GARLAND'S ESTIMATE REVISITED—LAPLACIANS OF SIMPLICIAL COMPLEXES
[GW16, SECTION 3.1]

Interpreted by Orr Paradise and Noam Wies – mistakes are likely ours.

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1. Reminders and definitions

Let's start with a few reminders. Let X be a pure k-dimensional simplical complex, and assume that X_0 is linearly ordered.

Definition 1.1. Let $F \in X_i$ and $G \in X_{i-1}$. Denote $F = \{v_0, \ldots, v_i\}$ where $v_0 < \cdots < v_i$. Then the **oriented incidence number** of G in F is

$$[F : G] \coloneqq \begin{cases} (-1)^j & F \setminus G = \{v_j\} \\ 0 & \text{else } (G \not\subseteq F) \end{cases}$$

The **boundary map** is the linear map ∂_i defined on basis vectors by

$$\partial_i : C^{i+1} \to C^i \quad \partial_i (e_H) = \sum_{F \in X_i} [H : F] \, e_F$$

The **coboundary map** is the linear map δ_i defined by

$$\delta_i : C^i \to C^{i+1} \quad (\delta_i f)(H) \coloneqq \sum_{F \in X_i} [H : F] \cdot f(F)$$

for $i \in \{-1, \ldots, \dim(X)\}$ and $\delta_i \coloneqq 0$ otherwise.

Definition 1.2. The **degree** of a face $F \in X$ is defined

$$\deg_X F \coloneqq |\{H \in X | F \subsetneq H\}|$$

Remark 1.3. If X is pure and k-dimensional, then for all $i < k$, for all $F \in X_i$ we have $\deg_X F > 0$.

Definition 1.4. For any weight function $w : X_i \to \mathbb{R}_{\geq 0}$ we may define an **weighted inner product** on C^i by

$$\langle f, g \rangle_w \coloneqq \sum_{F \in X_i} w(F) \, f(F) \, g(F)$$

Exercise 1.5. For each $F \in X_i$ we define $e_F \in C^i$ by $e_F(G) = \begin{cases} 1 & F = G \\ 0 & F \neq G \end{cases}$. Then $\{e_F\}_{F \in X_i}$ is an orthonormal basis for C^i w.r.t the weighted inner product when $w = \mathbf{1}$ (that is $w \equiv 1$), but it is not necisarily orthonormal w.r.t $w = \deg_X$.

Notation 1.6. We will be interested in the inner products derives when $w \equiv 1$ and when $w = \deg_X$, and specifically with adjoint maps with respect to these inner products. To avoid confusion, for a given $f \in C^i$ we denote its adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbf{1}}$ by $f^\dagger$ and its adjoint with respect to $\langle \cdot, \cdot \rangle_{\deg_X}$ with f^*.

Fact 1.7. $\delta_i^\dagger = \partial_{i+1}$

Definition 1.8. The i-**th upper laplacian** is the map defined

$$L_i^{up} : C^i \to C^i \quad L_i^{up} \coloneqq \partial_{i+1}\delta_i = \delta_i^\dagger \delta_i$$

The i-**th normalized laplacian** is the map defined

$$\Delta_i^{up} : C^i \to C^i \quad \Delta_i^{up} \coloneqq \delta_i^* \delta_i$$

2. Matrix representation of the normalized laplacian

We now turn to compute the matrices corresponding to L_i^{up}, Δ_i^{up} under the basis $\{e_F\}_{F\in X_i}$ where $e_F(G) = \begin{cases} 1 & F = G \\ 0 & F \neq G \end{cases}$

Definition 2.1. D_i, W_i are the diagonal $|X_i| \times |X_i|$ matrices defined

$$(D_i)_{F,F} \coloneqq |\{H \in X_{i+1} | F \subseteq H\}|$$

$$(W_i)_{F,F} \coloneqq \deg_X F$$

For $F, G \in X_i$ we say that $F \sim G$ if $F \cap G \in X_{i-1}$ and $F \cup G \in X_{i+1}$. We then define the i-**th adjacency matrix** to be

$$(A_i)_{F,G} \coloneqq \begin{cases} -[F \cup G : F][F \cup G : G] & F \sim G \\ 0 & \text{else} \end{cases}$$

Claim 2.2. The matrix corresponding to L_i^{up} under $\{e_F\}_{F\in X_i}$ is $D_i - A_i$.

