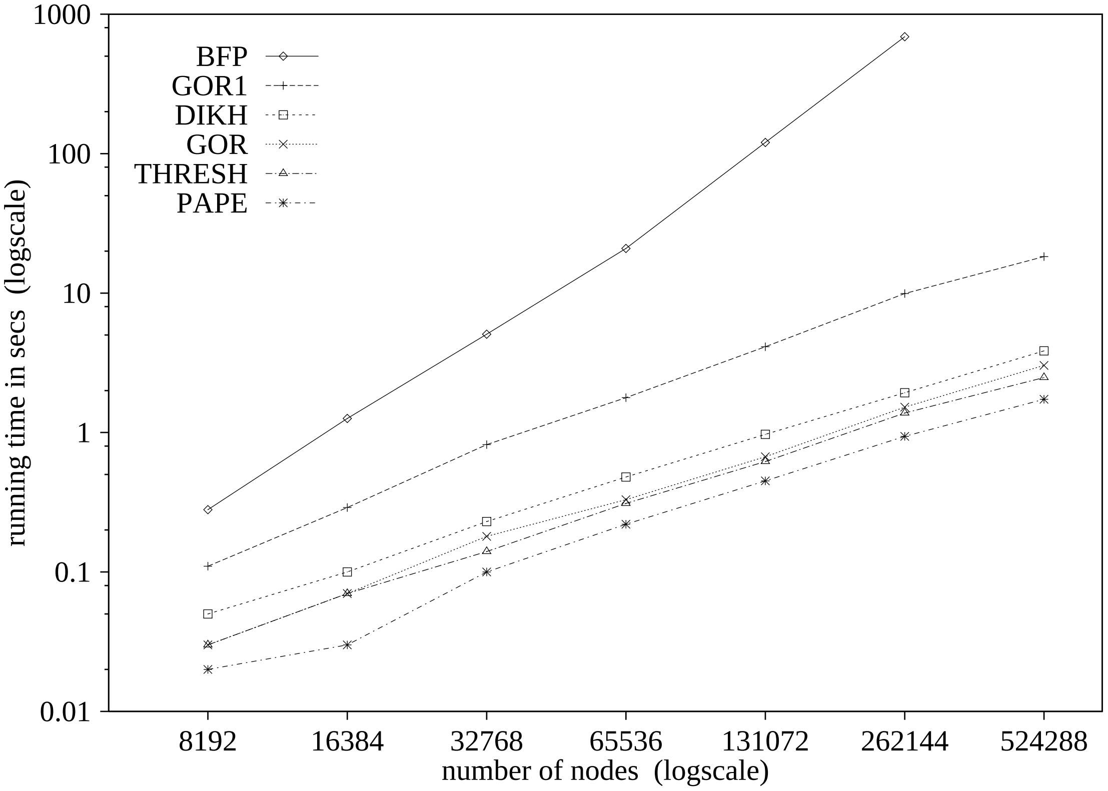
The computational results on the Grid-PHard family appear in Figure 11. Only four codes, GOR, GOR1, DIKH, and DIKBD, solve all problems in this family within the time limit.

The fastest code for this experiment is DIKBD, with DIKH close behind, losing by less than a factor of 2. The running time of these two codes seems to be close to linear in the number of nodes in Grid-PHard problems. The running time of GOR and TWO_Q also seems to be close



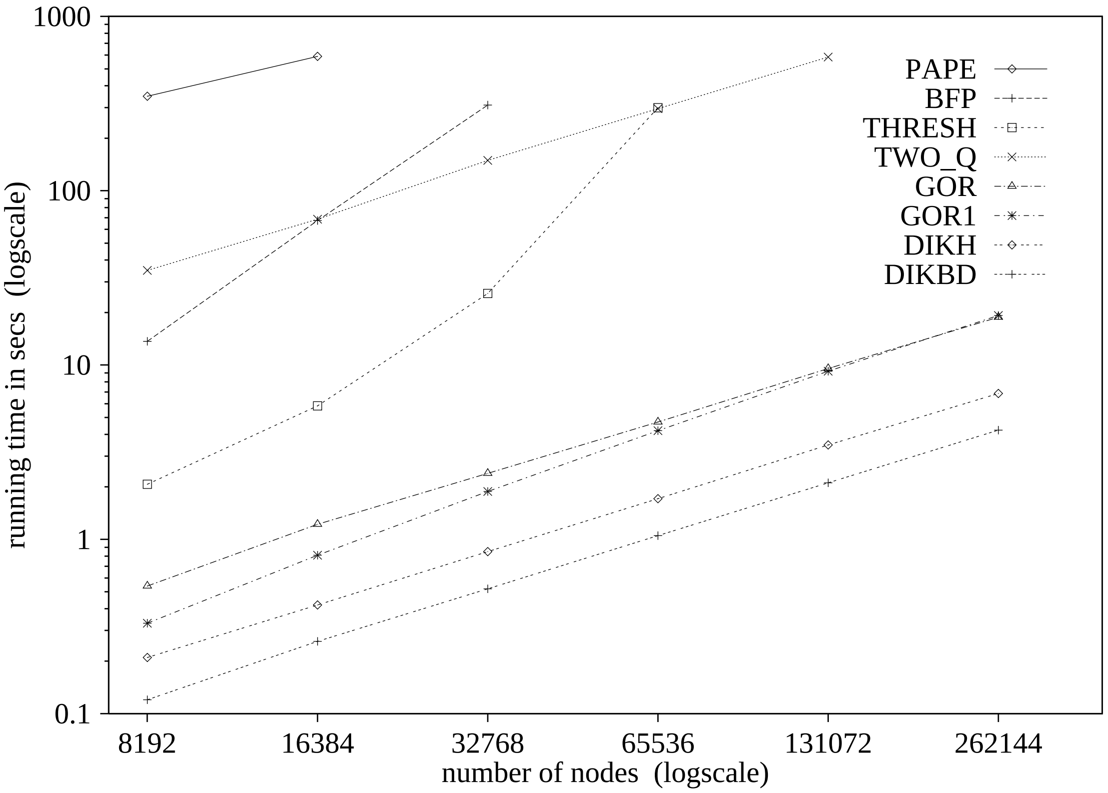
nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8193	**0.03**	**0.04**	**0.05**	**0.08**	**0.05**	**0.02**	**0.02**	**0.02**
24576	1.42	2.21	3.03	1.00	1.00	1.24	1.24	1.02
16385	**0.05**	**0.09**	**0.13**	**0.20**	**0.10**	**0.05**	**0.05**	**0.07**
49152	1.43	2.23	3.02	1.00	1.00	1.24	1.24	1.01
32769	**0.19**	**0.24**	**0.31**	**0.50**	**0.26**	**0.14**	**0.13**	**0.19**
98304	1.43	2.22	3.05	1.00	1.00	1.24	1.24	1.02
65537	**0.43**	**0.53**	**0.72**	**1.29**	**0.59**	**0.30**	**0.30**	**0.47**
196608	1.44	2.22	3.01	1.00	1.00	1.24	1.24	1.02
131073	**1.21**	**1.25**	**1.76**	**3.58**	**1.45**	**0.87**	**0.88**	**1.23**
393216	1.43	2.23	3.03	1.00	1.00	1.24	1.24	1.01
262145	**2.97**	**3.06**	**4.64**	**9.76**	**3.53**	**2.31**	**2.40**	**3.61**
786432	1.44	2.25	2.97	1.00	1.00	1.25	1.25	1.01
524289	**6.00**	**6.15**	**7.31**	**23.68**	**7.94**	**4.50**	**4.68**	**8.16**
1572864	1.44	2.25	3.04	1.00	1.00	1.25	1.25	1.01

FIGURE 9. *Grid-SWide family data.*



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8193	**0.28**	**0.03**	**0.11**	**0.05**	**0.05**	**0.02**	**0.02**	**0.03**
24576	19.84	2.26	7.41	1.00	1.00	1.25	1.25	1.39
16385	**1.26**	**0.07**	**0.29**	**0.10**	**0.10**	**0.03**	**0.04**	**0.07**
49152	36.04	2.27	9.09	1.00	1.00	1.26	1.26	1.40
32769	**5.07**	**0.18**	**0.82**	**0.23**	**0.22**	**0.10**	**0.10**	**0.14**
98304	70.51	2.27	9.88	1.00	1.00	1.26	1.26	1.46
65537	**20.89**	**0.33**	**1.78**	**0.48**	**0.45**	**0.22**	**0.23**	**0.31**
196608	154.44	2.27	10.57	1.00	1.00	1.26	1.26	1.47
131073	**120.33**	**0.67**	**4.12**	**0.97**	**0.92**	**0.45**	**0.45**	**0.62**
393216	318.07	2.27	11.65	1.00	1.00	1.26	1.26	1.50
262145	**689.46**	**1.52**	**9.92**	**1.93**	**1.82**	**0.94**	**0.96**	**1.38**
786432	666.76	2.27	12.42	1.00	1.00	1.26	1.26	1.50
524289		**3.03**	**18.25**	**3.85**	**3.68**	**1.73**	**1.82**	**2.48**
1572864		2.27	12.24	1.00	1.00	1.26	1.26	1.49

FIGURE 10. *Grid-SLong family data.*



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8193	**13.68**	**0.54**	**0.33**	**0.21**	**0.12**	**348.05**	**34.88**	**2.07**
63808	390.13	16.90	11.66	1.00	1.00	14988.77	1108.89	57.55
16385	**67.46**	**1.22**	**0.81**	**0.42**	**0.26**	**589.40**	**68.66**	**5.83**
129344	799.87	17.98	12.52	1.00	1.00	12694.03	1145.92	79.25
32769	**309.98**	**2.39**	**1.88**	**0.85**	**0.52**		**149.19**	**25.73**
260416	1612.36	17.87	13.07	1.00	1.00		1190.10	178.42
65537		**4.71**	**4.20**	**1.71**	**1.05**		**295.65**	**298.97**
522560		17.84	14.01	1.00	1.00		1190.76	803.15
131073		**9.54**	**9.21**	**3.48**	**2.11**		**584.69**	
1046848		17.98	14.73	1.00	1.00		1199.97	
262145		**18.82**	**19.25**	**6.86**	**4.23**			
2095424		17.82	15.12	1.00	1.00			

FIGURE 11. *Grid-PHard family data.*

to linear, but with bigger constant factors. While GOR is about five times slower than the Dijkstra's codes, and TWO_Q is two orders of magnitude slower.

The running time of GOR1 seems to grow a little faster than that of GOR. The latter code is a little slower on small problems but catches up on the biggest problems in our test. Both THRESH and BFP exhibit clearly superlinear rates of growth and exceed the time limit on the bigger problems.

In this test PAPE has the worst performance. In the set time, it is able to solve problems of the two smallest sizes only, losing to the best code by three orders of magnitude.

Figure 12 gives results of the Grid-NHard experiment. On this problem family, GOR1 and GOR are by far the best codes.

Remark. On all instances we tried, the number of scans done by GOR1 on Grid-NHard instances and the corresponding Grid-PHard ones are *exactly* the same. This code seems to be always able to figure out that the underlying problem structure is similar. The minor running time differences are mostly due to timing variations.

