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# ON COMPOSITIONS WHOSE PARTS ARE POLYGONAL NUMBERS

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**Abstract.** For a given strictly increasing sequence $\{a_n\}$ of natural numbers, let $g(n)$ be the number of compositions of $n$ all of whose parts belong to $\{a_n\}$. We derive a recurrence that enables the computation of $g(n)$, as well as an estimate for $g(n)$ for large $n$. We use these results to obtain useful data concerning compositions whose parts belong to the following 3 sequences: triangular, square, and pentagonal numbers.

## 1. Introduction

If $n$ is a natural number, let $g(n)$ be the number of compositions of $n$ all of whose parts belong to a given strictly increasing sequence of natural numbers, $\{a_n\}$, with $g(0) = 1$. We show how to compute $g(n)$ via a recurrence, and also how to estimate $g(n)$ for large $n$. In a previous effort, this was done for binary and Fibonacci numbers, that is, the cases $a_n = 2^{n-1}$ and $a_n = F_{n+1}$, where $F_n$ denotes the $n^{th}$ Fibonacci number. (See [1].) Here we obtain similar results concerning triangular, square, and pentagonal numbers. In these 3 cases, we tabulate $g(n)$ for $1 \le n \le 20$.

Let $p(n)$ denote the partition function. Let $\omega(k) = \frac{k(3k-1)}{2}$, where $k \in Z$. ($\omega(k)$ is the ($k^{th}$ pentagonal number). Let $x$ denote a complex variable such that $|x| < 1$.

$$p(n) = \sum_{k \neq 0} (-1)^{k-1} p(n - \omega(k)) \tag{1}$$

**Remark.** Identity (1) is Euler's well-known recurrence for $p(n)$.

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## 2. General theorems

Our first theorem concerns a recurrence that enables computation of $g(n)$.

**Theorem 1.** *Let $\{a_n\}$ be a strictly increasing sequence of natural numbers. If $n$ is a natural number, let $g(n)$ denote the number of compositions of $n$ whose parts all belong to $\{a_n\}$. Define $g(0) = 1$, and $g(\alpha) = 0$ if $\alpha$ is not a non)-negative integer. Then for all $n \geq 1$, we have*

$$g(n) = \sum_{k\geq 1} g(n - a_k) \ .$$

**Proof.** Let the generating function for compositions with exactly $k$ elements from $\{a_n\}$ be

$$g_k(x) = \Big(\sum_{i\geq 1} x^{a_i}\Big)^k \ .$$

Let $G(x)$ be the generating function for all compositions of $n$ with summands from $\{a_n\}$. Then we have

$$G(x) = 1 + \sum_{k=1}^{\infty} g_k(x) = \Big(1 - \sum_{i=1}^{\infty} x^{a_i}\Big)^{-1} \ ,$$

and also

$$G(x) = \sum_{n=0}^{\infty} g(n)x^n \to \Big(1 - \sum_{i=1}^{\infty} x^{a_i}\Big)\Big(\sum_{n=0}^{\infty} g(n)x^n\Big) = 1 \ .$$

The conclusion now follows if we equate coefficients of like powers of $x$. ■

The next theorem provides a convenient way to estimate $g(n)$ for large $n$.

**Theorem 2.** *Let*

$$a(x) = \sum_{j=1}^{\infty} x^{a_j} \ .$$

*Let $\rho$ be the unique positive root of $a(x) = 1$. Then, as $n \to \infty$, we have*

$$g(n) \sim \frac{\rho^{-n-1}}{a'(\rho)} \ .$$

**Proof.** The function $a(x)$ has the following properties; (i ) $a(0) = 0$; (ii) $a'(0) \neq 0$; (iii) there exists $\rho$ such that $0 < \rho < 1$ and $a(\rho) = 0$;
(iv) $a(x)$ is analytic at $x = \rho$. Therefore the function

$$B(x) = \frac{1}{1 - a(x)}$$

has a simple pole at $x = \rho$, and from the local expansion of $B(x)$ at this dominant pole, we have

$$B(x) \sim \left(\frac{1}{\rho a'(\rho)}\right)\left(\frac{1}{1-\frac{x}{\rho}}\right).$$

It therefore follows that

$$g(n) = [x^n]B(x) \sim \left(\frac{1}{\rho a'(\rho)}\right)\rho^{-n} \quad \text{as } n \to \infty. \qquad \blacksquare$$