Proof. Since $\{e_F\}_{F\in X_i}$ is an orthonormal basis w.r.t $\langle\cdot,\cdot\rangle_{\mathbf{1}}$, then $(L_i^{up})_{F,G} = \langle L_i^{up} e_F, e_G\rangle_{\mathbf{1}}$ for all $F, G \in X_i$. We have

$$\begin{aligned}
(L_i^{up})_{F,G} &= \langle L_i^{up} e_F, e_G\rangle_{\mathbf{1}} \\
&= \langle \partial_{i+1}\delta_i e_F, e_G\rangle_{\mathbf{1}} \\
\overset{1.7}{} &= \left\langle \delta_i^\dagger \delta_i e_F, e_G \right\rangle_{\mathbf{1}} = \langle \delta_i e_F, \delta_i e_F\rangle_{\mathbf{1}} \\
&= \sum_{H\in X_{i+1}} \delta_i e_F(H)\, \delta_i e_F(G) \\
&= \sum_{H\in X_{i+1}} \left(\sum_{F'\in X_i} [H : F']\, e_F(F') \right) \left(\sum_{G'\in X_i} [H : G']\, e_G(G') \right) \\
&= \sum_{H\in X_{i+1}} [H : F][H : G]
\end{aligned}$$

- If $F = G$, since $F \not\sim F$ always, then $(A_i)_{F,F} = 0$ and therefore

$$\sum_{H\in X_{i+1}} [H : F][H : F] = |\{H \in X_{i+1} | F \subseteq H\}| = (D_i)_{F,F} = (D_i - A_i)_{F,F}$$

- If $F \neq G$, then $(D_i)_{F,G} = 0$
  - If also $F \sim G$, then $[H : F]$ and $[H : G]$ are nonnzero only when $H = F \cup G$, and then

$$\begin{aligned}
\sum_{H\in X_{i+1}} [H : F][H : G] &= [F \cup G : F][F \cup G : G] \\
&= (-A_i)_{F,G} = (D_i - A_i)_{F,G}
\end{aligned}$$

  - Else $F \not\sim G$ and then $[H : F] \neq 0 \implies [H : G] = 0$ (because otherwise $H = F \cup G$ which implies $F \sim G$) and then

$$\sum_{H\in X_{i+1}} [H : F][H : G] = 0 = (D_i - A_i)_{F,G}$$

□

Claim 2.3. Denote the matrix corresponding to δ_i under the bases $\{e_G\}_{G\in X_i}, \{e_F\}_{F\in X_{i+1}}$ by M. Then $M_{F,G} = [F:G]$.

Proof. Since the bases are orthonormal

$$\begin{aligned} M_{F,G} &= \langle \delta_i e_G, e_F \rangle_{\mathbf{1}} \\ &= \sum_{F'\in X_{i+1}} (\delta_i e_G)(F')\, e_F(F') \\ &= (\delta_i e_G)(F) \\ &= \sum_{G'\in X_i} [F:G']\, e_G(G') \\ &= [F:G] \end{aligned}$$

□

Corollary 2.4. *Since the bases are orthonormal w.r.t $\langle\cdot,\cdot\rangle_{\mathbf{1}}$, then the the matrix corresponding to $\delta_i^\dagger$ under the bases $\{e_F\}_{F\in X_{i+1}}, \{e_G\}_{G\in X_i}$ is $M^\top$.*

Claim 2.5. For $i < k$,

$$\delta_i^*(f)(G) = \sum_{F\in X_{i+1}} \frac{\deg_X F}{\deg_X G}[F:G]\, f(F)$$

Proof. Denote the map on the right hand side of the equation with ρ. To prove that $\rho = \delta_i^*$ it suffices to show that ρ is adjoint to δ_i, since the adjoint map is unique. Indeed

$$\begin{aligned} \langle \delta_i f, g\rangle_{\deg_X} &= \sum_{F\in X_{i+1}} \deg_X F\, (\delta_i f)(F)\, g(F) \\ &= \sum_{F\in X_{i+1}} \deg_X F g(F) \sum_{G\in X_i} [F:G]\, f(G) \\ &= \sum_{F\in X_{i+1}} \sum_{G\in X_i} \frac{\deg_X G}{\deg_X G} \deg_X(F)\, g(F)\, [F:G]\, f(G) \\ &= \sum_{G\in X_i} \deg_X G f(G) \sum_{F\in X_{i+1}} \frac{\deg_X F}{\deg_X G}[F:G]\, g(F) \\ &= \langle f, \rho(g)\rangle_{\deg_X} \end{aligned}$$

