Maxima Manual

Version 5.49.0

Maxima is a computer algebra system, implemented in Lisp.

Maxima is derived from the Macsyma system, developed at MIT in the years 1968 through 1982 as part of Project MAC. MIT turned over a copy of the Macsyma source code to the Department of Energy in 1982; that version is now known as DOE Macsyma. A copy of DOE Macsyma was maintained by Professor William F. Schelter of the University of Texas from 1982 until his death in 2001. In 1998, Schelter obtained permission from the Department of Energy to release the DOE Macsyma source code under the GNU Public License, and in 2000 he initiated the Maxima project at SourceForge to maintain and develop DOE Macsyma, now called Maxima.

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1 Introduction to Maxima

Start Maxima with the command "maxima". Maxima will display version information and a prompt. End each Maxima command with a semicolon. End the session with the command "quit();". Here's a sample session:

```
$ maxima
Maxima 5.45.1 https://maxima.sourceforge.io
using Lisp SBCL 2.0.1.debian
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1) factor(10!);
                                 8  4  2
(%o1)                           2  3  5  7
(%i2) expand ((x + y)^6);
           6       5       2  4       3  3       4  2      5      6
(%o2)     y  + 6 x y  + 15 x  y  + 20 x  y  + 15 x  y  + 6 x  y + x
(%i3) factor (x^6 - 1);
                                   2           2
(%o3)              (x - 1) (x + 1) (x  - x + 1) (x  + x + 1)
(%i4) quit();
$
```

Maxima can search the info pages. Use the `describe` command to show information about the command or all the commands and variables containing a string. The question mark `?` (exact search) and double question mark `??` (inexact search) are abbreviations for `describe`:

```
(%i1) ?? integ
 0: Functions and Variables for Elliptic Integrals
 1: Functions and Variables for Integration
 2: Introduction to Elliptic Functions and Integrals
 3: Introduction to Integration
 4: askinteger  (Functions and Variables for Simplification)
 5: integerp  (Functions and Variables for Miscellaneous Options)
 6: integer_partitions  (Functions and Variables for Sets)
 7: integrate  (Functions and Variables for Integration)
 8: integrate_use_rootsof  (Functions and Variables for Integration)
 9: integration_constant_counter  (Functions and Variables for
    Integration)
 10: nonnegintegerp  (Functions and Variables for linearalgebra)
Enter space-separated numbers, `all' or `none': 5 4

 -- Function: integerp (<expr>)
     Returns `true' if <expr> is a literal numeric integer, otherwise
     `false'.

     `integerp' returns false if its argument is a symbol, even if the
```

```
     argument is declared integer.

     Examples:

          (%i1) integerp (0);
          (%o1)                         true
          (%i2) integerp (1);
          (%o2)                         true
          (%i3) integerp (-17);
          (%o3)                         true
          (%i4) integerp (0.0);
          (%o4)                         false
          (%i5) integerp (1.0);
          (%o5)                         false
          (%i6) integerp (%pi);
          (%o6)                         false
          (%i7) integerp (n);
          (%o7)                         false
          (%i8) declare (n, integer);
          (%o8)                         done
          (%i9) integerp (n);
          (%o9)                         false

 -- Function: askinteger (<expr>, integer)
 -- Function: askinteger (<expr>)
 -- Function: askinteger (<expr>, even)
 -- Function: askinteger (<expr>, odd)
     `askinteger (<expr>, integer)' attempts to determine from the
     `assume' database whether <expr> is an integer.  `askinteger'
     prompts the user if it cannot tell otherwise, and attempt to
     install the information in the database if possible.  `askinteger
     (<expr>)' is equivalent to `askinteger (<expr>, integer)'.

     `askinteger (<expr>, even)' and `askinteger (<expr>, odd)'
     likewise attempt to determine if <expr> is an even integer or odd
     integer, respectively.

(%o1)                                true
```

To use a result in later calculations, you can assign it to a variable or refer to it by its automatically supplied label. In addition, % refers to the most recent calculated result:

```
(%i1) u: expand ((x + y)^6);
       6        5       2  4       3  3       4  2      5      6
(%o1) y  + 6 x y  + 15 x  y  + 20 x  y  + 15 x  y  + 6 x  y + x
(%i2) diff (u, x);
         5        4       2  3       3  2       4       5
(%o2) 6 y  + 30 x y  + 60 x  y  + 60 x  y  + 30 x  y + 6 x
```

```
(%i3) factor (%o2);
                                    5
(%o3)                      6 (y + x)
```

Maxima knows about complex numbers and numerical constants:

```
(%i1) cos(%pi);
(%o1)                          - 1
(%i2) exp(%i*%pi);
(%o2)                          - 1
```

Maxima can do differential and integral calculus:

```
(%i1) u: expand ((x + y)^6);
       6        5       2  4       3  3       4  2      5      6
(%o1) y  + 6 x y  + 15 x  y  + 20 x  y  + 15 x  y  + 6 x  y + x
(%i2) diff (%, x);
         5         4       2  3       3  2       4        5
(%o2) 6 y  + 30 x y  + 60 x  y  + 60 x  y  + 30 x  y + 6 x
(%i3) integrate (1/(1 + x^3), x);
                                  2 x - 1
                2            atan(-------)
           log(x  - x + 1)        sqrt(3)    log(x + 1)
(%o3)    - --------------- + ------------- + ----------
                  6             sqrt(3)          3
```

Maxima can solve linear systems and cubic equations:

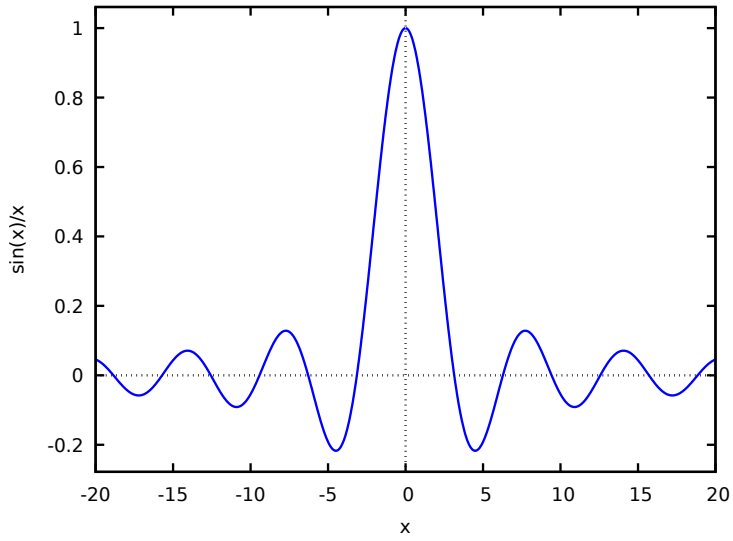
```
(%i1) linsolve ([3*x + 4*y = 7, 2*x + a*y = 13], [x, y]);
                        7 a - 52        25
(%o1)              [x = --------, y = -------]
                        3 a - 8       3 a - 8
(%i2) solve (x^3 - 3*x^2 + 5*x = 15, x);
(%o2)       [x = - sqrt(5) %i, x = sqrt(5) %i, x = 3]
```

Maxima can solve nonlinear sets of equations. Note that if you don't want a result printed, you can finish your command with *$* instead of *;*.

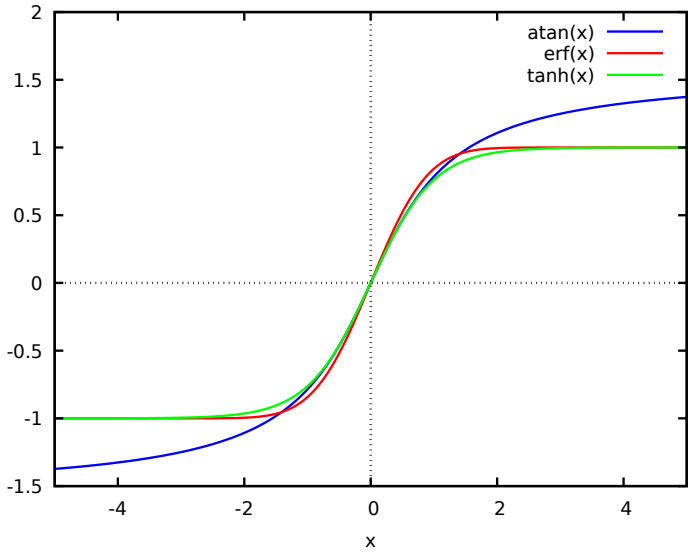
```
(%i1) eq_1: x^2 + 3*x*y + y^2 = 0$
(%i2) eq_2: 3*x + y = 1$
(%i3) solve ([eq_1, eq_2]);
              3 sqrt(5) + 7      sqrt(5) + 3
(%o3) [[y = - -------------, x = -----------],
                    2                 2
                                3 sqrt(5) - 7        sqrt(5) - 3
                          [y = -------------, x = - -----------]]
                                      2                   2
```

Maxima can generate plots of one or more functions:

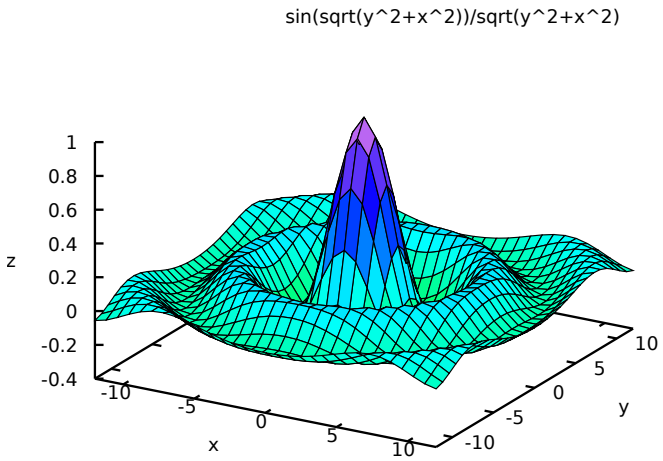
```
(%i1) plot2d (sin(x)/x, [x, -20, 20])$
```




```
(%i2) plot2d ([atan(x), erf(x), tanh(x)], [x, -5, 5], [y, -1.5, 2])$
```




```
(%i3) plot3d (sin(sqrt(x^2 + y^2))/sqrt(x^2 + y^2),
         [x, -12, 12], [y, -12, 12])$
```


sin(sqrt(y^2+x^2))/sqrt(y^2+x^2)
z
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
x
-10
-5
0
5
10
y
-10
-5
0
5
10

2 Command-line options

The following command line options are available for Maxima:

`-b <file>, --batch=<file>`
: Process maxima file <file> in batch mode.

`--batch-lisp=<file>`
: Process lisp file <file> in batch mode.

`--batch-string=<string>`
: Process maxima command(s) <string> in batch mode.

`-d, --directories`
: Display maxima internal directory information.

`--disable-readline`
: Disable readline support.

`-g, --enable-lisp-debugger`
: Enable underlying lisp debugger.

`-h, --help`
: Display this usage message.

`--userdir=<directory>`
: Use <directory> for user directory (default is %USERPROFILE%/maxima for Windows, and $HOME/.maxima for other operating systems).

`--init=<file>`
: Set the base name of the Maxima & Lisp initialization files (default is "maxima-init".) The last extension and any directory parts are removed to form the base name. The resulting files, <base>.mac and <base>.lisp are only searched for in userdir (see –userdir option). This may be specified for than once, but only the last is used.

`-l <lisp>, --lisp=<lisp>`
: Use lisp implementation <lisp>.

`--list-avail`
: List the installed version/lisp combinations.

`-p <file>, --preload=<file>, --preload-lisp=<file>, --init-mac=<file>, --init-lisp=<file>`
: Preload <file>, which may be any file time accepted by Maxima's LOAD function. The <file> is loaded before any other system initialization is done. This will be searched for in the locations given by file_search_maxima and file_search_lisp. This can be specified multiple times to load multiple files. The equivalent options –preload-lisp, –init-mac, and –init-lisp are deprecated.

`-q, --quiet`
: Suppress Maxima start-up message.

`-Q, --quit-on-error`
: Quit, and return an exit code 1, when Maxima encounters an error.

`-r <string>, --run-string=<string>`
: Process maxima command(s) <string> in interactive mode.

`-s <port>, --server=<port>`
: Connect Maxima to server on <port>.

`--suppress-input-echo`
: Do not print input expressions when processing noninteractively.

`-u <version>, --use-version=<version>`
: Use maxima version <version>.

`-v, --verbose`
: Display lisp invocation in maxima wrapper script.

`--version`
: Display the default installed version.

`--very-quiet`
: Suppress expression labels, Maxima start-up message and verification of html index.

`--very-very-quiet`
: In addition to –very-quiet, suppress most printed output by setting TTYOFF to T.

`-X <Lisp options>, --lisp-options=<Lisp options>`
: Options to be given to the underlying Lisp

`--no-init, --norc`
: Do not load the init file(s) on startup

`--verify-html-index`
: Verify on startup that the set of html topics is consistent with text topics.

3 Runtime Environment

3.1 Introduction for Runtime Environment

maxima-init.mac and maxima-init.lisp are loaded automatically when Maxima starts. maxima-init.mac contains Maxima code and is loaded using batchload, maxima-init.lisp contains Lisp code and is loaded using load.

You can use maxima-init.mac (and maxima-init.lisp) to customize your Maxima environment. These files typically placed in the directory named by maxima_userdir, although it can be in any directory searched by the function file_search.

Here is an example maxima-init.mac file:

```
setup_autoload ("specfun.mac", ultraspherical, assoc_legendre_p);
showtime:all;
```

In this example, setup_autoload tells Maxima to load the specified file (specfun.mac) if any of the functions (ultraspherical, assoc_legendre_p) are called but not yet defined. Thus you needn't remember to load the file before calling the functions.

The statement showtime: all tells Maxima to set the showtime variable. The maxima-init.mac file can contain any other assignments or other Maxima statements.

maxima-init.mac and maxima-init.lisp are loaded automatically when Maxima starts. maxima-init.mac contains Maxima code and is loaded using batchload, maxima-init.lisp contains Lisp code and is loaded using load.

maximarc is sourced by the maxima script at startup. It should be located in $MAXIMA_USERDIR. If Maxima was compiled with several Lisp compilers, maximarc can be used, e.g., to change the user's default lisp implementation. E.g. to select CMUCL create a maximarc file containing the line: MAXIMA_LISP=cmucl

You can also use the command option -l <lisp> or --lisp=<lisp> to select the Lisp when starting Maxima.

If you run Maxima using the graphical interface Xmaxima, the default configuration of that interface will be saved in the file xmaxima_default and the history of the last 200 commands will be saved in the file xmaxima_history in a sub-directory maxima (or Maxima in Windows) of the user's local configuration directory.

In Linux systems, the user's local configuration directory is usually /home/<username>/.config and in Windows systems it is usually <Drive_letter>:/Users/<username>/AppData/Local. Notice that in both systems those directories are hidden, and their default value can be changed with an environment variable (XDG_CONFIG_HOME in Linux or LOCALAPPDATA in Windows). If the local configuration directory is not found, those two files will be saved in the users home directory and with the names .xmaximarc and .xmaxima_history, as in older versions of Xmaxima.

3.2 Interrupts

The user can stop a time-consuming computation with the ^C (control-C) character. The default action is to stop the computation and print another user prompt. In this case, it is not possible to restart a stopped computation.

If the Lisp variable `*debugger-hook*` is set to `nil`, by executing

```
:lisp (setq *debugger-hook* nil)
```

then upon receiving ^C, Maxima will enter the Lisp debugger, and the user may use the debugger to inspect the Lisp environment. The stopped computation can be restarted by entering `continue` in the Lisp debugger. The means of returning to Maxima from the Lisp debugger (other than running the computation to completion) is different for each version of Lisp.

On Unix systems, the character ^Z (control-Z) causes Maxima to stop altogether, and control is returned to the shell prompt. The `fg` command causes Maxima to resume from the point at which it was stopped.

3.3 Functions and Variables for Runtime Environment

`maxima_tempdir` [System variable]

`maxima_tempdir` names the directory in which Maxima creates some temporary files. In particular, temporary files for plotting are created in `maxima_tempdir`.

The initial value of `maxima_tempdir` is the user's home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory.

`maxima_tempdir` may be assigned a string which names a directory.

`maxima_userdir` [System variable]

`maxima_userdir` names a directory which Maxima searches to find Maxima and Lisp files. (Maxima searches some other directories as well; `file_search_maxima` and `file_search_lisp` are the complete lists.)

The initial value of `maxima_userdir` is a subdirectory of the user's home directory, if Maxima can locate it; otherwise Maxima makes a guess about a suitable directory.

`maxima_userdir` may be assigned a string which names a directory. However, assigning to `maxima_userdir` does not automatically change `file_search_maxima` and `file_search_lisp`; those variables must be changed separately.

`room` [Function]

`room` ()
`room` (*true*)
`room` (*false*)

Prints out a description of the state of storage and stack management in Maxima. `room` calls the Lisp function of the same name.

- `room ()` prints out a moderate description.
- `room (true)` prints out a verbose description.
- `room (false)` prints out a terse description.

`sstatus` (*keyword*, *item*) [Function]

When *keyword* is the symbol `feature`, *item* is put on the list of system features. After `sstatus (keyword, item)` is executed, `status (feature, item)` returns `true`. If *keyword* is the symbol `nofeature`, *item* is deleted from the list of system features. This can be useful for package writers, to keep track of what features they have loaded in.

See also `status`.

status [Function]

status (feature)
status (feature, *item*)

Returns information about the presence or absence of certain system-dependent features.

- `status (feature)` returns a list of system features. These include Lisp version, operating system type, etc. The list may vary from one Lisp type to another.
- `status (feature, item)` returns `true` if *item* is on the list of items returned by `status (feature)` and `false` otherwise. `status` quotes the argument *item*. The quote-quote operator `''` defeats quotation. A feature whose name contains a special character, such as a hyphen, must be given as a string argument. For example, `status (feature, "ansi-cl")`.

See also `sstatus`.

The variable `features` contains a list of features which apply to mathematical expressions. See `features` and `featurep` for more information.

system [Function]

system (*command*, *arg_1*, ..., *arg_n*)
system (*command*)

Executes *command* as a separate process. The command is passed to the default shell for execution.

`system` is implemented by a command execution function in the Lisp implementation which compiled Maxima, and therefore the behavior of `system` varies with the operating system and Lisp implementation. `system` is known to work on Windows and Linux systems, and might also work on other systems.

All combinations of Lisp implementation and operating system allow command arguments as *arg_1*, ..., *arg_n*, and some allow command arguments as part of *command*. SBCL on Windows and Clozure CL on Windows are known to require command arguments to be specified as *arg_1*, ..., *arg_n*.

`system` does not attempt to quote or escape spaces or other characters in *command* or in *arg_1*, ..., *arg_n*; all arguments are supplied verbatim to the command execution function of the Lisp implementation.

Standard output from *command* is displayed on the Maxima console by default, and may be captured by `with_stdout`.

For most Lisp implementations, the call to `system` returns after *command* has completed. Job control operations, such as executing a command asynchronously with respect to Maxima, are not known to have the expected effect.

Examples:

`system` executes *command* as a separate process. The output of the command `dir` varies from one system to another.

```
(%i1) system ("dir", maxima_tempdir);
config-err-UsLLQM
gnome-software-0TNK22
MozillaUpdateLock-6939C585ADF59520
```

```
snap-private-tmp
systemd-private-169e359ab2d94b208622fa96dd88c05e-colord.service-ZP9Xn7
(%o1)                           0
```

All combinations of Lisp implementation and operating system allow command arguments as *arg_1*, ..., *arg_n*.

```
(%i1) system ("echo", "Hello", "world", "glad", "to", "meet", "you");
Hello world glad to meet you
(%o1)                           0
```

Standard output from *command* is displayed on the Maxima console by default, and may be captured by `with_stdout`.

```
(%i1) my_output: sconcat (maxima_tempdir, "/tmp.out");
(%o1)                     /tmp/tmp.out
(%i2) with_stdout (my_output, system ("dir"));
(%o2)                           0
(%i3) S: openr (my_output);
(%o3)               #<FILE-STREAM {7B500975}>
(%i4) readline (S);
(%o4)                      aclocal.m4
(%i5) readline (S);
(%o5)                         admin
(%i6) readline (S);
(%o6)                       archive
```

For most Lisp implementations, the call to `system` returns after *command* has completed. `xfontsel` is a utility to inspect fonts for the X Windows system; `system` returns after the user clicks the "quit" button.

```
(%i1) system ("xfontsel");
(%o1)                           0
```

`time` (*%o1, %o2, %o3, ...*) [Function]

Returns a list of the times, in seconds, taken to compute the output lines `%o1`, `%o2`, `%o3`, ... The time returned is Maxima's estimate of the internal computation time, not the elapsed time. `time` can only be applied to output line variables; for any other variables, `time` returns `unknown`.

Set `showtime: true` to make Maxima print out the computation time and elapsed time with each output line.

`timedate` [Function]

`timedate (`*T*, *tz_offset*`)`
`timedate (`*T*`)`
`timedate ()`

`timedate(`*T*, *tz_offset*`)` returns a string representing the time *T* in the time zone *tz_offset*. The string format is `YYYY-MM-DD HH:MM:SS.NNN[+|-]ZZ:ZZ` (using as many digits as necessary to represent the fractional part) if *T* has a nonzero fractional part, or `YYYY-MM-DD HH:MM:SS[+|-]ZZ:ZZ` if its fractional part is zero.

T measures time, in seconds, since midnight, January 1, 1900, in the GMT time zone.

tz_offset measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. *tz_offset* must be an integer, rational, or float between -24 and 24, inclusive. If *tz_offset* is not a multiple of 1/60, it is rounded to the nearest multiple of 1/60.

`timedate(T)` is equivalent to `timedate(T, tz_offset)` with *tz_offset* equal to the offset of the local time zone.

`timedate()` is equivalent to `timedate(absolute_real_time())`. That is, it returns the current time in the local time zone.

Example:

`timedate` with no argument returns a string representing the current time and date.

```
(%i1) d : timedate ();
(%o1)                       2010-06-08 04:08:09+01:00
(%i2) print ("timedate reports current time", d) $
timedate reports current time 2010-06-08 04:08:09+01:00
```

`timedate` with an argument returns a string representing the argument.

```
(%i1) timedate (0);
(%o1)                       1900-01-01 01:00:00+01:00
(%i2) timedate (absolute_real_time () - 7*24*3600);
(%o2)                       2010-06-01 04:19:51+01:00
```

`timedate` with optional timezone offset.

```
(%i1) timedate (1000000000, -9.5);
(%o1)               1931-09-09 16:16:40-09:30
```

`parse_timedate` [Function]

`parse_timedate (S)`

Parses a string S representing a date or date and time of day and returns the number of seconds since midnight, January 1, 1900 GMT. If there is a nonzero fractional part, the value returned is a rational number, otherwise, it is an integer. `parse_timedate` returns `false` if it cannot parse S according to any of the allowed formats.

The string S must have one of the following formats, optionally followed by a timezone designation:

- `YYYY-MM-DD[ T]hh:mm:ss[,.]nnn`
- `YYYY-MM-DD[ T]hh:mm:ss`
- `YYYY-MM-DD`

where the fields are year, month, day, hours, minutes, seconds, and fraction of a second, and square brackets indicate acceptable alternatives. The fraction may contain one or more digits.

Except for the fraction of a second, each field must have exactly the number of digits indicated: four digits for the year, and two for the month, day of the month, hours, minutes, and seconds.

A timezone designation must have one of the following forms:

- `[+-]hh:mm`
- `[+-]hhmm`

- `[+-]hh`
- `Z`

where `hh` and `mm` indicate hours and minutes east (`+`) or west (`-`) of GMT. The timezone may be from +24 hours (inclusive) to -24 hours (inclusive).

A literal character `Z` is equivalent to `+00:00` and its variants, indicating GMT.

If no timezone is indicated, the time is assumed to be in the local time zone.

Any leading or trailing whitespace (space, tab, newline, and carriage return) is ignored, but any other leading or trailing characters cause `parse_timedate` to fail and return `false`.

See also `timedate` and `absolute_real_time`.

Examples:

Midnight, January 1, 1900, in the local time zone, in each acceptable format. The result is the number of seconds the local time zone is ahead (negative result) or behind (positive result) GMT. In this example, the local time zone is 8 hours behind GMT.

```
(%i1) parse_timedate ("1900-01-01 00:00:00,000");
(%o1)                          28800
(%i2) parse_timedate ("1900-01-01 00:00:00.000");
(%o2)                          28800
(%i3) parse_timedate ("1900-01-01T00:00:00,000");
(%o3)                          28800
(%i4) parse_timedate ("1900-01-01T00:00:00.000");
(%o4)                          28800
(%i5) parse_timedate ("1900-01-01 00:00:00");
(%o5)                          28800
(%i6) parse_timedate ("1900-01-01T00:00:00");
(%o6)                          28800
(%i7) parse_timedate ("1900-01-01");
(%o7)                          28800
```

Midnight, January 1, 1900, GMT, in different indicated time zones.

```
(%i1) parse_timedate ("1900-01-01 19:00:00+19:00");
(%o1)                            0
(%i2) parse_timedate ("1900-01-01 07:00:00+07:00");
(%o2)                            0
(%i3) parse_timedate ("1900-01-01 01:00:00+01:00");
(%o3)                            0
(%i4) parse_timedate ("1900-01-01Z");
(%o4)                            0
(%i5) parse_timedate ("1899-12-31 21:00:00-03:00");
(%o5)                            0
(%i6) parse_timedate ("1899-12-31 13:00:00-11:00");
(%o6)                            0
(%i7) parse_timedate ("1899-12-31 08:00:00-16:00");
(%o7)                            0
```

encode_time [Function]

encode_time (*year*, *month*, *day*, *hours*, *minutes*, *seconds*, *tz_offset*)
encode_time (*year*, *month*, *day*, *hours*, *minutes*, *seconds*)

Given a time and date specified by *year*, *month*, *day*, *hours*, *minutes*, and *seconds*, `encode_time` returns the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT.

year must be an integer greater than or equal to 1899. However, 1899 is allowed only if the resulting encoded time is greater than or equal to 0.

month must be an integer from 1 to 12, inclusive.

day must be an integer from 1 to *n*, inclusive, where *n* is the number of days in the month specified by *month*.

hours must be an integer from 0 to 23, inclusive.

minutes must be an integer from 0 to 59, inclusive.

seconds must be an integer, rational, or float greater than or equal to 0 and less than 60. When *seconds* is not an integer, `encode_time` returns a rational, such that the fractional part of the return value is equal to the fractional part of *seconds*. Otherwise, *seconds* is an integer, and the return value is likewise an integer.

tz_offset measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. *tz_offset* must be an integer, rational, or float between -24 and 24, inclusive. If *tz_offset* is not a multiple of 1/3600, it is rounded to the nearest multiple of 1/3600.

If *tz_offset* is not present, the offset of the local time zone is assumed.

See also `decode_time`.

Examples:

```
(%i1) encode_time (1900, 1, 1, 0, 0, 0, 0);
(%o1)                               0
(%i2) encode_time (1970, 1, 1, 0, 0, 0, 0);
(%o2)                         2208988800
(%i3) encode_time (1970, 1, 1, 8, 30, 0, 8.5);
(%o3)                         2208988800
(%i4) encode_time (1969, 12, 31, 16, 0, 0, -8);
(%o4)                         2208988800
(%i5) encode_time (1969, 12, 31, 16, 0, 1/1000, -8);
                             2208988800001
(%o5)                        -------------
                                  1000
(%i6) % - 2208988800;
                                   1
(%o6)                              ----
                                   1000
```

`decode_time` [Function]

`decode_time (`*T*`, `*tz_offset*`)`
`decode_time (`*T*`)`

Given the number of seconds (possibly including a fractional part) since midnight, January 1, 1900 GMT, returns the date and time as represented by a list comprising the year, month, day of the month, hours, minutes, seconds, and time zone offset.

tz_offset measures the offset of the time zone, in hours, east (positive) or west (negative) of GMT. *tz_offset* must be an integer, rational, or float between -24 and 24, inclusive. If *tz_offset* is not a multiple of 1/3600, it is rounded to the nearest multiple of 1/3600.