Although the performance of BFP, PAPE, and TWO_Q codes is not exactly the same in this experiment as in the previous one, it is quite similar. Much worse performance is exhibited by THRESH, DIKBD, and DIKH. The latter code is the worst, exceeding the time limit on all test problems.

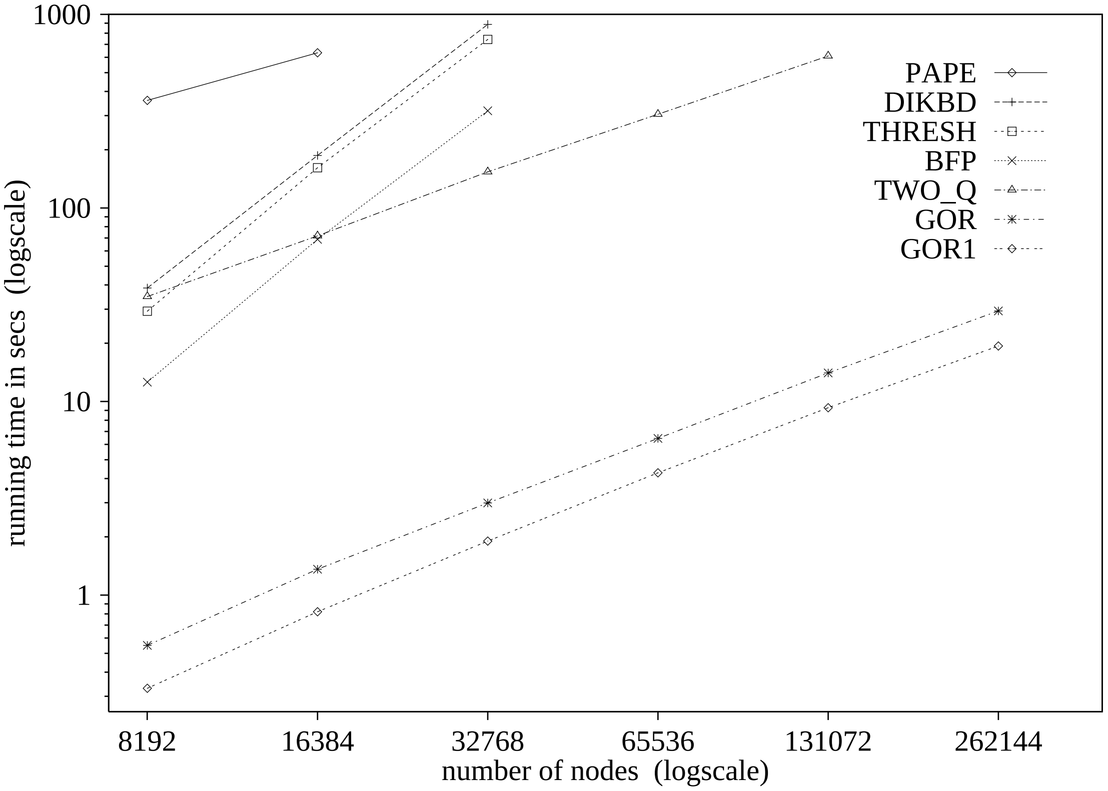
One should note similar behavior of DIKBD and THRESH on Grid-NHard problems. Moreover, their behavior is analogous to BFP, that is, DIKBD and THRESH differ from BFP by roughly the same factor for all problem sizes.

8. Experiments with SPRAND Families

In this section we study performance of the codes on graphs produced by the SPRAND generator. All graphs we consider are constructed by creating a hamiltonian cycle and then adding arcs with distinct random end points. In our experiments we set the length of the arcs on the cycle to 1 and pick the lengths of other arcs uniformly at random from a certain interval. For all problem families except Rand-Len, this interval is $[0, 10000]$.

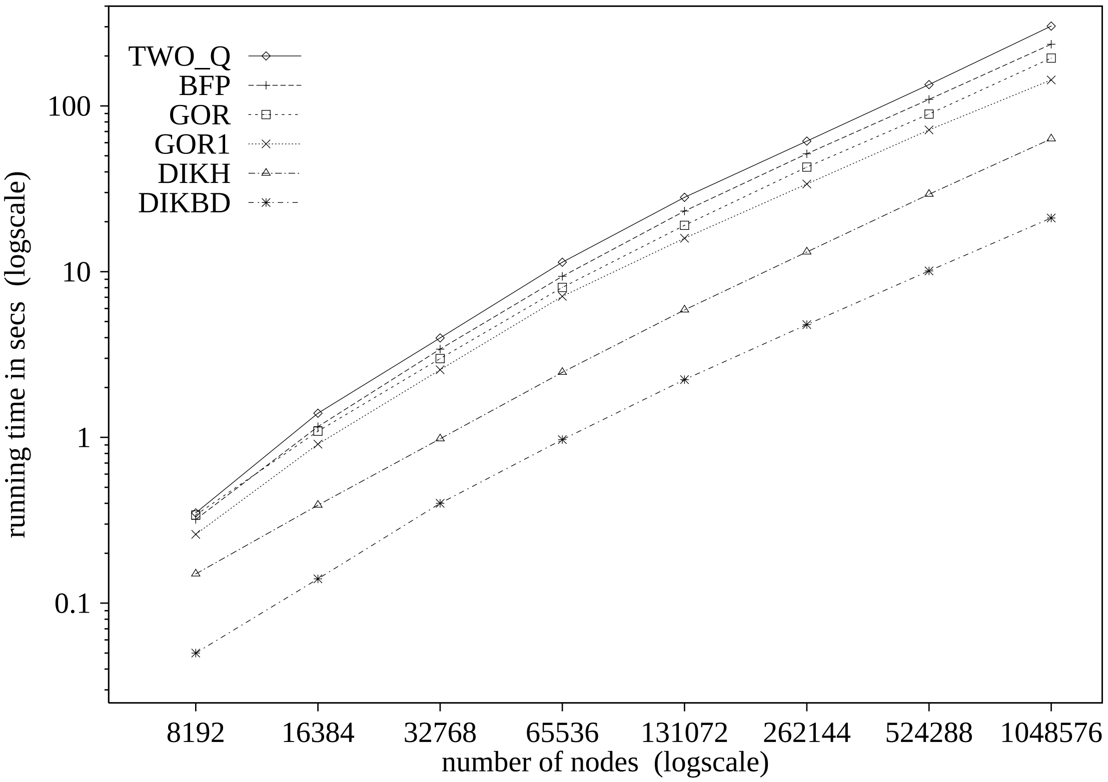
Note that if we were to pick the cycle arcs lengths in the same ways as the other arc lengths, the resulting graphs would be essentially random. We found, however, that the resulting problems are easy for all the codes. Setting the cycle arc lengths to 1 makes the problems more interesting and the experiments more insightful.

8.1. Sparse and Dense Networks. The graphs in Rand-4 family have $m = 4n$. These are sparse graphs. As one can see in Figure 13, the Dijkstra's codes are the best on these problems, with DIKBD clearly the fastest code and DIKH slower by a factor of about 2 for the smaller problems and a factor of about 3 for the bigger problems. Other codes are noticeably slower, with TWO_Q and PAPE being the slowest.



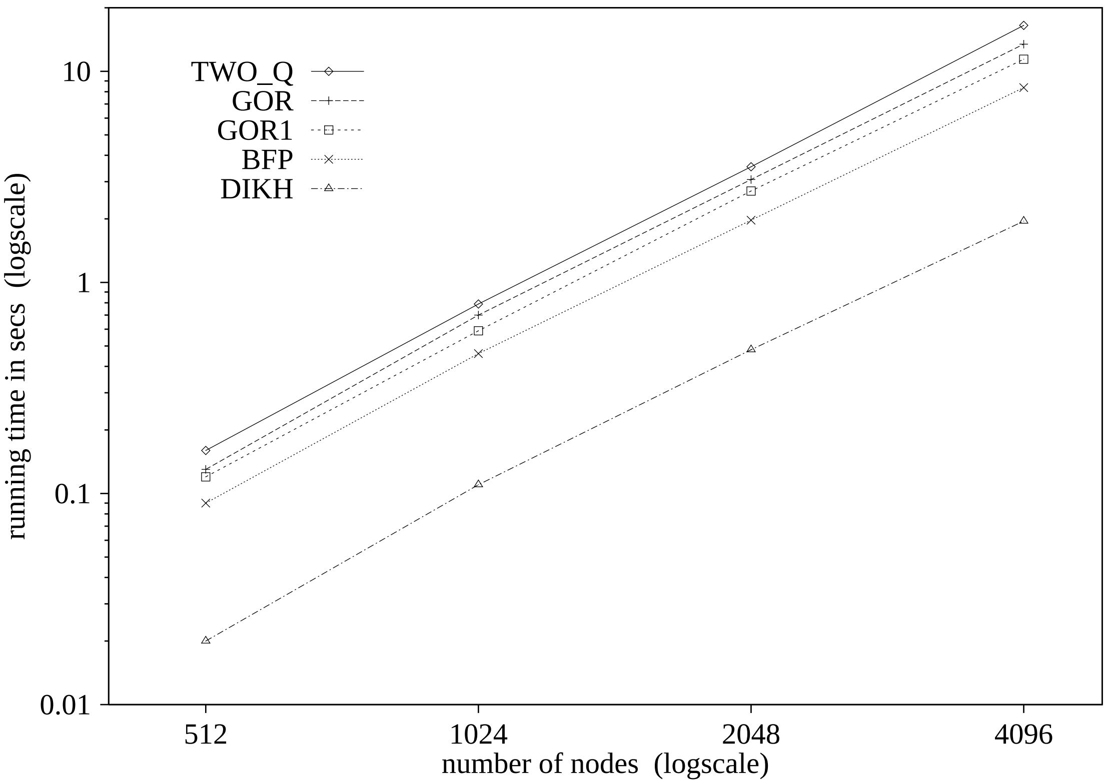
nodes/arcs	BFP	GOR	GOR1	DIKBD	PAPE	TWO_Q	THRESH
8193	**12.59**	**0.55**	**0.33**	**38.56**	**359.73**	**34.88**	**29.26**
63808	392.09	17.88	11.66	876.92	16092.24	1143.77	912.61
16385	**68.86**	**1.36**	**0.82**	**186.71**	**633.44**	**71.75**	**161.48**
129344	803.31	19.89	12.52	1850.02	13846.24	1187.05	1854.19
32769	**317.84**	**2.99**	**1.90**	**888.04**		**153.64**	**741.89**
260416	1619.02	20.66	13.07	3844.69		1231.74	3859.87
65537		**6.45**	**4.28**			**305.02**	
522560		21.30	14.01			1231.16	
131073		**14.05**	**9.28**			**609.94**	
1046848		22.58	14.73			1239.82	
262145		**29.38**	**19.34**				
2095424		23.47	15.12				

FIGURE 12. *Grid-NHard family data.* DIKH *exceeded the time limit on all problems.*