□

Corollary 2.6. *The F,G-th entry in the matrix corresponding to δ_i^* under $\{e_F\}_{F\in X_{i+1}}$,$\{e_G\}_{G\in X_i}$ is $\frac{\deg_X F}{\deg_X G}[F:G]$*

Proof. From the above

$$\begin{aligned} (\delta_i^* e_F)(G) &= \sum_{F'\in X_{i+1}} \frac{\deg_X F'}{\deg_X G}[F':G]\, e_F(F') \\ &= \frac{\deg_X F}{\deg_X G}[F:G] \end{aligned}$$

and therefore

$$\delta_i^* e_F = \sum_{G\in X_i} \frac{\deg_X F}{\deg_X G}[F:G]\, e_G$$

□

Claim 2.7. The matrix corresponding to Δ_i^{up} under $\{e_F\}_{F\in X_i}$ is $W_i^{-1}\delta_i^\top W_{i+1}\delta_i$

Proof. Recall that W_i^{-1} is the $|X_i| \times |X_i|$ diagonal matrix where $\left(W_i^{-1}\right)_{F,F} = \frac{1}{\deg_X F}$, and notice that $\left(\delta_i^\top\right)_{F,H} = [H:F]$ and so since W_i^{-1}, W_{i+1} are diagonal

$$\left(W_i^{-1}\delta_i^\top W_{i+1}\right)_{F,H} = \frac{1}{\deg_X F}[H:F]\deg_X H = \frac{\deg_X H}{\deg_X F}[H:F]$$

therefore

$$\begin{aligned}\left(W_i^{-1}\delta_i^\top W_{i+1}\delta_i\right)_{F,G} &= \sum_{H\in X_{i+1}} \left(W_i^{-1}\delta_i^\top W_{i+1}\right)_{F,H}(\delta_i)_{H,G} \\ &= \sum_{H\in X_{i+1}} \frac{\deg_X H}{\deg_X F}[H:F][H:G] \\ &= \sum_{H\in X_i+1} (\delta_i^*)_{F,H}(\delta_i)_{H,G} \\ &= (\delta_i^*\delta_i)_{F,G}\end{aligned}$$

□

Corollary 2.8. *When $i = k-1$ in the above, then the matrix is $I - D_{k-1}^{-1}A_{k-1}$*

Proof. Checking the definition of W_i we immediately get that $W_k = I$ and $W_{k-1} = D_{k-1}$. Therefore

$$\begin{aligned}\Delta_{k-1}^{up} &= W_{k-1}^{-1}\delta_{k-1}^\top W_k\delta_{k-1} = D_{k-1}^{-1}\delta_{k-1}^\dagger I\delta_{k-1} \\ &= D_{k-1}^{-1}\delta_{k-1}^\dagger\delta_{k-1} \\ &= D_{k-1}^{-1}L_{k-1}^{up} \\ &= D_{k-1}^{-1}(D_{k-1} - A_{k-1}) \\ &= I - D_{k-1}^{-1}A_{k-1}\end{aligned}$$

□

3. Garland's estimate revisited

From this point onwards, we are only interested in the inner product derived from $w = \deg_X$. That is, we let $\langle\cdot,\cdot\rangle \coloneqq \langle\cdot,\cdot\rangle_{\deg_X}$ and say that $f \perp g$ when $\langle f, g\rangle = 0$.

Definition 3.1. For a given $F \in X$, the **link** of F is defines

$$\mathrm{lk}F \coloneqq \{G \in X | F \sqcup G \in X\}$$

If $F \in X_{k-2}$ then $\dim(\mathrm{lk}F) \leq 1$. That is, $\mathrm{lk}F$ is a graph. In this case we identify $A(\mathrm{lk}F) = A_0(\mathrm{lk}F)$, $D(\mathrm{lk}F) = D_0(\mathrm{lk}F)$, and $\Delta(\mathrm{lk}F) = \Delta_0(\mathrm{lk}F)$.