If *tz_offset* is not present, the offset of the local time zone is assumed.

See also `encode_time`.

Examples:

```
(%i1) decode_time (0, 0);
(%o1)                [1900, 1, 1, 0, 0, 0, 0]
(%i2) decode_time (0);
(%o2)              [1899, 12, 31, 16, 0, 0, - 8]
(%i3) decode_time (2208988800, 9.25);
                                              37
(%o3)               [1970, 1, 1, 9, 15, 0, --]
                                              4
(%i4) decode_time (2208988800);
(%o4)              [1969, 12, 31, 16, 0, 0, - 8]
(%i5) decode_time (2208988800 + 1729/1000, -6);
                                         1729
(%o5)           [1969, 12, 31, 18, 0, ----, - 6]
                                         1000
(%i6) decode_time (2208988800 + 1729/1000);
                                         1729
(%o6)           [1969, 12, 31, 16, 0, ----, - 8]
                                         1000
```

`absolute_real_time ()` [Function]

Returns the number of seconds since midnight, January 1, 1900 GMT. The return value is an integer.

See also `elapsed_real_time` and `elapsed_run_time`.

Example:

```
(%i1) absolute_real_time ();
(%o1)                       3385045277
(%i2) 1900 + absolute_real_time () / (365.25 * 24 * 3600);
(%o2)                    2007.265612087104
```

`elapsed_real_time ()` [Function]

Returns the number of seconds (including fractions of a second) since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_run_time`.

Example:

```
(%i1) elapsed_real_time ();
(%o1)                       2.559324
(%i2) expand ((a + b)^500)$
(%i3) elapsed_real_time ();
(%o3)                       7.552087
```

`elapsed_run_time` () [Function]

Returns an estimate of the number of seconds (including fractions of a second) which Maxima has spent in computations since Maxima was most recently started or restarted. The return value is a floating-point number.

See also `absolute_real_time` and `elapsed_real_time`.

Example:

```
(%i1) elapsed_run_time ();
(%o1)                         0.04
(%i2) expand ((a + b)^500)$
(%i3) elapsed_run_time ();
(%o3)                         1.26
```

4 Bug Detection and Reporting

4.1 Functions and Variables for Bug Detection and Reporting

`bug_report` () [Function]

Prints out Maxima and Lisp version numbers, and gives a link to the Maxima project bug report web page. The version information is the same as reported by `build_info`.

When a bug is reported, it is helpful to copy the Maxima and Lisp version information into the bug report.

`bug_report` returns an empty string `""`.

`build_info` () [Function]

Returns a summary of the parameters of the Maxima build, as a Maxima structure (defined by `defstruct`). The fields of the structure are: `version`, `timestamp`, `host`, `lisp_name`, and `lisp_version`. When the pretty-printer is enabled (via `display2d`), the structure is displayed as a short table.

See also `bug_report`.

Examples:

```
(%i1) build_info ();
(%o1)
Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
(%i2) x : build_info ()$
(%i3) x@version;
(%o3)                                5.36.1
(%i4) x@timestamp;
(%o4)                         2015-06-02 11:26:48
(%i5) x@host;
(%o5)                       x86_64-unknown-linux-gnu
(%i6) x@lisp_name;
(%o6)                         GNU Common Lisp (GCL)
(%i7) x@lisp_version;
(%o7)                              GCL 2.6.12
(%i8) x;
(%o8)
Maxima version: "5.36.1"
Maxima build date: "2015-06-02 11:26:48"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "GNU Common Lisp (GCL)"
Lisp implementation version: "GCL 2.6.12"
```

The Maxima version string (here 5.36.1) can look very different:

```
(%i1) build_info();
(%o1)
Maxima version: "branch_5_37_base_331_g8322940_dirty"
Maxima build date: "2016-01-01 15:37:35"
Host type: "x86_64-unknown-linux-gnu"
Lisp implementation type: "CLISP"
Lisp implementation version: "2.49 (2010-07-07) (built 3605577779) (memory 366064
```

In that case, Maxima was not build from a released sourcecode, but directly from the GIT-checkout of the sourcecode. In the example, the checkout is 331 commits after the latest GIT tag (usually a Maxima (major) release (5.37 in our example)) and the abbreviated commit hash of the last commit was "8322940".

Front-ends for maxima can add information about currently being used by setting the variables `maxima_frontend` and `maxima_frontend_version` accordingly.

`run_testsuite` ([*options*]) [Function]

Run the Maxima test suite. Tests producing the desired answer are considered "passes," as are tests that do not produce the desired answer, but are marked as known bugs.

`run_testsuite` takes the following optional keyword arguments

`display_all`
: Display all tests. Normally, the tests are not displayed, unless the test fails. (Defaults to `false`).

`display_known_bugs`
: Displays tests that are marked as known bugs. (Default is `false`).

`tests`
: This is a single test or a list of tests that should be run. Each test can be specified by either a string or a symbol. By default, all tests are run. The complete set of tests is specified by `testsuite_files`.

`time`
: Display time information. If `true`, the time taken for each test file is displayed. If `all`, the time for each individual test is shown if `display_all` is `true`. The default is `false`, so no timing information is shown.

`share_tests`
: Load additional tests for the `share` directory. If `true`, these additional tests are run as a part of the testsuite. If `false`, no tests from the `share` directory are run. If `only`, only the tests from the `share` directory are run. Of course, the actual set of test that are run can be controlled by the `tests` option. The default is `false`.

`answers_from_file`
: Read answers to interactive questions from the source file. May only be `false` or `true` (default). See also `batch_answers_from_file`.

For example `run_testsuite(display_known_bugs = true, tests=[rtest5])` runs just test `rtest5` and displays the test that are marked as known bugs.

`run_testsuite(display_all = true, tests=["rtest1", rtest1a])` will run tests `rtest1` and `rtest2`, and displays each test.

`run_testsuite` changes the Maxima environment. Typically a test script executes `kill` to establish a known environment (namely one without user-defined functions and variables) and then defines functions and variables appropriate to the test.

`run_testsuite` returns `done`.

`testsuite_files` [Option variable]

`testsuite_files` is the set of tests to be run by `run_testsuite`. It is a list of names of the files containing the tests to run. If some of the tests in a file are known to fail, then instead of listing the name of the file, a list containing the file name and the test numbers that fail is used.

For example, this is a part of the default set of tests:

```
["rtest13s", ["rtest14", 57, 63]]
```

This specifies the testsuite consists of the files "rtest13s" and "rtest14", but "rtest14" contains two tests that are known to fail: 57 and 63.

`share_testsuite_files` [Option variable]

`share_testsuite_files` is the set of tests from the `share` directory that is run as a part of the test suite by `run_testsuite`..

5 Help

5.1 Documentation

The Maxima on-line user's manual can be viewed in different forms. From the Maxima interactive prompt, the user's manual is viewed as plain text by the `?` command (i.e., the `describe` function). Command `?` can also display its result in a Web browser, if the variable `output_format_for_help` is set to "html"; furthermore, the HTML page shown can be the official documentation in Maxima's website, as explained in `url_base`. The user's manual can be viewed as `info` hypertext with the `info` viewer program, or as Web pages by any ordinary Web browser.

`example` displays examples for many Maxima functions. For example,

```
(%i1) example (integrate);
```

yields

```
(%i2) test(f):=block([u],u:integrate(f,x),ratsimp(f-diff(u,x)))
(%o2) test(f) := block([u], u : integrate(f, x),
                                          ratsimp(f - diff(u, x)))
(%i3) test(sin(x))
(%o3)                          0
(%i4) test(1/(x+1))
(%o4)                          0
(%i5) test(1/(x^2+1))
(%o5)                          0
```

and additional output.

5.2 Functions and Variables for Help

`apropos` (*name*) [Function]

Searches for Maxima names which have *name* appearing anywhere within them; *name* must be a string or symbol. Thus, `apropos (exp)` returns a list of all the flags and functions which have `exp` as part of their names, such as `expand`, `exp`, and `exponentialize`. So, if you can only remember part of the name of a Maxima command or variable, you can use this command to find the rest of the name. Similarly, you can type `apropos (tr_)` to find a list of many of the switches relating to the translator, most of which begin with `tr_`.

`apropos("")` returns a list with all Maxima names.

`apropos` returns the empty list `[]`, if no name is found.

Example:

Show all Maxima symbols which have `gamma` in the name:

```
(%i1) apropos("gamma");
(%o1) [%gamma, Gamma, gamma_expand, gammalim, makegamma,
prefer_gamma_incomplete, gamma, gamma-incomplete, gamma_incomplete,
gamma_incomplete_generalized, gamma_incomplete_generalized_regularized,
gamma_incomplete_lower, gamma_incomplete_regularized, log_gamma]
```

```
The same example, using the symbol gamma, rather than the string:
(%i2) apropos(gamma);
(%o2) [%gamma, Gamma, gamma_expand, gammalim, makegamma,
prefer_gamma_incomplete, gamma, gamma-incomplete, gamma_incomplete,
gamma_incomplete_generalized, gamma_incomplete_generalized_regularized,
gamma_incomplete_lower, gamma_incomplete_regularized, log_gamma]

The number of symbols in the current Maxima session. This will vary.
(%i3) length(apropos(""));
(%o3)                                2338
```

demo (*filename*) [Function]

Evaluates Maxima expressions in *filename* and displays the results. `demo` pauses after evaluating each expression and continues after the user enters a carriage return. (If running in Xmaxima, `demo` may need to see a semicolon `;` followed by a carriage return.)

`demo` searches the list of directories `file_search_demo` to find `filename`. If the file has the suffix `dem`, the suffix may be omitted. See also `file_search`.

`demo` evaluates its argument. `demo` returns the name of the demonstration file.

Example:

```
(%i1) demo ("disol");

batching /home/wfs/maxima/share/simplification/disol.dem
 At the _ prompt, type ';' followed by enter to get next demo
(%i2)                      load("disol")


_
(%i3)           exp1 : a (e (g + f) + b (d + c))
(%o3)               a (e (g + f) + b (d + c))


_
(%i4)                disolate(exp1, a, b, e)
(%t4)                         d + c

(%t5)                         g + f

(%o5)                    a (%t5 e + %t4 b)


_
```

describe [Function]

describe (*string*)
describe (*string*, *exact*)
describe (*string*, *inexact*)

`describe(`*string*`)` is equivalent to `describe(`*string*`, exact)`.

`describe(`*string*`, exact)` finds an item with title equal (case-insensitive) to *string*, if there is any such item.

`describe(`*string*`, inexact)` finds all documented items which contain *string* in their titles. If there is more than one such item, Maxima asks the user to select an item or items to display.

At the interactive prompt, `? foo` (with a space between `?` and `foo`) is equivalent to `describe("foo", exact)`, and `?? foo` is equivalent to `describe("foo", inexact)`.

`describe("", inexact)` yields a list of all topics documented in the on-line manual.

`describe` quotes its argument. `describe` returns `true` if some documentation is found, otherwise `false`.

To display the topics using a Web browser see `output_format_for_help`. Also see `browser` and `url_base` to configure how to display the HTML files.

See also Section 5.1 [Documentation], page 23.

Example:

```
(%i1) ?? integ
 0: Functions and Variables for Elliptic Integrals
 1: Functions and Variables for Integration
 2: Introduction to Elliptic Functions and Integrals
 3: Introduction to Integration
 4: askinteger  (Functions and Variables for Simplification)
 5: integerp  (Functions and Variables for Miscellaneous Options)
 6: integer_partitions  (Functions and Variables for Sets)
 7: integrate  (Functions and Variables for Integration)
 8: integrate_use_rootsof  (Functions and Variables for
    Integration)
 9: integration_constant_counter  (Functions and Variables for
    Integration)
 10: nonnegintegerp  (Functions and Variables for linearalgebra)
Enter space-separated numbers, `all' or `none': 7 8

 -- Function: integrate (<expr>, <x>)
 -- Function: integrate (<expr>, <x>, <a>, <b>)
     Attempts to symbolically compute the integral of <expr> with
     respect to <x>.  `integrate (<expr>, <x>)' is an indefinite
     integral, while `integrate (<expr>, <x>, <a>, <b>)' is a
     definite integral, [...]

 -- Option variable: integrate_use_rootsof
     Default value: `false'

     When `integrate_use_rootsof' is `true' and the denominator of
     a rational function cannot be factored, `integrate' returns
     the integral in a form which is a sum over the roots (not yet
     known) of the denominator.
     [...]
```

In this example, items 7 and 8 were selected (output is shortened as indicated by `[...]`). All or none of the items could have been selected by entering `all` or `none`, which can be abbreviated `a` or `n`, respectively.

`output_format_for_help` [Option variable]

Default value: `text`

`output_format_for_help` controls how `describe` displays help.

`output_format_for_help` can be set to one of the following values:

`text` Help is displayed as plain text sent to a terminal. This is the default.

`html` Help is displayed using a Web browser to display the HTML version of the manual.

`frontend` When Maxima is being run from a graphical interface (for example, wxMaxima or xmaxima), lets that program decide how to display the help results. If no frontend is running then an error is signaled.

Any other value is a error.

See also `browser`, and `url_base`.

`browser` [Option variable]

This specifies the command to use to open an HTML file. This is a format string of the form `"browser_command"` that corresponds to a valid command that when given a URL as argument, as in `'browser_command URL'`, it will open up a browser to the given URL.

The default setting is `"start"` on Windows, `"xdg-open"` on Linux/Unix, and `"open"` on MacOS, all of which will open the default Web browser. In other systems, the default value of `browser` is set as `"firefox"`, which will open the Firefox browser if it is installed (if it is not installed, the user should change the value of `browser` to some other valid browser).

You may replace the default value of `browser` with other valid browser in your system, e.g. `"chrome"` or `"iexplore"`.

See also `output_format_for_help`, and `url_base`.

`url_base` [Option variable]

When displaying help using a browser, `url_base` defines the URL to use. It defaults to a `file://` path pointing to the directory containing the html files for documentation; something such as, `file:///home/user/.local/share/maxima/5.48.1/doc/html`. However, you could change the value of `url_base` to any valid URL that has the HTML help files of the manual. For instance to see the official manual in Maxima's website instead of the local copy in your disk, set `url_base` to `"https://maxima.sourceforge.io/docs/manual"`.

But keep in mind that the URL to where `url_base` points must have exactly the same HTML files as in the Maxima version that you are using, otherwise the help topics you are searching might not be found.

See also [output_format_for_help], page 26, and `browser`.

example [Function]

example (*topic*)
example ()

`example (topic)` displays some examples of *topic*, which is a symbol or a string. To get examples for operators like `if`, `do`, or `lambda` the argument must be a string, e.g. `example ("do")`. `example` is not case sensitive. Most topics are function names.

`example ()` returns the list of all recognized topics.

The name of the file containing the examples is given by the global option variable `manual_demo`, which defaults to `"manual.demo"`.

`example` quotes its argument. `example` returns `done` unless no examples are found or there is no argument, in which case `example` returns the list of all recognized topics.

Examples:

```
(%i1) example(append);
(%i2) append([y+x,0,-3.2],[2.5e+20,x])
(%o2)              [y + x, 0, - 3.2, 2.5e+20, x]
(%o2)                           done
(%i3) example("lambda");
(%i4) lambda([x,y,z],x^2+y^2+z^2)
                                          2    2    2
(%o4)             lambda([x, y, z], x  + y  + z )
(%i5) %(1,2,a)
                                 2
(%o5)                           a  + 5
(%i6) 1+2+a
(%o6)                            a + 3
(%o6)                            done
```

manual_demo [Option variable]

Default value: `"manual.demo"`

`manual_demo` specifies the name of the file containing the examples for the function `example`. See `example`.

6 Command Line

6.1 Introduction to Command Line

This section documents Maxima's interactive command-line interface, called a read-eval-print loop (REPL).

For information on command-line options, see command_line_options.

6.2 Functions and Variables for Command Line

__ [System variable]

__ is the input expression currently being evaluated. That is, while an input expression *expr* is being evaluated, __ is *expr*.

__ is assigned the input expression before the input is simplified or evaluated. However, the value of __ is simplified (but not evaluated) when it is displayed.

__ is recognized by batch and load. In a file processed by batch, __ has the same meaning as at the interactive prompt. In a file processed by load, __ is bound to the input expression most recently entered at the interactive prompt or in a batch file; __ is not bound to the input expressions in the file being processed. In particular, when load (*filename*) is called from the interactive prompt, __ is bound to load (*filename*) while the file is being processed.

See also _ and %.

Examples:

```
(%i1) print ("I was called as", __);
I was called as print(I was called as, __)
(%o1)              print(I was called as, __)
(%i2) foo (__);
(%o2)                      foo(foo(__))
(%i3) g (x) := (print ("Current input expression =", __), 0);
(%o3) g(x) := (print("Current input expression =", __), 0)
(%i4) [aa : 1, bb : 2, cc : 3];
(%o4)                         [1, 2, 3]
(%i5) (aa + bb + cc)/(dd + ee + g(x));
                               cc + bb + aa
Current input expression = --------------
                              g(x) + ee + dd
                                   6
(%o5)                         -------
                              ee + dd
```

_ [System variable]

_ is the most recent input expression (e.g., %i1, %i2, %i3, ...).

_ is assigned the input expression before the input is simplified or evaluated. However, the value of _ is simplified (but not evaluated) when it is displayed.

_ is recognized by batch and load. In a file processed by batch, _ has the same meaning as at the interactive prompt. In a file processed by load, _ is bound to the input expression most recently evaluated at the interactive prompt or in a batch file; _ is not bound to the input expressions in the file being processed.

See also __ and %.

Examples:

```
(%i1) 13 + 29;
(%o1)                          42
(%i2) :lisp $_
((MPLUS) 13 29)
(%i2) _;
(%o2)                          42
(%i3) sin (%pi/2);
(%o3)                           1
(%i4) :lisp $_
((%SIN) ((MQUOTIENT) $%PI 2))
(%i4) _;
(%o4)                           1
(%i5) a: 13$
(%i6) b: 29$
(%i7) a + b;
(%o7)                          42
(%i8) :lisp $_
((MPLUS) $A $B)
(%i8) _;
(%o8)                        b + a
(%i9) a + b;
(%o9)                          42
(%i10) ev (_);
(%o10)                         42
```

% [System variable]

% is the output expression (e.g., %o1, %o2, %o3, ...) most recently computed by Maxima, whether or not it was displayed.

% is recognized by batch and load. In a file processed by batch, % has the same meaning as at the interactive prompt. In a file processed by load, % is bound to the output expression most recently computed at the interactive prompt or in a batch file; % is not bound to output expressions in the file being processed.

See also _, %%, and %th.

%% [System variable]

In compound statements, namely block, lambda, or (*s_1*, ..., *s_n*), %% is the value of the previous statement.

At the first statement in a compound statement, or outside of a compound statement, %% is undefined.

%% is recognized by batch and load, and it has the same meaning as at the interactive prompt.

See also %.

Examples:

The following two examples yield the same result.

```
(%i1) block (integrate (x^5, x), ev (%%, x=2) - ev (%%, x=1));
                                 21
(%o1)                            --
                                 2
(%i2) block ([prev], prev: integrate (x^5, x),
              ev (prev, x=2) - ev (prev, x=1));
                                 21
(%o2)                            --
                                 2
```

A compound statement may comprise other compound statements. Whether a statement be simple or compound, %% is the value of the previous statement.

```
(%i1) block (block (a^n, %%*42), %%/6);
                                   n
(%o1)                            7 a
```

Within a compound statement, the value of %% may be inspected at a break prompt, which is opened by executing the break function. For example, entering %%; in the following example yields 42.

```
(%i4) block (a: 42, break ())$

Entering a Maxima break point. Type 'exit;' to resume.
_%%;
42
_
```

%th (*i*) [Function]

The value of the *i*'th previous output expression. That is, if the next expression to be computed is the *n*'th output, %th (*m*) is the (*n* - *m*)'th output.

%th is recognized by batch and load. In a file processed by batch, %th has the same meaning as at the interactive prompt. In a file processed by load, %th refers to output expressions most recently computed at the interactive prompt or in a batch file; %th does not refer to output expressions in the file being processed.

See also % and %%.

Example:

%th is useful in batch files or for referring to a group of output expressions. This example sets s to the sum of the last five output expressions.

```
(%i1) 1;2;3;4;5;
(%o1)                           1
(%o2)                           2
(%o3)                           3
(%o4)                           4
(%o5)                           5
(%i6) block (s: 0, for i:1 thru 5 do s: s + %th(i), s);
(%o6)                          15
```

`?` [Special symbol]

As prefix to a function or variable name, `?` signifies that the name is a Lisp name, not a Maxima name. For example, `?round` signifies the Lisp function `ROUND`. See Section 33.1 [Lisp and Maxima], page 621, for more on this point.

The notation `? word` (a question mark followed a word, separated by whitespace) is equivalent to `describe("word")`. The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also `describe`.

`??` [Special symbol]

The notation `?? word` (`??` followed a word, separated by whitespace) is equivalent to `describe("word", inexact)`. The question mark must occur at the beginning of an input line; otherwise it is not recognized as a request for documentation. See also `describe`.

`$` [Input terminator]

The dollar sign `$` terminates an input expression, and the most recent output `%` and an output label, e.g. `%o1`, are assigned the result, but the result is not displayed.

See also `;`.

Example:

```
(%i1) 1 + 2 + 3 $
(%i2) %;
(%o2)                           6
(%i3) %o1;
(%o3)                           6
```

`;` [Input terminator]

The semicolon `;` terminates an input expression, and the resulting output is displayed.

See also `$`.

Example:

```
(%i1) 1 + 2 + 3;
(%o1)                           6
```

`inchar` [Option variable]

Default value: `%i`

`inchar` is the prefix of the labels of expressions entered by the user. Maxima automatically constructs a label for each input expression by concatenating `inchar` and `linenum`.

`inchar` may be assigned any string or symbol, not necessarily a single character. A string is coerced to a symbol with the same printed representation. Because Maxima internally takes into account only the first char of the prefix, the prefixes `inchar`, `outchar`, and `linechar` should have a different first char. Otherwise some commands like `kill(inlabels)` do not work as expected.

See also `labels`.

Example:

```
(%i1) inchar: "input";
(%o1)                          input
(input2) expand((a+b)^3);
                      3        2      2      3
(%o2)                b  + 3 a b  + 3 a  b + a
```

`infolists` [System variable]

Default value: `[]`

`infolists` is a list of the names of all of the information lists in Maxima. These are:

`labels` All bound `%i`, `%o`, and `%t` labels.

`values` All bound atoms which are user variables, not Maxima options or switches, created by `:` or `::` or functional binding.

`functions` All user-defined functions, created by `:=` or `define`.

`arrays` All arrays, `hashed arrays` and `memoizing functions`.

`macros` All user-defined macro functions, created by `::=`.

`myoptions` All options ever reset by the user (whether or not they are later reset to their default values).

`rules` All user-defined pattern matching and simplification rules, created by `tellsimp`, `tellsimpafter`, `defmatch`, or `defrule`.

`aliases` All atoms which have a user-defined alias, created by the `alias`, `ordergreat`, `orderless` functions or by declaring the atom as a `noun` with `declare`.

`dependencies` All atoms which have functional dependencies, created by the `depends`, `dependencies`, or `gradef` functions.

`gradefs` All functions which have user-defined derivatives, created by the `gradef` function.

`props` All atoms which have any property other than those mentioned above, such as properties established by `atvalue` or `matchdeclare`, etc., as well as properties established in the `declare` function.

`structures` All structs defined using `defstruct`.

`let_rule_packages`
: All user-defined `let` rule packages plus the special package `default_let_rule_package`. (`default_let_rule_package` is the name of the rule package used when one is not explicitly set by the user.)

`kill` [Function]

`kill` (`a_1`, ..., `a_n`)
`kill` (*labels*)
`kill` (*inlabels*, *outlabels*, *linelabels*)
`kill` (`n`)
`kill` ([`m`, `n`])
`kill` (*values*, *functions*, *arrays*, ...)
`kill` (*all*)
`kill` (*allbut* (`a_1`, ..., `a_n`))

Removes all bindings (value, function, array, or rule) from the arguments *a_1*, ..., *a_n*. An argument *a_k* may be a symbol or a single array element. When *a_k* is a single array element, `kill` unbinds that element without affecting any other elements of the array.

Several special arguments are recognized. Different kinds of arguments may be combined, e.g., `kill (inlabels, functions, allbut (foo, bar))`.

`kill (labels)` unbinds all input, output, and intermediate expression labels created so far. `kill (inlabels)` unbinds only input labels which begin with the current value of `inchar`. Likewise, `kill (outlabels)` unbinds only output labels which begin with the current value of `outchar`, and `kill (linelabels)` unbinds only intermediate expression labels which begin with the current value of `linechar`.

`kill (n)`, where *n* is an integer, unbinds the *n* most recent input and output labels.

`kill ([m, n])` unbinds input and output labels *m* through *n*.

`kill (infolist)`, where *infolist* is any item in `infolists` (such as `values`, `functions`, or `arrays`) unbinds all items in *infolist*. See also `infolists`.

`kill (all)` unbinds all items on all infolists. `kill (all)` does not reset global variables to their default values; see `reset` on this point.

`kill (allbut (a_1, ..., a_n))` unbinds all items on all infolists except for *a_1*, ..., *a_n*. `kill (allbut (infolist))` unbinds all items except for the ones on *infolist*, where *infolist* is `values`, `functions`, `arrays`, etc.

The memory taken up by a bound property is not released until all symbols are unbound from it. In particular, to release the memory taken up by the value of a symbol, one unbinds the output label which shows the bound value, as well as unbinding the symbol itself.

`kill` quotes its arguments. The quote-quote operator `''` defeats quotation.

`kill (symbol)` unbinds all properties of *symbol*. In contrast, the functions `remvalue`, `remfunction`, `remarray`, and `remrule` unbind a specific property. Note that facts declared by `assume` don't require a symbol they apply to, therefore aren't stored as properties of symbols and therefore aren't affected by `kill`.

`kill` always returns `done`, even if an argument has no binding.

`labels (symbol)` [Function]

Returns the list of input, output, or intermediate expression labels which begin with *symbol*. Typically *symbol* is the value of `inchar`, `outchar`, or `linechar`. If no labels begin with *symbol*, `labels` returns an empty list.

By default, Maxima displays the result of each user input expression, giving the result an output label. The output display is suppressed by terminating the input with `$` (dollar sign) instead of `;` (semicolon). An output label is constructed and bound to the result, but not displayed, and the label may be referenced in the same way as displayed output labels. See also `%`, `%%`, and `%th`.

Intermediate expression labels can be generated by some functions. The option variable `programmode` controls whether `solve` and some other functions generate intermediate expression labels instead of returning a list of expressions. Some other functions, such as `ldisplay`, always generate intermediate expression labels.

See also `inchar`, `outchar`, `linechar`, and `infolists`.

`labels` [System variable]

The variable `labels` is the list of input, output, and intermediate expression labels, including all previous labels if `inchar`, `outchar`, or `linechar` were redefined.

`linechar` [Option variable]

Default value: `%t`

`linechar` is the prefix of the labels of intermediate expressions generated by Maxima. Maxima constructs a label for each intermediate expression (if displayed) by concatenating `linechar` and `linenum`.