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8192	**0.32**	**0.34**	**0.26**	**0.15**	**0.05**	**0.32**	**0.35**	**0.20**
32768	12.23	14.58	10.92	1.00	1.00	16.32	15.90	6.31
16384	**1.16**	**1.09**	**0.91**	**0.39**	**0.14**	**1.22**	**1.40**	**0.79**
65536	13.45	16.19	11.72	1.00	1.00	20.14	18.83	7.22
32768	**3.41**	**2.99**	**2.56**	**0.98**	**0.40**	**3.56**	**3.98**	**2.40**
131072	13.73	16.47	12.00	1.00	1.00	20.08	19.38	7.22
65536	**9.37**	**8.03**	**7.09**	**2.47**	**0.97**	**10.60**	**11.39**	**7.12**
262144	15.51	18.21	13.29	1.00	1.00	25.16	23.38	8.56
131072	**23.20**	**19.04**	**15.91**	**5.87**	**2.23**	**25.45**	**28.06**	**17.45**
524288	16.75	19.48	14.15	1.00	1.00	27.31	26.07	9.16
262144	**51.39**	**42.74**	**33.81**	**13.15**	**4.79**	**54.73**	**61.36**	**40.43**
1048576	17.61	20.76	14.62	1.00	1.00	27.80	26.95	10.02
524288	**109.36**	**89.36**	**71.58**	**29.28**	**10.11**	**118.04**	**134.44**	**86.66**
2097152	18.19	21.02	15.02	1.00	1.00	28.91	28.41	10.09
1048576	**235.63**	**194.31**	**143.54**	**63.27**	**21.09**	**257.31**	**302.76**	**186.23**
4194304	19.12	22.40	15.18	1.00	1.00	31.23	30.70	10.56

FIGURE 13. *Rand-4 family data.*



nodes/arcs	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
512	**0.09**	**0.13**	**0.12**	**0.02**	**0.02**	**0.15**	**0.16**	**0.08**
65536	5.32	7.91	6.22	1.00	1.00	9.03	8.70	4.26
1024	**0.46**	**0.70**	**0.59**	**0.11**	**0.12**	**0.76**	**0.79**	**0.46**
262144	5.10	7.91	6.21	1.00	1.00	9.11	8.88	4.78
2048	**1.97**	**3.07**	**2.71**	**0.48**	**0.49**	**3.39**	**3.53**	**1.94**
1048576	4.65	7.23	6.01	1.00	1.00	8.43	8.33	4.45
4096	**8.37**	**13.45**	**11.41**	**1.95**	**2.02**	**16.18**	**16.51**	**9.21**
4194304	4.65	7.24	5.87	1.00	1.00	9.12	8.92	4.99

FIGURE 14. *Rand-1:4 family data.*

The graphs in Rand-1:4 family have $m = n^2/4$. These are dense graphs. As one can see in Figure 14, there is little difference in relative performance of the codes, except DIKH improves relative to DIKBD and becomes the fastest code, although DIKBD is only slightly slower.

8.2. Dependency on Arc Lengths. Problems in the Rand-Len family are the same except for the interval from which the arc lengths are selected. The arc length is fixed to 1 for the first problem in the family and selected from an interval $[0, U]$ for the other problems. See Figure 15. Note that because the path arcs' lengths are set to 1, the structure of the shortest paths tree changes as U increases. For bigger values of U, the path arcs are more likely to be in the tree and the tree is likely to be taller.

On the unit length problems, BFP, DIKH, DIKBD, PAPE, TWO_Q, and THRESH make one scan per node. The running times of BFP, PAPE, and TWO_Q are the fastest (and almost the same). Other codes that perform well are GOR, DIKH, DIKBD, and THRESH. These codes loose to the fastest codes by less than a factor of 2. The worst code, GOR1, loses by about a factor of 5.

As the length range expands, the algorithms become slower. DIKBD shows very little dependence on the arc length range and is the fastest except for the unit length case. The performance of GOR1 and DIKH is also affected very little. Other codes are significantly affected; their performance decreases by over an order of magnitude for the $[0, 1000000]$ length range (compared to the unit length case).

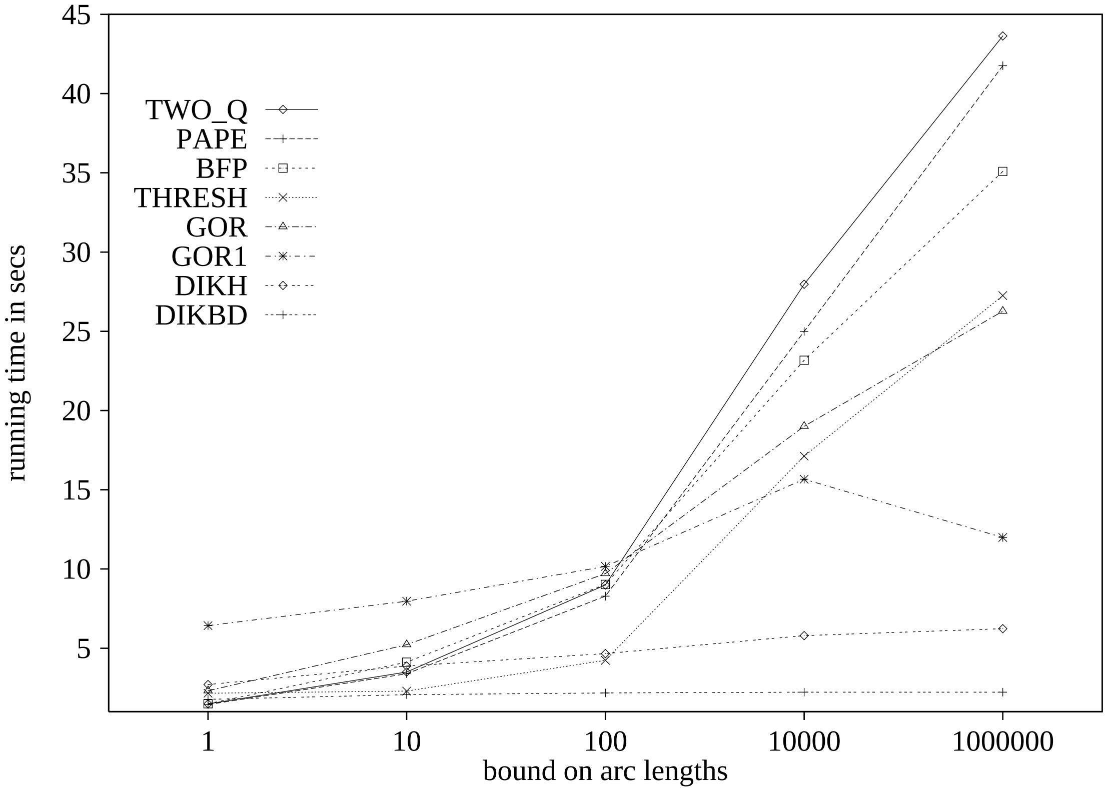
8.3. Node Potentials. Problems in the Rand-P family are the same except the length function ℓ is modified by assigning each node v a potential $p(v)$ chosen uniformly at random from the interval $[0, P]$ and replacing ℓ by ℓ_p. (For $P = 0$, the problems are the same as the 131072 node problems of the Rand-4 family.) While ℓ is nonnegative, ℓ_p can take on negative values. However, for small P, the expected fraction of negative length arcs is small.

Note that BFP, GOR1, PAPE, and TWO_Q make the same number of scans regardless of the potentials. This observation is justified by Theorem 11.1.

9. Experiments with SPACYC Families

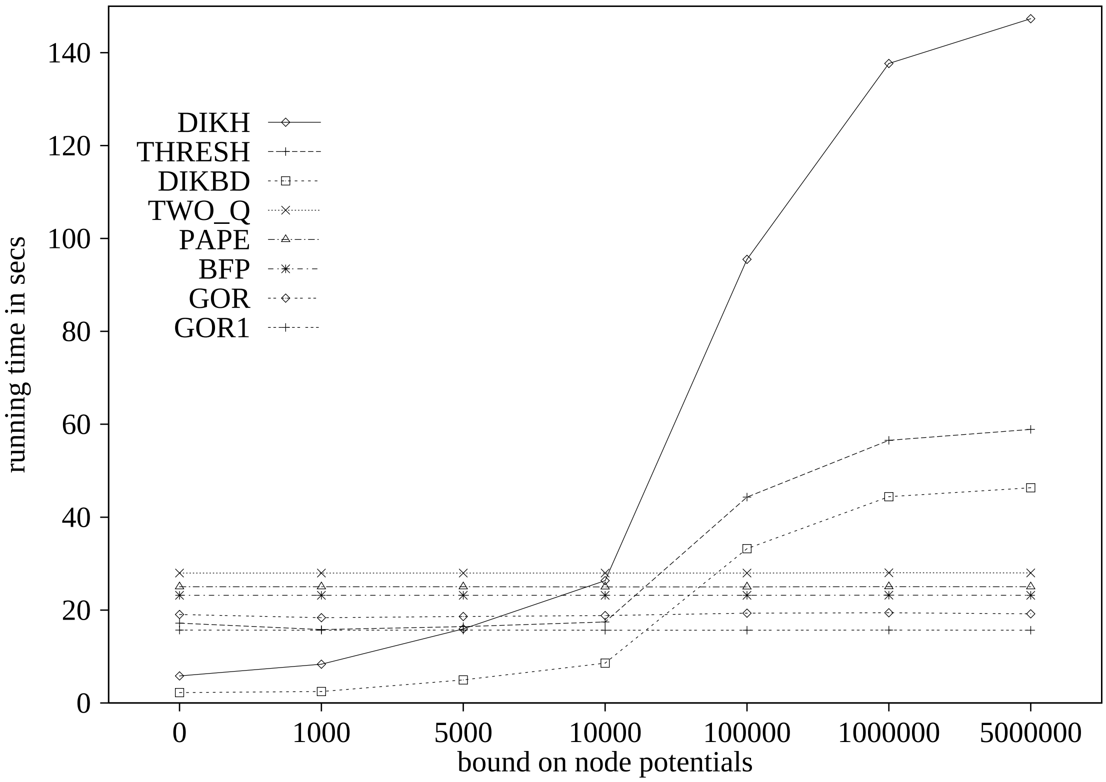
In this section we study performance of the codes on acyclic networks. The shortest paths problem on an acyclic graph can be solved in linear time (see *e.g.* [4]) and the experiments of this section include the linear time algorithm for acyclic graphs, ACC.

Experiments with acyclic graphs are interesting for several reasons. Shortest paths problems in acyclic graphs come up in applications, such as PERT network analysis (see *e.g.* [17]). Furthermore, some networks that come up in applications have large acyclic subgraphs (*e.g.* electric networks) and an algorithm that behaves poorly on acyclic networks is likely to behave poorly on networks with large acyclic subgraphs. Acyclic networks are also easy to use in certain experiments because negative length cycles are not a problem for these networks.