Exercise 3.2. Check that the definitions for A, D, Δ indeed coincide for graphs and simplical complexes of dimension ≤ 1.

Definition 3.3. For a pure k-dimensional complex X, let $F \in X_{k-2}$. Define $\Delta_{k-1}^{up,F}$ by nullifying all entries of Δ_{k-1}^{up} residing in a row or column corresponding to a face that does not contain F.

Additionally, for $f \in C^{k-1}(X)$ we define $f_F \in C^0(\mathrm{lk}F)$ by

$$f_F(\{u\}) \coloneqq [F \sqcup \{u\} : F]\, f(F \sqcup \{u\})$$

Remark 3.4. Formally, $\Delta_{k-1}^{up,F}$ is defined by defining the $|X_{k-1}|\times|X_{k-1}|$ diagonal matrix $(\rho_F)_{G,G} = \begin{cases}1 & F \subseteq G \\ 0 & \text{else}\end{cases}$ and letting

$$\Delta_{k-1}^{up,F} \coloneqq \rho_F\Delta_{k-1}^{up}\rho_F$$

3.1. **Lemmata (Lemma 10 in the paper).** Before proving the theorem we wish to explore the relation between the local and global properties of the normalized laplacian.

Lemma 3.5.

$$\sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F}(X) = \Delta_{k-1}^{up}(X) + (k-1)\, I$$

Proof. Let $G, H \in X_{k-1}$.

If $G = H$, since $(A_{k-1})_{G,G} = 0$ then $\left(\Delta_{k-1}^{up}\right)_{G,G} = 1$ and therefore $\left(\Delta_{k-1}^{up}(X) + (k-1)\, I\right)_{G,G} = k$. On the other hand,

$$|\{F \in X_{k-2} | F \subseteq G\}| = |G| = k$$

since each F is obtained by removing an element of G. Since $F \not\subseteq G \implies \left(\Delta_{k-1}^{up,F}\right)_{G,G} = 0$ and otherwise $\left(\Delta_{k-1}^{up,F}\right)_{G,G} = 1$ (because as stated above the diagonal is always all 1s), we get that

$$\left(\sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F}(X)\right)_{G,G} = k$$

If $G \neq H$ then

$$\left(\Delta_{k-1}^{up}(X) + (k-1)\, I\right)_{G,H} = \left(\Delta_{k-1}^{up}\right)_{G,H}$$

and one of the following holds:

- If $\left(\Delta_{k-1}^{up}\right)_{G,H} = 0$ then $\forall F \in X_{k-2}$ $\left(\Delta_{k-1}^{up,F}\right)_{G,H} = 0$ since $\Delta_{k-1}^{up,F}$ is obtained by nullifying some elements of Δ_{k-1}^{up}. So the equality holds.
- If $\left(\Delta_{k-1}^{up}\right)_{G,H} \neq 0$, since

  $$\left(\Delta_{k-1}^{up}\right)_{G,H} = \left(I - D_{k-1}^{-1} A_{k-1}\right)_{G,H} = \left(-D_{k-1}^{-1} A_{k-1}\right)_{G,H}$$

  we have $(A_{k-1})_{G,H} \neq 0$, therefore $G \sim H$ which means that $G \cap H \in X_{k-2}$. For all $F \neq G \cap H$, $F \not\subseteq G$ or $F \not\subseteq H$ and so $\left(\Delta_{k-1}^{up,F}\right)_{G,H} = 0$. Also

  $$\left(\Delta_{k-1}^{up,G\cap H}\right)_{G,H} = \left(\Delta_{k-1}^{up}\right)_{G,H}$$

  and therefore

  $$\left(\sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F}(X)\right)_{G,H} = \sum_{F\in X_{k-2}} \left(\Delta_{k-1}^{up,F}(X)\right)_{G,H} = \left(\Delta_{k-1}^{up}\right)_{G,H}$$