`linechar` may be assigned any string or symbol, not necessarily a single character. A string is coerced to a symbol with the same printed representation. Because Maxima internally takes into account only the first char of the prefix, the prefixes `inchar`, `outchar`, and `linechar` should have a different first char. Otherwise some commands like `kill(inlabels)` do not work as expected.

Intermediate expressions might or might not be displayed. See `programmode` and `labels`.

`linenum` [System variable]

The line number of the current pair of input and output expressions.

`myoptions` [System variable]

Default value: `[]`

`myoptions` is the list of all options ever reset by the user, whether or not they get reset to their default value.

`nolabels` [Option variable]

Default value: `false`

When `nolabels` is `true`, input and output result labels (`%i` and `%o`, respectively) are displayed, but the labels are not bound to results, and the labels are not appended to the `labels` list. Since labels are not bound to results, garbage collection can recover the memory taken up by the results.

Otherwise input and output result labels are bound to results, and the labels are appended to the `labels` list.

Intermediate expression labels (`%t`) are not affected by `nolabels`; whether `nolabels` is `true` or `false`, intermediate expression labels are bound and appended to the `labels` list.

See also `batch`, `load`, and `labels`.

`optionset` [Option variable]

Default value: `false`

When `optionset` is `true`, Maxima prints out a message whenever a Maxima option is reset. This is useful if the user is doubtful of the spelling of some option and wants to make sure that the variable he assigned a value to was truly an option variable.

Example:

```
(%i1) optionset:true;
(%o1)                          true
(%i2) gamma_expand:true;
assignment: assigning to option gamma_expand
(%o2)                          true
```

`outchar` [Option variable]

Default value: `%o`

`outchar` is the prefix of the labels of expressions computed by Maxima. Maxima automatically constructs a label for each computed expression by concatenating `outchar` and `linenum`.

`outchar` may be assigned any string or symbol, not necessarily a single character. A string is coerced to a symbol with the same printed representation. Because Maxima internally takes into account only the first char of the prefix, the prefixes `inchar`, `outchar` and `linechar` should have a different first char. Otherwise some commands like `kill(inlabels)` do not work as expected.

See also `labels`.

Example:

```
(%i1) outchar: "output";
(output1)                     output
(%i2) expand((a+b)^3);
                       3        2      2      3
(output2)             b  + 3 a b  + 3 a  b + a
```

playback [Function]

playback ()
playback (*n*)
playback ([*m*, *n*])
playback ([*m*])
playback (*input*)
playback (*slow*)
playback (*time*)
playback (*grind*)

Displays input, output, and intermediate expressions, without recomputing them. playback only displays the expressions bound to labels; any other output (such as text printed by print or describe, or error messages) is not displayed. See also labels.

playback quotes its arguments. The quote-quote operator '' defeats quotation. playback always returns done.

playback () (with no arguments) displays all input, output, and intermediate expressions generated so far. An output expression is displayed even if it was suppressed by the $ terminator when it was originally computed.

playback (*n*) displays the most recent *n* expressions. Each input, output, and intermediate expression counts as one.

playback ([*m*, *n*]) displays input, output, and intermediate expressions with numbers from *m* through *n*, inclusive.

playback ([*m*]) is equivalent to playback ([*m*, *m*]); this usually prints one pair of input and output expressions.

playback (input) displays all input expressions generated so far.

playback (slow) pauses between expressions and waits for the user to press enter. This behavior is similar to demo. playback (slow) is useful in conjunction with save or stringout when creating a secondary-storage file in order to pick out useful expressions.

playback (time) displays the computation time for each expression.

playback (grind) displays input expressions in the same format as the grind function. Output expressions are not affected by the grind option. See grind.

Arguments may be combined, e.g., playback ([5, 10], grind, time, slow).

prompt [Option variable]

Default value: _

prompt is the prompt symbol of the demo function, playback (slow) mode, and the Maxima break loop (as invoked by break).

quit ([*exit-code*]) [Function]

Terminates the Maxima session. Note that the function must be invoked as quit(); or quit()$, not quit by itself. quit supports returning an exit code to the shell for Lisps and OSes that support exit codes. The default exit code is 0 (usually indicating no errors encountered). Thus quit(1) indicates to the shell that maxima exited with

some kind of failure. This is useful in scripts where maxima can indicate to the shell that maxima failed to compute something or some other bad thing happened.

To stop a lengthy computation, type `control-C`. The default action is to return to the Maxima prompt. If `*debugger-hook*` is `nil`, `control-C` opens the Lisp debugger. See also Chapter 34 [Debugging], page 637.

`read` (*expr_1*, ..., *expr_n*) [Function]

Prints *expr_1*, ..., *expr_n*, then reads one expression from the console and returns the evaluated expression. The expression is terminated with a semicolon `;` or dollar sign `$`.

See also `readonly`

Example:

```
(%i1) foo: 42$
(%i2) foo: read ("foo is", foo, " -- enter new value.")$
foo is 42  -- enter new value.
(a+b)^3;
(%i3) foo;
                                        3
(%o3)                          (b + a)
```

`readonly` (*expr_1*, ..., *expr_n*) [Function]

Prints *expr_1*, ..., *expr_n*, then reads one expression from the console and returns the expression (without evaluation). The expression is terminated with a `;` (semicolon) or `$` (dollar sign).

See also `read`.

Examples:

```
(%i1) aa: 7$
(%i2) foo: readonly ("Enter an expression:");
Enter an expression:
2^aa;
                                     aa
(%o2)                               2
(%i3) foo: read ("Enter an expression:");
Enter an expression:
2^aa;
(%o3)                               128
```

`reset` () [Function]

Resets many global variables and options, and some other variables, to their default values.

`reset` processes the variables on the Lisp list `*variable-initial-values*`. The Lisp macro `defmvar` puts variables on this list (among other actions). Many, but not all, global variables and options are defined by `defmvar`, and some variables defined by `defmvar` are not global variables or options.

`showtime` [Option variable]

Default value: `false`

When `showtime` is `true`, the computation time and elapsed time is printed with each output expression.

The computation time is always recorded, so `time` and `playback` can display the computation time even when `showtime` is `false`.

See also `timer`.

`to_lisp ()` [Function]

Enters the Lisp system under Maxima. `(to-maxima)` returns to Maxima.

Example:

Define a function and enter the Lisp system under Maxima. The definition is inspected on the property list, then the function definition is extracted, factored and stored in the variable `$result`. The variable can be used in Maxima after returning to Maxima.

```
(%i1) f(x):=x^2+x;
                                    2
(%o1)                      f(x) := x  + x
(%i2) to_lisp();
Type (to-maxima) to restart, ($quit) to quit Maxima.
MAXIMA> (symbol-plist '$f)
(MPROPS (NIL MEXPR ((LAMBDA) ((MLIST) $X)
                             ((MPLUS) ((MEXPT) $X 2) $X))))
MAXIMA> (setq $result ($factor (caddr (mget '$f 'mexpr))))
((MTIMES SIMP FACTORED) $X ((MPLUS SIMP IRREDUCIBLE) 1 $X))
MAXIMA> (to-maxima)
Returning to Maxima
(%o2)                           true
(%i3) result;
(%o3)                        x (x + 1)
```

`eval_string_lisp (`*str*`)` [Function]

Sequentially read lisp forms from the string *str* and evaluate them. Any values produced from the last form are returned as a Maxima list.

Examples:

```
(%i1) eval_string_lisp ("");
(%o1)                            []
(%i2) eval_string_lisp ("(values)");
(%o2)                            []
(%i3) eval_string_lisp ("69");
(%o3)                           [69]
(%i4) eval_string_lisp ("1 2 3");
(%o4)                            [3]
(%i5) eval_string_lisp ("(values 1 2 3)");
(%o5)                        [1, 2, 3]
(%i6) eval_string_lisp ("(defun $foo (x) (* 2 x))");
(%o6)                           [foo]
```

```
(%i7) foo (5);
(%o7)                          10
```

See also [eval_string], page 1293.

`values` [System variable]

Initial value: `[]`

`values` is a list of all bound user variables (not Maxima options or switches). The list comprises symbols bound by `:`, or `::`.

If the value of a variable is removed with the commands `kill`, `remove`, or `remvalue` the variable is deleted from `values`.

See `functions` for a list of user defined functions.

Examples:

First, `values` shows the symbols `a`, `b`, and `c`, but not `d`, it is not bound to a value, and not the user function `f`. The values are removed from the variables. `values` is the empty list.

```
(%i1) [a:99, b:: a-90, c:a-b, d, f(x):=x^2];
                                                  2
(%o1)              [99, 9, 90, d, f(x) := x ]
(%i2) values;
(%o2)                        [a, b, c]
(%i3) [kill(a), remove(b,value), remvalue(c)];
(%o3)                    [done, done, [c]]
(%i4) values;
(%o4)                           []
```

6.3 Functions and Variables for Display

`%edispflag` [Option variable]

Default value: `false`

When `%edispflag` is `true`, Maxima displays `%e` to a negative exponent as a quotient. For example, `%e^-x` is displayed as `1/%e^x`. See also `exptdispflag`.

Example:

```
(%i1) %e^-10;
                                 - 10
(%o1)                           %e
(%i2) %edispflag:true$
(%i3) %e^-10;
                                  1
(%o3)                            ----
                                   10
                                 %e
```

`absboxchar` [Option variable]

Default value: `!`

`absboxchar` is the character used to draw absolute value signs around expressions which are more than one line tall.

`absboxchar` is only used when `display2d_unicode` is `false`.

Example:

```
(%i1) display2d_unicode: false $
(%i2) abs((x^3+1));
                              | 3    |
(%o2)                         |x  + 1|
```

`declare_index_properties` (*a*, [*p_1*, *p_2*, *p_3*, ...]) [Function]
`declare_index_properties` ([*a*, *b*, *c*, ...], [*p_1*, *p_2*, *p_3*, ...]) [Function]
`postsubscript` [Symbol]
`postsuperscript` [Symbol]
`presuperscript` [Symbol]
`presubscript` [Symbol]

Declares the properties of indices applied to the symbol *a* or each of the of symbols *a*, *b*, *c*, ... If multiple symbols are given, the whole list of properties applies to each symbol.

Given a symbol with indices, `a[i_1, i_2, i_3, ...]`, the `k`-th property *p_k* applies to the `k`-th index *i_k*. There may be any number of index properties, in any order.

Each property *p_k* must one of these four recognized properties: `postsubscript`, `postsuperscript`, `presuperscript`, or `presubscript`, to denote indices which are displayed, respectively, to the right and below, to the right and above, to the left and above, or to the left and below.

Index properties apply only to the 2-dimensional display of indexed variables (i.e., when `display2d` is `true`) and TeX output via `tex`. Otherwise, index properties are ignored. Index properties do not change the input of indexed variables, do not change the algebraic properties of indexed variables, and do not change the 1-dimensional display of indexed variables.

`declare_index_properties` quotes (does not evaluate) its arguments.

`remove_index_properties` removes index properties. `kill` also removes index properties (and all other properties).

`get_index_properties` retrieves index properties.

Examples:

Given a symbol with indices, `a[i_1, i_2, i_3, ...]`, the `k`-th property *p_k* applies to the `k`-th index *i_k*. There may be any number of index properties, in any order.

```
(%i1) declare_index_properties (A, [presubscript, postsubscript]);
(%o1)                         done
(%i2) declare_index_properties (B, [postsuperscript, postsuperscript,
 presuperscript]);
(%o2)                         done
(%i3) declare_index_properties (C, [postsuperscript, presubscript,
 presubscript, presuperscript]);
(%o3)                         done
```

```
(%i4) A[w, x];
(%o4)                                A
                                    w x
(%i5) B[w, x, y];
                                  y w, x
(%o5)                              B
(%i6) C[w, x, y, z];
                                     z w
(%o6)                                 C
                                  x, y
```

Index properties apply only to the 2-dimensional display of indexed variables and TeX output. Otherwise, index properties are ignored.

```
(%i1) declare_index_properties (A, [presubscript, postsubscript]);
(%o1)                              done
(%i2) A[w, x];
(%o2)                                A
                                    w x
(%i3) tex (A[w, x]);
$${}_{w}A_{x}$$
(%o3)                              false
(%i4) display2d: false $
(%i5) A[w, x];
(%o5) A[w,x]
(%i6) display2d: true $
(%i7) grind (A[w, x]);
A[w,x]$
(%o7)                              done
(%i8) stringdisp: true $
(%i9) string (A[w, x]);
(%o9)                            "A[w,x]"
```

`get_index_properties (a)` [Function]

Returns the properties for *a* established by `declare_index_properties`.

See also `remove_index_properties`.

`remove_index_properties (a, b, c, ...)` [Function]

Removes the properties established by `declare_index_properties`. All index properties are removed from each symbol *a*, *b*, *c*, ...

`remove_index_properties` quotes (does not evaluate) its arguments.

`display_index_separator` [Symbol property]

When a symbol *A* has index display properties declared via `declare_index_properties`, the value of the property `display_index_separator` is the string or other expression which is displayed between indices.

The value of `display_index_separator` is assigned by `put(A, S, display_index_separator)`, where *S* is a string or other expression. The assigned value is retrieved by `get(A, display_index_separator)`.

The display index separator S can be a string, including an empty string, or `false`, indicating the default separator, or any expression. If not a string and not `false`, the property value is coerced to a string via `string`.

If no display index separator is assigned, the default separator is used. The default separator is a comma. There is no way to change the default separator.

Each symbol has its own value of `display_index_separator`.

See also `put`, `get`, and `declare_index_properties`.

Examples:

When a symbol A has index display properties, the value of the property `display_index_separator` is the string or other expression which is displayed between indices. The value is assigned by `put(A, S, display_index_separator)`,

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript,
 presubscript, presubscript]);
(%o1)                          done
(%i2) put (A, ";", display_index_separator);
(%o2)                            ;
(%i3) A[w, x, y, z];
                                  w;x
(%o3)                            A
                                y;z
```

The assigned value is retrieved by `get(A, display_index_separator)`.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript,
 presubscript, presubscript]);
(%o1)                          done
(%i2) put (A, ";", display_index_separator);
(%o2)                            ;
(%i3) get (A, display_index_separator);
(%o3)                            ;
```

The display index separator S can be a string, including an empty string, or `false`, indicating the default separator, or any expression.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript,
 presubscript, presubscript]);
(%o1)                          done
(%i2) A[w, x, y, z];
                                  w, x
(%o2)                            A
                               y, z
(%i3) put (A, "-", display_index_separator);
(%o3)                            -
(%i4) A[w, x, y, z];
                                  w-x
(%o4)                            A
                                y-z
(%i5) put (A, " ", display_index_separator);
(%o5)
```

```
(%i6) A[w, x, y, z];
                                  w x
(%o6)                            A
                                  y z
(%i7) put (A, "", display_index_separator);
(%o7)
(%i8) A[w, x, y, z];
                                  wx
(%o8)                            A
                                  yz
(%i9) put (A, false, display_index_separator);
(%o9)                            false
(%i10) A[w, x, y, z];
                                   w, x
(%o10)                           A
                                  y, z
(%i11) put (A, 'foo, display_index_separator);
(%o11)                            foo
(%i12) A[w, x, y, z];
                                   wfoox
(%o12)                           A
                                 yfooz
```

If no display index separator is assigned, the default separator is used. The default separator is a comma.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript,
 presubscript, presubscript]);
(%o1)                            done
(%i2) A[w, x, y, z];
                                   w, x
(%o2)                            A
                                  y, z
```

Each symbol has its own value of `display_index_separator`.

```
(%i1) declare_index_properties (A, [postsuperscript, postsuperscript,
 presubscript, presubscript]);
(%o1)                            done
(%i2) put (A, " ", display_index_separator);
(%o2)
(%i3) declare_index_properties (B, [presuperscript, presuperscript,
 postsubscript, postsubscript]);
(%o3)                            done
(%i4) put (B, ";", display_index_separator);
(%o4)                             ;
(%i5) A[w, x, y, z] + B[w, x, y, z];
                          w;x           w x
(%o5)                        B    +    A
                              y;z   y z
```

disp (*expr_1*, *expr_2*, ...) [Function]

is like `display` but only the value of the arguments are displayed rather than equations. This is useful for complicated arguments which don't have names or where only the value of the argument is of interest and not the name.

See also `ldisp` and `print`.

Example:

```
(%i1) b[1,2]:x-x^2$
(%i2) x:123$
(%i3) disp(x, b[1,2], sin(1.0));
                                  123

                                     2
                                x - x

                          0.8414709848078965

(%o3)                            done
```

display (*expr_1*, *expr_2*, ...) [Function]

Displays equations whose left side is *expr_i* unevaluated, and whose right side is the value of the expression centered on the line. This function is useful in blocks and `for` statements in order to have intermediate results displayed. The arguments to `display` are usually atoms, subscripted variables, or function calls.

See also `ldisplay`, `disp`, and `ldisp`.

Example:

```
(%i1) b[1,2]:x-x^2$
(%i2) x:123$
(%i3) display(x, b[1,2], sin(1.0));
                                x = 123

                                         2
                            b     = x - x
                             1, 2

                      sin(1.0) = 0.8414709848078965

(%o3)                            done
```

display2d [Option variable]

Default value: `true`

When `display2d` is `true`, the console display is an attempt to present mathematical expressions as they might appear in books and articles, using only letters, numbers, and some punctuation characters. This display is sometimes called the "pretty printer" display.

When `display2d` is `true`, Maxima attempts to honor the global variable for line length, `linel`. When an atom (symbol, number, or string) would otherwise cause

a line to exceed `linel`, the atom may be printed in pieces on successive lines, with a continuation character (backslash, \) at the end of the leading piece; however, in some cases, such atoms are printed without a line break, and the length of the line is greater than `linel`.

When `display2d` is `false`, the console display is a 1-dimensional or linear form which is the same as the output produced by `grind`.

When `display2d` is `false`, the value of `stringdisp` is ignored, and strings are always displayed with quote marks.

When `display2d` is `false`, Maxima attempts to honor `linel`, but atoms are not broken across lines, and the actual length of an output line may exceed `linel`.

See also `leftjust` to switch between a left justified and a centered display of equations.

Example:

```
(%i1) x/(x^2+1);
                                  x
(%o1)                           ------
                                 2
                                x  + 1
(%i2) display2d:false$
(%i3) x/(x^2+1);
(%o3) x/(x^2+1)
```

`display_format_internal` [Option variable]

Default value: `false`

When `display_format_internal` is `true`, expressions are displayed without being transformed in ways that hide the internal mathematical representation. The display then corresponds to what `inpart` returns rather than `part`.

Examples:

```
User     part       inpart
a-b;      a - b     a + (- 1) b

           a            - 1
a/b;       -         a b
           b
                      1/2
sqrt(x);   sqrt(x)    x

          4 X        4
X*4/3;    ---        - X
           3         3
```

`with_default_2d_display` (*expr*) [Function]

While maxima by default realizes 2d Output using ASCII-Art some frontend change that to TeX, MathML or a specific XML dialect that better suits the needs for this specific frontend. `with_default_2d_display` temporarily switches maxima to the default 2D ASCII Art formatter for outputting the result of `expr`.

See also `set_alt_display` and `display2d`.

`display2d_unicode` [Option variable]

Default value: `true`

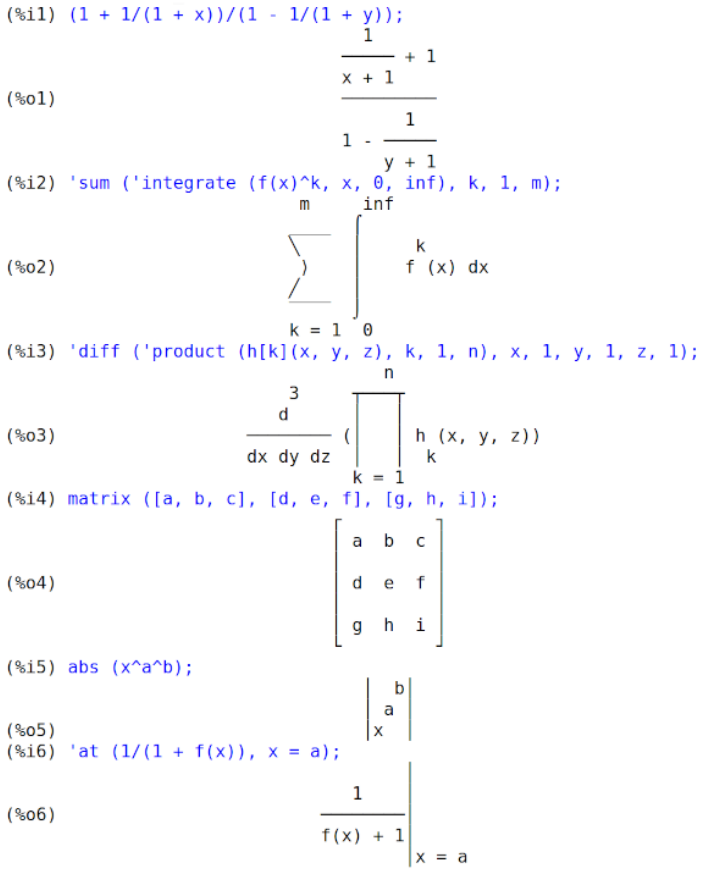
When `display2d_unicode` is `true`, the 2-d pretty printer (enabled by the global flag `display2d`) uses Unicode drawing characters [1] to display integrals, summations, products, matrices, ratios, derivatives, `box` expressions, `at` expressions, and absolute value expressions.

Otherwise, the pretty printer uses only ASCII characters to display every kind of expression.

In addition to displaying expressions in console interaction (as `%o` labeled expressions), the 2-d pretty printer is invoked to display expressions for `print`, and `printf` with the `~m` format specifier.

Examples:

Expressions displayed by 2-d pretty printer using Unicode drawing characters (`display2d_unicode` equal to `true`), shown as an image:



Same expressions, displayed using only ASCII characters (`display2d_unicode` equal to `false`), shown as an image:

```
(%i1) (1 + 1/(1 + x))/(1 - 1/(1 + y));
                                    1
                                 ----- + 1
                                 x + 1
(%o1)                            ---------
                                       1
                                 1 - -----
                                     y + 1
(%i2) 'sum ('integrate (f(x)^k, x, 0, inf), k, 1, m);
                              m     inf
                             ====  /
                             \     [     k
(%o2)                         >    I    f (x) dx
                             /     ]
                             ====  /
                             k = 1  0
(%i3) 'diff ('product (h[k](x, y, z), k, 1, n), x, 1, y, 1, z, 1);
                                     n
                             3     =====
                            d      !   !
(%o3)                    -------- (!   ! h (x, y, z))
                         dx dy dz  !   !  k
                                   k = 1
(%i4) matrix ([a, b, c], [d, e, f], [g, h, i]);
                                 [ a  b  c ]
                                 [         ]
(%o4)                            [ d  e  f ]
                                 [         ]
                                 [ g  h  i ]
(%i5) abs (x^a^b);
                                    !  b!
                                    ! a !
(%o5)                               !x  !
(%i6) 'at (1/(1 + f(x)), x = a);
                                        !
                                    1    !
(%o6)                            --------!
                                 f(x) + 1!
                                        !x = a
```

Footnotes:

[1] https://en.wikipedia.org/wiki/Box-drawing_character

dispterms (*expr*) [Function]

Displays *expr* in parts one below the other. That is, first the operator of *expr* is displayed, then each term in a sum, or factor in a product, or part of a more general expression is displayed separately. This is useful if *expr* is too large to be otherwise displayed. For example if `P1`, `P2`, ... are very large expressions then the display program may run out of storage space in trying to display `P1 + P2 + ...` all at once. However, `dispterms (P1 + P2 + ...)` displays `P1`, then below it `P2`, etc. When not using `dispterms`, if an exponential expression is too wide to be displayed as `A^B` it appears as `expt (A, B)` (or as `ncexpt (A, B)` in the case of `A^^B`).

Example:

```
(%i1) dispterms(2*a*sin(x)+%e^x);
+

2 a sin(x)


  x
%e


(%o1)                         done
```

`expt` (`a`, *b*) [Special symbol]
`ncexpt` (`a`, *b*) [Special symbol]

If an exponential expression is too wide to be displayed as `a^b` it appears as `expt (a, b)` (or as `ncexpt (a, b)` in the case of `a^^b`).

`expt` and `ncexpt` are not recognized in input.

`exptdispflag` [Option variable]

Default value: `true`

When `exptdispflag` is `true`, Maxima displays expressions with negative exponents using quotients. See also `%edispflag`.

Example:

```
(%i1) exptdispflag:true;
(%o1)                         true
(%i2) 10^-x;
                                1
(%o2)                          ---
                                 x
                               10
(%i3) exptdispflag:false;
(%o3)                         false
(%i4) 10^-x;
                                 - x
(%o4)                         10
```

`grind` (*expr*) [Function]

The function `grind` prints *expr* to the console in a form suitable for input to Maxima. `grind` always returns `done`.

When *expr* is the name of a function or macro, `grind` prints the function or macro definition instead of just the name.

See also `string`, which returns a string instead of printing its output. `grind` attempts to print the expression in a manner which makes it slightly easier to read than the output of `string`.

`grind` evaluates its argument.

Examples:

```
(%i1) aa + 1729;
(%o1)                         aa + 1729
(%i2) grind (%);
aa+1729$
(%o2)                           done
(%i3) [aa, 1729, aa + 1729];
(%o3)                  [aa, 1729, aa + 1729]
(%i4) grind (%);
[aa,1729,aa+1729]$
(%o4)                           done
(%i5) matrix ([aa, 17], [29, bb]);
                              [ aa  17 ]
(%o5)                         [        ]
                              [ 29  bb ]
(%i6) grind (%);
matrix([aa,17],[29,bb])$
(%o6)                           done
(%i7) set (aa, 17, 29, bb);
(%o7)                     {17, 29, aa, bb}
(%i8) grind (%);
{17,29,aa,bb}$
(%o8)                           done
(%i9) exp (aa / (bb + 17)^29);
                                  aa
                               -----------
                                        29
                              (bb + 17)
(%o9)                        %e
(%i10) grind (%);
%e^(aa/(bb+17)^29)$
(%o10)                          done
(%i11) expr: expand ((aa + bb)^10);
         10           9        2   8         3   7         4   6
(%o11) bb   + 10 aa bb  + 45 aa  bb  + 120 aa  bb  + 210 aa  bb
         5   5         6   4         7   3        8   2
 + 252 aa  bb  + 210 aa  bb  + 120 aa  bb  + 45 aa  bb
        9        10
 + 10 aa  bb + aa
(%i12) grind (expr);
bb^10+10*aa*bb^9+45*aa^2*bb^8+120*aa^3*bb^7+210*aa^4*bb^6
     +252*aa^5*bb^5+210*aa^6*bb^4+120*aa^7*bb^3+45*aa^8*bb^2
     +10*aa^9*bb+aa^10$
(%o12)                          done
```

```
(%i13) string (expr);
(%o13) bb^10+10*aa*bb^9+45*aa^2*bb^8+120*aa^3*bb^7+210*aa^4*bb^6\
+252*aa^5*bb^5+210*aa^6*bb^4+120*aa^7*bb^3+45*aa^8*bb^2+10*aa^9*\
bb+aa^10
(%i14) cholesky (A):= block ([n : length (A), L : copymatrix (A),
  p : makelist (0, i, 1, length (A))],
  for i thru n do for j : i thru n do
  (x : L[i, j], x : x - sum (L[j, k] * L[i, k], k, 1, i - 1),
  if i = j then p[i] : 1 / sqrt(x) else L[j, i] : x * p[i]),
  for i thru n do L[i, i] : 1 / p[i],
  for i thru n do for j : i + 1 thru n do L[i, j] : 0, L)$
define: warning: redefining the built-in operator cholesky
(%i15) grind (cholesky);
cholesky(A):=block(
         [n:length(A),L:copymatrix(A),
          p:makelist(0,i,1,length(A))],
         for i thru n do
             (for j from i thru n do
                  (x:L[i,j],x:x-sum(L[j,k]*L[i,k],k,1,i-1),
                   if i = j then p[i]:1/sqrt(x)
                       else L[j,i]:x*p[i])),
         for i thru n do L[i,i]:1/p[i],
         for i thru n do (for j from i+1 thru n do L[i,j]:0),L)$
(%o15)                          done
(%i16) string (fundef (cholesky));
(%o16) cholesky(A):=block([n:length(A),L:copymatrix(A),p:makelis\
t(0,i,1,length(A))],for i thru n do (for j from i thru n do (x:L\
[i,j],x:x-sum(L[j,k]*L[i,k],k,1,i-1),if i = j then p[i]:1/sqrt(x\
) else L[j,i]:x*p[i])),for i thru n do L[i,i]:1/p[i],for i thru \
n do (for j from i+1 thru n do L[i,j]:0),L)
```

`grind` [Option variable]

When the variable `grind` is `true`, the output of `string` and `stringout` has the same format as that of `grind`; otherwise no attempt is made to specially format the output of those functions. The default value of the variable `grind` is `false`.