$[L,U]$	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
$[1,1]$	**1.50**	**2.31**	**6.43**	**2.71**	**1.77**	**1.45**	**1.49**	**2.16**
	1.00	1.61	4.48	1.00	1.00	1.00	1.00	1.00
$[0,10]$	**4.12**	**5.22**	**7.97**	**3.88**	**2.07**	**3.39**	**3.50**	**2.30**
	2.67	4.36	5.94	1.00	1.00	2.85	2.84	1.01
$[0,100]$	**9.03**	**9.71**	**10.16**	**4.66**	**2.18**	**8.29**	**8.99**	**4.24**
	6.15	8.93	8.21	1.00	1.00	8.07	8.05	1.78
$[0,10000]$	**23.17**	**19.01**	**15.67**	**5.80**	**2.23**	**24.99**	**27.97**	**17.12**
	16.75	19.48	14.15	1.00	1.00	27.31	26.07	9.16
$[0,1000000]$	**35.09**	**26.27**	**11.99**	**6.23**	**2.23**	**41.76**	**43.64**	**27.25**
	26.77	27.96	12.26	1.00	1.00	47.60	41.89	16.03

FIGURE 15. *Rand-Len family data. All problems have 131072 nodes and 524288 arcs.*



P	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
0	**23.18**	**19.04**	**15.68**	**5.82**	**2.24**	**25.00**	**27.98**	**17.17**
	16.75	19.48	14.15	1.00	1.00	27.31	26.07	9.16
1000	**23.18**	**18.35**	**15.67**	**8.36**	**2.48**	**25.00**	**27.96**	**15.80**
	16.75	18.89	14.15	1.28	1.28	27.31	26.07	7.77
5000	**23.18**	**18.62**	**15.68**	**15.96**	**4.95**	**25.02**	**27.97**	**16.45**
	16.75	19.21	14.15	2.99	2.99	27.31	26.07	8.00
10000	**23.18**	**18.83**	**15.67**	**26.33**	**8.57**	**24.99**	**27.97**	**17.45**
	16.75	19.42	14.15	5.55	5.55	27.31	26.07	8.49
100000	**23.18**	**19.33**	**15.66**	**95.49**	**33.23**	**24.99**	**27.97**	**44.33**
	16.75	19.76	14.15	23.72	22.85	27.31	26.07	20.86
1000000	**23.21**	**19.43**	**15.68**	**137.67**	**44.42**	**25.05**	**28.02**	**56.53**
	16.75	19.78	14.15	33.80	30.44	27.31	26.07	26.77
5000000	**23.18**	**19.20**	**15.65**	**147.31**	**46.32**	**25.01**	**28.00**	**58.88**
	16.75	19.53	14.15	35.93	31.40	27.31	26.07	27.88

FIGURE 16. *Rand-P family data. All problems have 131072 nodes and 524288 arcs.*

The networks used in the experiments of this section are generated as follows. The nodes are numbered from 1 to n, and there is a path of arcs $(i, i+1)$, $1 \leq i < n$. These arcs are called the *path arcs*. Additional arcs are generated by picking two distinct nodes at random and creating an arc from the lower to the higher numbered node. The lengths of the additional arcs are selected uniformly at random from the interval $[L, U]$.

9.1. Positive Arc Length. For the Acyc-Pos family, the path arcs' length is set to 1 and the other arc lengths are selected from the interval $[0, 10000]$. The unit length of the path arcs makes these problems more difficult for some of the codes. Figure 17 shows how the codes perform on this problem family.

The fastest codes for this family are DIKBD and ACC. These codes perform similarly, but the former is a little faster on bigger problems, in spite of the fact that ACC is especially designed for acyclic graphs. These two algorithms make the same number of scans; the additional overhead of ACC is a topological sort of the graph and the additional overhead of DIKBD is in maintaining the bucket data structure. The latter overhead is smaller than the former for large Acyc-Pos problems. The performance of GOR1 is only slightly worse than that of ACC; DIKH also performs well, losing to DIKBD by about a factor of two.

The other codes are an order of magnitude slower than the fastest ones. Worst performers are GOR, PAPE, and TWO_Q.

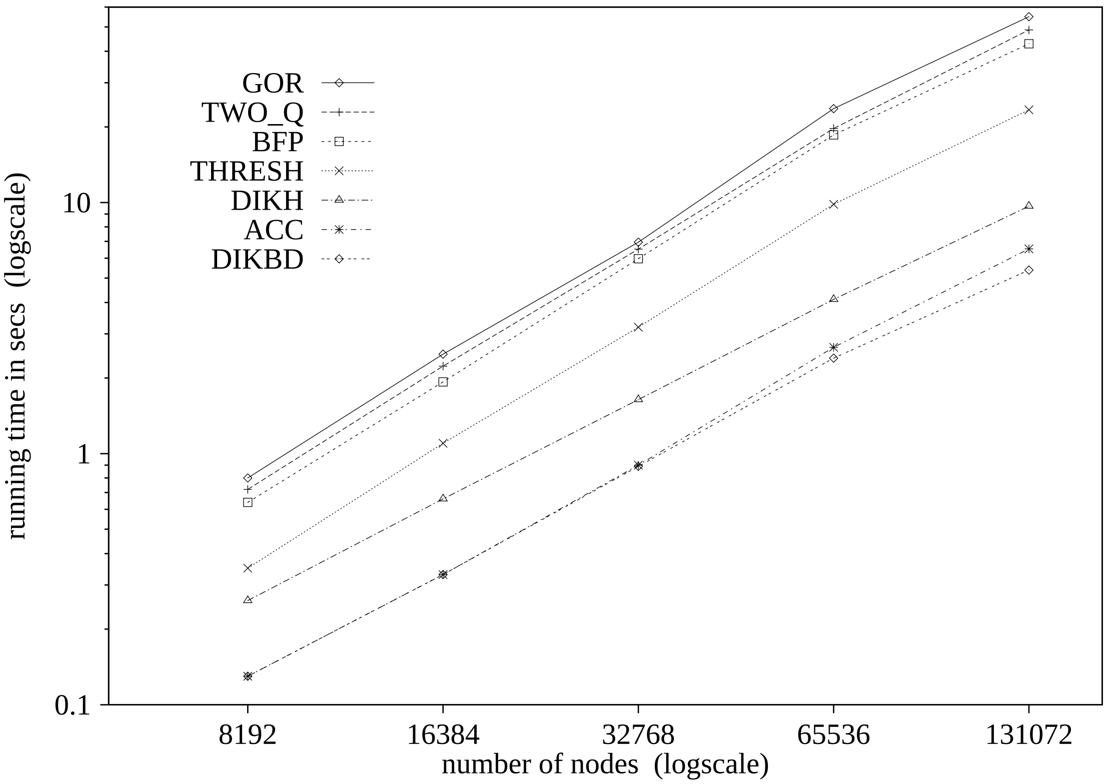
9.2. Negative Arc Length. For the Acyc-Neg family, the path arc length is set to -1 and the other arc lengths are selected uniformly at random from the interval $[-10000, 0]$. We would like to note that Acyc-Neg problems are very natural. For example, to solve a problem of finding a longest path in an acyclic graph, one negates arc lengths and looks for a shortest paths. In applications such as PERT, lengths are nonnegative, and the resulting problems are similar to the Acyc-Neg problems. Figure 18 shows how the codes perform on this problem family.

In this experiment, ACC and GOR1 perform similarly to the previous experiment, and GOR performs better than in the previous experiment, matching GOR1.

All other codes perform worse by a very wide margin. Within the time limit, BFP finishes on three smallest problem sizes, THRESH, DIKBD, and TWO_Q on two, DIKH and PAPE only on one.

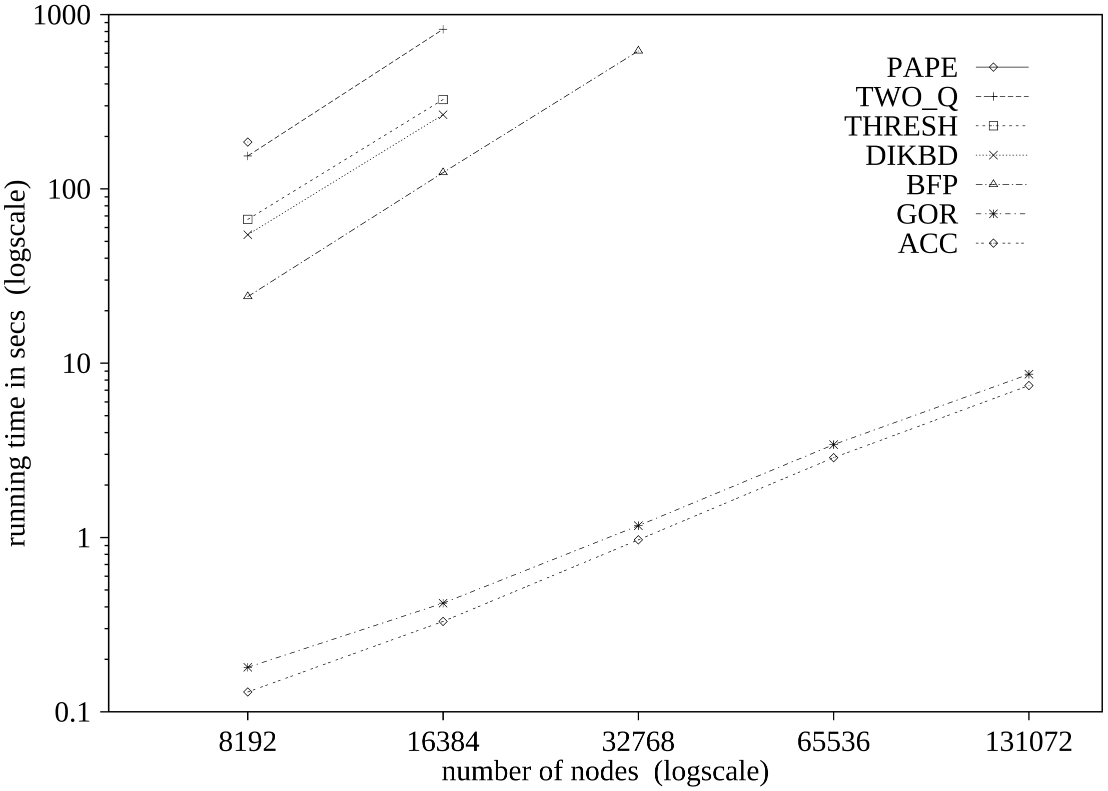
9.3. Variable Fraction of Negative Arcs. The previous experiments with acyclic graphs show that performance of many algorithms changes dramatically if arc lengths in an acyclic graph are negated. We study this phenomenon further by varying the fraction of negative length arcs.