## 3. Compositions whose parts are polygonal numbers

First, let $g_s(n)$ denote the number of compositions of $n$ whose parts are square numbers. In this case, we have

$$a_n = n^2\ ,\ a(x) = \sum_{j=1}^{\infty} x^{j^2}\ ,\ \rho = .70534668\ ,\ a'(\rho) = 3.046149872\ .$$

**Theorem 3.**

$$\text{(a)}\quad g_s(n) = \sum_{k\geq 1} g_s(n-k^2);\quad \text{(b)}\quad \textit{as } n\to\infty,\ g_s(n) \sim \frac{.70534668^{-n-1}}{3.046149872}\ .$$

**Proof.** (a) follows from Theorem 1; (b) follows from Theorem 2. ■

Next, let $g_t(n)$ denote the number of compositions of $n$ whose parts are triangular numbers. In this case, we have

$$a_n = \frac{n(n+1)}{2}\ ,\ a(x) = \sum_{j=1}^{\infty} x^{\frac{j(j+1)}{2}}\ ,\ \rho = .645222707\ ,\ a'(\rho) = 3.149722816\ .$$

**Theorem 4.**

$$\text{(a)}\ g_t(n) = \sum_{k\geq 1} g_t(n - \frac{k(k+1)}{2});\quad \text{(b)}\ \textit{as } n\to\infty,\ g_t(n) \sim \frac{.645222707^{-n-1}}{3.149722816}\ .$$

**Proof.** Same as proof of Theorem 3. ■

Finally, let $g_p(n)$ denote the number of compositions of $n$ whose parts are pentagonal numbers. In this case, we have

$$a_n = \frac{n(3n\mp 1)}{2}\ ,\ a(x) = \sum_{k\neq 0} x^{\frac{k(3k-1)}{2}}\ ,\ \rho = .578044621\ ,\ a'(\rho) = 3.011573540\ .$$

**Theorem 5.**

$$\text{(a)}\quad g_p(n) = \sum_{k\neq 0} g_p(n-\omega(k)); \quad \text{(b)}\quad as\ n\to\infty,\ g_p(n) \sim \frac{.578044621^{-n-1}}{3.0115735}\ .$$

**Proof.** Same as proof of Theorem 3. ■

The table below lists $g_s(n)$ , $g_t(n)$ , $g_p(n)$ for $1 \leq n \leq 20$.

| $n$ | $g_s(n)$ | $g_t(n)$ | $g_p(n)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 |
| 3 | 1 | 2 | 3 |
| 4 | 2 | 3 | 5 |
| 5 | 3 | 4 | 9 |
| 6 | 4 | 7 | 15 |
| 7 | 5 | 11 | 27 |
| 8 | 7 | 16 | 46 |
| 9 | 11 | 25 | 80 |
| 10 | 16 | 40 | 138 |
| 11 | 22 | 61 | 238 |
| 12 | 30 | 93 | 413 |
| 13 | 43 | 147 | 713 |
| 14 | 62 | 227 | 1235 |
| 15 | 88 | 351 | 2136 |
| 16 | 124 | 546 | 3695 |
| 17 | 175 | 845 | 6393 |
| 18 | 249 | 1308 | 11057 |
| 19 | 354 | 2029 | 19130 |
| 20 | 502 | 3145 | 33091 |

Glancing at the table above, it appears that $g_p(n)$ has the same parity as the partition function, $p(n)$. This is shown in Theorem 6 below:

**Theorem 6.** $g_p(n) \equiv p(n) \pmod 2$ for all $n \geq 0$.

**Proof.** (Induction on $n$). We have $p(0) = g_p(0) = 1$. If $n \geq 1$, then by (1), induction, and Theorem 5, part (a), we have

$$p(n) \equiv \sum_{k\neq 0} p(n-\omega(k)) \equiv \sum_{k\neq 0} g_p(n-\omega(k) \equiv g_p(n) \pmod 2\,.$$ ■

The function $g_p(n)$ appears to share the following property with $p(n)$:

**Conjecture.** $g_p(5n+4) \equiv 0 \pmod 5$ for all $n \geq 0$.

## References

[1] **Knopfmacher, A. and N. Robbins,** On binary and Fibonacci compositions, *Annales Univ. Sci. Budapest., Sect. Comp.*, **22** (2003), 193–206.

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