`grind` can also be specified as an argument of `playback`. When `grind` is present, `playback` prints input expressions in the same format as the `grind` function. Otherwise, no attempt is made to specially format input expressions.

`ibase` [Option variable]

Default value: `10`

`ibase` is the base for integers read by Maxima.

`ibase` may be assigned any integer between 2 and 36 (decimal), inclusive. When `ibase` is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus letters of the alphabet `A`, `B`, `C`, . . . , as needed to make `ibase` digits in all. Letters are interpreted as digits only if the first digit is 0 through 9.

Uppercase and lowercase letters are not distinguished. The numerals for base 36, the largest acceptable base, comprise 0 through 9 and `A` through `Z`.

Whatever the value of `ibase`, when an integer is terminated by a decimal point, it is interpreted in base 10.

See also `obase`.

Examples:

`ibase` less than 10 (for example binary numbers).

```
(%i1) ibase : 2 $
(%i2) obase;
(%o2)                          10
(%i3) 1111111111111111;
(%o3)                        65535
```

`ibase` greater than 10. Letters are interpreted as digits only if the first digit is 0 through 9 which means that hexadecimal numbers might need to be prepended by a 0.

```
(%i1) ibase : 16 $
(%i2) obase;
(%o2)                          10
(%i3) 1000;
(%o3)                        4096
(%i4) abcd;
(%o4)                        abcd
(%i5) symbolp (abcd);
(%o5)                        true
(%i6) 0abcd;
(%o6)                        43981
(%i7) symbolp (0abcd);
(%o7)                        false
```

When an integer is terminated by a decimal point, it is interpreted in base 10.

```
(%i1) ibase : 36 $
(%i2) obase;
(%o2)                          10
(%i3) 1234;
(%o3)                        49360
(%i4) 1234.;
(%o4)                        1234
```

`ldisp` (*expr_1*, ..., *expr_n*) [Function]

Displays expressions *expr_1*, ..., *expr_n* to the console as printed output. `ldisp` assigns an intermediate expression label to each argument and returns the list of labels.

See also `disp`, `display`, and `ldisplay`.

Examples:

```
(%i1) e: (a+b)^3;
                                    3
(%o1)                          (b + a)
(%i2) f: expand (e);
                       3        2      2      3
(%o2)                 b  + 3 a b  + 3 a  b + a
(%i3) ldisp (e, f);
                                    3
(%t3)                          (b + a)

                       3        2      2      3
(%t4)                 b  + 3 a b  + 3 a  b + a

(%o4)                        [%t3, %t4]
(%i5) %t3;
                                    3
(%o5)                          (b + a)
(%i6) %t4;
                       3        2      2      3
(%o6)                 b  + 3 a b  + 3 a  b + a
```

`ldisplay` (*expr_1*, ..., *expr_n*) [Function]

Displays expressions *expr_1*, ..., *expr_n* to the console as printed output. Each expression is printed as an equation of the form `lhs = rhs` in which `lhs` is one of the arguments of `ldisplay` and `rhs` is its value. Typically each argument is a variable. `ldisp` assigns an intermediate expression label to each equation and returns the list of labels.

See also `display`, `disp`, and `ldisp`.

Examples:

```
(%i1) e: (a+b)^3;
                                    3
(%o1)                          (b + a)
(%i2) f: expand (e);
                       3        2      2      3
(%o2)                 b  + 3 a b  + 3 a  b + a
(%i3) ldisplay (e, f);
                                      3
(%t3)                        e = (b + a)

                         3        2      2      3
(%t4)              f = b  + 3 a b  + 3 a  b + a

(%o4)                        [%t3, %t4]
(%i5) %t3;
                                      3
(%o5)                        e = (b + a)
```

```
(%i6) %t4;
                          3        2      2      3
(%o6)             f = b  + 3 a b  + 3 a  b + a
```

`leftjust` [Option variable]

Default value: `false`

When `leftjust` is `true`, equations in 2D-display are drawn left justified rather than centered.

See also `display2d` to switch between 1D- and 2D-display.

Example:

```
(%i1) expand((x+1)^3);
                           3      2
(%o1)                     x  + 3 x  + 3 x + 1
(%i2) leftjust:true$
(%i3) expand((x+1)^3);
       3      2
(%o3) x  + 3 x  + 3 x + 1
```

`linel` [Option variable]

Default value: `79`

`linel` is the assumed width (in characters) of the console display for the purpose of displaying expressions. `linel` may be assigned any value by the user, although very small or very large values may be impractical. Text printed by built-in Maxima functions, such as error messages and the output of `describe`, is not affected by `linel`.

`lispdisp` [Option variable]

Default value: `false`

When `lispdisp` is `true`, Lisp symbols are displayed with a leading question mark `?`. Otherwise, Lisp symbols are displayed with no leading mark. This has the same effect for 1-d and 2-d display.

Examples:

```
(%i1) lispdisp: false$
(%i2) ?foo + ?bar;
(%o2)                          foo + bar
(%i3) lispdisp: true$
(%i4) ?foo + ?bar;
(%o4)                         ?foo + ?bar
```

`negsumdispflag` [Option variable]

Default value: `true`

When `negsumdispflag` is `true`, `x - y` displays as `x - y` instead of as `- y + x`. Setting it to `false` causes the special check in display for the difference of two expressions to not be done. One application is that thus `a + %i*b` and `a - %i*b` may both be displayed the same way.

`obase` [Option variable]

Default value: `10`

`obase` is the base for integers displayed by Maxima.

`obase` may be assigned any integer between 2 and 36 (decimal), inclusive. When `obase` is greater than 10, the numerals comprise the decimal numerals 0 through 9 plus capital letters of the alphabet A, B, C, ..., as needed. A leading 0 digit is displayed if the leading digit is otherwise a letter. The numerals for base 36, the largest acceptable base, comprise 0 through 9, and A through Z.

See also `ibase`.

Examples:

```
(%i1) obase : 2;
(%o1)                          10
(%i2) 2^8 - 1;
(%o2)                      11111111
(%i3) obase : 8;
(%o3)                          10
(%i4) 8^8 - 1;
(%o4)                      77777777
(%i5) obase : 16;
(%o5)                          10
(%i6) 16^8 - 1;
(%o6)                      0FFFFFFFF
(%i7) obase : 36;
(%o7)                          10
(%i8) 36^8 - 1;
(%o8)                      0ZZZZZZZZ
```

`pfeformat` [Option variable]

Default value: `false`

When `pfeformat` is `true`, a ratio of integers is displayed with the solidus (forward slash) character, and an integer denominator `n` is displayed as a leading multiplicative term `1/n`.

Examples:

```
(%i1) pfeformat: false$
(%i2) 2^16/7^3;
                                65536
(%o2)                           -----
                                 343
(%i3) (a+b)/8;
                                b + a
(%o3)                           -----
                                  8
(%i4) pfeformat: true$
(%i5) 2^16/7^3;
(%o5)                      65536/343
```

```
(%i6) (a+b)/8;
(%o6)                        (1/8) (b + a)
```

`powerdisp` [Option variable]

Default value: `false`

When `powerdisp` is `true`, a sum is displayed with its terms in order of increasing power. Thus a polynomial is displayed as a truncated power series, with the constant term first and the highest power last.

By default, terms of a sum are displayed in order of decreasing power.

Example:

```
(%i1) powerdisp:true;
(%o1)                             true
(%i2) x^2+x^3+x^4;
                               2    3    4
(%o2)                         x  + x  + x
(%i3) powerdisp:false;
(%o3)                             false
(%i4) x^2+x^3+x^4;
                               4    3    2
(%o4)                         x  + x  + x
```

`print` (*expr_1*, ..., *expr_n*) [Function]

Evaluates and displays *expr_1*, ..., *expr_n* one after another, from left to right, starting at the left edge of the console display.

The value returned by `print` is the value of its last argument. `print` does not generate intermediate expression labels.

See also `display`, `disp`, `ldisplay`, and `ldisp`. Those functions display one expression per line, while `print` attempts to display two or more expressions per line.

To display the contents of a file, see `printfile`.

Examples:

```
(%i1) r: print ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is", radcan (log (a
            3        2      2      3
(a+b)^3 is b  + 3 a b  + 3 a  b + a  log (a^10/b) is
                                                 10 log(a) - log(b)
(%i2) r;
(%o2)                    10 log(a) - log(b)
(%i3) disp ("(a+b)^3 is", expand ((a+b)^3), "log (a^10/b) is", radcan (log (a^10/
                              (a+b)^3 is

                           3        2      2      3
                          b  + 3 a b  + 3 a  b + a

                              log (a^10/b) is

                             10 log(a) - log(b)
```

`sqrtdispflag` [Option variable]

Default value: `true`

When `sqrtdispflag` is `false`, causes `sqrt` to display with exponent 1/2.

`stardisp` [Option variable]

Default value: `false`

When `stardisp` is `true`, multiplication is displayed with an asterisk `*` between operands.

`ttyoff` [Option variable]

Default value: `false`

When `ttyoff` is `true`, output expressions are not displayed. Output expressions are still computed and assigned labels. See `labels`.

Text printed by built-in Maxima functions, such as error messages and the output of `describe`, is not affected by `ttyoff`.

7 Data Types and Structures

7.1 Numbers

7.1.1 Introduction to Numbers

Complex numbers

A complex expression is specified in Maxima by adding the real part of the expression to `%i` times the imaginary part. Thus the roots of the equation `x^2 - 4*x + 13 = 0` are `2 + 3*%i` and `2 - 3*%i`. Note that simplification of products of complex expressions can be effected by expanding the product. Simplification of quotients, roots, and other functions of complex expressions can usually be accomplished by using the `realpart`, `imagpart`, `rectform`, `polarform`, `abs`, `carg` functions.

7.1.2 Functions and Variables for Numbers

`bfloat` (*expr*) [Function]

`bfloat` replaces integers, rationals, floating point numbers, and some symbolic constants in *expr* with bigfloat (variable-precision floating point) numbers.

The constants `%e`, `%gamma`, `%phi`, and `%pi` are replaced by a numerical approximation. However, `%e` in `%e^x` is not replaced by a numeric value unless `bfloat(x)` is a number.

`bfloat` also causes numerical evaluation of some built-in functions, namely trigonometric functions, exponential functions, `abs`, and `log`.

The number of significant digits in the resulting bigfloats is specified by the global variable `fpprec`. Bigfloats already present in *expr* are replaced with values which have precision specified by the current value of `fpprec`.

When `float2bf` is `false`, a warning message is printed when a floating point number is replaced by a bigfloat number with less precision.

Examples:

`bfloat` replaces integers, rationals, floating point numbers, and some symbolic constants in *expr* with bigfloat numbers.

```
(%i1) bfloat([123, 17/29, 1.75]);
(%o1)         [1.23b2, 5.862068965517241b-1, 1.75b0]
(%i2) bfloat([%e, %gamma, %phi, %pi]);
(%o2) [2.718281828459045b0, 5.772156649015329b-1,
                        1.618033988749895b0, 3.141592653589793b0]
(%i3) bfloat((f(123) + g(h(17/29)))/(x + %gamma));
           1.0b0 (g(h(5.862068965517241b-1)) + f(1.23b2))
(%o3)      ----------------------------------------------
                       x + 5.772156649015329b-1
```

`bfloat` also causes numerical evaluation of some built-in functions.

```
(%i1) bfloat(sin(17/29));
(%o1)                  5.532051841609784b-1
(%i2) bfloat(exp(%pi));
```

```
(%o2)                  2.314069263277927b1
(%i3) bfloat(abs(-%gamma));
(%o3)                 5.772156649015329b-1
(%i4) bfloat(log(%phi));
(%o4)                 4.812118250596035b-1
```

`bfloatp` (*expr*) [Function]

Returns `true` if *expr* is a bigfloat number, otherwise `false`.

`bftorat` [Option variable]

Default value: `false`

`bftorat` controls the conversion of bfloats to rational numbers. When `bftorat` is `false`, `ratepsilon` will be used to control the conversion (this results in relatively small rational numbers). When `bftorat` is `true`, the rational number generated will accurately represent the bfloat.

Note: `bftorat` has no effect on the transformation to rational numbers with the function `rationalize`.

Example:

```
(%i1) ratepsilon:1e-4;
(%o1)                          1.0e-4
(%i2) rat(bfloat(11111/111111)), bftorat:false;
'rat' replaced 9.99990999991B-2 by 1/10 = 1.0B-1
                                1
(%o2)/R/                        --
                                10
(%i3) rat(bfloat(11111/111111)), bftorat:true;
'rat' replaced 9.99990999991B-2 by 11111/111111 = 9.99990999991B-2
                               11111
(%o3)/R/                       ------
                               111111
```

`bftrunc` [Option variable]

Default value: `true`

`bftrunc` causes trailing zeroes in non-zero bigfloat numbers not to be displayed. Thus, if `bftrunc` is `false`, `bfloat (1)` displays as `1.000000000000000B0`. Otherwise, this is displayed as `1.0B0`.

`bigfloat_bits` () [Function]

Returns the number of bits of precision in a bigfloat number. This value depends, of course, on the value of `fpprec`.

```
(%i1) fpprec:16;
(%o1)                                16
(%i2) bigfloat_bits();
(%o2)                                56
(%i3) fpprec:32;
(%o3)                                32
(%i4) bigfloat_bits();
(%o4)                                109
```

`bigfloat_eps ()` [Function]

Returns the smallest bigfloat value, `eps`, such that `1+eps` is not equal to 1. The value depends on `fpprec`, of course.

```
(%i1) fpprec:16;
(%o1)                                16
(%i2) bigfloat_eps();
(%o2)                       1.387778780781446b-17
(%i3) fpprec:32;
(%o3)                                32
(%i4) bigfloat_eps();
(%o4)            1.5407439555097886824447823540679b-33
```

`decode_float (f)` [Function]

`decode_float` takes a float *f* and returns a list of three values that characterizes *f*, which must be either a `float` or `bfloat`. The first value has the same type as *f*, but is a number in the range `[1, 2)`. The second value is an exponent. The third value is a float of the same type as *f* and has the value of 1 if *f* is greater than or equal to 0; otherwise, -1.

If the returned list is `[mantissa, expo, sign]`, then `scale_float(mantissa, exp)*sign` is identical to *f*.

```
(%i1) decode_float(4e0);
(%o1)                          [1.0, 2, 1.0]
(%i2) decode_float(4b0);
(%o2)                        [1.0b0, 2, 1.0b0]
(%i3) decode_float(%pi);

decode_float is only defined for floats and bfloats: %pi
 -- an error. To debug this try: debugmode(true);
(%i4) decode_float(float(%pi));
(%o4)                  [1.570796326794897, 1, 1.0]
(%i5) decode_float(1.1e-5);
(%o5)                     [1.441792, - 17, 1.0]
(%i6) %[1]*2^%[2];
(%o6)                               1.1e-5
```

This is a relatively simple interface to Common Lisp decode_float. However we return a signficand in the range `[1,2)` instead of `[0.5, 1)`. The former matches IEEE-754. Of course, this is extended to support bfloats.

`evenp (expr)` [Function]

Returns `true` if *expr* is a literal even integer, otherwise `false`.

`evenp` returns `false` if *expr* is a symbol, even if *expr* is declared `even`.

`float (expr)` [Function]

Converts integers, rational numbers and bigfloats in *expr* to floating point numbers. It is also an `evflag`, `float` causes non-integral rational numbers and bigfloat numbers to be converted to floating point.

`float2bf` [Option variable]

Default value: `true`

When `float2bf` is `false`, a warning message is printed when a floating point number is replaced by a bigfloat number with less precision.

`float_bits ()` [Function]

Returns the number of bits of precision of a floating-point number.

`float_eps ()` [Function]

Returns the smallest floating-point value, `eps`, such that `1+eps` is not equal to 1.

`float_precision (`*f*`)` [Function]

Returns the number of bits of precision of a floating-point number, which can be either a float or bigfloat. This is basically the number of bits used to represent the mantissa of a floating-point number. For floats, this is 53 (for IEEE double-floats), but can be less when denormal numbers occur. For bigfloats, this is equal to `fpprec`, when converted from digits to bits.

`float_sign (`*f*`)` [Function]

Returns the sign of *f*. It is +1 or −1 of the same type as *f*. It is an error if *f* is not a float or bigfloat. Note that some lisps do not support signed zeros for floating-point numbers. Bigfloats do not support signed zeroes. The examples below assume signed zeroes are supported.

```
(%i1) float_sign(1.0);
(%o1)                                1.0
(%i2) float_sign(-5.0);
(%o2)                               - 1.0
(%i3) float_sign(-0.0);
(%o3)                               - 1.0
(%i4) float_sign(1b0);
(%o4)                               1.0b0
(%i5) float_sign(-5b0);
(%o5)                              - 1.0b0
(%o6) float_sign(-0b0);
(%o6)                               1.0b0
(%i7) float_sign(%pi);

float_sign is only defined for floats and bfloats: %pi
 -- an error. To debug this try: debugmode(true);
```

`float_infinity_p (`*x*`)` [Function]

Returns `true` if *x* is floating point positive infinity or floating point negative infinity, and returns `false` for all other arguments; arguments which are not numbers are allowed, and `float_infinity_p` returns `false` for all such arguments.

Positive and negative floating point infinity may be distinguished by `sign`, which returns `pos` for positive infinity and `neg` for negative infinity.

`float_infinity_p` is defined whether or not the Lisp implementation supports float infinity. When float infinity does not exist in the Lisp implementation's number system, `float_infinity_p` returns `false` for all arguments.

A Lisp implementation may support more than one precision of floating point numbers. `float_infinity_p` only recognizes double precision floating point infinity, and not any other precision.

`float_nan_p` (*x*) [Function]

Returns `true` if *x* is a floating point not-a-number (NaN) value, and returns `false` for all other arguments; arguments which are not numbers are allowed, and `float_nan_p` returns `false` for all such arguments.

`float_nan_p` is defined whether or not the Lisp implementation supports floating point not-a-number values. When floating point not-a-number does not exist in the Lisp implementation's number system, `float_nan_p` returns `false` for all arguments.

A Lisp implementation may support more than one precision of floating point numbers. `float_nan_p` only recognizes double precision floating point not-a-number, and not any other precision.

`floatnump` (*expr*) [Function]

Returns `true` if *expr* is a floating point number, otherwise `false`.

`fpprec` [Option variable]

Default value: 16

`fpprec` is the number of significant digits for arithmetic on bigfloat numbers. `fpprec` does not affect computations on ordinary floating point numbers.

See also `bfloat` and `fpprintprec`.

`fpprintprec` [Option variable]

Default value: 0

`fpprintprec` is the number of digits to print when printing an ordinary float or bigfloat number.

For ordinary floating point numbers, when `fpprintprec` has a value between 2 and 16 (inclusive), the number of digits printed is equal to `fpprintprec`. Otherwise, `fpprintprec` is 0, or greater than 16, and the number is printed "readably": that is, it is printed with sufficient digits to exactly reconstruct the number on input.

For bigfloat numbers, when `fpprintprec` has a value between 2 and `fpprec` (inclusive), the number of digits printed is equal to `fpprintprec`. Otherwise, `fpprintprec` is 0, or greater than `fpprec`, and the number of digits printed is equal to `fpprec`.

For both ordinary floats and bigfloats, trailing zero digits are suppressed. The actual number of digits printed is less than `fpprintprec` if there are trailing zero digits.

`fpprintprec` cannot be 1.

`integerp` (*expr*) [Function]

Returns `true` if *expr* is a literal numeric integer, otherwise `false`.

`integerp` returns `false` if *expr* is a symbol, even if *expr* is declared `integer`.

Examples:

```
(%i1) integerp (0);
(%o1)                         true
(%i2) integerp (1);
```

```
(%o2)                          true
(%i3) integerp (-17);
(%o3)                          true
(%i4) integerp (0.0);
(%o4)                          false
(%i5) integerp (1.0);
(%o5)                          false
(%i6) integerp (%pi);
(%o6)                          false
(%i7) integerp (n);
(%o7)                          false
(%i8) declare (n, integer);
(%o8)                          done
(%i9) integerp (n);
(%o9)                          false
```

`integer_decode_float` (*f*) [Function]

`integer_decode_float` takes a float *f* and returns a list of three values that characterizes *f*, which must be either a `float` or `bfloat`. The first value is an integer. The second value is an exponent. The third value is 1 if *f* is positive or zero; otherwise, -1.

If the returned list is `[mantissa, expo, sign]`, then `scale_float(fl(mantissa), expo)*sign` is identical to *f*. Here, `fl` is either `float` or `bfloat` depending on whether *f* is a `float` or a `bfloat`.

```
(%i1) integer_decode_float(4.0);
(%o1)                     [4503599627370496, - 50, 1]
(%i2) integer_decode_float(4b0);
(%o2)                    [36028797018963968, - 53, 1]
(%i3) scale_float(float(%o1[1]), %o1[2]);
(%o3)                                 4.0
(%i4) scale_float(bfloat(%o2[1]), %o2[2]);
(%o4)                                4.0b0
(%i5) integer_decode_float(4);

decode_float is only defined for floats and bfloats: 4
 -- an error. To debug this try: debugmode(true);
(%i6) integer_decode_float(1e-7);
(%o6)                     [7555786372591432, - 76, 1]
(%i7) integer_decode_float(1b-7);
(%o7)                    [60446290980731459, - 79, 1]
(%i8) scale_float(float(%o6[1]), %o6[2]);
(%o8)                                1.0e-7
```

For lisps that support denormal numbers, we have the following results.

```
(%i1) integer_decode_float(least_positive_float);
(%o1)                           [1, - 1074, 1]
(%i2) integer_decode_float(100*least_positive_float);
```

```
(%o2)                          [100, - 1074, 1]
(%i3) integer_decode_float(least_positive_normalized_float);
(%o3)                    [4503599627370496, - 1074, 1]
```

The number of bits in the integer part decreases as the denormal number decreases. Bfloat numbers do not have denormals because the exponent is not bounded.

This is a relatively simple interface to Common Lisp integer_decode_float. However, the integer part can vary depending on the Lisp implementation; we return the same value, independent of the Lisp implementation. Of course, this is extended to support bfloats.

`is_power_of_two` (*n*) [Function]

`is_power_to_two` returns `true` if *n* is a power of two and `false` otherwise. *n* may be an integer, a rational, a float, or a big float.

Some examples:

```
(%i1) is_power_of_two(0);
(%o1)                                false
(%i2) is_power_of_two(4);
(%o2)                                true
(%i3) is_power_of_two(355/113);
(%o3)                                false
(%i4) is_power_of_two(1/32);
(%o4)                                true
(%i5) is_power_of_two(1048576);
(%o5)                                true
(%i6) is_power_of_two(1048575);
(%o6)                                false
(%i7) is_power_of_two(0.0);
(%o7)                                false
(%i8) is_power_of_two(1048576.0);
(%o8)                                true
(%i9) is_power_of_two(1048575.0);
(%o9)                                false
(%i10) is_power_of_two(1/256.0);
(%o10)                               true
(%i11) is_power_of_two(0b0);
(%o11)                               false
(%i12) is_power_of_two(1048576b0);
(%o12)                               true
(%i13) is_power_of_two(1048575b0);
(%o13)                               false
(%i14) is_power_of_two(1/256b0);
(%o14)                               true
```

`m1pbranch` [Option variable]

Default value: `false`

`m1pbranch` is the principal branch for `-1` to a power. Quantities such as `(-1)^(1/3)` (that is, an "odd" rational exponent) and `(-1)^(1/4)` (that is, an "even" rational exponent) are handled as follows:

```
                 domain:real

(-1)^(1/3):      -1
(-1)^(1/4):   (-1)^(1/4)

              domain:complex
m1pbranch:false          m1pbranch:true
(-1)^(1/3)               1/2+%i*sqrt(3)/2
(-1)^(1/4)              sqrt(2)/2+%i*sqrt(2)/2
```

`nonnegintegerp` (*n*) [Function]

Return `true` if and only if *n* `>= 0` and *n* is an integer.

`numberp` (*expr*) [Function]

Returns `true` if *expr* is a literal integer, rational number, floating point number, or bigfloat, otherwise `false`.

`numberp` returns `false` if *expr* is a symbol, even if *expr* is a symbolic number such as `%pi` or `%i`, or declared to be `even`, `odd`, `integer`, `rational`, `irrational`, `real`, `imaginary`, or `complex`.

Examples:

```
(%i1) numberp (42);
(%o1)                         true
(%i2) numberp (-13/19);
(%o2)                         true
(%i3) numberp (3.14159);
(%o3)                         true
(%i4) numberp (-1729b-4);
(%o4)                         true
(%i5) map (numberp, [%e, %pi, %i, %phi, inf, minf]);
(%o5)      [false, false, false, false, false, false]
(%i6) declare (a, even, b, odd, c, integer, d, rational,
     e, irrational, f, real, g, imaginary, h, complex);
(%o6)                         done
(%i7) map (numberp, [a, b, c, d, e, f, g, h]);
(%o7) [false, false, false, false, false, false, false, false]
```

`numer` [Option variable]

`numer` causes some mathematical functions (including exponentiation) with numerical arguments to be evaluated in floating point. It causes variables in `expr` which have been given numerals to be replaced by their values. It also sets the `float` switch on.

See also `%enumer`.

Examples:

```
(%i1) [sqrt(2), sin(1), 1/(1+sqrt(3))];
                                                1
(%o1)            [sqrt(2), sin(1), -----------]
                                             sqrt(3) + 1
(%i2) [sqrt(2), sin(1), 1/(1+sqrt(3))],numer;
(%o2) [1.414213562373095, 0.8414709848078965, 0.3660254037844387]
```

`numer_pbranch` [Option variable]

Default value: `false`

The option variable `numer_pbranch` controls the numerical evaluation of the power of a negative integer, rational, or floating point number. When `numer_pbranch` is `true` and the exponent is a floating point number or the option variable `numer` is `true` too, Maxima evaluates the numerical result using the principal branch. Otherwise a simplified, but not an evaluated result is returned.

Examples:

```
(%i1) (-2)^0.75;
                                    0.75
(%o1)                         (- 2)
(%i2) (-2)^0.75,numer_pbranch:true;
(%o2)       1.189207115002721 %i - 1.189207115002721
(%i3) (-2)^(3/4);
                                  3/4  3/4
(%o3)                        (- 1)    2
(%i4) (-2)^(3/4),numer;
                                              0.75
(%o4)               1.681792830507429 (- 1)
(%i5) (-2)^(3/4),numer,numer_pbranch:true;
(%o5)       1.189207115002721 %i - 1.189207115002721
```

`numerval` (*x_1*, *expr_1*, ..., *var_n*, *expr_n*) [Function]

Declares the variables `x_1`, ..., *x_n* to have numeric values equal to `expr_1`, ..., `expr_n`. The numeric value is evaluated and substituted for the variable in any expressions in which the variable occurs if the `numer` flag is `true`. See also `ev`.