For the Acyc-P2N family, the problem size is fixed and all arc lengths are selected uniformly at random from the interval $[L, U]$, where the values of L and U determine the expected fraction



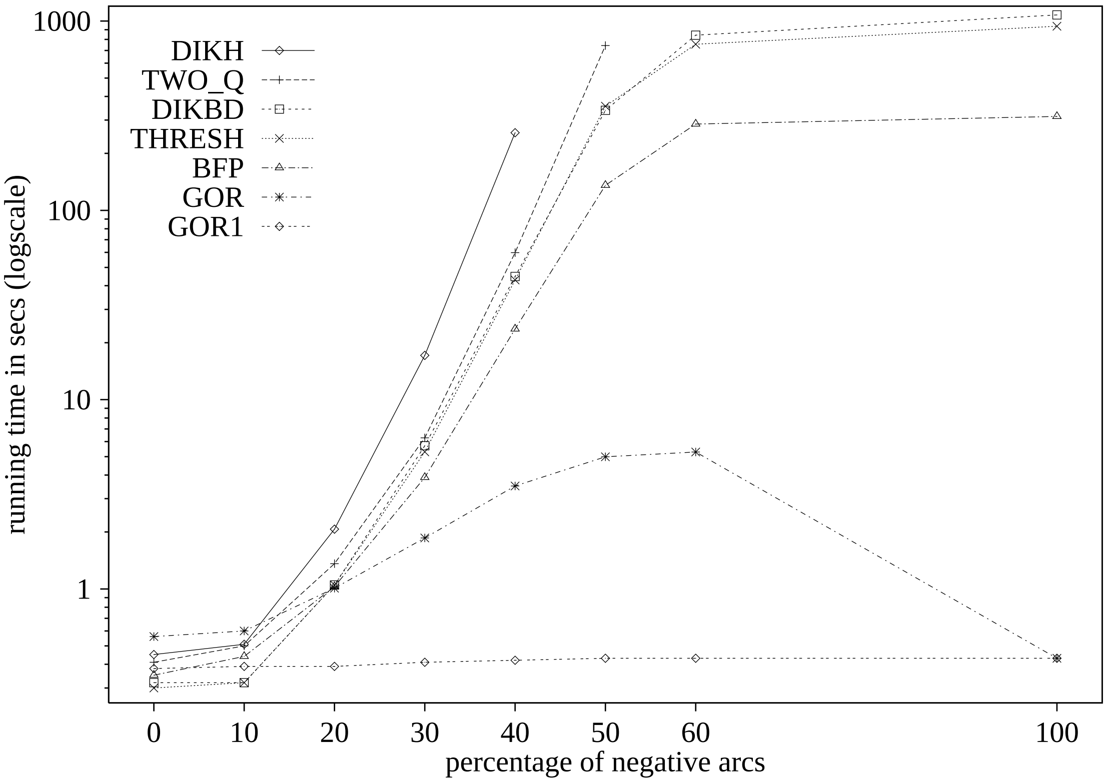
nodes/arcs	ACC	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8192	**0.13**	**0.64**	**0.80**	**0.14**	**0.26**	**0.13**	**0.63**	**0.72**	**0.35**
131072	1.00	8.80	12.51	2.00	1.00	1.00	11.87	11.63	5.15
16384	**0.33**	**1.93**	**2.49**	**0.39**	**0.66**	**0.33**	**1.87**	**2.23**	**1.10**
262144	1.00	9.76	14.52	2.00	1.00	1.00	13.92	13.50	5.78
32768	**0.90**	**5.97**	**6.95**	**1.12**	**1.64**	**0.89**	**5.51**	**6.52**	**3.19**
524288	1.00	10.09	14.55	2.00	1.00	1.00	14.33	14.22	5.89
65536	**2.65**	**18.58**	**23.66**	**2.99**	**4.11**	**2.40**	**18.38**	**19.72**	**9.84**
1048576	1.00	11.36	17.23	2.00	1.00	1.00	15.47	15.17	6.60
131072	**6.54**	**42.86**	**54.96**	**7.24**	**9.67**	**5.38**	**46.43**	**48.66**	**23.39**
2097152	1.00	11.44	16.78	2.00	1.00	1.00	16.29	15.87	6.53

FIGURE 17. *Acyc-Pos family data.*



nodes/arcs	ACC	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
8192	**0.13**	**24.12**	**0.18**	**0.17**	**1047.35**	**54.60**	**185.55**	**154.47**	**66.83**
131072	1.00	466.95	2.00	2.00	9037.71	964.79	6188.81	4590.70	1453.33
16384	**0.33**	**123.98**	**0.42**	**0.42**		**266.25**		**823.41**	**325.61**
262144	1.00	887.44	2.00	2.00		1841.51		9699.84	2797.25
32768	**0.97**	**618.04**	**1.17**	**1.15**					
524288	1.00	1724.82	2.00	2.00					
65536	**2.87**		**3.41**	**3.28**					
1048576	1.00		2.00	2.00					
131072	**7.44**		**8.65**	**8.51**					
2097152	1.00		2.00	2.00					

FIGURE 18. *Acyc-Neg family data.*



f (%)	ACC	BFP	GOR	GOR1	DIKH	DIKBD	PAPE	TWO_Q	THRESH
0	**0.35**	**0.35**	**0.56**	**0.38**	**0.45**	**0.32**	**0.39**	**0.41**	**0.30**
	1.00	1.66	2.61	2.00	1.00	1.00	1.83	1.83	1.01
10	**0.35**	**0.44**	**0.60**	**0.39**	**0.51**	**0.32**	**0.49**	**0.50**	**0.32**
	1.00	2.21	3.01	2.00	1.14	1.14	2.64	2.64	1.29
20	**0.36**	**1.03**	**1.01**	**0.39**	**2.07**	**1.05**	**1.25**	**1.36**	**1.05**
	1.00	6.78	6.29	2.00	6.58	7.28	10.92	10.53	7.06
30	**0.37**	**3.88**	**1.86**	**0.41**	**17.13**	**5.71**	**5.78**	**6.28**	**5.30**
	1.00	28.65	12.49	2.00	63.54	41.31	69.27	59.79	40.28
40	**0.38**	**23.62**	**3.50**	**0.42**	**257.13**	**44.78**	**66.90**	**59.80**	**42.87**
	1.00	179.75	22.81	2.00	978.74	325.84	940.89	641.19	339.58
50	**0.37**	**135.64**	**4.99**	**0.43**		**337.62**	**874.90**	**742.54**	**355.35**
	1.00	1056.25	30.81	2.00		2450.04	12088.96	9510.17	2862.66
60	**0.40**	**285.63**	**5.29**	**0.43**		**843.39**			**754.70**
	1.00	2149.88	33.52	2.00		5634.02			5918.05
100	**0.38**	**313.34**	**0.43**	**0.43**		**1078.63**			**939.29**
	1.00	2349.75	2.00	2.00		6975.15			7114.46

FIGURE 19. *Acyc-P2N family data.*

f of negative length arcs. Note that unlike the previous experiments with acyclic networks, the path arc lengths are random in this experiment.

Figure 19 summarizes the results. As expected from theory and previous experiments, the performance of ACC and GOR1 shows almost no dependence on f. Also as expected, the implementations of Dijkstra's algorithms perform poorly when the fraction of negative length arcs is large.

Performance of PAPE and TWO_Q in this test is similar. The performance degrades dramatically as the fraction of negative length arcs increases. Performance of BFP and THRESH also significantly degrades, although not as dramatically.

Performance of GOR degrades somewhat until the fraction of negative length arcs becomes very large, at which point the performance improves.

10. Experiments With Variations of Dijkstra's Algorithm

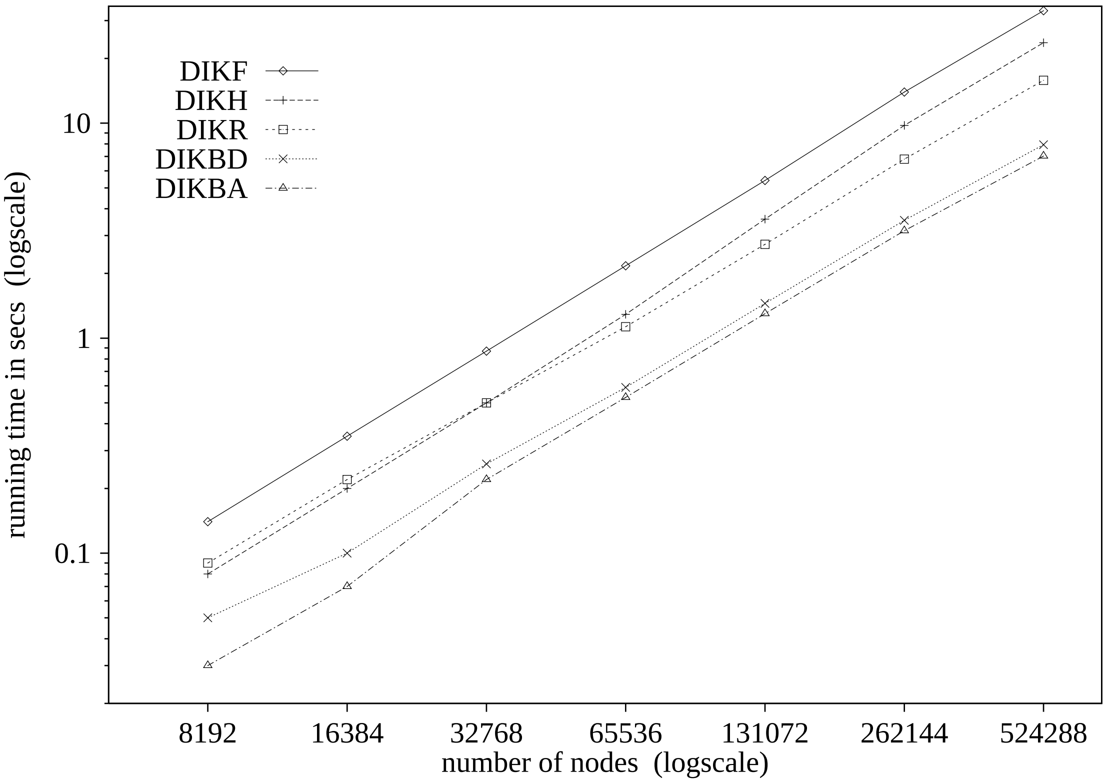
The above experiments involve two implementations of Dijkstra's algorithm, the "classical" k-ary heap implementation DIKH and our double bucket implementation DIKBD. In this section we compare these implementations with several other implementations on problem families Grid-SWide, Grid-SLong, Grid-SSquare-S, Grid-PHard, and Rand-Len. The problem families are chosen to emphasize differences in the codes' performance. The additional implementations we evaluate are the R-heap implementation DIKR, the Fibonacci heap implementation DIKF, Dial's implementation DIKB, the overflow bag implementation DIKM, and the approximate bucket implementation DIKBA.

Figure 20 presents data for the Grid-SWide family. Here DIKBA performs best, with DIKB, DIKBD, and DIKBM close behind. Note that DIKBA makes only one scan per node on these problems. The heap implementations DIKR and DIKH are somewhat slower than the bucket implementations, with DIKR is a little faster than DIKH except for the smaller problem sizes. The slowest code in this test is DIKF.

Figure 21 presents data for the Grid-SLong family. On this family, DIKH and DIKBD are the fastest codes. The third-fastest code is DIKBA, with DIKR close behind it and not far behind the fastest codes. Only slightly slower than DIKR is DIKF. The remaining two codes, DIKB and DIKBM, are significantly slower.