□

Lemma 3.6. *For $u, v \in \mathrm{lk}F$ let $F_u := F \sqcup \{u\}, F_v := F \sqcup \{v\}$. Then*

$$\left(\Delta_{k-1}^{up,F}(X)\right)_{F_u,F_v} = [F_u : F]\,[F_v : F]\,(\Delta(\mathrm{lk}F))_{u,v}$$

Proof. Notice that for all $w \in \mathrm{lk}F$ $\deg_X F_w = \deg_{\mathrm{lk}F} w$, because $F \in X_{k-2}$ and so $\dim_X(F_w) = k-1$ so it is $\subsetneq$ only in maximal faces, so $F_w \subsetneq G$ for $G \in X_k$ iff $G = F_w \sqcup \{w'\} \in X_k$ iff $\{w, w'\} \in \mathrm{lk}F$.

Assume that $u \neq v$, let $F_{u,v} := F \sqcup \{u, v\}$ and assume $F_{u,v} \in X$. We claim that

$$[F_{u,v} : F_u]\,[F_{u,v} : F_v] = -[F_u : F]\,[F_v : F]$$

Indeed, assume WLOG $u < v$ and let $F_{u,v} = \{w_0 < \cdots < w_i = u < \cdots < w_j = v < \cdots < w_k\}$, then direct computation shows that

$$[F_{u,v} : F_u] = (-1)^j, [F_{u,v} : F_v] = (-1)^i, [F_u : F] = (-1)^i, [F_v : F] = (-1)^{j-1}$$

and so

$$[F_{u,v} : F_u]\,[F_{u,v} : F_v] = (-1)^j (-1)^i = -(-1)^i (-1)^{j-1} = -[F_u : F]\,[F_v : F]$$

And now, noticing that $(\Delta \mathrm{lk}F)_{u,v} = \left(I - D^{-1}(\mathrm{lk}F) A(\mathrm{lk}F)\right) = -\frac{1}{\deg_{\mathrm{lk}F} u}$ and that $F_u \sim F_v$,

$$\begin{aligned}
\left(\Delta_{k-1}^{up,F}(X)\right)_{F_u,F_v} &= \left(I - D_{k-1}^{-1}(X) A_{k-1}(X)\right)_{F_u,F_v} = \left(-D_{k-1}^{-1}(X) A_{k-1}(X)\right)_{F_u,F_v} \\
&= -\frac{1}{\deg_X F_u} \left(A_{k-1}(X)\right)_{F_u,F_v} \\
&= -\frac{-[F_u \cup F_v : F_u][F_u \cup F_v : F_v]}{\deg_X F_u} \\
&= \frac{[F_{u,v} : F_u][F_{u,v} : F_v]}{\deg_X F_u} \\
&= \frac{-[F_u : F][F_v : F]}{\deg_{\mathrm{lk}F} u} = [F_u : F][F_v : F](\Delta \mathrm{lk}F)_{u,v}
\end{aligned}$$

For the case when $u \neq v$ and $F_{u,v} \notin X_k$, we have that $\{u,v\} \notin \mathrm{lk}F$ and thus $(A(\mathrm{lk}F))_{u,v} = 0$ so $(\Delta \mathrm{lk}F)_{u,v} = 0$. On the left side, $\left(\Delta_{k-1}^{up,F}(X)\right)_{F_u,F_v} = 0$ since $F_u \not\sim F_v$.

The final case is that $u = v$. Then $\left(\Delta_{k-1}^{up,F}(X)\right)_{F_u,F_u} = 1$ because we are on the diagonal and $F \subseteq F_u$, and on the right side we similarly have $(\Delta \mathrm{lk}F)_{u,u} = 1$ and so

$$[F_u : F][F_u : F](\Delta \mathrm{lk}F)_{u,u} = 1 \cdot 1 = 1$$

□

Corollary 3.7. *For $f \in C^{k-1}(X)$,*

$$\left\langle \Delta_{k-1}^{up,F}(X) f, f \right\rangle = \langle \Delta(\mathrm{lk}F) f_F, f_F \rangle$$