The expressions `expr_1`, ..., `expr_n` can be any expressions, not necessarily numeric.

`oddp` (*expr*) [Function]

Returns `true` if *expr* is a literal odd integer, otherwise `false`.

`oddp` returns `false` if *expr* is a symbol, even if *expr* is declared `odd`.

`ratepsilon` [Option variable]

Default value: `2.0e-15`

`ratepsilon` is the tolerance used in the conversion of floating point numbers to rational numbers, when the option variable `bftorat` has the value `false`. See `bftorat` for an example.

`rationalize` (*expr*) [Function]

Convert all double floats and big floats in the Maxima expression *expr* to their exact rational equivalents. If you are not familiar with the binary representation of floating

point numbers, you might be surprised that `rationalize (0.1)` does not equal 1/10. This behavior isn't special to Maxima – the number 1/10 has a repeating, not a terminating, binary representation.

```
(%i1) rationalize (0.5);
                                1
(%o1)                           -
                                2
(%i2) rationalize (0.1);
                        3602879701896397
(%o2)                   -----------------
                        36028797018963968
(%i3) fpprec : 5$
(%i4) rationalize (0.1b0);
                             209715
(%o4)                        -------
                             2097152
(%i5) fpprec : 20$
(%i6) rationalize (0.1b0);
                      236118324143482260685
(%o6)                 ----------------------
                      236118324143482606848
(%i7) rationalize (sin (0.1*x + 5.6));
                 3602879701896397 x   3152519739159347
(%o7)      sin(------------------ + ----------------)
                 36028797018963968    562949953421312
```

`ratnump` (*expr*) [Function]

Returns `true` if *expr* is a literal integer or ratio of literal integers, otherwise `false`.

`scale_float` (*f*, *n*) [Function]

`scale_float` scales the float *f* by the value `2^n`. This is done carefully so that no round-off every occurs. If *f* is a float, then it is possible to underflow to 0 or overflow, depending on the value of *f* and *n*. Bigfloats cannot underflow or overflow.

```
(%i1) scale_float(2d0, 2);
(%o1)                                8.0
(%i2) scale_float(2d0, -2);
(%o2)                                0.5
(%i3) scale_float(-2d0, -10);
(%o3)                          - 0.001953125
(%i4) scale_float(1d0, -2000);
(%o4)                                0.0
(%i5) scale_float(2b0, 2);
(%o5)                               8.0b0
(%i6) scale_float(1b0, -2000);
(%o6)                    8.709809816217217b-603
(%i7) scale_float(1, 5);
```

```
scale_float: first arg must be a float or bfloat: 1
 -- an error. To debug this try: debugmode(true);
(%i8) scale_float(1.0, n);

scale_float: second arg must be an integer: n
 -- an error. To debug this try: debugmode(true);
```

This is a relatively simple interface to Common Lisp scale_float. Of course, this is extended to support bfloats.

`unit_in_last_place` (*n*) [Function]

`unit_in_last_place` returns a value that is the gap between *n* and the nearest other number. See, for example, Kahan, FOOTNOTE 1. `unit_in_last_place` supports rational numbers, floating-point numbers and bigfloat numbers. For integer, the result is always 1, and for rational numbers the result is always 0.

The examples below assume IEEE-754 arithmetic that supports denormal numbers. Some lisps like Clisp do not have denormal numbers.

```
(%i1) unit_in_last_place(0);
(%o1)                                1
(%i2) unit_in_last_place(-123);
(%o2)                                1
(%i3) unit_in_last_place(2/3);
(%o3)                                0
(%i4) unit_in_last_place(355/113);
(%o4)                                0
(%i5) unit_in_last_place(0b0);
(%o5)                             0.0b0
(%i6) unit_in_last_place(0.0);
(%o6)                    4.940656458412465e-324
(%i7) unit_in_last_place(1.0);
(%o7)                     1.110223024625157e-16
(%i8) unit_in_last_place(1b0);
(%o8)                     1.387778780781446b-17
(%i9) unit_in_last_place(100.0);
(%o9)                      1.4210854715202e-14
(%i10) unit_in_last_place(100b0);
(%o10)                    1.77635683940025b-15
(%i11) fpprec:32;
(%o11)                               32
(%i12) unit_in_last_place(1b0);
(%o12)           1.5407439555097886824447823540679b-33
(%i13) unit_in_last_place(100b0);
(%o13)           1.9721522630525295135293214132O7b-31
```

7.2 Strings

7.2.1 Introduction to Strings

Strings (quoted character sequences) are enclosed in double quote marks " for input, and displayed with or without the quote marks, depending on the global variable stringdisp.

Strings may contain any characters, including embedded tab, newline, and carriage return characters. The sequence \" is recognized as a literal double quote, and \\ as a literal backslash. When backslash appears at the end of a line, the backslash and the line termination (either newline or carriage return and newline) are ignored, so that the string continues with the next line. No other special combinations of backslash with another character are recognized; when backslash appears before any character other than ", \, or a line termination, the backslash is ignored. There is no way to represent a special character (such as tab, newline, or carriage return) except by embedding the literal character in the string.

There is no character type in Maxima; a single character is represented as a one-character string.

The stringproc add-on package contains many functions for working with strings.

Examples:

```
(%i1) s_1 : "This is a string.";
(%o1)                   This is a string.
(%i2) s_2 : "Embedded \"double quotes\" and backslash \\ characters.";
(%o2) Embedded "double quotes" and backslash \ characters.
(%i3) s_3 : "Embedded line termination
in this string.";
(%o3) Embedded line termination
in this string.
(%i4) s_4 : "Ignore the \
line termination \
characters in \
this string.";
(%o4) Ignore the line termination characters in this string.
(%i5) stringdisp : false;
(%o5)                          false
(%i6) s_1;
(%o6)                   This is a string.
(%i7) stringdisp : true;
(%o7)                          true
(%i8) s_1;
(%o8)                  "This is a string."
```

7.2.2 Functions and Variables for Strings

concat (*arg_1*, *arg_2*, ...) [Function]

Concatenates its arguments. The arguments must evaluate to atoms. The return value is a symbol if the first argument is a symbol and a string otherwise.

concat evaluates its arguments. The single quote ' prevents evaluation.

See also sconcat, that works on non-atoms, too, simplode, string and eval_string. For complex string conversions see also printf.

```
(%i1) y: 7$
(%i2) z: 88$
(%i3) concat (y, z/2);
(%o3)                          744
(%i4) concat ('y, z/2);
(%o4)                          y44
```

A symbol constructed by concat may be assigned a value and appear in expressions. The :: (double colon) assignment operator evaluates its left-hand side.

```
(%i5) a: concat ('y, z/2);
(%o5)                          y44
(%i6) a:: 123;
(%o6)                          123
(%i7) y44;
(%o7)                          123
(%i8) b^a;
                               y44
(%o8)                         b
(%i9) %, numer;
                               123
(%o9)                         b
```

Note that although concat (1, 2) looks like a number, it is a string.

```
(%i10) concat (1, 2) + 3;
(%o10)                      12 + 3
```

sconcat (*arg_1*, *arg_2*, ...) [Function]

Concatenates its arguments into a string. Unlike concat, the arguments do *not* need to be atoms.

See also concat, simplode, string and eval_string. For complex string conversions see also printf.

```
(%i1) sconcat ("xx[", 3, "]:", expand ((x+y)^3));
(%o1)             xx[3]:y^3+3*x*y^2+3*x^2*y+x^3
```

Another purpose for sconcat is to convert arbitrary objects to strings.

```
(%i1) sconcat (x);
(%o1)                           x
(%i2) stringp(%);
(%o2)                          true
```

string (*expr*) [Function]

Converts expr to Maxima's linear notation just as if it had been typed in.

The return value of string is a string, and thus it cannot be used in a computation.

See also concat, sconcat, simplode and eval_string.

`stringdisp` [Option variable]

Default value: `false`

When `stringdisp` is `true`, strings are displayed enclosed in double quote marks. Otherwise, quote marks are not displayed.

`stringdisp` is always `true` when displaying a function definition.

Examples:

```
(%i1) stringdisp: false$
(%i2) "This is an example string.";
(%o2)              This is an example string.
(%i3) foo () :=
      print ("This is a string in a function definition.");
(%o3) foo() :=
                print("This is a string in a function definition.")
(%i4) stringdisp: true$
(%i5) "This is an example string.";
(%o5)             "This is an example string."
```

7.3 Constants

7.3.1 Functions and Variables for Constants

`%catalan` [Constant]

`%catalan` represents Catalan's constant, G, defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

(It is also sometimes denoted by C).

The numeric value of `%catalan` is approximately 0.915965594177219. (See DLMF 25.11.E40).

`%e` [Constant]

`%e` represents the base of the natural logarithm, also known as Euler's number. The numeric value of `%e` is the double-precision floating-point value 2.718281828459045d0. (See A&S eqn 4.1.16, A&S 4.1.17.)

`%i` [Constant]

`%i` represents the imaginary unit, $\sqrt{-1}$.

`false` [Constant]

`false` represents the Boolean constant of the same name. Maxima implements `false` by the value `NIL` in Lisp.

`%gamma` [Constant]

The Euler-Mascheroni constant, 0.5772156649015329.... It is defined by (A&S eqn 6.1.3 and DLMF 5.2.ii)

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$$

`ind` [Constant]

`ind` represents a bounded, indefinite result.

See also `limit`.

Example:

```
(%i1) limit (sin(1/x), x, 0);
(%o1)                          ind
```

`inf` [Constant]

`inf` represents real positive infinity.

`infinity` [Constant]

`infinity` represents complex infinity.

`least_negative_float` [Constant]

The least negative floating-point number in Maxima. That is, the negative floating-point number closest to 0. It is approximately -4.94065e-324, when denormal numbers are supported. Otherwise it is the same as `least_negative_normalized_float`.

`least_negative_normalized_float` [Constant]

The least negative normalized floating-point number in Maxima. That is, the negative normalized floating-point number closest to 0. It is approximately -2.22507e-308.

`least_positive_float` [Constant]

The least positive floating-point number in Maxima. That is, the positive floating-point number closest to 0. It is approximately 4.94065e-324, when denormal numbers are supported. Otherwise it is the same as `least_positive_normalized_float`.

`least_positive_normalized_float` [Constant]

The least positive normalized floating-point number in Maxima. That is, the positive normalized floating-point number closest to 0. It is approximately 2.22507e-308.

`minf` [Constant]

`minf` represents real minus (i.e., negative) infinity.

`most_negative_float` [Constant]

The most negative floating-point number in Maxima. It is approximately -1.79769e+308.

`most_positive_float` [Constant]

The most positive floating-point number in Maxima. It is approximately 1.797693e+308.

`%phi` [Constant]

`%phi` represents the so-called *golden mean*, $(1+\sqrt{5})/2$. The numeric value of `%phi` is the double-precision floating-point value 1.618033988749895d0.

`fibtophi` expresses Fibonacci numbers `fib(n)` in terms of `%phi`.

By default, Maxima does not know the algebraic properties of `%phi`. After evaluating `tellrat(%phi^2 - %phi - 1)` and `algebraic: true`, `ratsimp` can simplify some expressions containing `%phi`.

Examples:

`fibtophi` expresses Fibonacci numbers `fib(n)` in terms of `%phi`.

```
(%i1) fibtophi (fib (n));
                              n             n
                          %phi  - (1 - %phi)
(%o1)                     ------------------
                              2 %phi - 1
(%i2) fib (n-1) + fib (n) - fib (n+1);
(%o2)           - fib(n + 1) + fib(n) + fib(n - 1)
(%i3) fibtophi (%);
              n + 1             n + 1       n             n
          %phi      - (1 - %phi)         %phi  - (1 - %phi)
(%o3) - --------------------------- + ------------------
                 2 %phi - 1                2 %phi - 1
                                           n - 1             n - 1
                                       %phi      - (1 - %phi)
                                     + ---------------------------
                                                2 %phi - 1
```

```
(%i4) ratsimp (%);
(%o4)                          0
```

By default, Maxima does not know the algebraic properties of `%phi`. After evaluating `tellrat (%phi^2 - %phi - 1)` and `algebraic: true`, `ratsimp` can simplify some expressions containing `%phi`.

```
(%i1) e : expand ((%phi^2 - %phi - 1) * (A + 1));
                  2                       2
(%o1)        %phi  A - %phi A - A + %phi  - %phi - 1
(%i2) ratsimp (e);
                   2                      2
(%o2)        (%phi  - %phi - 1) A + %phi  - %phi - 1
(%i3) tellrat (%phi^2 - %phi - 1);
                              2
(%o3)                   [%phi  - %phi - 1]
(%i4) algebraic : true;
(%o4)                         true
(%i5) ratsimp (e);
(%o5)                          0
```

`%pi` [Constant]

`%pi` represents the ratio of the perimeter of a circle to its diameter. The numeric value of `%pi` is the double-precision floating-point value 3.141592653589793d0.

`true` [Constant]

`true` represents the Boolean constant of the same name. Maxima implements `true` by the value `T` in Lisp.

`und` [Constant]

`und` represents an undefined result.

See also `limit`.

Example:

```
(%i1) limit (x*sin(x), x, inf);
(%o1)                         und
```

`zeroa` [Constant]

`zeroa` represents an infinitesimal above zero. `zeroa` can be used in expressions. `limit` simplifies expressions which contain infinitesimals.

See also `zerob` and `limit`.

Example:

`limit` simplifies expressions which contain infinitesimals:

```
(%i1) limit(zeroa);
(%o1)                          0
(%i2) limit(zeroa+x);
(%o2)                          x
```

`zerob` [Constant]

`zerob` represents an infinitesimal below zero. `zerob` can be used in expressions. `limit` simplifies expressions which contain infinitesimals.

See also `zeroa` and `limit`.

7.4 Lists

7.4.1 Introduction to Lists

Lists are the basic building block for Maxima and Lisp. All data types other than arrays, hashed arrays and numbers are represented as Lisp lists, These Lisp lists have the form

```
((MPLUS) $A 2)
```

to indicate an expression a+2. At Maxima level one would see the infix notation a+2. Maxima also has lists which are printed as

```
[1, 2, 7, x+y]
```

for a list with 4 elements. Internally this corresponds to a Lisp list of the form

```
((MLIST) 1 2 7 ((MPLUS) $X $Y))
```

The flag which denotes the type field of the Maxima expression is a list itself, since after it has been through the simplifier the list would become

```
((MLIST SIMP) 1 2 7 ((MPLUS SIMP) $X $Y))
```

7.4.2 Functions and Variables for Lists

[[Operator]
] [Operator]

[and] mark the beginning and end, respectively, of a list.

[and] also enclose the subscripts of a list, array, hashed array, or memoizing function. Note that other than for arrays accessing the nth element of a list may need an amount of time that is roughly proportional to n, See Section 7.4.3 [Performance considerations for Lists], page 94.

Note that if an element of a subscripted variable is written to before a list or an array of this name is declared a hashed array (see Section 7.5 [Arrays], page 96) is created, not a list.

Examples:

```
(%i1) x: [a, b, c];
(%o1)                         [a, b, c]
(%i2) x[3];
(%o2)                             c
(%i3) array (y, fixnum, 3);
(%o3)                             y
(%i4) y[2]: %pi;
(%o4)                            %pi
(%i5) y[2];
(%o5)                            %pi
(%i6) z['foo]: 'bar;
(%o6)                            bar
(%i7) z['foo];
(%o7)                            bar
```

```
(%i8) g[k] := 1/(k^2+1);
                                    1
(%o8)                      g  := ------
                            k      2
                                  k  + 1
(%i9) g[10];
                                  1
(%o9)                            ---
                                 101
```

append (*list_1*, ..., *list_n*) [Function]

Returns a single list of the elements of *list_1* followed by the elements of *list_2*, ... `append` also works on general expressions, e.g. `append (f(a,b), f(c,d,e));` yields `f(a,b,c,d,e)`.

See also `addrow`, `addcol` and `join`.

Do `example(append);` for an example.

assoc [Function]

`assoc` (*key*, *e*, *default*)
`assoc` (*key*, *e*)

`assoc` searches for *key* as the first part of an argument of *e* and returns the second part of the first match, if any.

key may be any expression. *e* must be a nonatomic expression, and every argument of *e* must have exactly two parts. `assoc` returns the second part of the first matching argument of *e*. Matches are determined by `is(`*key* `= first(`*a*`))` where *a* is an argument of *e*.

If there are two or more matches, only the first is returned. If there are no matches, *default* is returned, if specified. Otherwise, `false` is returned.

See also `sublist` and `member`.

Examples:

key may be any expression. *e* must be a nonatomic expression, and every argument of *e* must have exactly two parts. `assoc` returns the second part of the first matching argument of *e*.

```
(%i1) assoc (f(x), foo(g(x) = y, f(x) = z + 1, h(x) = 2*u));
(%o1)                          z + 1
```

If there are two or more matches, only the first is returned.

```
(%i1) assoc (yy, [xx = 111, yy = 222, yy = 333, yy = 444]);
(%o1)                           222
```

If there are no matches, *default* is returned, if specified. Otherwise, `false` is returned.

```
(%i1) assoc (abc, [[x, 111], [y, 222], [z, 333]], none);
(%o1)                          none
(%i2) assoc (abc, [[x, 111], [y, 222], [z, 333]]);
(%o2)                          false
```

cons [Function]

cons (*expr*, *list*)
cons (*expr_1*, *expr_2*)

`cons (expr, list)` returns a new list constructed of the element *expr* as its first element, followed by the elements of *list*. This is analogous to the Lisp language construction operation "cons".

The Maxima function `cons` can also be used where the second argument is other than a list and this might be useful. In this case, `cons (expr_1, expr_2)` returns an expression with same operator as *expr_2* but with argument `cons(expr_1, args(expr_2))`. Examples:

```
(%i1) cons(a,[b,c,d]);
(%o1)                      [a, b, c, d]
(%i2) cons(a,f(b,c,d));
(%o2)                      f(a, b, c, d)
```

In general, `cons` applied to a nonlist doesn't make sense. For instance, `cons(a,b^c)` results in an illegal expression, since '^' cannot take three arguments.

When `inflag` is true, `cons` operates on the internal structure of an expression, otherwise `cons` operates on the displayed form. Especially when `inflag` is true, `cons` applied to a nonlist sometimes gives a surprising result; for example

```
(%i1) cons(a,-a), inflag : true;
                                    2
(%o1)                            - a
(%i2) cons(a,-a), inflag : false;
(%o2)                              0
```

copylist (*list*) [Function]

Returns a copy of the list *list*.

create_list (*form*, *x_1*, *list_1*, ..., *x_n*, *list_n*) [Function]

Create a list by evaluating *form* with *x_1* bound to each element of *list_1*, and for each such binding bind *x_2* to each element of *list_2*, ... The number of elements in the result will be the product of the number of elements in each list. Each variable *x_i* must actually be a symbol – it will not be evaluated. The list arguments will be evaluated once at the beginning of the iteration.

```
(%i1) create_list (x^i, i, [1, 3, 7]);
                                   3   7
(%o1)                         [x, x , x ]
```

With a double iteration:

```
(%i1) create_list ([i, j], i, [a, b], j, [e, f, h]);
(%o1)   [[a, e], [a, f], [a, h], [b, e], [b, f], [b, h]]
```

Instead of *list_i* two args may be supplied each of which should evaluate to a number. These will be the inclusive lower and upper bounds for the iteration.

```
(%i1) create_list ([i, j], i, [1, 2, 3], j, 1, i);
(%o1)   [[1, 1], [2, 1], [2, 2], [3, 1], [3, 2], [3, 3]]
```

Note that the limits or list for the `j` variable can depend on the current value of `i`.

delete [Function]

delete (*expr_1*, *expr_2*)
delete (*expr_1*, *expr_2*, *n*)

`delete(expr_1, expr_2)` removes from *expr_2* any arguments of its top-level operator which are the same (as determined by "=") as *expr_1*. Note that "=" tests for formal equality, not equivalence. Note also that arguments of subexpressions are not affected.

expr_1 may be an atom or a non-atomic expression. *expr_2* may be any non-atomic expression. `delete` returns a new expression; it does not modify *expr_2*.

`delete(expr_1, expr_2, n)` removes from *expr_2* the first *n* arguments of the top-level operator which are the same as *expr_1*. If there are fewer than *n* such arguments, then all such arguments are removed.

Examples:

Removing elements from a list.

```
(%i1) delete (y, [w, x, y, z, z, y, x, w]);
(%o1)                  [w, x, z, z, x, w]
```

Removing terms from a sum.

```
(%i1) delete (sin(x), x + sin(x) + y);
(%o1)                           y + x
```

Removing factors from a product.

```
(%i1) delete (u - x, (u - w)*(u - x)*(u - y)*(u - z));
(%o1)                (u - w) (u - y) (u - z)
```

Removing arguments from an arbitrary expression.

```
(%i1) delete (a, foo (a, b, c, d, a));
(%o1)                      foo(b, c, d)
```

Limit the number of removed arguments.

```
(%i1) delete (a, foo (a, b, a, c, d, a), 2);
(%o1)                     foo(b, c, d, a)
```

Whether arguments are the same as *expr_1* is determined by "=". Arguments which are `equal` but not "=" are not removed.

```
(%i1) [is (equal (0, 0)), is (equal (0, 0.0)), is (equal (0, 0b0))];
(%o1)                   [true, true, true]
(%i2) [is (0 = 0), is (0 = 0.0), is (0 = 0b0)];
(%o2)                  [true, false, false]
(%i3) delete (0, [0, 0.0, 0b0]);
(%o3)                       [0.0, 0.0b0]
(%i4) is (equal ((x + y)*(x - y), x^2 - y^2));
(%o4)                           true
(%i5) is ((x + y)*(x - y) = x^2 - y^2);
(%o5)                           false
(%i6) delete ((x + y)*(x - y), [(x + y)*(x - y), x^2 - y^2]);
                                2    2
(%o6)                         [x  - y ]
```

`eighth` (*expr*) [Function]

Returns the 8th item of expression or list *expr*. See `first` for more details.

`endcons` [Function]

`endcons` (*expr*, *list*)
`endcons` (*expr_1*, *expr_2*)

`endcons` (*expr*, *list*) returns a new list constructed of the elements of *list* followed by *expr*. The Maxima function `endcons` can also be used where the second argument is other than a list and this might be useful. In this case, `endcons` (*expr_1*, *expr_2*) returns an expression with same operator as *expr_2* but with argument `endcons(expr_1, args(expr_2))`. Examples:

```
(%i1) endcons(a,[b,c,d]);
(%o1)                      [b, c, d, a]
(%i2) endcons(a,f(b,c,d));
(%o2)                     f(b, c, d, a)
```

In general, `endcons` applied to a nonlist doesn't make sense. For instance, `endcons(a,b^c)` results in an illegal expression, since '^' cannot take three arguments.

When `inflag` is true, `endcons` operates on the internal structure of an expression, otherwise `endcons` operates on the displayed form. Especially when `inflag` is true, `endcons` applied to a nonlist sometimes gives a surprising result; for example

```
(%i1) endcons(a,-a), inflag : true;
                                  2
(%o1)                           - a
(%i2) endcons(a,-a), inflag : false;
(%o2)                             0
```

`fifth` (*expr*) [Function]

Returns the 5th item of expression or list *expr*. See `first` for more details.

`first` (*expr*) [Function]

Returns the first part of *expr* which may result in the first element of a list, the first row of a matrix, the first term of a sum, etc.:

```
(%i1) matrix([1,2],[3,4]);
                                    [ 1  2 ]
(%o1)                               [      ]
                                    [ 3  4 ]
(%i2) first(%);
(%o2)                               [1,2]
(%i3) first(%);
(%o3)                               1
(%i4) first(a*b/c+d+e/x);
                                    a b
(%o4)                               ---
                                     c
(%i5) first(a=b/c+d+e/x);
(%o5)                               a
```

Note that `first` and its related functions, `rest` and `last`, work on the form of *expr* which is displayed not the form which is typed on input. If the variable `inflag` is set to `true` however, these functions will look at the internal form of *expr*. One reason why this may make a difference is that the simplifier re-orders expressions:

```
(%i1) x+y;
(%o1)                                y+1
(%i2) first(x+y),inflag : true;
(%o2)                                x
(%i3) first(x+y),inflag : false;
(%o3)                                y
```

The functions `second` ... `tenth` yield the second through the tenth part of their input argument.

See also `firstn` and `part`.

`firstn` (*expr*, *count*) [Function]

Returns the first *count* arguments of *expr*, if *expr* has at least *count* arguments. Returns *expr* if *expr* has less than *count* arguments.

expr may be any nonatomic expression. When *expr* is something other than a list, `firstn` returns an expression which has the same operator as *expr*. *count* must be a nonnegative integer.

`firstn` honors the global flag `inflag`, which governs whether the internal form of an expression is processed (when `inflag` is true) or the displayed form (when `inflag` is false).

Note that `firstn(`*expr*`, 1)`, which returns a nonatomic expression containing the first argument, is not the same as `first(`*expr*`)`, which returns the first argument by itself.

See also `lastn` and `rest`.

Examples:

`firstn` returns the first *count* elements of *expr*, if *expr* has at least *count* elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 0);
(%o2)                            []
(%i3) firstn (mylist, 1);
(%o3)                            [1]
(%i4) firstn (mylist, 2);
(%o4)                         [1, a]
(%i5) firstn (mylist, 7);
(%o5)               [1, a, 2, b, 3, x, 4 - y]
```

`firstn` returns *expr* if *expr* has less than *count* elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) firstn (mylist, 100);
(%o2)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

expr may be any nonatomic expression.

```
(%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o1)      foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i2) firstn (myfoo, 4);
(%o2)                    foo(1, a, 2, b)
(%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o3)    bar    (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
           m, n
(%i4) firstn (mybar, 4);
(%o4)                  bar    (1, a, 2, b)
                         m, n
(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
(%i6) firstn (mymatrix, 3);
                       [ 11  12  13  14 ]
                       [                ]
(%o6)                  [ 21  22  23  24 ]
                       [                ]
                       [ 31  32  33  34 ]
```

`firstn` honors the global flag `inflag`.

```
(%i1) myexpr : a + b + c + d + e;
(%o1)                    e + d + c + b + a
(%i2) firstn (myexpr, 3), inflag=true;
(%o2)                        c + b + a
(%i3) firstn (myexpr, 3), inflag=false;
(%o3)                        e + d + c
```

Note that `firstn(`*expr*`, 1)` is not the same as `first(`*expr*`)`.