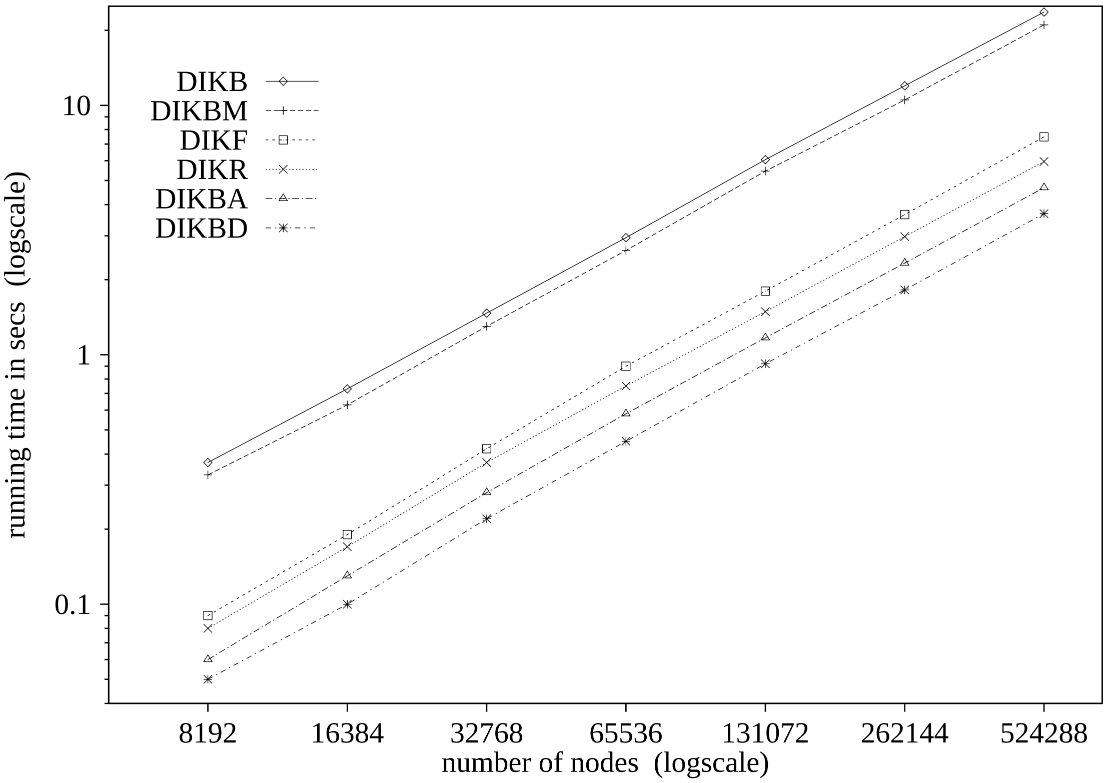
Figure 22 presents data for the Grid-SSquare-S family. Here DIKBA performs best and DIKBD is somewhat worse on smaller problems but catches up with DIKBA on the larger problems. The code DIKR is somewhat slower; DIKF and DIKH are significantly slower than the fastest codes, and DIKBM is slower than DIKH.

Figure 23 presents data for the Grid-PHard family. Here DIKR performs best, with DIKBD a very close second. Another code that does very well on these problems is DIKBM. The



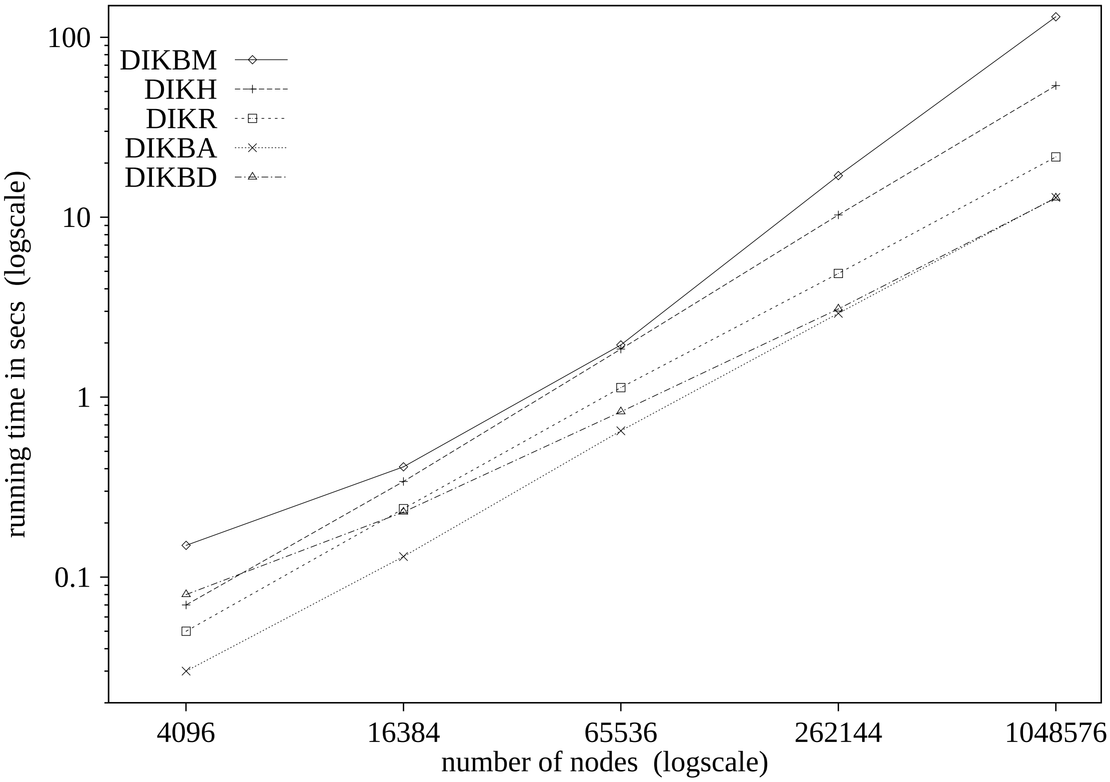
nodes/arcs	DIKH	DIKF	DIKR	DIKB	DIKBM	DIKBA	DIKBD
8193	**0.08**	**0.14**	**0.09**	**0.05**	**0.05**	**0.03**	**0.05**
24576	1.00	1.00	1.00	1.00	1.00	1.00	1.00
16385	**0.20**	**0.35**	**0.22**	**0.08**	**0.11**	**0.07**	**0.10**
49152	1.00	1.00	1.00	1.00	1.00	1.00	1.00
32769	**0.50**	**0.87**	**0.50**	**0.22**	**0.27**	**0.22**	**0.26**
98304	1.00	1.00	1.00	1.00	1.00	1.00	1.00
65537	**1.29**	**2.17**	**1.13**	**0.53**	**0.62**	**0.53**	**0.59**
196608	1.00	1.00	1.00	1.00	1.00	1.00	1.00
131073	**3.58**	**5.41**	**2.73**	**1.32**	**1.52**	**1.30**	**1.45**
393216	1.00	1.00	1.00	1.00	1.00	1.00	1.00
262145	**9.76**	**13.95**	**6.80**	**3.27**	**3.91**	**3.16**	**3.53**
786432	1.00	1.00	1.00	1.00	1.00	1.00	1.00
524289	**23.68**	**33.32**	**15.83**	**7.57**	**9.24**	**7.03**	**7.94**
1572864	1.00	1.00	1.00	1.00	1.00	1.00	1.00

FIGURE 20. *Performance of Dijkstra's implementations on Grid-SWide problems.*



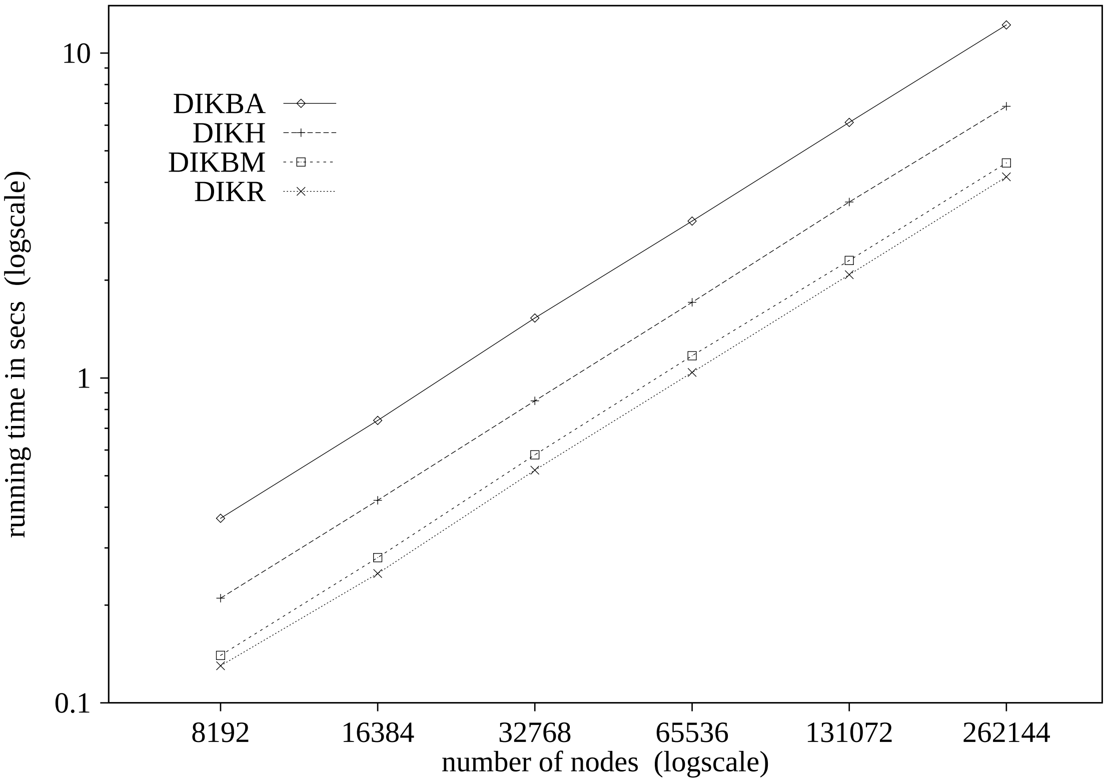
nodes/arcs	DIKH	DIKF	DIKR	DIKB	DIKBM	DIKBA	DIKBD
8193	**0.05**	**0.09**	**0.08**	**0.37**	**0.33**	**0.06**	**0.05**
24576	1.00	1.00	1.00	1.00	1.00	1.00	1.00
16385	**0.10**	**0.19**	**0.17**	**0.73**	**0.63**	**0.13**	**0.10**
49152	1.00	1.00	1.00	1.00	1.00	1.00	1.00
32769	**0.23**	**0.42**	**0.37**	**1.47**	**1.30**	**0.28**	**0.22**
98304	1.00	1.00	1.00	1.00	1.00	1.00	1.00
65537	**0.48**	**0.90**	**0.75**	**2.95**	**2.62**	**0.58**	**0.45**
196608	1.00	1.00	1.00	1.00	1.00	1.00	1.00
131073	**0.97**	**1.80**	**1.49**	**6.06**	**5.45**	**1.17**	**0.92**
393216	1.00	1.00	1.00	1.00	1.00	1.00	1.00
262145	**1.93**	**3.65**	**2.98**	**11.99**	**10.52**	**2.33**	**1.82**
786432	1.00	1.00	1.00	1.00	1.00	1.00	1.00
524289	**3.85**	**7.48**	**5.96**	**23.67**	**21.03**	**4.68**	**3.68**
1572864	1.00	1.00	1.00	1.00	1.00	1.00	1.00