Proof. Identifying $f \in C^{k-1}$ with its matrix representation under $\{e_F\}_{F \in X_{k-1}}$, for all $G \in X_{k-1}$ we have

$$\left(\Delta_{k-1}^{up,F}(X) f\right)(G) = \sum_{H \in X_{k-1}} \left(\Delta_{k-1}^{up,F}\right)_{G,H} (f)_H = \sum_{H \in X_{k-1}} \left(\Delta_{k-1}^{up,F}\right)_{G,H} f(H)$$

and similarly

$$(\Delta \mathrm{lk}F f_F)(u) = \sum_{v \in \mathrm{lk}F} (\Delta \mathrm{lk}F)_{u,v} f_F(v)$$

and so

$$\begin{aligned}
\left\langle \Delta_{k-1}^{up,F}(X) f, f \right\rangle &= \sum_{G \in X_{k-1}} \deg_X G \left(\Delta_{k-1}^{up,F}(X) f\right)(G) f(G) \\
&= \sum_{G \in X_{k-1}} \deg_X G \sum_{H \in X_{k-1}} \left(\Delta_{k-1}^{up,F}\right)_{G,H} f(H) f(G) \\
(*) \quad &= \sum_{u \in \mathrm{lk}F} \deg_X F_u \sum_{v \in \mathrm{lk}F} \left(\Delta_{k-1}^{up,F}\right)_{F_u,F_v} f(F_v) f(F_u) \\
\text{def of } f_F \text{ and since } \deg_X F_w = \deg_{\mathrm{lk}F} w \quad &= \sum_{u \in \mathrm{lk}F} \deg_{\mathrm{lk}F} F_u \sum_{v \in \mathrm{lk}F} \left(\Delta_{k-1}^{up,F}\right)_{F_u,F_v} [F_v : F] f_F(v) [F_u : F] f_F(u) \\
3.7 \quad &= \sum_{u \in \mathrm{lk}F} \deg_{\mathrm{lk}F} F_u f_F(u) \sum_{v \in \mathrm{lk}F} [F_u : F]^2 [F_v : F]^2 (\Delta(\mathrm{lk}F))_{u,v} f_F(v) \\
&= \sum_{u \in \mathrm{lk}F} \deg_{\mathrm{lk}F} F_u f_F(u) \sum_{v \in \mathrm{lk}F} (\Delta(\mathrm{lk}F))_{u,v} f_F(v) \\
&= \sum_{u \in \mathrm{lk}F} \deg_{\mathrm{lk}F} F_u f_F(u) (\Delta(\mathrm{lk}F) f_F)(u) \\
&= \langle \Delta(\mathrm{lk}F) f_F, f_F \rangle
\end{aligned}$$

(∗) we may sum only on $G = F_u$'s because if $G \neq F_u$ then $F \not\subseteq G$ and then $\left(\Delta_{k-1}^{up,F}\right)_{G,H} = 0$ for all $H \in X_{k-1}$. □

Lemma 3.8. *If $f \in B^{k-1}(X)^{\perp}$ then $f_F \perp \mathbf{1}$, where $\mathbf{1}$ denotes the all-ones vector.*

Proof. As seen in the proof of 2.3 $\delta_{k-2}e_F(G) = [G:F]$, and by definition $\delta_{k-2}e_F \in B^{k-1}(X)$ for all $F \in X_{k-2}$, therefore

$$\begin{aligned}
\langle f_F, \mathbf{1}\rangle &= \sum_{u\in \mathrm{lk}F} (\deg_{\mathrm{lk}F} u) f_F(u) \\
&= \sum_{u\in \mathrm{lk}F} \deg_X F_u \, [F_u : F] f(F_u) \\
&= \sum_{G\in X_{k-1}} \deg_X G \, [G:F] f(G) \\
&= \sum_{G\in X_{k-1}} \deg_X G \delta_{k-2}e_F(G) f(G) \\
&= \langle \delta_{k-2}e_F, f\rangle = 0
\end{aligned}$$

□

3.2. **Theorem (Theorem 9 in the paper).**

Fact 3.9. *(Courant-Fischer theorem) Let $M \in \mathbb{R}^{m\times m}$ be a symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m$. Then for all $j \in [m]$*

$$\lambda_j = \min_{w_1,\ldots,w_{m-j}\in\mathbb{R}^n} \max_{0\neq x\perp w_1,\ldots,w_{m-j}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle} = \max_{w_1,\ldots,w_{j-1}\in\mathbb{R}^n} \min_{0\neq x\perp w_1,\ldots,w_{j-1}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle}$$

specifically

$$\lambda_1 = \min_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle}, \lambda_m = \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle}$$