```
(%i1) firstn ([w, x, y, z], 1);
(%o1)                           [w]
(%i2) first ([w, x, y, z]);
(%o2)                            w
```

`fourth` (*expr*) [Function]

Returns the 4th item of expression or list *expr*. See `first` for more details.

`join` (*l*, *m*) [Function]

Creates a new list containing the elements of lists *l* and *m*, interspersed. The result has elements `[`*l*`[1], `*m*`[1], `*l*`[2], `*m*`[2], ...]`. The lists *l* and *m* may contain any type of elements.

If the lists are different lengths, `join` ignores elements of the longer list.

Maxima complains if *l* or *m* is not a list.

See also `append`.

Examples:

```
(%i1) L1: [a, sin(b), c!, d - 1];
(%o1)                 [a, sin(b), c!, d - 1]
(%i2) join (L1, [1, 2, 3, 4]);
(%o2)          [a, 1, sin(b), 2, c!, 3, d - 1, 4]
```

```
(%i3) join (L1, [aa, bb, cc, dd, ee, ff]);
(%o3)        [a, aa, sin(b), bb, c!, cc, d - 1, dd]
```

`last` (*expr*) [Function]

Returns the last part (term, row, element, etc.) of the *expr*.

See also `lastn`.

`lastn` (*expr*, *count*) [Function]

Returns the last *count* arguments of *expr*, if *expr* has at least *count* arguments. Returns *expr* if *expr* has less than *count* arguments.

expr may be any nonatomic expression. When *expr* is something other than a list, `lastn` returns an expression which has the same operator as *expr*. *count* must be a nonnegative integer.

`lastn` honors the global flag `inflag`, which governs whether the internal form of an expression is processed (when `inflag` is true) or the displayed form (when `inflag` is false).

Note that `lastn(`*expr*`, 1)`, which returns a nonatomic expression containing the last argument, is not the same as `last(`*expr*`)`, which returns the last argument by itself.

See also `firstn` and `rest`.

Examples:

`lastn` returns the last *count* elements of *expr*, if *expr* has at least *count* elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) lastn (mylist, 0);
(%o2)                          []
(%i3) lastn (mylist, 1);
(%o3)                     [2 z + sin(u)]
(%i4) lastn (mylist, 2);
(%o4)                  [4 - y, 2 z + sin(u)]
(%i5) lastn (mylist, 7);
(%o5)         [a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

`lastn` returns *expr* if *expr* has less than *count* elements.

```
(%i1) mylist : [1, a, 2, b, 3, x, 4 - y, 2*z + sin(u)];
(%o1)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
(%i2) lastn (mylist, 100);
(%o2)        [1, a, 2, b, 3, x, 4 - y, 2 z + sin(u)]
```

expr may be any nonatomic expression.

```
(%i1) myfoo : foo(1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o1)      foo(1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
(%i2) lastn (myfoo, 4);
(%o2)            foo(3, x, 4 - y, 2 z + sin(u))
(%i3) mybar : bar[m, n](1, a, 2, b, 3, x, 4 - y, 2*z + sin(u));
(%o3)    bar    (1, a, 2, b, 3, x, 4 - y, 2 z + sin(u))
              m, n
```

```
(%i4) lastn (mybar, 4);
(%o4)          bar    (3, x, 4 - y, 2 z + sin(u))
                  m, n
(%i5) mymatrix : genmatrix (lambda ([i, j], 10*i + j), 10, 4) $
(%i6) lastn (mymatrix, 3);
                      [ 81   82   83   84  ]
                      [                    ]
(%o6)                 [ 91   92   93   94  ]
                      [                    ]
                      [ 101  102  103  104 ]
```

`lastn` honors the global flag `inflag`.

```
(%i1) myexpr : a + b + c + d + e;
(%o1)                    e + d + c + b + a
(%i2) lastn (myexpr, 3), inflag=true;
(%o2)                         e + d + c
(%i3) lastn (myexpr, 3), inflag=false;
(%o3)                         c + b + a
```

Note that `lastn(`*expr*`, 1)` is not the same as `last(`*expr*`)`.

```
(%i1) lastn ([w, x, y, z], 1);
(%o1)                            [z]
(%i2) last ([w, x, y, z]);
(%o2)                             z
```

`length (`*expr*`)` [Function]

Returns (by default) the number of parts in the external (displayed) form of *expr*. For lists this is the number of elements, for matrices it is the number of rows, and for sums it is the number of terms (see `dispform`).

The `length` command is affected by the `inflag` switch. So, e.g. `length(a/(b*c));` gives 2 if `inflag` is `false` (Assuming `exptdispflag` is `true`), but 3 if `inflag` is `true` (the internal representation is essentially `a*b^-1*c^-1`).

Determining a list's length typically needs an amount of time proportional to the number of elements in the list. If the length of a list is used inside a loop it therefore might drastically increase the performance if the length is calculated outside the loop instead.

`listarith` [Option variable]

Default value: `true`

If `false` causes any arithmetic operations with lists to be suppressed; when `true`, list-matrix operations are contagious causing lists to be converted to matrices yielding a result which is always a matrix. However, list-list operations should return lists.

`listp (`*expr*`)` [Function]

Returns `true` if *expr* is a list else `false`.

`lreduce` [Function]

`lreduce (`*F*, *s*`)`
`lreduce (`*F*, *s*, *s_0*`)`

Extends the binary function F to an n-ary function by composition, where s is a list.

`lreduce(F, s)` returns `F(... F(F(s_1, s_2), s_3), ... s_n)`. When the optional argument *s_0* is present, the result is equivalent to `lreduce(F, cons(s_0, s))`.

The function *F* is first applied to the *leftmost* list elements, thus the name "lreduce".

See also `rreduce`, `xreduce`, and `tree_reduce`.

Examples:

`lreduce` without the optional argument.

```
(%i1) lreduce (f, [1, 2, 3]);
(%o1)                     f(f(1, 2), 3)
(%i2) lreduce (f, [1, 2, 3, 4]);
(%o2)                  f(f(f(1, 2), 3), 4)
```

`lreduce` with the optional argument.

```
(%i1) lreduce (f, [1, 2, 3], 4);
(%o1)                  f(f(f(4, 1), 2), 3)
```

`lreduce` applied to built-in binary operators. `/` is the division operator.

```
(%i1) lreduce ("^", [a, b, c, d]);
                                 b c d
(%o1)                         ((a ) )
(%i2) lreduce ("/", [a, b, c, d]);
                                  a
(%o2)                            -----
                                 b c d
```

`makelist` [Function]

`makelist ()`
`makelist (expr)`
`makelist (expr, n)`
`makelist (expr, i, i_max)`
`makelist (expr, i, i_0, i_max)`
`makelist (expr, i, i_0, i_max, step)`
`makelist (expr, x, list)`

The first form, `makelist ()`, creates an empty list. The second form, `makelist (expr)`, creates a list with *expr* as its single element. `makelist (expr, n)` creates a list of *n* elements generated from *expr*.

The most general form, `makelist (expr, i, i_0, i_max, step)`, returns the list of elements obtained when `ev (expr, i=j)` is applied to the elements *j* of the sequence: *i_0*, *i_0* + *step*, *i_0* + 2**step*, ..., with `|j|` less than or equal to `|i_max|`.

The increment *step* can be a number (positive or negative) or an expression. If it is omitted, the default value 1 will be used. If both *i_0* and *step* are omitted, they will both have a default value of 1.

`makelist (expr, x, list)` returns a list, the `j`th element of which is equal to `ev (expr, x=list[j])` for `j` equal to 1 through `length (list)`.

Examples:

```
(%i1) makelist (concat (x,i), i, 6);
(%o1)                [x1, x2, x3, x4, x5, x6]
```

```
(%i2) makelist (x=y, y, [a, b, c]);
(%o2)                  [x = a, x = b, x = c]
(%i3) makelist (x^2, x, 3, 2*%pi, 2);
(%o3)                          [9, 25]
(%i4) makelist (random(6), 4);
(%o4)                      [2, 0, 2, 5]
(%i5) flatten (makelist (makelist (i^2, 3), i, 4));
(%o5)        [1, 1, 1, 4, 4, 4, 9, 9, 9, 16, 16, 16]
(%i6) flatten (makelist (makelist (i^2, i, 3), 4));
(%o6)         [1, 4, 9, 1, 4, 9, 1, 4, 9, 1, 4, 9]
```

`member` (*expr_1*, *expr_2*) [Function]

Returns `true` if `is(`*expr_1* `= a)` for some element *a* in `args(`*expr_2*`)`, otherwise returns `false`.

`expr_2` is typically a list, in which case `args(`*expr_2*`) =` *expr_2* and `is(`*expr_1* `= a)` for some element *a* in `expr_2` is the test.

`member` does not inspect parts of the arguments of `expr_2`, so it may return `false` even if `expr_1` is a part of some argument of `expr_2`.

See also `elementp`.

Examples:

```
(%i1) member (8, [8, 8.0, 8b0]);
(%o1)                          true
(%i2) member (8, [8.0, 8b0]);
(%o2)                          false
(%i3) member (b, [a, b, c]);
(%o3)                          true
(%i4) member (b, [[a, b], [b, c]]);
(%o4)                          false
(%i5) member ([b, c], [[a, b], [b, c]]);
(%o5)                          true
(%i6) F (1, 1/2, 1/4, 1/8);
                                1  1  1
(%o6)                    F(1, -, -, -)
                                2  4  8
(%i7) member (1/8, %);
(%o7)                          true
(%i8) member ("ab", ["aa", "ab", sin(1), a + b]);
(%o8)                          true
```

`ninth` (*expr*) [Function]

Returns the 9th item of expression or list *expr*. See `first` for more details.

`pop` (*list*) [Function]

`pop` removes and returns the first element from the list *list*. The argument *list* must be a mapatom that is bound to a nonempty list. If the argument *list* is not bound to a nonempty list, Maxima signals an error. For examples, see `push`.

push (*item*, *list*) [Function]

push prepends the item *item* to the list *list* and returns a copy of the new list. The second argument *list* must be a mapatom that is bound to a list. The first argument *item* can be any Maxima symbol or expression. If the argument *list* is not bound to a list, Maxima signals an error.

To remove the first item from a list, see pop.

Examples:

```
(%i1) ll: [];
(%o1)                           []
(%i2) push (x, ll);
(%o2)                           [x]
(%i3) push (x^2+y, ll);
                                  2
(%o3)                       [y + x , x]
(%i4) a: push ("string", ll);
                                      2
(%o4)                  [string, y + x , x]
(%i5) pop (ll);
(%o5)                        string
(%i6) pop (ll);
                                   2
(%o6)                        y + x
(%i7) pop (ll);
(%o7)                           x
(%i8) ll;
(%o8)                           []
(%i9) a;
                                      2
(%o9)                  [string, y + x , x]
```

rest [Function]

rest (*expr*, *n*)
rest (*expr*)

Returns *expr* with its first *n* elements removed if *n* is positive and its last - *n* elements removed if *n* is negative. If *n* is 1 it may be omitted. The first argument *expr* may be a list, matrix, or other expression. When *expr* is an atom, rest signals an error; when *expr* is an empty list and partswitch is false, rest signals an error. When *expr* is an empty list and partswitch is true, rest returns end.

Applying rest to expression such as f(a,b,c) returns f(b,c). In general, applying rest to a nonlist doesn't make sense. For example, because '^' requires two arguments, rest(a^b) results in an error message. The functions args and op may be useful as well, since args(a^b) returns [a,b] and op(a^b) returns ^.

See also firstn and lastn.

```
(%i1) rest(a+b+c);
(%o1) b+a
(%i2) rest(a+b+c,2);
(%o2) a
(%i3) rest(a+b+c,-2);
(%o3) c
```

`reverse` (*list*) [Function]

Reverses the order of the members of the *list* (not the members themselves). `reverse` also works on general expressions, e.g. `reverse(a=b);` gives `b=a`.

See also `sreverse`.

`rreduce` [Function]

`rreduce` (*F*, *s*)
`rreduce` (*F*, *s*, *s_{n + 1}*)

Extends the binary function *F* to an n-ary function by composition, where *s* is a list. `rreduce(`*F*`,` *s*`)` returns `F(s_1, ... F(s_{n - 2}, F(s_{n - 1}, s_n)))`. When the optional argument *s_{n + 1}* is present, the result is equivalent to `rreduce(`*F*`, endcons(`*s_{n + 1}*`,` *s*`))`.

The function *F* is first applied to the *rightmost* list elements, thus the name "rreduce".

See also `lreduce`, `tree_reduce`, and `xreduce`.

Examples:

`rreduce` without the optional argument.

```
(%i1) rreduce (f, [1, 2, 3]);
(%o1)                     f(1, f(2, 3))
(%i2) rreduce (f, [1, 2, 3, 4]);
(%o2)                  f(1, f(2, f(3, 4)))
```

`rreduce` with the optional argument.

```
(%i1) rreduce (f, [1, 2, 3], 4);
(%o1)                  f(1, f(2, f(3, 4)))
```

`rreduce` applied to built-in binary operators. `/` is the division operator.

```
(%i1) rreduce ("^", [a, b, c, d]);
                                    d
                                   c
                                  b
(%o1)                            a
(%i2) rreduce ("/", [a, b, c, d]);
                                   a c
(%o2)                              ---
                                   b d
```

`second` (*expr*) [Function]

Returns the 2nd item of expression or list *expr*. See `first` for more details.

`seventh` (*expr*) [Function]

Returns the 7th item of expression or list *expr*. See `first` for more details.

`sixth` (*expr*) [Function]

Returns the 6th item of expression or list *expr*. See `first` for more details.

`sort` [Function]

`sort` (*L*, *P*)
`sort` (*L*)

`sort(L, P)` sorts a list *L* according to a predicate `P` of two arguments which defines a strict weak order on the elements of *L*. If *P*`(a, b)` is `true`, then `a` appears before `b` in the result. If neither *P*`(a, b)` nor *P*`(b, a)` are `true`, then `a` and `b` are equivalent, and appear in the result in the same order as in the input. That is, `sort` is a stable sort.

If *P*`(a, b)` and *P*`(b, a)` are both `true` for some elements of *L*, then *P* is not a valid sort predicate, and the result is undefined. If *P*`(a, b)` is something other than `true` or `false`, `sort` signals an error.

The predicate may be specified as the name of a function or binary infix operator, or as a `lambda` expression. If specified as the name of an operator, the name must be enclosed in double quotes.

The sorted list is returned as a new object; the argument *L* is not modified.

`sort(L)` is equivalent to `sort(L, orderlessp)`.

The default sorting order is ascending, as determined by `orderlessp`. The predicate `ordergreatp` sorts a list in descending order.

All Maxima atoms and expressions are comparable under `orderlessp` and `ordergreatp`.

Operators `<` and `>` order numbers, constants, and constant expressions by magnitude. Note that `orderlessp` and `ordergreatp` do not order numbers, constants, and constant expressions by magnitude.

`ordermagnitudep` orders numbers, constants, and constant expressions the same as `<`, and all other elements the same as `orderlessp`.

Examples:

`sort` sorts a list according to a predicate of two arguments which defines a strict weak order on the elements of the list.

```
(%i1) sort ([1, a, b, 2, 3, c], 'orderlessp);
(%o1)                  [1, 2, 3, a, b, c]
(%i2) sort ([1, a, b, 2, 3, c], 'ordergreatp);
(%o2)                  [c, b, a, 3, 2, 1]
```

The predicate may be specified as the name of a function or binary infix operator, or as a `lambda` expression. If specified as the name of an operator, the name must be enclosed in double quotes.

```
(%i1) L : [[1, x], [3, y], [4, w], [2, z]];
(%o1)           [[1, x], [3, y], [4, w], [2, z]]
(%i2) foo (a, b) := a[1] > b[1];
(%o2)                 foo(a, b) := a  > b
                                     1    1
(%i3) sort (L, 'foo);
(%o3)           [[4, w], [3, y], [2, z], [1, x]]
```

```
(%i4) infix (">>");
(%o4)                          >>
(%i5) a >> b := a[1] > b[1];
(%o5)                  (a >> b) := a  > b
                                      1    1
(%i6) sort (L, ">>");
(%o6)           [[4, w], [3, y], [2, z], [1, x]]
(%i7) sort (L, lambda ([a, b], a[1] > b[1]));
(%o7)           [[4, w], [3, y], [2, z], [1, x]]
```

`sort(L)` is equivalent to `sort(L, orderlessp)`.

```
(%i1) L : [a, 2*b, -5, 7, 1 + %e, %pi];
(%o1)             [a, 2 b, - 5, 7, %e + 1, %pi]
(%i2) sort (L);
(%o2)             [- 5, 7, %e + 1, %pi, a, 2 b]
(%i3) sort (L, 'orderlessp);
(%o3)             [- 5, 7, %e + 1, %pi, a, 2 b]
```

The default sorting order is ascending, as determined by `orderlessp`. The predicate `ordergreatp` sorts a list in descending order.

```
(%i1) L : [a, 2*b, -5, 7, 1 + %e, %pi];
(%o1)             [a, 2 b, - 5, 7, %e + 1, %pi]
(%i2) sort (L);
(%o2)             [- 5, 7, %e + 1, %pi, a, 2 b]
(%i3) sort (L, 'ordergreatp);
(%o3)             [2 b, a, %pi, %e + 1, 7, - 5]
```

All Maxima atoms and expressions are comparable under `orderlessp` and `ordergreatp`.

```
(%i1) L : [11, -17, 29b0, 9*c, 7.55, foo(x, y), -5/2, b + a];
                                                    5
(%o1)  [11, - 17, 2.9b1, 9 c, 7.55, foo(x, y), - -, b + a]
                                                    2
(%i2) sort (L, orderlessp);
                5
(%o2)  [- 17, - -, 7.55, 11, 2.9b1, b + a, 9 c, foo(x, y)]
                2
(%i3) sort (L, ordergreatp);
                                                     5
(%o3)  [foo(x, y), 9 c, b + a, 2.9b1, 11, 7.55, - -, - 17]
                                                     2
```

Operators `<` and `>` order numbers, constants, and constant expressions by magnitude. Note that `orderlessp` and `ordergreatp` do not order numbers, constants, and constant expressions by magnitude.

```
(%i1) L : [%pi, 3, 4, %e, %gamma];
(%o1)                [%pi, 3, 4, %e, %gamma]
(%i2) sort (L, ">");
(%o2)                [4, %pi, 3, %e, %gamma]
```

```
(%i3) sort (L, ordergreatp);
(%o3)                 [%pi, %gamma, %e, 4, 3]
```

`ordermagnitudep` orders numbers, constants, and constant expressions the same as `<`, and all other elements the same as `orderlessp`.

```
(%i1) L: [%i, 1+%i, 2*x, minf, inf, %e, sin(1), 0, 1,2,3, 1.0, 1.0b0];
(%o1) [%i, %i + 1, 2 x, minf, inf, %e, sin(1), 0, 1, 2, 3, 1.0,
                                                          1.0b0]
(%i2) sort (L, ordermagnitudep);
(%o2) [minf, 0, sin(1), 1, 1.0, 1.0b0, 2, %e, 3, inf, %i,
                                                  %i + 1, 2 x]
(%i3) sort (L, orderlessp);
(%o3) [0, 1, 1.0, 2, 3, sin(1), 1.0b0, %e, %i, %i + 1, inf,
                                                    minf, 2 x]
```

`sublist` (*list*, *p*) [Function]

Returns the list of elements of *list* for which the predicate `p` returns `true`.

Example:

```
(%i1) L: [1, 2, 3, 4, 5, 6];
(%o1)                  [1, 2, 3, 4, 5, 6]
(%i2) sublist (L, evenp);
(%o2)                        [2, 4, 6]
```

`sublist_indices` (*L*, *P*) [Function]

Returns the indices of the elements `x` of the list *L* for which the predicate `maybe(P(x))` returns `true`; this excludes `unknown` as well as `false`. *P* may be the name of a function or a lambda expression. *L* must be a literal list.

Examples:

```
(%i1) sublist_indices ('[a, b, b, c, 1, 2, b, 3, b],
                       lambda ([x], x='b));
(%o1)                     [2, 3, 7, 9]
(%i2) sublist_indices ('[a, b, b, c, 1, 2, b, 3, b], symbolp);
(%o2)                  [1, 2, 3, 4, 7, 9]
(%i3) sublist_indices ([1 > 0, 1 < 0, 2 < 1, 2 > 1, 2 > 0],
                       identity);
(%o3)                       [1, 4, 5]
(%i4) assume (x < -1);
(%o4)                       [x < - 1]
(%i5) map (maybe, [x > 0, x < 0, x < -2]);
(%o5)                [false, true, unknown]
(%i6) sublist_indices ([x > 0, x < 0, x < -2], identity);
(%o6)                          [2]
```

`tenth` (*expr*) [Function]

Returns the 10th item of expression or list *expr*. See `first` for more details.

`third` (*expr*) [Function]

Returns the 3rd item of expression or list *expr*. See `first` for more details.

tree_reduce [Function]

tree_reduce (*F*, *s*)
tree_reduce (*F*, *s*, *s_0*)

Extends the binary function F to an n-ary function by composition, where s is a set or list.

`tree_reduce` is equivalent to the following: Apply F to successive pairs of elements to form a new list `[F(s_1, s_2), F(s_3, s_4), ...]`, carrying the final element unchanged if there are an odd number of elements. Then repeat until the list is reduced to a single element, which is the return value.

When the optional argument *s_0* is present, the result is equivalent `tree_reduce(F, cons(s_0, s))`.

For addition of floating point numbers, `tree_reduce` may return a sum that has a smaller rounding error than either `rreduce` or `lreduce`.

The elements of s and the partial results may be arranged in a minimum-depth binary tree, thus the name "tree_reduce".

Examples:

`tree_reduce` applied to a list with an even number of elements.

```
(%i1) tree_reduce (f, [a, b, c, d]);
(%o1)                  f(f(a, b), f(c, d))
```

`tree_reduce` applied to a list with an odd number of elements.

```
(%i1) tree_reduce (f, [a, b, c, d, e]);
(%o1)               f(f(f(a, b), f(c, d)), e)
```

unique (*L*) [Function]

Returns the unique elements of the list L.

When all the elements of L are unique, `unique` returns a shallow copy of L, not L itself.

If L is not a list, `unique` returns L.

Example:

```
(%i1) unique ([1, %pi, a + b, 2, 1, %e, %pi, a + b, [1]]);
(%o1)              [1, 2, %e, %pi, [1], b + a]
```

xreduce [Function]

xreduce (*F*, *s*)
xreduce (*F*, *s*, *s_0*)

Extends the function F to an n-ary function by composition, or, if F is already n-ary, applies F to s. When F is not n-ary, `xreduce` is the same as `lreduce`. The argument s is a list.

Functions known to be n-ary include addition `+`, multiplication `*`, `and`, `or`, `max`, `min`, and `append`. Functions may also be declared n-ary by `declare(F, nary)`. For these functions, `xreduce` is expected to be faster than either `rreduce` or `lreduce`.

When the optional argument *s_0* is present, the result is equivalent to `xreduce(s, cons(s_0, s))`.

Floating point addition is not exactly associative; be that as it may, `xreduce` applies Maxima's n-ary addition when *s* contains floating point numbers.

Examples:

`xreduce` applied to a function known to be n-ary. `F` is called once, with all arguments.

```
(%i1) declare (F, nary);
(%o1)                           done
(%i2) F ([L]) := L;
(%o2)                        F([L]) := L
(%i3) xreduce (F, [a, b, c, d, e]);
(%o3)                      [a, b, c, d, e]
```

`xreduce` applied to a function not known to be n-ary. `G` is called several times, with two arguments each time.

```
(%i1) G ([L]) := L;
(%o1)                        G([L]) := L
(%i2) xreduce (G, [a, b, c, d, e]);
(%o2)                  [[[[a, b], c], d], e]
(%i3) lreduce (G, [a, b, c, d, e]);
(%o3)                  [[[[a, b], c], d], e]
```

7.4.3 Performance considerations for Lists

Lists provide efficient ways of appending and removing elements. They can be created without knowing their final dimensions. Lisp provides efficient means of copying and handling lists. Also nested lists do not need to be strictly rectangular. These advantages over declared arrays come with the drawback that the amount of time needed for accessing a random element within a list may be roughly proportional to the element's distance from its beginning. Efficient traversal of lists is still possible, though, by using the list as a stack or a fifo:

```
(%i1) l:[Test,1,2,3,4];
(%o1)                  [Test, 1, 2, 3, 4]
(%i2) while l # [] do
   disp(pop(l));
                               Test

                                 1

                                 2

                                 3

                                 4

(%o2)                          done
```

Another even faster example would be:

```
(%i1) l:[Test,1,2,3,4];
(%o1)                  [Test, 1, 2, 3, 4]
```

```
(%i2) for i in l do
   disp(pop(l));
                                Test

                                  1

                                  2

                                  3

                                  4

(%o2)                            done
```

Beginning traversal with the last element of a list is possible after reversing the list using `reverse ()`. If the elements of a long list need to be processed in a different order performance might be increased by converting the list into a declared array first.

Note also that the ending condition of `for` loops is tested for every iteration which means that the result of a `length` should be cached if it is used in the ending condition:

```
(%i1) l:makelist(i,i,1,100000)$
(%i2) lngth:length(l);
(%o2)                           100000
(%i3) x:1;
(%o3)                             1
(%i4) for i:1 thru lngth do
    x:x+1$
(%i5) x;
(%o5)                           100001
```

7.5 Arrays

Maxima supports 3 array-like constructs:

- If one tries to write to an indexed variable without creating a list first an undeclared array (also named hashed array) is created that grows dynamically and allows numbers, symbols and strings as indices:

```
(%i1) a["feww"]:1;
(%o1)                           1
(%i2) a[qqwdqwd]:3;
(%o2)                           3
(%i3) a[5]:99;
(%o3)                          99
(%i4) a[qqwdqwd];
(%o4)                           3
(%i5) a[5];
(%o5)                          99
(%i6) a["feww"];
(%o6)                           1
```

  Since lisp handles hashed arrays and `memoizing functions` similar to arrays many of the functions that can be applied to arrays can be applied to them, as well.
- Lists (see `makelist` allow for fast addition and removal of elements, can be created without knowing their final size.
- Declared arrays that allow fast access to random elements at the cost that their size needs to be known at construction time. (See Section 7.4.3 [Performance considerations for Lists], page 94.)

7.5.1 Functions and Variables for Arrays

`array` [Function]

`array` (*name*, *dim_1*, ..., *dim_n*)
`array` (*name*, *type*, *dim_1*, ..., *dim_n*)
`array` ([*name_1*, ..., *name_m*], *dim_1*, ..., *dim_n*)

Creates an n-dimensional array. n may be less than or equal to 5. The subscripts for the i'th dimension are the integers running from 0 to *dim_i*.

`array (name, dim_1, ..., dim_n)` creates a general array.

`array (name, type, dim_1, ..., dim_n)` creates an array, with elements of a specified type. *type* can be `fixnum` for integers of limited size or `flonum` for floating-point numbers.

`array ([name_1, ..., name_m], dim_1, ..., dim_n)` creates m arrays, all of the same dimensions.

See also `arraymake`, `arrayinfo` and `make_array`.

`arrayapply` (*A*, [*i_1*, ..., *i_n*]) [Function]

Evaluates `A [i_1, ..., i_n]`, where A is an array and *i_1*, ..., *i_n* are integers.

This is reminiscent of `apply`, except the first argument is an array instead of a function.

`arrayinfo (A)` [Function]

Returns information about the array A. The argument A may be a declared array, a hashed array, a memoizing function, or a subscripted function.

For declared arrays, `arrayinfo` returns a list comprising the atom `declared`, the number of dimensions, and the size of each dimension. The elements of the array, both bound and unbound, are returned by `listarray`.

For undeclared arrays (hashed arrays), `arrayinfo` returns a list comprising the atom `hashed`, the number of subscripts, and the subscripts of every element which has a value. The values are returned by `listarray`.

For memoizing functions, `arrayinfo` returns a list comprising the atom `hashed`, the number of subscripts, and any subscript values for which there are stored function values. The stored function values are returned by `listarray`.

For subscripted functions, `arrayinfo` returns a list comprising the atom `hashed`, the number of subscripts, and any subscript values for which there are lambda expressions. The lambda expressions are returned by `listarray`.

See also `listarray`.

Examples:

`arrayinfo` and `listarray` applied to a declared array.

```
(%i1) array (aa, 2, 3);
(%o1)                          aa
(%i2) aa [2, 3] : %pi;
(%o2)                          %pi
(%i3) aa [1, 2] : %e;
(%o3)                          %e
(%i4) arrayinfo (aa);
(%o4)                 [declared, 2, [2, 3]]
(%i5) listarray (aa);
(%o5) [#####, #####, #####, #####, #####, #####, %e, #####,
                                          #####, #####, #####, %pi]
```

`arrayinfo` and `listarray` applied to an undeclared array (hashed array.).