FIGURE 21. *Performance of Dijkstra's implementations on Grid-SLong problems.*



nodes/arcs	DIKH	DIKF	DIKR	DIKBM	DIKBA	DIKBD
4098	**0.07**	**0.07**	**0.05**	**0.15**	**0.03**	**0.08**
16385	1.00	1.00	1.00	1.00	1.82	1.00
16386	**0.34**	**0.37**	**0.24**	**0.41**	**0.13**	**0.23**
65537	1.00	1.00	1.00	1.00	1.75	1.00
65538	**1.85**	**2.00**	**1.13**	**1.95**	**0.65**	**0.83**
262145	1.00	1.00	1.00	1.00	1.72	1.00
262146	**10.29**	**9.51**	**4.87**	**17.03**	**2.92**	**3.09**
1048577	1.00	1.00	1.00	1.00	1.70	1.00
1048578	**53.86**	**45.60**	**21.62**	**129.90**	**12.88**	**12.78**
4194305	1.00	1.00	1.00	1.00	1.70	1.00

FIGURE 22. *Performance of Dijkstra's implementations on Grid-SSquare-S problems. On these problems,* DIKB *requires too many buckets and does not run.*



nodes/arcs	DIKH	DIKF	DIKR	DIKBM	DIKBA	DIKBD
8193	**0.21**	**0.20**	**0.13**	**0.14**	**0.37**	**0.12**
63808	1.00	1.00	1.00	1.00	6.50	1.00
16385	**0.42**	**0.42**	**0.25**	**0.28**	**0.74**	**0.26**
129344	1.00	1.00	1.00	1.00	6.42	1.00
32769	**0.85**	**0.88**	**0.52**	**0.58**	**1.53**	**0.52**
260416	1.00	1.00	1.00	1.00	6.54	1.00
65537	**1.71**	**1.75**	**1.04**	**1.17**	**3.04**	**1.05**
522560	1.00	1.00	1.00	1.00	6.47	1.00
131073	**3.48**	**3.57**	**2.08**	**2.30**	**6.12**	**2.11**
1046848	1.00	1.00	1.00	1.00	6.51	1.00
262145	**6.86**	**7.13**	**4.16**	**4.59**	**12.21**	**4.23**
2095424	1.00	1.00	1.00	1.00	6.47	1.00

FIGURE 23. *Performance of Dijkstra's implementations on Grid-PHard problems. On these problems,* DIKB *requires too many buckets and does not run.*

performance of DIKH and DIKF is reasonably good, and these code perform very similarly. The worst code, DIKBA, loses to the best by about a factor of 3.

Figure 24 presents data for the Rand-Len family. On problems with small lengths, DIKB, DIKBA, and DIKBD are the fastest codes and on problems with big lengths, DIKBM is the fastest. However, the difference among all these codes is small, except that DIKB exceeds its limit on the number of bucket and does not run on the problems with the biggest arc length range. Somewhat slower than the fastest codes is DIKH. The code DIKF is the slowest except on the problem with the biggest arc lengths, where it is the second slowest.

11. Discussion

The purpose of this section is to evaluate the experimental results and to provide theoretical explanations of the observed behavior of the algorithms.

Our experimental data motivated an interesting theoretical discovery which we describe next. We say that two instances of the shortest path problem are *equivalent* if the underlying networks, including their representations, are identical and the two length functions, ℓ' and ℓ'', satisfy $\ell'_d = \ell''$ for some potential function d. (If networks are given in the adjacency list representations, identical representations have the corresponding nodes and arcs appearing in the same order.) A labeling shortest paths algorithm is *potential-invariant* if it performs the same sequence of node scans on two equivalent problem instances. Figure 16 shows that GOR, DIKH, DIKBD, and THRESH algorithms are not potential-invariant and suggests that the other algorithms in the figure are potential-invariant.

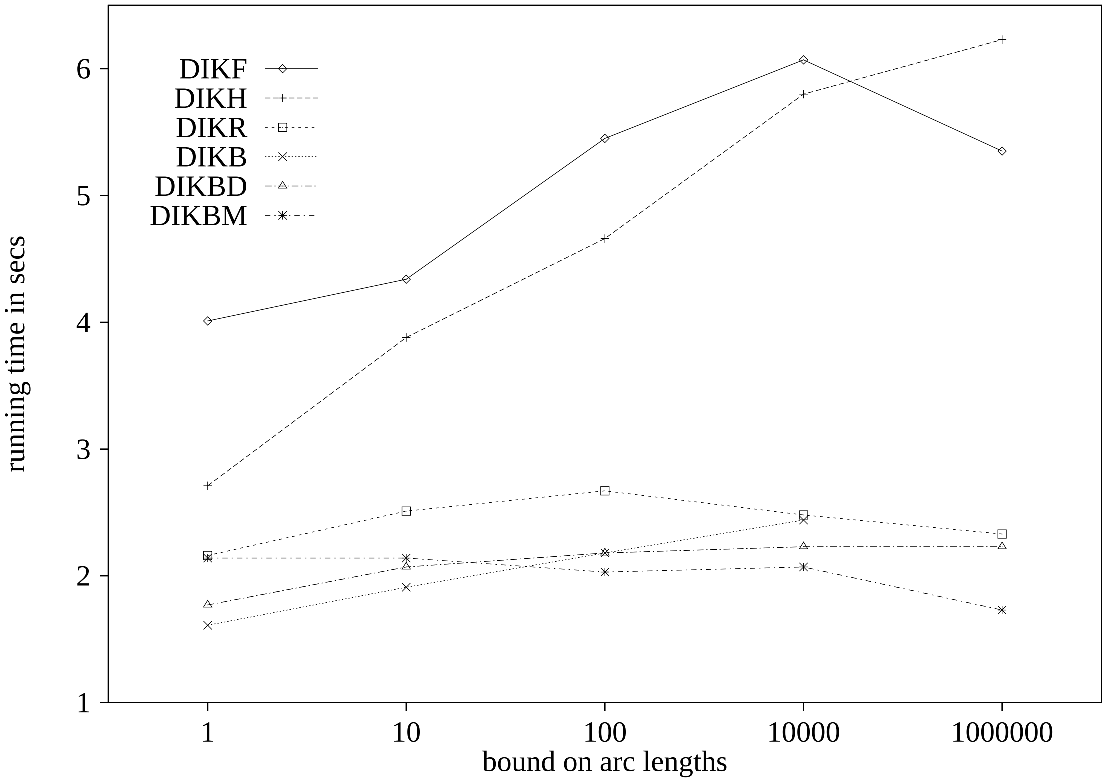
Theorem 11.1. *Algorithms* BF, BFP, GOR1, PAPE, *and* TWO_Q *are potential-invariant.*

The proof of this theorem is straightforward from the following lemma. The lemma follows from the fact that replacing arc lengths by reduced costs with respect to a potential function does not change the difference in lengths of two paths between the same pair of nodes.

Lemma 11.2. *If on any fixed graph the behavior of a labeling algorithm depends only on the relative lengths of paths from the source node to other nodes, than the algorithm is potential-invariant.*

Note that GOR is not potential-invariant because, for example, during the first depth-first search an arc may or may not belong to the admissible graph depending on its input length.

Theorem 11.1 is powerful and useful. For example, the theorem shows that no heuristic for computing a "good" initial potential function can improve performance of a potential-invariant algorithm such as BF. Note that any feasible shortest paths problem has an equivalent one with nonnegative arc lengths. If the problem with nonnegative arc lengths is computationally



$[L,U]$	DIKH	DIKF	DIKR	DIKB	DIKBM	DIKBA	DIKBD
$[1,1]$	**2.71**	**4.01**	**2.16**	**1.61**	**2.14**	**1.62**	**1.77**
	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$[0,10]$	**3.88**	**4.34**	**2.51**	**1.91**	**2.14**	**1.91**	**2.07**
	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$[0,100]$	**4.66**	**5.45**	**2.67**	**2.18**	**2.03**	**2.20**	**2.18**
	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$[0,10000]$	**5.80**	**6.07**	**2.48**	**2.44**	**2.07**	**2.30**	**2.23**
	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$[0,1000000]$	**6.23**	**5.35**	**2.33**		**1.73**	**2.27**	**2.23**
	1.00	1.00	1.00		1.00	1.05	1.00

FIGURE 24. *Performance of Dijkstra's implementations on Rand-Len problems. For the largest length interval,* DIKB *requires too many buckets and does not run.*

simpler than the general problem, the theorem suggests that a potential-invariant algorithm cannot be superior to all other algorithms on problems with nonnegative arc lengths.

Next we discuss performance of individual algorithms.

11.1. Bellman-Ford-Moore Algorithm. In this section we discuss the BFP code. This discussion also applies to BF.

Theorem 4.1 suggests that the number of passes of BFP depends on the depth of the shortest paths tree. The wide and long grid experiments (Figures 9 and 10) show how much the tree depth affects the performance. For the wide grids, the tree is likely to be very shallow, while for the long grids with $n+1$ nodes the tree depth must be at least $n/16$. The performance difference is as the theory suggests: BFP is asymptotically much faster on the wide grids than on the long ones.

The number of node scans is usually a good measure of performance of BF and BFP. The number of scans depends on both the number of passes over the queue (related to the shortest paths tree depth) and on the average number of nodes scanned during a pass. Our parent-checking heuristic tries to reduce the latter parameter.

On problems with unit arc lengths, BF behaves like breadth-first search and does one scan per node reachable from the source, and BFP behaves in exactly the same way but is slightly slower because of the parent checks (which always come out negative). See Figure 4. The number of scans does not depend on the number of passes; if the number of passes is large, the average number of nodes in the queue is small.

The Bellman-Ford-Moore algorithm works well on networks with small shortest paths tree depth. This algorithm also works well on networks with highly "metric"arc lengths, such as small nonnegative lengths. (See Section 11.3 for a discussion of "metric" length functions.) In general, however, the algorithm does not perform very well relative to the best codes. It performs especially poorly on Grid-SLong, Grid-PHard, Grid-NHard, and Acyc-Neg problem families. We note that GOR never loses to BFP by more than a factor of 2 in our experiments and performs reasonably where BFP does poorly.

11.2. Dijkstra's Algorithm. First we discuss relative performance of the implementations of Dijkstra's algorithm on networks with nonnegative length functions (Figures 21 – 24). On these networks, all implementations we consider except for DIKBA do one scan per node reachable from the source. The difference in the running time of these implementations is due to the different work involved in selecting a labeled node with the minimum distance label. Note that on dense graphs this work is small compared to the work involved in the node scans, so the code performance is nearly identical. (Compare DIKH and DIKBD on the Rand-1:4 family.)