Claim 3.10.

$$\sum_{F\in X_{k-2}} \langle f_F, f_F\rangle = k\langle f, f\rangle$$

Proof.

$$\begin{aligned}
\sum_{F\in X_{k-2}} \langle f_F, f_F\rangle &= \sum_{F\in X_{k-2}} \sum_{u\in \mathrm{lk}F} (\deg_{\mathrm{lk}F} u) f_F(u) f_F(u) \\
&= \sum_{F\in X_{k-2}} \sum_{u\in \mathrm{lk}F} (\deg_X F_u) [F_u : F] f(F_u) [F_u : F] f(F_u) \\
&= \sum_{F\in X_{k-2}} \sum_{u\in \mathrm{lk}F} (\deg_X F_u) f(F_u) f(F_u) \\
&= \sum_{G\in X_{k-1}} k (\deg_X G) f(G) f(G) \\
&= k\langle f, f\rangle
\end{aligned}$$

where the penultimate equality is because each $G \in X_{k-1}$ appears exactly k times in the two $\sum$'s, since for every $u \in G$ there is exactly one $F = G \setminus \{u\}$ such that $G = F_u$, and on the other hand each G may be obtained in such a way. Since $|G| = k$, we have that for each G, $\deg_X Gf(G)^2$ is summed over exactly k times. □

Theorem 3.11. *Let X be a pure k-dimensional complex and let $\Delta_{k-1}^{up} = \Delta_{k-1}^{up}(X)$ be its normalized Laplacian. Assume that there exist $\lambda_{\min}, \lambda_{\max}$ such that for all $F \in X_{k-2}$*

$$\lambda_{\min} \leq \lambda_2(\Delta(\mathrm{lk}F)) \leq \lambda_{n-k+1}(\Delta(\mathrm{lk}F)) \leq \lambda_{\max}$$

Then for all $f \in B^{k-1}(X)^{\perp}$

$$(1+k\lambda_{\min}-k)(\langle f,f\rangle) \leq \left\langle \Delta_{k-1}^{up} f, f\right\rangle \leq (1+k\lambda_{\max}-k)(\langle f,f\rangle)$$

and therefore all nontrivial eigenvalues of Δ_{k-1}^{up} *on* $B^{k-1}(X)^{\perp}$ *lie in*

$$[1+k\lambda_{\min}-k, 1+k\lambda_{\max}-k]$$

Proof. Let $f \in B^{k-1}(X)^{\perp}$. Define

$$S_f \coloneqq \{F \in X_{k-2} | \exists G \in X_{k-1}\ F \subseteq G \wedge f(G) \neq 0\}$$

Let $F \in S_f$. We prove that

$$\lambda_{\min} \leq \frac{\langle \Delta(\mathrm{lk}F) f_F, f_F\rangle}{\langle f_F, f_F\rangle} \leq \lambda_{\max}$$

- By definition of S_f $f_F \neq 0$ so $f_F \in C^0(\mathrm{lk}F) \setminus \{0\}$ so from 3.9

$$\lambda_{\max} \geq \lambda_{n-k+1}(\Delta(\mathrm{lk}F)) = \max_{h \in C^0(\mathrm{lk}F)\setminus\{0\}} \frac{\langle \Delta(\mathrm{lk}F) h, h\rangle}{\langle h,h\rangle} \geq \frac{\langle \Delta(\mathrm{lk}F) f_F, f_F\rangle}{\langle f_F, f_F\rangle}$$

- Observe that $D^{-1}(\mathrm{lk}F) A(\mathrm{lk}F) \mathbf{1} = \mathbf{1}$ and therefore

$$\Delta(\mathrm{lk}F)\mathbf{1} = \left(I - D^{-1}(\mathrm{lk}F) A(\mathrm{lk}F)\right)\mathbf{1} = I\mathbf{1} - D^{-1}(\mathrm{lk}F) A(\mathrm{lk}F)\mathbf{1} = \mathbf{1}-\mathbf{1} = 0$$