```
(%i1) bb [FOO] : (a + b)^2;
                                    2
(%o1)                        (b + a)
(%i2) bb [BAR] : (c - d)^3;
                                    3
(%o2)                        (c - d)
(%i3) arrayinfo (bb);
(%o3)               [hashed, 1, [BAR], [FOO]]
(%i4) listarray (bb);
                                3         2
(%o4)                 [(c - d) , (b + a) ]
```

`arrayinfo` and `listarray` applied to a memoizing function.

```
(%i1) cc [x, y] := y / x;
                                        y
(%o1)                        cc     := -
                              x, y     x
(%i2) cc [u, v];
                                    v
(%o2)                               -
                                    u
(%i3) cc [4, z];
                                    z
(%o3)                               -
                                    4
(%i4) arrayinfo (cc);
(%o4)                [hashed, 2, [4, z], [u, v]]
(%i5) listarray (cc);
                                  z  v
(%o5)                            [-, -]
                                  4  u
```

Using `arrayinfo` in order to convert an undeclared array to a declared array:

```
(%i1) for i:0 thru 10 do a[i]:i^2$
(%i2) indices:map(first,rest(rest(arrayinfo(a))));
(%o2)          [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
(%i3) array(A,fixnum,length(indices)-1)$
(%i4) fillarray(A,map(lambda([x],a[x]),indices))$
(%i5) listarray(A);
(%o5)       [0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
```

`arrayinfo` and `listarray` applied to a subscripted function.

```
(%i1) dd [x] (y) := y ^ x;
                                       x
(%o1)                        dd (y) := y
                                 x
(%i2) dd [a + b];
                                     b + a
(%o2)                    lambda([y], y     )
(%i3) dd [v - u];
                                     v - u
(%o3)                    lambda([y], y     )
(%i4) arrayinfo (dd);
(%o4)               [hashed, 1, [b + a], [v - u]]
(%i5) listarray (dd);
                            b + a                 v - u
(%o5)       [lambda([y], y     ), lambda([y], y     )]
```

`arraymake` (*A*, [*i_1*, ..., *i_n*]) [Function]

Returns the expression *A*[*i_1*, ..., *i_n*]. The result is an unevaluated array reference.

`arraymake` is reminiscent of `funmake`, except the return value is an unevaluated array reference instead of an unevaluated function call.

Examples:

```
(%i1) arraymake (A, [1]);
(%o1)                           A
                                 1
(%i2) arraymake (A, [k]);
(%o2)                           A
                                 k
(%i3) arraymake (A, [i, j, 3]);
(%o3)                        A
                              i, j, 3
(%i4) array (A, fixnum, 10);
(%o4)                            A
(%i5) fillarray (A, makelist (i^2, i, 1, 11));
(%o5)                            A
(%i6) arraymake (A, [5]);
(%o6)                           A
                                 5
(%i7) ''%;
(%o7)                           36
(%i8) L : [a, b, c, d, e];
(%o8)                     [a, b, c, d, e]
(%i9) arraymake ('L, [n]);
(%o9)                           L
                                 n
(%i10) ''%, n = 3;
(%o10)                           c
(%i11) A2 : make_array (fixnum, 10);
(%o11)        {Lisp Array: #(0 0 0 0 0 0 0 0 0 0)}
(%i12) fillarray (A2, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]);
(%o12)        {Lisp Array: #(1 2 3 4 5 6 7 8 9 10)}
(%i13) arraymake ('A2, [8]);
(%o13)                          A2
                                  8
(%i14) ''%;
(%o14)                           9
```

`arrays` [System variable]

Default value: `[]`

`arrays` is a list of arrays that have been allocated. These comprise arrays declared by `array`, `hashed arrays` that can be constructed by implicit definition (assigning something to an element that isn't yet declared as a list or an array), and `memoizing functions` defined by `:=` and `define`. Arrays defined by `make_array` are not included.

See also `array`, `arrayapply`, `arrayinfo`, `arraymake`, `fillarray`, `listarray`, and `rearray`.

Examples:

```
(%i1) array (aa, 5, 7);
(%o1)                           aa
(%i2) bb [FOO] : (a + b)^2;
                                          2
(%o2)                        (b + a)
(%i3) cc [x] := x/100;
                                          x
(%o3)                       cc  := ---
                              x    100
(%i4) dd : make_array ('any, 7);
(%o4)     {Lisp Array: #(NIL NIL NIL NIL NIL NIL NIL)}
(%i5) arrays;
(%o5)                      [aa, bb, cc]
```

`arraysetapply` (*A*, [*i_1*, ..., *i_n*], *x*) [Function]

Assigns *x* to `A[i_1, ..., i_n]`, where *A* is an array and *i_1*, ..., *i_n* are integers.

`arraysetapply` evaluates its arguments.

`fillarray` (*A*, *B*) [Function]

Fills array *A* from *B*, which is a list or an array.

If a specific type was declared for *A* when it was created, it can only be filled with elements of that same type; it is an error if an attempt is made to copy an element of a different type.

If the dimensions of the arrays *A* and *B* are different, *A* is filled in row-major order. If there are not enough elements in *B* the last element is used to fill out the rest of *A*. If there are too many, the remaining ones are ignored.

`fillarray` returns its first argument.

Examples:

Create an array of 9 elements and fill it from a list.

```
(%i1) array (a1, fixnum, 8);
(%o1)                           a1
(%i2) listarray (a1);
(%o2)              [0, 0, 0, 0, 0, 0, 0, 0, 0]
(%i3) fillarray (a1, [1, 2, 3, 4, 5, 6, 7, 8, 9]);
(%o3)                           a1
(%i4) listarray (a1);
(%o4)              [1, 2, 3, 4, 5, 6, 7, 8, 9]
```

When there are too few elements to fill the array, the last element is repeated. When there are too many elements, the extra elements are ignored.

```
(%i1) a2 : make_array (fixnum, 8);
(%o1)           {Lisp Array: #(0 0 0 0 0 0 0 0)}
(%i2) fillarray (a2, [1, 2, 3, 4, 5]);
(%o2)           {Lisp Array: #(1 2 3 4 5 5 5 5)}
(%i3) fillarray (a2, [4]);
(%o3)           {Lisp Array: #(4 4 4 4 4 4 4 4)}
```

```
(%i4) fillarray (a2, makelist (i, i, 1, 100));
(%o4)           {Lisp Array: #(1 2 3 4 5 6 7 8)}
```

Multiple-dimension arrays are filled in row-major order.

```
(%i1) a3 : make_array (fixnum, 2, 5);
(%o1)      {Lisp Array: #2A((0 0 0 0 0) (0 0 0 0 0))}
(%i2) fillarray (a3, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]);
(%o2)      {Lisp Array: #2A((1 2 3 4 5) (6 7 8 9 10))}
(%i3) a4 : make_array (fixnum, 5, 2);
(%o3)   {Lisp Array: #2A((0 0) (0 0) (0 0) (0 0) (0 0))}
(%i4) fillarray (a4, a3);
(%o4)   {Lisp Array: #2A((1 2) (3 4) (5 6) (7 8) (9 10))}
```

`listarray` (*A*) [Function]

Returns a list of the elements of the array *A*. The argument *A* may be an array, an undeclared array (hashed array), a memoizing function, or a subscripted function.

Elements are listed in row-major order. That is, elements are sorted according to the first index, then according to the second index, and so on. The sorting order of index values is the same as the order established by `orderless`.

For undeclared arrays (hashed arrays), memoizing functions, and subscripted functions, the elements correspond to the index values returned by `arrayinfo`.

Unbound elements of general arrays (that is, not `fixnum` and not `flonum`) are returned as `#####`. Unbound elements of `fixnum` or `flonum` arrays are returned as 0 or 0.0, respectively. Unbound elements of hashed arrays, memoizing functions, and subscripted functions are not returned.

Examples:

`listarray` and `arrayinfo` applied to a declared array.

```
(%i1) array (aa, 2, 3);
(%o1)                          aa
(%i2) aa [2, 3] : %pi;
(%o2)                          %pi
(%i3) aa [1, 2] : %e;
(%o3)                          %e
(%i4) listarray (aa);
(%o4) [#####, #####, #####, #####, #####, #####, %e, #####,
                                          #####, #####, #####, %pi]
(%i5) arrayinfo (aa);
(%o5)                [declared, 2, [2, 3]]
```

`listarray` and `arrayinfo` applied to an undeclared array (hashed array).

```
(%i1) bb [FOO] : (a + b)^2;
                                        2
(%o1)                         (b + a)
(%i2) bb [BAR] : (c - d)^3;
                                        3
(%o2)                         (c - d)
```

```
(%i3) listarray (bb);
                                3         2
(%o3)                  [(c - d) , (b + a) ]
(%i4) arrayinfo (bb);
(%o4)               [hashed, 1, [BAR], [FOO]]
```

`listarray` and `arrayinfo` applied to a `memoizing function`.

```
(%i1) cc [x, y] := y / x;
                                      y
(%o1)                      cc     := -
                             x, y    x
(%i2) cc [u, v];
                                   v
(%o2)                              -
                                   u
(%i3) cc [4, z];
                                   z
(%o3)                              -
                                   4
(%i4) listarray (cc);
                                  z  v
(%o4)                            [-, -]
                                  4  u
(%i5) arrayinfo (cc);
(%o5)              [hashed, 2, [4, z], [u, v]]
```

`listarray` and `arrayinfo` applied to a subscripted function.

```
(%i1) dd [x] (y) := y ^ x;
                                        x
(%o1)                       dd (y) := y
                              x
(%i2) dd [a + b];
                                     b + a
(%o2)                   lambda([y], y     )
(%i3) dd [v - u];
                                     v - u
(%o3)                   lambda([y], y     )
(%i4) listarray (dd);
                           b + a                 v - u
(%o4)      [lambda([y], y     ), lambda([y], y     )]
(%i5) arrayinfo (dd);
(%o5)             [hashed, 1, [b + a], [v - u]]
```

`make_array` (*type*, *dim_1*, ..., *dim_n*) [Function]

Creates and returns a Lisp array. *type* may be `any`, `flonum`, `fixnum`, `hashed` or `functional`. There are n indices, and the i'th index runs from 0 to $dim\_i - 1$.

The advantage of `make_array` over `array` is that the return value doesn't have a name, and once a pointer to it goes away, it will also go away. For example, if `y`:

`make_array (...)` then `y` points to an object which takes up space, but after `y: false`, `y` no longer points to that object, so the object can be garbage collected.

Examples:

```
(%i1) A1 : make_array (fixnum, 10);
(%o1)         {Lisp Array: #(0 0 0 0 0 0 0 0 0 0)}
(%i2) A1 [8] : 1729;
(%o2)                          1729
(%i3) A1;
(%o3)        {Lisp Array: #(0 0 0 0 0 0 0 0 1729 0)}
(%i4) A2 : make_array (flonum, 10);
(%o4) {Lisp Array: #(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
(%i5) A2 [2] : 2.718281828;
(%o5)                       2.718281828
(%i6) A2;
(%o6)
 {Lisp Array: #(0.0 0.0 2.718281828 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}
(%i7) A3 : make_array (any, 10);
(%o7) {Lisp Array: #(NIL NIL NIL NIL NIL NIL NIL NIL NIL NIL)}
(%i8) A3 [4] : x - y - z;
(%o8)                       (- z) - y + x
(%i9) A3;
(%o9) {Lisp Array: #(NIL NIL NIL NIL
                ((MPLUS SIMP) $X ((MTIMES SIMP) -1 $Y) ((MTIMES S\
IMP) -1 $Z))
                NIL NIL NIL NIL NIL)}
(%i10) A4 : make_array (fixnum, 2, 3, 5);
(%o10) {Lisp Array: #3A(((0 0 0 0 0) (0 0 0 0 0) (0 0 0 0 0))
                  ((0 0 0 0 0) (0 0 0 0 0) (0 0 0 0 0)))}
(%i11) fillarray (A4, makelist (i, i, 1, 2*3*5));
(%o11) {Lisp Array: #3A(((1 2 3 4 5) (6 7 8 9 10) (11 12 13 14 1\
5))
                  ((16 17 18 19 20) (21 22 23 24 25) (26 27 28 29\
 30)))}
(%i12) A4 [0, 2, 1];
(%o12)                          12
```

`rearray (A, dim_1, ..., dim_n)` [Function]

Changes the dimensions of an array. The new array will be filled with the elements of the old one in row-major order. If the old array was too small, the remaining elements are filled with `false`, `0.0` or `0`, depending on the type of the array. The type of the array cannot be changed.

remarray [Function]

remarray (*A_1*, ..., *A_n*)

remarray (*all*)

Removes arrays and array associated functions and frees the storage occupied. The arguments may be declared arrays, hashed arrays, array functions, and subscripted functions.

remarray (all) removes all items in the global list arrays.

It may be necessary to use this function if it is desired to clear the cache of a memoizing function.

remarray returns the list of arrays removed.

remarray quotes its arguments.

subvar (*x*, *i*) [Function]

Evaluates the subscripted expression *x*[*i*].

subvar evaluates its arguments.

arraymake (*x*, [*i*]) constructs the expression *x*[*i*], but does not evaluate it.

Examples:

```
(%i1) x : foo $
(%i2) i : 3 $
(%i3) subvar (x, i);
(%o3)                          foo
                                  3
(%i4) foo : [aa, bb, cc, dd, ee]$
(%i5) subvar (x, i);
(%o5)                           cc
(%i6) arraymake (x, [i]);
(%o6)                          foo
                                  3
(%i7) ''%;
(%o7)                           cc
```

subvarp (*expr*) [Function]

Returns true if *expr* is a subscripted variable, for example a[i].

use_fast_arrays [Option variable]

Default value: false

When use_fast_arrays is true, arrays declared by array are values instead of properties, and undeclared arrays (hashed arrays) are implemented as Lisp hashed arrays.

When use_fast_arrays is false, arrays declared by array are properties, and undeclared arrays are implemented with Maxima's own hashed array implementation.

Note that the code use_fast_arrays switches to is not necessarily faster than the default one; Arrays created by make_array are not affected by use_fast_arrays.

See also translate_fast_arrays.

`translate_fast_arrays` [Option variable]

Default value: `false`

When `translate_fast_arrays` is `true`, the Maxima-to-Lisp translator generates code that assumes arrays are values instead of properties, as if `use_fast_arrays` were `true`.

When `translate_fast_arrays` is `false`, the Maxima-to-Lisp translator generates code that assumes arrays are properties, as if `use_fast_arrays` were `false`.

7.6 Structures

7.6.1 Introduction to Structures

Maxima provides a simple data aggregate called a structure. A structure is an expression in which arguments are identified by name (the field name) and the expression as a whole is identified by its operator (the structure name). A field value can be any expression.

A structure is defined by the `defstruct` function; the global variable `structures` is the list of user-defined structures. The function `new` creates instances of structures. The `@` operator refers to fields. `kill(`*S*`)` removes the structure definition *S*, and `kill(`*x*`@ a)` unbinds the field *a* of the structure instance *x*.

In the pretty-printing console display (with `display2d` equal to `true`), structure instances are displayed with the value of each field represented as an equation, with the field name on the left-hand side and the value on the right-hand side. (The equation is only a display construct; only the value is actually stored.) In 1-dimensional display (via `grind` or with `display2d` equal to `false`), structure instances are displayed without the field names.

There is no way to use a field name as a function name, although a field value can be a lambda expression. Nor can the values of fields be restricted to certain types; any field can be assigned any kind of expression. There is no way to make some fields accessible or inaccessible in different contexts; all fields are always visible.

7.6.2 Functions and Variables for Structures

`structures` [Global variable]

`structures` is the list of user-defined structures defined by `defstruct`.

`defstruct` [Function]

`defstruct (`*S*`(`*a_1*`, ..., `*a_n*`))`
`defstruct (`*S*`(`*a_1* `=` *v_1*`, ..., `*a_n* `=` *v_n*`))`

Define a structure, which is a list of named fields *a_1*, ..., *a_n* associated with a symbol *S*. An instance of a structure is just an expression which has operator *S* and exactly `n` arguments. `new(`*S*`)` creates a new instance of structure *S*.

An argument which is just a symbol *a* specifies the name of a field. An argument which is an equation `a =` *v* specifies the field name *a* and its default value *v*. The default value can be any expression.

`defstruct` puts *S* on the list of user-defined structures, `structures`.

`kill(`*S*`)` removes *S* from the list of user-defined structures, and removes the structure definition.

Examples:

```
(%i1) defstruct (foo (a, b, c));
(%o1)                     [foo(a, b, c)]
(%i2) structures;
(%o2)                     [foo(a, b, c)]
(%i3) new (foo);
(%o3)                      foo(a, b, c)
(%i4) defstruct (bar (v, w, x = 123, y = %pi));
```

```
(%o4)             [bar(v, w, x = 123, y = %pi)]
(%i5) structures;
(%o5)      [foo(a, b, c), bar(v, w, x = 123, y = %pi)]
(%i6) new (bar);
(%o6)              bar(v, w, x = 123, y = %pi)
(%i7) kill (foo);
(%o7)                          done
(%i8) structures;
(%o8)             [bar(v, w, x = 123, y = %pi)]
```

new [Function]

new (*S*)
new (*S* (*v_1*, ..., *v_n*))

new creates new instances of structures.

new(*S*) creates a new instance of structure *S* in which each field is assigned its default value, if any, or no value at all if no default was specified in the structure definition.

new(*S*(*v_1*, ..., *v_n*)) creates a new instance of *S* in which fields are assigned the values *v_1*, ..., *v_n*.

Examples:

```
(%i1) defstruct (foo (w, x = %e, y = 42, z));
(%o1)              [foo(w, x = %e, y = 42, z)]
(%i2) new (foo);
(%o2)               foo(w, x = %e, y = 42, z)
(%i3) new (foo (1, 2, 4, 8));
(%o3)            foo(w = 1, x = 2, y = 4, z = 8)
```

@ [Operator]

@ is the structure field access operator. The expression *x*@ a refers to the value of field *a* of the structure instance *x*. The field name is not evaluated.

If the field *a* in *x* has not been assigned a value, *x*@ a evaluates to itself.

kill(*x*@ a) removes the value of field *a* in *x*.

Examples:

```
(%i1) defstruct (foo (x, y, z));
(%o1)                     [foo(x, y, z)]
(%i2) u : new (foo (123, a - b, %pi));
(%o2)           foo(x = 123, y = a - b, z = %pi)
(%i3) u@z;
(%o3)                           %pi
(%i4) u@z : %e;
(%o4)                           %e
(%i5) u;
(%o5)            foo(x = 123, y = a - b, z = %e)
(%i6) kill (u@z);
(%o6)                          done
(%i7) u;
(%o7)              foo(x = 123, y = a - b, z)
```

```
(%i8) u@z;
(%o8)                              u@z
```

The field name is not evaluated.

```
(%i1) defstruct (bar (g, h));
(%o1)                          [bar(g, h)]
(%i2) x : new (bar);
(%o2)                           bar(g, h)
(%i3) x@h : 42;
(%o3)                              42
(%i4) h : 123;
(%o4)                              123
(%i5) x@h;
(%o5)                              42
(%i6) x@h : 19;
(%o6)                              19
(%i7) x;
(%o7)                        bar(g, h = 19)
(%i8) h;
(%o8)                              123
```

8 Expressions

8.1 Introduction to Expressions

There are a number of reserved words which should not be used as variable names. Their use would cause a possibly cryptic syntax error.

```
integrate           next          from              diff
in                  at            limit             sum
for                 and           elseif            then
else                do            or                if
unless              product       while             thru
step
```

Most things in Maxima are expressions. A sequence of expressions can be made into an expression by separating them by commas and putting parentheses around them. This is similar to the **C** *comma expression*.

```
(%i1) x: 3$
(%i2) (x: x+1, x: x^2);
(%o2)                          16
(%i3) (if (x > 17) then 2 else 4);
(%o3)                           4
(%i4) (if (x > 17) then x: 2 else y: 4, y+x);
(%o4)                          20
```

Even loops in Maxima are expressions, although the value they return is the not too useful `done`.

```
(%i1) y: (x: 1, for i from 1 thru 10 do (x: x*i))$
(%i2) y;
(%o2)                         done
```

Whereas what you really want is probably to include a third term in the *comma expression* which actually gives back the value.

```
(%i1) y: (x: 1, for i from 1 thru 10 do (x: x*i), x)$
(%i2) y;
(%o2)                       3628800
```

8.2 Nouns and Verbs

Maxima distinguishes between operators which are "nouns" and operators which are "verbs". A verb is an operator which can be executed. A noun is an operator which appears as a symbol in an expression, without being executed. By default, function names are verbs. A verb can be changed into a noun by quoting the function name or applying the `nounify` function. A noun can be changed into a verb by applying the `verbify` function. The evaluation flag `nouns` causes `ev` to evaluate nouns in an expression.

The verb form is distinguished by a leading dollar sign `$` on the corresponding Lisp symbol. In contrast, the noun form is distinguished by a leading percent sign `%` on the corresponding Lisp symbol. Some nouns have special display properties, such as `'integrate` and `'derivative` (returned by `diff`), but most do not. By default, the noun and verb

forms of a function are identical when displayed. The global flag noundisp causes Maxima to display nouns with a leading quote mark '.

See also noun, nouns, nounify, and verbify.

Examples:

```
(%i1) foo (x) := x^2;
                                      2
(%o1)                      foo(x) := x
(%i2) foo (42);
(%o2)                           1764
(%i3) 'foo (42);
(%o3)                          foo(42)
(%i4) 'foo (42), nouns;
(%o4)                           1764
(%i5) declare (bar, noun);
(%o5)                           done
(%i6) bar (x) := x/17;
                                    x
(%o6)                      bar(x) := --
                                    17
(%i7) bar (52);
(%o7)                          bar(52)
(%i8) bar (52), nouns;
(%o8)                          bar(52)
(%i9) integrate (1/x, x, 1, 42);
(%o9)                          log(42)
(%i10) 'integrate (1/x, x, 1, 42);
                               42
                              /
                              |   1
(%o10)                        |   - dx
                              |   x
                              /
                               1
(%i11) ev (%, nouns);
(%o11)                         log(42)
```

8.3 Identifiers

Maxima identifiers may comprise alphabetic characters, plus the numerals 0 through 9, plus any other character preceded by the backslash \ character.

A numeral may be the first character of an identifier if it is preceded by a backslash. Numerals which are the second or later characters need not be preceded by a backslash.

The alphabetic characters are initially %, _, and all characters for which the Lisp function ALPHA-CHAR-P returns true. Characters may be declared alphabetic by the declare function. If so declared, they need not be preceded by a backslash in an identifier.

Maxima is case-sensitive. The identifiers `foo`, `FOO`, and `Foo` are distinct. See Section 33.1 [Lisp and Maxima], page 621, for more on this point.

A Maxima identifier is a Lisp symbol which begins with a dollar sign `$`. Any other Lisp symbol is preceded by a question mark `?` when it appears in Maxima. See Section 33.1 [Lisp and Maxima], page 621, for more on this point.

Examples:

```
(%i1) %an_ordinary_identifier42;
(%o1)               %an_ordinary_identifier42
(%i2) embedded\ spaces\ in\ an\ identifier;
(%o2)           embedded spaces in an identifier
(%i3) symbolp (%);
(%o3)                          true
(%i4) [foo+bar, foo\+bar];
(%o4)                  [foo + bar, foo+bar]
(%i5) [1729, \1729];
(%o5)                      [1729, 1729]
(%i6) [symbolp (foo\+bar), symbolp (\1729)];
(%o6)                      [true, true]
(%i7) [is (foo\+bar = foo+bar), is (\1729 = 1729)];
(%o7)                    [false, false]
(%i8) baz\~quux;
(%o8)                        baz~quux
(%i9) declare ("~", alphabetic);
(%o9)                          done
(%i10) baz~quux;
(%o10)                       baz~quux
(%i11) [is (foo = FOO), is (FOO = Foo), is (Foo = foo)];
(%o11)                [false, false, false]
(%i12) :lisp (defvar *my-lisp-variable* '$foo)
*MY-LISP-VARIABLE*
(%i12) ?\*my\-lisp\-variable\*;
(%o12)                          foo
```

8.4 Inequality

Maxima has the inequality operators `<`, `<=`, `>=`, `>`, `#`, and `notequal`. See [if], page 629, for a description of conditional expressions.

8.5 Functions and Variables for Expressions

`alias` (*new_name_1*, *old_name_1*, ..., *new_name_n*, *old_name_n*) [Function]
provides an alternate name for a (user or system) function, variable, array, etc. Any even number of arguments may be used.

`aliases` [System variable]
Default value: `[]`

`aliases` is the list of atoms which have a user defined alias (set up by the `alias`, `ordergreat`, `orderless` functions or by declaring the atom a `noun` with `declare`.)

`allbut` [Keyword]

works with the `part` commands (i.e. `part`, `inpart`, `substpart`, `substinpart`, `dpart`, and `lpart`). For example,

```
(%i1) expr : e + d + c + b + a;
(%o1)                    e + d + c + b + a
(%i2) part (expr, [2, 5]);
(%o2)                           d + a
```

while

```
(%i1) expr : e + d + c + b + a;
(%o1)                    e + d + c + b + a
(%i2) part (expr, allbut (2, 5));
(%o2)                         e + c + b
```

`allbut` is also recognized by `kill`.