The k-ary heap implementation, DIKH, is the second-worst on Grid-SWide, Grid-SSquare-S, and Rand-Len problems. This is because the heap operations are relatively expensive unless the number of elements on the heap is small. The number of elements on the heap is large on Grid-SWide and Grid-SSquare-S problems and small on Grid-SLong problems. On the latter problems, the implementation works very well, being just a little slower than the fastest code. The implementation performs reasonably on Grid-PHard problems.

The R-heap implementation, DIKR, is usually better than the DIKH implementation except on GRID-SLONG family. This implementation is the best on Grid-PHard problems. The implementation, however, is noticeably worse than the best ones on Grid-SWide and Grid-SSquare-S problems.

The Fibonacci heap code, DIKF, is usually slower than the DIKH code and is always slower than DIKR in our tests.

The potential for large memory requirements is one of the problems of the bucket implementation DIKB. Because of this, the implementation does not run on Grid-SSquare-S problems, Grid-PHard problems, and the Rand-Len problems with the biggest length range. Another problem of this implementation is that it may examine a large number of empty buckets. This is the case, for example, on Grid-SLong problems, where DIKB is the slowest code. The code worked reasonably well on Grid-SWide and Grid-PHard families, and on those Rand-Len problems on which it ran.

The drawback of the overflow bag implementation, DIKBM, is that the bag size can be large and the bag may be examined many times. This happens on Grid-SSquare-S problems where all nodes (except for the source) are placed in the bag at the beginning of the computation and relatively few are removed at each stage. This also happens on Grid-SLong problems where the graph has long paths. The implementation performs poorly on these problems. The implementation works very well on Rand-Len problems, and reasonably well on Grid-SWide and Grid-PHard problems.

The DIKBA implementation works very well on all the problem families except Grid-PHard. Unlike DIKB, this implementation has to look at fewer buckets. On the negative side, nodes may be scanned more than once, but on most of our problem classes the number of scans per node was small. Grid-PHard problems have many arcs of small length and DIKBA makes about 6.5 scans per node on these problems. As a result, DIKBA performed poorly on this family.

The DIKBD is the best or nearly the best code on all problems except Grid-SSquare-S problems of small sizes, where it is slower than DIKBA. But even on these problems DIKBD loses by less than a factor of 3. This code works well because if a high-level bucket is empty, the code skips it, and if the bucket is full, the code deals with it in a way that is in general more efficient compared to DIKBA. The reason for the relatively poor performance of DIKBD on small Grid-SSquare-S problems is that the value of C is very large only because of the artificial arcs, and the choice of Δ made by the implementation is much larger than it should be ideally. As a result, on small problems the work involved in examining empty buckets dominates.

The DIKBD code seems the best overall implementation of Dijkstra's algorithm in our study. If the length function is nonnegative, DIKBD performs well. It is fastest or nearly fastest on the Grid-PHard, Rand-4, Rand-1:4, Rand-Len, Acyc-Pos families and the large Grid-SSquare-S problems. On other problems with nonnegative arc lengths, DIKBD is always within a factor of 4 from the fastest code.

On problems with many negative arcs DIKBD may be extremely slow. It happens on the Grid-NHard and Acyc-Neg problems. However, if the fraction of negative arcs is small, DIKBD may work well, as Rand-P and Acyc-P2N experiments show.

In practice, a good bound on the maximum arc length is often available. Some other characteristics of the length function, such as the minimum arc length and the fractions of big and small length arcs, may also be known. This information can be used to select better values for the parameters in the bucket-based implementations of Dijkstra's algorithm and improve performance of these implementations.

11.3. The Incremental Graph Algorithms. The performance of the incremental graph codes PAPE and TWO_Q is mixed: excellent on some problem families and terrible on others.

These codes perform extremely well on simple grid problems without the artificial source, where they average at about 1 to 1.5 scans per node. Since in these codes the overhead of selecting the next node to be scanned is very small, it is hard to beat these algorithms by more than 30% on such a family.

For unit arc length networks in the Rand-Len experiments both incremental graph algorithms, and also the threshold algorithm, make one scan per node. In general, these algorithms make at most one scan per node on networks with arc unit length. One can show this using the fact that the low priority set is maintained as a FIFO queue and checking that the high priority set is irrelevant in this case. (The high priority set in PAPE and TWO_Q is always empty. If the high priority set NOW in THRESH becomes empty, it acquires all nodes from the low priority set NEXT.) Thus on networks with unit arc length these algorithms work essentially in the same way as BF.

On the other hand, the incremental graph codes perform poorly on Grid-SSquare-S, Grid-PHard, Grid-NHard, Rand-4, Rand-1:4, and acyclic graph problems. The poor performance of the codes on the Grid-SSquare-S family is due to the fact that all nodes become labeled during the scan of the artificial source, which is the first scan performed by the algorithms. As a result, on this family (and any other problem with an artificial source), PAPE works like STACK and TWO_Q works like BF. Since STACK and BF work poorly on Grid-SSquare-S problems, so do PAPE and TWO_Q.

In general, PAPE and TWO_Q seem to perform poorly on graphs with highly "nonmetric" length functions, *i.e.,* length functions with many violations of the triangle inequality. For example, on Acyc-Pos graphs, a violation of the triangle inequality is possible since a sum of two random numbers can be less then the third number picked from the same nonnegative distribution. For Acyc-Neg graphs this violation is much more likely, however, because the distribution is nonpositive, and the algorithms perform much worse. Intuitively, if $\ell(u, v) + \ell(v, w) < \ell(u, w)$ and an incremental graph algorithm places u and w into the high-priority set before v, adding v to the set is likely to cause the algorithm to recompute the distance label values of w and its successors in the current shortest path tree. If the number of violations of the triangle inequality is large, the number of scans per node is likely to be high. Although we are unable to prove formally that nonmetric length functions are bad for PAPE and TWO_Q, this seems to be the case.

We would like to note that when PAPE and TWO_Q perform well, they seem to do a similar number of scans per node and their running times are close, with PAPE usually slightly faster because of a simpler low-level implementation. When the codes perform poorly, TWO_Q is significantly faster than PAPE.

11.4. The Threshold Algorithm. The performance of THRESH is also mixed. This code performed well on the simple grid networks and the unit length networks. However, the code performed poorly on Grid-PHard, Grid-NHard, Rand-4, Rand-1:4, and acyclic graph problems.

We would like to note that since THRESH examines the NEXT list at every iteration but does not, in general, scan all the nodes on NEXT, the running time of THRESH is not necessarily proportional to the number of scans. The threshold parameter, however, is computed in such a way that the algorithm tends to scan a constant fraction of the nodes on NEXT at each iteration, so often the number of scans is a good the algorithm performance.

In a sense, THRESH is a compromise between the Bellman-Ford-Moore algorithm (which scans all labeled nodes at each iteration) and Dijkstra's algorithm (which scans a labeled node with the minimum distance label). Although THRESH compares favorably with the former algorithm, never loosing to BFP by more than a factor of 2 except for the Acyc-Neg and Acyc-P2N families in our tests, it does not look as good when compared with DIKBD. While DIKBD is always within a factor of 3 from THRESH, DIKBD is orders of magnitude faster on problem families such as Grid-PHard, Rand-4, and Rand-1:4.

We did not attempt to improve the performance of THRESH by adjusting its parameters, and it may be possible to improve the overall performance of the algorithm by fine-tuning. It is unlikely, however, that this will make the algorithm competitive on the problems where it performs poorly in our tests.

11.5. The Topological Ordering Algorithms. The topological ordering algorithms GOR and GOR1 are the most robust algorithms in our study. These are the only algorithms, for example, that solved all Grid-NHard problems within the time limit.

An examination of Figures 7 – 19 shows that GOR never loses to BFP, PAPE, or TWO_Q by more than a factor of 3 while it often wins by orders of magnitude. The performance of GOR is good on all SPGRID families we consider except the Grid-PHard family. The code also works very well on the Acyc-Neg family; it can be shown that GOR does at most two scans per node on an acyclic network with nonpositive length function.

Although GOR never loses by two orders of magnitude or more in our tests, it is slower than the fastest codes by about an order of magnitude for some problem sizes on the Rand-4, Rand-1:4, Rand-Len, Acyc-Neg, and Acyc-P2N problems.

On acyclic networks, GOR1 works well and Theorem 4.12 proves that this must be the case. The code is also the best on Grid-NHard problems and it performs reasonably well on Grid-SSquare-S and Grid-SWide problems. The code performs poorly on Grid-SSquare, Grid-SLong, Rand-4, and Rand-1:4 families, where it loses by an order of magnitude for some problem sizes.

12. Concluding Remarks

Our study does not produce a single best code for all classes of shortest paths problems. We can, however, suggest two algorithms, one for networks with negative arcs and one for networks without negative arcs. These algorithms may not be the best on a particular problem class, but their running time is likely to be of the same order of magnitude as that of the fastest algorithm and often will be much closer.

For problems with nonnegative arc lengths, Dijkstra's algorithm is robust and an appropriate implementation of this algorithm is usually quite competitive. In our tests, the double bucket implementation, DIKBD, is the best overall. This implementation also seems to work reasonably well if the network has a small number of negative length arcs.

For problems with many negative length arcs, GOR1 appears to be a good choice. This code also works well on graphs that have large node-induced acyclic subgraphs.

In practice, problems often have a very specific structure, and algorithms that can take advantage of this structure may perform very well. For example, practical problems are often quite "metric" and incremental graph algorithms may work well on these problems. Our experiments suggest, however, that extra care is needed if one decides to use these algorithms because small changes (such as addition of an artificial source) may drastically decrease performance of these algorithms. Our experiments give strong evidence that TWO_Q is more robust than PAPE and is a safer choice in practice.

The relatively good performance of the R-heap and the double-bucket implementations compared to the k-ary heap and bucket implementations, respectively, show that sophisticated data structures may be worth implementing. R-heaps are very promising for other algorithms using the priority queue data structure, such as the minimum-cost spanning tree algorithms. On the other hand, the relatively poor performance of the Fibonacci heap implementation compared to the k-ary heap implementation shows that a sophisticated data structure with a better theoretical worst-case bound is not neccesarily better in practice.

We evaluated the classical algorithms and the new algorithms that we considered to be most interesting and promising. We also implemented a scaling algorithm of [13]. Performance of our implementation was not especially good, but a better implementation may be possible. A careful experimental study of several other methods, such as variations of the threshold algorithm [12, 10] and the auction algorithm [3], may produce interesting results as well.

We experimented with networks without negative cycles. An interesting question is which algorithms are best at detecting a negative cycle if there is one.

Code Availability

The codes of our implementations and generators, the generator inputs used in our experiments, and a description of our network representation format are available via a mail server. To obtain the codes and the other data, send mail to `ftp-request@theory.stanford.edu` and put `send splib.tar` as the subject line. A reply will be a uuencoded tar file containing the codes, generator inputs, and documentation.

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