From 3.8 $f_F \perp \mathbf{1}$, so from 3.9

$$\lambda_{\min} \leq \lambda_2(\Delta(\mathrm{lk}F)) = \max_{g \in C^0(\mathrm{lk}F)} \min_{0 \neq h \perp g} \frac{\langle \Delta(\mathrm{lk}F) h, h\rangle}{\langle h,h\rangle}$$

Notice that if $g \perp \mathbf{1}$ then

$$\min_{0\neq h\perp g} \frac{\langle \Delta(\mathrm{lk}F) h, h\rangle}{\langle h,h\rangle} \leq \frac{\langle \Delta(\mathrm{lk}F)\mathbf{1},\mathbf{1}\rangle}{\langle \mathbf{1},\mathbf{1}\rangle} = 0$$

If $\lambda_{\min} \leq 0$ the result is trivial so assuming $\lambda_{\min} > 0$ this implies that any $g \perp \mathbf{1}$ cannot be the maximizer. So the maximum is obtained at some $g \not\perp \mathbf{1}$, but then

Summing over all $F \in S_f$ we get

$$\lambda_{\min} \sum_{F\in S_f} \langle f_F, f_F\rangle \leq \sum_{F\in S_f} \langle \Delta(\mathrm{lk}F) f_F, f_F\rangle \leq \lambda_{\max} \sum_{F\in S_f} \langle f_F, f_F\rangle$$

Notice that if $F \notin S_f$ then

$$G \in X_{k-1} \wedge F \subseteq G \implies f(G) = 0$$

therefore $f_F = 0$. Similarly if $F \in S_f$ then $f_F \neq 0$. In other words, $F \in S_f \iff f_F \neq 0$. Therefore

$$\lambda_{\min} \sum_{F\in X_{k-2}} \langle f_F, f_F\rangle \leq \sum_{F\in X_{k-2}} \langle \Delta(\mathrm{lk}F) f_F, f_F\rangle \leq \lambda_{\max} \sum_{F\in X_{k-2}} \langle f_F, f_F\rangle$$

so from 3.10

$$k\lambda \langle f,f\rangle \leq \sum_{F\in X_{k-2}} \langle \Delta(\mathrm{lk}F) f_F, f_F\rangle \leq k\lambda_{\max}\langle f,f\rangle$$

and since $F \in S_f \iff f_F \neq 0$ we have

$$\left\langle \sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F} f, f\right\rangle = \sum_{F\in X_{k-2}} \left\langle \Delta_{k-1}^{up,F} f, f\right\rangle \underset{3.7}{=} \sum_{F\in X_{k-2}} \langle \Delta(\mathrm{lk}F) f_F, f_F\rangle = \sum_{F\in S_f} \langle \Delta(\mathrm{lk}F) f_F, f_F\rangle$$

and in the context of our inequalities that means that

$$k\lambda_{\min}\langle f,f\rangle \leq \left\langle \sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F} f, f\right\rangle \leq k\lambda_{\max}\langle f,f\rangle$$

From 3.5

$$\left\langle \sum_{F\in X_{k-2}} \Delta_{k-1}^{up,F} f, f\right\rangle = \left\langle \left(\Delta_{k-1}^{up}(X) + (k-1)I\right) f, f\right\rangle = \left\langle \Delta_{k-1}^{up}(X) f, f\right\rangle + (k-1)\langle f,f\rangle$$

so

$$k\lambda_{\min}\langle f,f\rangle \leq \left\langle \Delta_{k-1}^{up}(X)\, f,f\right\rangle + (k-1)\langle f,f\rangle \leq k\lambda_{\max}\langle f,f\rangle$$

rearranging we get

$$(1+k\lambda_{\min}-k)\langle f,f\rangle \leq \left\langle \Delta_{k-1}^{up}(X)\, f,f\right\rangle \leq (1+k\lambda_{\max}-k)\langle f,f\rangle$$

which is the first required result.

To conclude the proof, the above implies that for $f \neq 0$,

$$\frac{\left\langle \Delta_{k-1}^{up}(X)\, f,f\right\rangle}{\langle f,f\rangle} \in [1+k\lambda_{\min}-k, 1+k\lambda_{\max}-k]$$

and this holds when min/max-imizing over all $f \neq 0$. 3.9 then yields the required result. □

References

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