```
(%i1) [aa : 11, bb : 22, cc : 33, dd : 44, ee : 55];
(%o1)                  [11, 22, 33, 44, 55]
(%i2) kill (allbut (cc, dd));
(%o0)                           done
(%i1) [aa, bb, cc, dd];
(%o1)                     [aa, bb, 33, 44]
```

`kill(allbut(a_1, a_2, ...))` has the effect of `kill(all)` except that it does not kill the symbols *a_1*, *a_2*, ...

`args (expr)` [Function]

Returns the list of arguments of `expr`, which may be any kind of expression other than an atom. Only the arguments of the top-level operator are extracted; subexpressions of `expr` appear as elements or subexpressions of elements of the list of arguments.

The order of the items in the list may depend on the global flag `inflag`.

`args (expr)` is equivalent to `substpart ("[", expr, 0)`. See also `substpart`, `apply`, `funmake`, and `op`.

How to convert a matrix to a nested list:

```
(%i1) M:matrix([1,2],[3,4]);
                               [ 1  2 ]
(%o1)                          [      ]
                               [ 3  4 ]
(%i2) args(M);
(%o2)                     [[1, 2], [3, 4]]
```

Since maxima internally treats a sum of `n` terms as a summation command with `n` arguments args() can extract the list of terms in a sum:

```
(%i1) a+b+c;
(%o1)                          c + b + a
(%i2) args(%);
(%o2)                          [c, b, a]
```

`atom` (*expr*) [Function]

Returns `true` if *expr* is atomic (i.e. a number, name or string) else `false`. Thus `atom(5)` is `true` while `atom(a[1])` and `atom(sin(x))` are `false` (assuming `a[1]` and `x` are unbound).

`display_box_double_lines` [Option variable]

Default value: `true`

When `display_box_double_lines` is `true`, `box` expressions are displayed with Unicode double-line characters.

When `display_box_double_lines` is `false`, `box` expressions are displayed with Unicode single-line characters.

`display_box_double_lines` only has any effect when `display2d_unicode` is `true`.

`box` [Function]

`box` (*expr*)
`box` (*expr*, *a*)

Returns *expr* enclosed in a box. The return value is an expression with `box` as the operator and *expr* as the argument. A box is drawn on the display when `display2d` is `true`.

`box` (*expr*, *a*) encloses *expr* in a box labelled by the symbol *a*. The label is truncated if it is longer than the width of the box.

`box` evaluates its argument. However, a boxed expression does not evaluate to its content, so boxed expressions are effectively excluded from computations. `rembox` removes the box again.

`boxchar` is the character used to draw the box in `box` and in the `dpart` and `lpart` functions.

See also `rembox`, `dpart`, `lpart`, and `display_box_double_lines`.

Examples:

```
(%i1) box (a^2 + b^2);
                                  """"""""""
                                  " 2    2"
(%o1)                             "b  + a "
                                  """"""""""
(%i2) a : 1234;
(%o2)                               1234
(%i3) b : c - d;
(%o3)                               c - d
(%i4) box (a^2 + b^2);
                            """""""""""""""""""""
                            "       2          "
(%o4)                       "(c - d)  + 1522756"
                            """""""""""""""""""""
(%i5) box (a^2 + b^2, term_1);
                            term_1"""""""""""""""
                            "       2          "
(%o5)                       "(c - d)  + 1522756"
                            """""""""""""""""""""
```

```
(%i6) 1729 - box (1729);
                                      """""""
(%o6)                       1729 - "1729"
                                      """""""
(%i7) boxchar: "-";
(%o7)                                 -
(%i8) box (sin(x) + cos(y));
                            -----------------
(%o8)                       -cos(y) + sin(x)-
                            -----------------
```

`boxchar` [Option variable]

Default value: `"`

`boxchar` is the character used to draw the box in the `box` and in the `dpart` and `lpart` functions.

`boxchar` is only used when `display2d_unicode` is `false`.

All boxes in an expression are drawn with the current value of `boxchar`; the drawing character is not stored with the box expression.

`collapse` (*expr*) [Function]

Collapses *expr* by causing all of its common (i.e., equal) subexpressions to share (i.e., use the same cells), thereby saving space. (`collapse` is a subroutine used by the `optimize` command.) Thus, calling `collapse` may be useful after loading in a `save` file. You can collapse several expressions together by using `collapse ([`*expr_1*`, ..., `*expr_n*`])`. Similarly, you can collapse the elements of the array `A` by doing `collapse (listarray ('A))`.

`copy` (*e*) [Function]

Return a copy of the Maxima expression *e*. Although *e* can be any Maxima expression, the copy function is the most useful when *e* is either a list or a matrix; consider:

```
(%i1) m : [1,[2,3]]$
(%i2) mm : m$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4)                       [1, [x, 3]]
(%i5) mm;
(%o5)                       [1, [x, 3]]
```

Let's try the same experiment, but this time let *mm* be a copy of *m*

```
(%i1) m : [1,[2,3]]$
(%i2) mm : copy(m)$
(%i3) mm[2][1] : x$
(%i4) m;
(%o4)                       [1, [2, 3]]
(%i5) mm;
(%o5)                       [1, [x, 3]]
```

This time, the assignment to *mm* does not change the value of *m*.

disolate (*expr*, *x_1*, ..., *x_n*) [Function]

is similar to isolate (*expr*, *x*) except that it enables the user to isolate more than one variable simultaneously. This might be useful, for example, if one were attempting to change variables in a multiple integration, and that variable change involved two or more of the integration variables. This function is autoloaded from simplification/disol.mac. A demo is available by demo("disol")$.

dispform [Function]

dispform (*expr*)
dispform (*expr*, *all*)

Returns the external representation of *expr*.

dispform(*expr*) returns the external representation with respect to the main (top-level) operator. dispform(*expr*, all) returns the external representation with respect to all operators in *expr*.

See also part, inpart, and inflag.

Examples:

The internal representation of - x is "negative one times x" while the external representation is "minus x".

```
(%i1) - x;
(%o1)                          - x
(%i2) ?format (true, "~S~%", %);
((MTIMES SIMP) -1 $X)
(%o2)                           false
(%i3) dispform (- x);
(%o3)                          - x
(%i4) ?format (true, "~S~%", %);
((MMINUS SIMP) $X)
(%o4)                           false
```

The internal representation of sqrt(x) is "x to the power 1/2" while the external representation is "square root of x".

```
(%i1) sqrt (x);
(%o1)                         sqrt(x)
(%i2) ?format (true, "~S~%", %);
((MEXPT SIMP) $X ((RAT SIMP) 1 2))
(%o2)                          false
(%i3) dispform (sqrt (x));
(%o3)                         sqrt(x)
(%i4) ?format (true, "~S~%", %);
((%SQRT SIMP) $X)
(%o4)                          false
```

Use of the optional argument all.

```
(%i1) expr : sin (sqrt (x));
(%o1)                     sin(sqrt(x))
(%i2) freeof (sqrt, expr);
(%o2)                          true
```

```
(%i3) freeof (sqrt, dispform (expr));
(%o3)                          true
(%i4) freeof (sqrt, dispform (expr, all));
(%o4)                          false
```

dpart (*expr*, *n_1*, ..., *n_k*) [Function]

Selects the same subexpression as part, but instead of just returning that subexpression as its value, it returns the whole expression with the selected subexpression displayed inside a box. The box is actually part of the expression.

```
(%i1) dpart (x+y/z^2, 1, 2, 1);
                                y
(%o1)                         ---- + x
                                2
                               """
                               "z"
                               """
```

exptisolate [Option variable]

Default value: false

exptisolate, when true, causes isolate (expr, var) to examine exponents of atoms (such as %e) which contain var.

exptsubst [Option variable]

Default value: false

exptsubst, when true, permits substitutions such as y for %e^x in %e^(a x).

```
(%i1) %e^(a*x);
                                  a x
(%o1)                            %e
(%i2) exptsubst;
(%o2)                            false
(%i3) subst(y, %e^x, %e^(a*x));
                                  a x
(%o3)                            %e
(%i4) exptsubst: not exptsubst;
(%o4)                            true
(%i5) subst(y, %e^x, %e^(a*x));
                                   a
(%o5)                              y
```

freeof (*x_1*, ..., *x_n*, *expr*) [Function]

freeof (*x_1*, *expr*) returns true if no subexpression of *expr* is equal to *x_1* or if *x_1* occurs only as a dummy variable in *expr*, or if *x_1* is neither the noun nor verb form of any operator in *expr*, and returns false otherwise.

freeof (*x_1*, ..., *x_n*, *expr*) is equivalent to freeof (*x_1*, *expr*) and ... and freeof (*x_n*, *expr*).

The arguments *x_1*, ..., *x_n* may be names of functions and variables, subscripted names, operators (enclosed in double quotes), or general expressions. freeof evaluates its arguments.

`freeof` operates only on *expr* as it stands (after simplification and evaluation) and does not attempt to determine if some equivalent expression would give a different result. In particular, simplification may yield an equivalent but different expression which comprises some different elements than the original form of *expr*.

A variable is a dummy variable in an expression if it has no binding outside of the expression. Dummy variables recognized by `freeof` are the index of a sum or product, the limit variable in `limit`, the integration variable in the definite integral form of `integrate`, the original variable in `laplace`, formal variables in `at` expressions, and arguments in `lambda` expressions.

The indefinite form of `integrate` is *not* free of its variable of integration.

Examples:

Arguments are names of functions, variables, subscripted names, operators, and expressions. `freeof (a, b, expr)` is equivalent to `freeof (a, expr) and freeof (b, expr)`.

```
(%i1) expr: z^3 * cos (a[1]) * b^(c+d);
                                    d + c  3
(%o1)                    cos(a ) b      z
                              1
(%i2) freeof (z, expr);
(%o2)                           false
(%i3) freeof (cos, expr);
(%o3)                           false
(%i4) freeof (a[1], expr);
(%o4)                           false
(%i5) freeof (cos (a[1]), expr);
(%o5)                           false
(%i6) freeof (b^(c+d), expr);
(%o6)                           false
(%i7) freeof ("^", expr);
(%o7)                           false
(%i8) freeof (w, sin, a[2], sin (a[2]), b*(c+d), expr);
(%o8)                           true
```

`freeof` evaluates its arguments.

```
(%i1) expr: (a+b)^5$
(%i2) c: a$
(%i3) freeof (c, expr);
(%o3)                           false
```

`freeof` does not consider equivalent expressions. Simplification may yield an equivalent but different expression.

```
(%i1) expr: (a+b)^5$
(%i2) expand (expr);
          5        4       2  3       3  2      4      5
(%o2)    b  + 5 a b  + 10 a  b  + 10 a  b  + 5 a  b + a
(%i3) freeof (a+b, %);
(%o3)                           true
```

```
(%i4) freeof (a+b, expr);
(%o4)                          false
(%i5) exp (x);
                                  x
(%o5)                            %e
(%i6) freeof (exp, exp (x));
(%o6)                          true
```

A summation or definite integral is free of its dummy variable. An indefinite integral is not free of its variable of integration.

```
(%i1) freeof (i, 'sum (f(i), i, 0, n));
(%o1)                          true
(%i2) freeof (x, 'integrate (x^2, x, 0, 1));
(%o2)                          true
(%i3) freeof (x, 'integrate (x^2, x));
(%o3)                          false
```

`inflag` [Option variable]

Default value: `false`

When `inflag` is `true`, functions for part extraction inspect the internal form of `expr`.

Note that the simplifier re-orders expressions. Thus `first (x + y)` returns `x` if `inflag` is `true` and `y` if `inflag` is `false`. (`first (y + x)` gives the same results.)

Also, setting `inflag` to `true` and calling `part` or `substpart` is the same as calling `inpart` or `substinpart`.

Functions affected by the setting of `inflag` are: `part`, `substpart`, `first`, `rest`, `last`, `length`, the `for ... in` construct, `map`, `fullmap`, `maplist`, `reveal`, `pickapart`, `args` and `op`.

`inpart` (*expr*, *n_1*, ..., *n_k*) [Function]

is similar to `part` but works on the internal representation of the expression rather than the displayed form and thus may be faster since no formatting is done. Care should be taken with respect to the order of subexpressions in sums and products (since the order of variables in the internal form is often different from that in the displayed form) and in dealing with unary minus, subtraction, and division (since these operators are removed from the expression). `part (x+y, 0)` or `inpart (x+y, 0)` yield `+`, though in order to refer to the operator it must be enclosed in "s. For example `... if inpart (%o9,0) = "+" then ...`.

Examples:

```
(%i1) x + y + w*z;
(%o1)                        w z + y + x
(%i2) inpart (%, 3, 2);
(%o2)                             z
(%i3) part (%th (2), 1, 2);
(%o3)                             z
(%i4) 'limit (f(x)^g(x+1), x, 0, minus);
                                      g(x + 1)
(%o4)                  limit   f(x)
                       x -> 0-
```

```
(%i5) inpart (%, 1, 2);
(%o5)                        g(x + 1)
```

isolate (*expr*, *x*) [Function]

Returns *expr* with subexpressions which are sums and which do not contain *var* replaced by intermediate expression labels (these being atomic symbols like %t1, %t2, ...). This is often useful to avoid unnecessary expansion of subexpressions which don't contain the variable of interest. Since the intermediate labels are bound to the subexpressions they can all be substituted back by evaluating the expression in which they occur.

exptisolate (default value: false) if true will cause isolate to examine exponents of atoms (like %e) which contain *var*.

isolate_wrt_times if true, then isolate will also isolate with respect to products. See isolate_wrt_times. See also disolate.

Do example (isolate) for examples.

isolate_wrt_times [Option variable]

Default value: false

When isolate_wrt_times is true, isolate will also isolate with respect to products. E.g. compare both settings of the switch on

```
(%i1) isolate_wrt_times: true$
(%i2) isolate (expand ((a+b+c)^2), c);

(%t2)                          2 a


(%t3)                          2 b

                             2           2
(%t4)                       b  + 2 a b + a

                        2
(%o4)                  c  + %t3 c + %t2 c + %t4
(%i4) isolate_wrt_times: false$
(%i5) isolate (expand ((a+b+c)^2), c);
                        2
(%o5)                  c  + 2 b c + 2 a c + %t4
```

listconstvars [Option variable]

Default value: false

When listconstvars is true the list returned by listofvars contains constant variables, such as %e, %pi, %i or any variables declared as constant that occur in *expr*. A variable is declared as constant type via declare, and constantp returns true for all variables declared as constant. The default is to omit constant variables from listofvars return value.

`listdummyvars` [Option variable]

Default value: `true`

When `listdummyvars` is `false`, "dummy variables" in the expression will not be included in the list returned by `listofvars`. (The meaning of "dummy variables" is as given in `freeof`. "Dummy variables" are mathematical things like the index of a sum or product, the limit variable, and the definite integration variable.)

Example:

```
(%i1) listdummyvars: true$
(%i2) listofvars ('sum(f(i), i, 0, n));
(%o2)                         [i, n]
(%i3) listdummyvars: false$
(%i4) listofvars ('sum(f(i), i, 0, n));
(%o4)                           [n]
```

`listofvars` (*expr*) [Function]

Returns a list of the variables in *expr*.

`listconstvars` if `true` causes `listofvars` to include `%e`, `%pi`, `%i`, and any variables declared constant in the list it returns if they appear in *expr*. The default is to omit these.

See also the option variable `listdummyvars` to exclude or include "dummy variables" in the list of variables.

```
(%i1) listofvars (f (x[1]+y) / g^(2+a));
(%o1)                      [g, a, x , y]
                                    1
```

`lfreeof` (*list*, *expr*) [Function]

For each member *m* of *list*, calls `freeof (m, expr)`. It returns `false` if any call to `freeof` does and `true` otherwise.

Example:

```
(%i1) lfreeof ([ a, x], x^2+b);
(%o1)                          false
(%i2) lfreeof ([ b, x], x^2+b);
(%o2)                          false
(%i3) lfreeof ([ a, y], x^2+b);
(%o3)                          true
```

`lpart` (*label*, *expr*, *n_1*, ..., *n_k*) [Function]

is similar to `dpart` but uses a labelled box. A labelled box is similar to the one produced by `dpart` but it has a name in the top line.

`mainvar` [Property]

You may declare variables to be `mainvar`. The ordering scale for atoms is essentially: numbers < constants (e.g., `%e`, `%pi`) < scalars < other variables < mainvars. E.g., compare `expand ((X+Y)^4)` with `(declare (x, mainvar), expand ((x+y)^4))`. (Note: Care should be taken if you elect to use the above feature. E.g., if you subtract an

expression in which `x` is a `mainvar` from one in which `x` isn't a `mainvar`, resimplification e.g. with `ev (expr, simp)` may be necessary if cancellation is to occur. Also, if you save an expression in which `x` is a `mainvar`, you probably should also save `x`.)

`noun` [Property]

`noun` is one of the options of the `declare` command. It makes a function so declared a "noun", meaning that it won't be evaluated automatically.

Example:

```
(%i1) factor (12345678);
                                2
(%o1)                      2 3  47 14593
(%i2) declare (factor, noun);
(%o2)                          done
(%i3) factor (12345678);
(%o3)                    factor(12345678)
(%i4) ''%, nouns;
                                2
(%o4)                      2 3  47 14593
```

`noundisp` [Option variable]

Default value: `false`

When `noundisp` is `true`, nouns display with a single quote. This switch is always `true` when displaying function definitions.

`nounify (`*f*`)` [Function]

Returns the noun form of the function name *f*. This is needed if one wishes to refer to the name of a verb function as if it were a noun. Note that some verb functions will return their noun forms if they can't be evaluated for certain arguments. This is also the form returned if a function call is preceded by a quote.

See also `verbify`.

`nterms (`*expr*`)` [Function]

Returns the number of terms that *expr* would have if it were fully expanded out and no cancellations or combination of terms occurred. Note that expressions like `sin (`*expr*`)`, `sqrt (`*expr*`)`, `exp (`*expr*`)`, etc. count as just one term regardless of how many terms *expr* has (if it is a sum).

`op (`*expr*`)` [Function]

Returns the main operator of the expression *expr*. This is equivalent to `part (`*expr*`, 0)` with `partswitch` set to `false`.

`op` returns a string if the main operator is a built-in or user-defined prefix, binary or n-ary infix, postfix, matchfix, or nofix operator. Otherwise, if *expr* is a subscripted function expression, `op` returns the subscripted function; in this case the return value is not an atom. Otherwise, *expr* is a `memoizing function` or ordinary function expression, and `op` returns a symbol.

`op` observes the value of the global flag `inflag`.

`op` evaluates it argument.

See also args.

Examples:

```
(%i1) stringdisp: true$
(%i2) op (a * b * c);
(%o2)                          "*"
(%i3) op (a * b + c);
(%o3)                          "+"
(%i4) op ('sin (a + b));
(%o4)                          sin
(%i5) op (a!);
(%o5)                          "!"
(%i6) op (-a);
(%o6)                          "-"
(%i7) op ([a, b, c]);
(%o7)                          "["
(%i8) op ('(if a > b then c else d));
(%o8)                         "if"
(%i9) op ('foo (a));
(%o9)                          foo
(%i10) prefix (foo);
(%o10)                        "foo"
(%i11) op (foo a);
(%o11)                        "foo"
(%i12) op (F [x, y] (a, b, c));
(%o12)                        F
                               x, y
(%i13) op (G [u, v, w]);
(%o13)                         G
```

operatorp [Function]

operatorp (*expr*, *op*)
operatorp (*expr*, [*op_1*, ..., *op_n*])

operatorp (*expr*, *op*) returns true if *op* is equal to the operator of *expr*.

operatorp (*expr*, [*op_1*, ..., *op_n*]) returns true if some element *op_1*, ..., *op_n* is equal to the operator of *expr*.

operatorp observes the value of the global flag inflag.

opsubst [Option variable]

Default value: true

When opsubst is false, subst does not attempt to substitute into the operator of an expression. E.g., (opsubst: false, subst (x^2, r, r+r[0])) will work.

```
(%i1) r+r[0];
(%o1)                        r + r
                                  0
(%i2) opsubst;
(%o2)                         true
```

```
(%i3) subst (x^2, r, r+r[0]);
                              2     2
(%o3)                        x  + (x )
                                     0
(%i4) opsubst: not opsubst;
(%o4)                          false
(%i5) subst (x^2, r, r+r[0]);
                               2
(%o5)                         x  + r
                                   0
```

`optimize` (*expr*) [Function]

Returns an expression that produces the same value and side effects as *expr* but does so more efficiently by avoiding the recomputation of common subexpressions. `optimize` also has the side effect of "collapsing" its argument so that all common subexpressions are shared. Do `example (optimize)` for examples.

`optimprefix` [Option variable]

Default value: `%`

`optimprefix` is the prefix used for generated symbols by the `optimize` command.

`ordergreat` (*v_1*, ..., *v_n*) [Function]

`orderless` (*v_1*, ..., *v_n*) [Function]

`ordergreat` changes the canonical ordering of Maxima expressions such that *v_1* succeeds *v_2* succeeds ... succeeds *v_n*, and *v_n* succeeds any other symbol not mentioned as an argument.

`orderless` changes the canonical ordering of Maxima expressions such that *v_1* precedes *v_2* precedes ... precedes *v_n*, and *v_n* precedes any other variable not mentioned as an argument.

The order established by `ordergreat` and `orderless` is dissolved by `unorder`. `ordergreat` and `orderless` can be called only once each, unless `unorder` is called; only the last call to `ordergreat` and `orderless` has any effect.

See also `ordergreatp`.

`ordergreatp` (*expr_1*, *expr_2*) [Function]

`orderlessp` (*expr_1*, *expr_2*) [Function]

`ordergreatp` returns `true` if *expr_1* succeeds *expr_2* in the canonical ordering of Maxima expressions, and `false` otherwise.

`orderlessp` returns `true` if *expr_1* precedes *expr_2* in the canonical ordering of Maxima expressions, and `false` otherwise.

All Maxima atoms and expressions are comparable under `ordergreatp` and `orderlessp`, although there are isolated examples of expressions for which these predicates are not transitive; that is a bug.

The canonical ordering of atoms (symbols, literal numbers, and strings) is the following.

(integers and floats) precede (bigfloats) precede (declared constants) precede (strings) precede (declared scalars) precede (first argument to orderless) precedes ... precedes (last argument to orderless) precedes (other symbols) precede (last argument to ordergreat) precedes ... precedes (first argument to ordergreat) precedes (declared main variables)

For non-atomic expressions, the canonical ordering is derived from the ordering for atoms. For the built-in + * and ^ operators, the ordering is not easily summarized. For other built-in operators and all other functions and operators, expressions are ordered by their arguments (beginning with the first argument), then by the name of the operator or function. In the case of subscripted expressions, the subscripted symbol is considered the operator and the subscript is considered an argument.

The canonical ordering of expressions is modified by the functions ordergreat and orderless, and the mainvar, constant, and scalar declarations.

See also sort.

Examples:

Ordering ordinary symbols and constants. Note that %pi is not ordered according to its numerical value.

```
(%i1) stringdisp : true;
(%o1)                          true
(%i2) sort ([%pi, 3b0, 3.0, x, X, "foo", 3, a, 4, "bar", 4.0, 4b0]);
(%o2) [3, 3.0, 4, 4.0, 3.0b0, 4.0b0, %pi, "bar", "foo", X, a, x]
```

Effect of ordergreat and orderless functions.

```
(%i1) sort ([M, H, K, T, E, W, G, A, P, J, S]);
(%o1)           [A, E, G, H, J, K, M, P, S, T, W]
(%i2) ordergreat (S, J);
(%o2)                          done
(%i3) orderless (M, H);
(%o3)                          done
(%i4) sort ([M, H, K, T, E, W, G, A, P, J, S]);
(%o4)           [M, H, A, E, G, K, P, T, W, J, S]
```

Effect of mainvar, constant, and scalar declarations.

```
(%i1) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o1)   [A1, B1, C1, aa, bar, baz, bb, cc, dd, foo, quux]
(%i2) declare (aa, mainvar);
(%o2)                          done
(%i3) declare ([baz, quux], constant);
(%o3)                          done
(%i4) declare ([A1, B1], scalar);
(%o4)                          done
(%i5) sort ([aa, foo, bar, bb, baz, quux, cc, dd, A1, B1, C1]);
(%o5)   [baz, quux, A1, B1, C1, bar, bb, cc, dd, foo, aa]
```

Ordering non-atomic expressions.

```
(%i1) sort ([1, 2, n, f(1), f(2), f(2, 1), g(1), g(1, 2), g(n),
       f(n, 1)]);
(%o1) [1, 2, f(1), g(1), g(1, 2), f(2), f(2, 1), n, g(n),
                                                        f(n, 1)]
(%i2) sort ([foo(1), X[1], X[k], foo(k), 1, k]);
(%o2)            [1, X , foo(1), k, X , foo(k)]
                      1              k
```

part (*expr*, *n_1*, ..., *n_k*) [Function]

Returns parts of the displayed form of `expr`. It obtains the part of `expr` as specified by the indices *n_1*, ..., *n_k*. First part *n_1* of `expr` is obtained, then part *n_2* of that, etc. The result is part *n_k* of ... part *n_2* of part *n_1* of `expr`. If no indices are specified `expr` is returned.

`part` can be used to obtain an element of a list, a row of a matrix, etc.

If the last argument to a `part` function is a list of indices then several subexpressions are picked out, each one corresponding to an index of the list. Thus `part (x + y + z, [1, 3])` is `z+x`.

`piece` holds the last expression selected when using the `part` functions. It is set during the execution of the function and thus may be referred to in the function itself as shown below.

If `partswitch` is set to `true` then `end` is returned when a selected part of an expression doesn't exist, otherwise an error message is given.

See also `inpart`, `substpart`, `substinpart`, `dpart`, and `lpart`.

Examples:

```
(%i1) part(z+2*y+a,2);
(%o1)                          2 y
(%i2) part(z+2*y+a,[1,3]);
(%o2)                         z + a
(%i3) part(z+2*y+a,2,1);
(%o3)                           2
```

`example (part)` displays additional examples.

partition (*expr*, *x*) [Function]

Returns a list of two expressions. They are (1) the factors of *expr* (if it is a product), the terms of *expr* (if it is a sum), or the list (if it is a list) which don't contain *x* and, (2) the factors, terms, or list which do.

Examples:

```
(%i1) partition (2*a*x*f(x), x);
(%o1)                      [2 a, x f(x)]
(%i2) partition (a+b, x);
(%o2)                        [b + a, 0]
(%i3) partition ([a, b, f(a), c], a);
(%o3)                   [[b, c], [a, f(a)]]
```

partswitch [Option variable]

Default value: `false`

When `partswitch` is `true`, `end` is returned when a selected part of an expression doesn't exist, otherwise an error message is given.

`pickapart` (*expr*, *n*) [Function]

Assigns intermediate expression labels to subexpressions of *expr* at depth *n*, an integer. Subexpressions at greater or lesser depths are not assigned labels. `pickapart` returns an expression in terms of intermediate expressions equivalent to the original expression *expr*.

See also `part`, `dpart`, `lpart`, `inpart`, and `reveal`.

Examples:

```
(%i1) expr: (a+b)/2 + sin (x^2)/3 - log (1 + sqrt(x+1));
                                              2
                                         sin(x )   b + a
(%o1)       - log(sqrt(x + 1) + 1) + ------- + -----
                                           3       2
(%i2) pickapart (expr, 0);
                                           2
                                         sin(x )   b + a
(%t2)       - log(sqrt(x + 1) + 1) + ------- + -----
                                           3       2
(%o2)                           %t2
(%i3) pickapart (expr, 1);

(%t3)                - log(sqrt(x + 1) + 1)


                                 2
                              sin(x )
(%t4)                         -------
                                 3


                               b + a
(%t5)                          -----
                                 2

(%o5)                    %t5 + %t4 + %t3
(%i5) pickapart (expr, 2);

(%t6)                 log(sqrt(x + 1) + 1)


                                 2
(%t7)                         sin(x )


(%t8)                          b + a
```

```
                          %t8   %t7
(%o8)                     --- + --- - %t6
                           2     3
(%i8) pickapart (expr, 3);

(%t9)                     sqrt(x + 1) + 1


                                2
(%t10)                         x

                   b + a                sin(%t10)
(%o10)             ----- - log(%t9) + ---------
                     2                     3
(%i10) pickapart (expr, 4);

(%t11)                       sqrt(x + 1)
                        2
                   sin(x )   b + a
(%o11)             ------- + ----- - log(%t11 + 1)
                      3        2
(%i11) pickapart (expr, 5);

(%t12)                          x + 1

                     2
                sin(x )   b + a
(%o12)          ------- + ----- - log(sqrt(%t12) + 1)
                   3        2
(%i12) pickapart (expr, 6);
                    2
               sin(x )   b + a
(%o12)         ------- + ----- - log(sqrt(x + 1) + 1)
                  3        2
```

`piece` [System variable]

Holds the last expression selected when using the `part` functions. It is set during the execution of the function and thus may be referred to in the function itself.

`psubst` [Function]

`psubst` (*list*, *expr*)
`psubst` (*a*, *b*, *expr*)

`psubst(a, b, expr)` is similar to `subst`. See `subst`.

In distinction from `subst` the function `psubst` makes parallel substitutions, if the first argument *list* is a list of equations.

See also sublis for making parallel substitutions and let and letsimp for others ways to do substitutions.

Example:

The first example shows parallel substitution with psubst. The second example shows the result for the function subst, which does a serial substitution.

```
(%i1) psubst ([a^2=b, b=a], sin(a^2) + sin(b));
(%o1)                     sin(b) + sin(a)
(%i2) subst ([a^2=b, b=a], sin(a^2) + sin(b));
(%o2)                       2 sin(a)
```

rembox [Function]

rembox (*expr*, *unlabelled*)
rembox (*expr*, *label*)
rembox (*expr*)

Removes boxes from *expr*.

rembox (*expr*, unlabelled) removes all unlabelled boxes from *expr*.

rembox (*expr*, *label*) removes only boxes bearing *label*.

rembox (*expr*) removes all boxes, labelled and unlabelled.

Boxes are drawn by the box, dpart, and lpart functions.

Examples:

```
(%i1) expr: (a*d - b*c)/h^2 + sin(%pi*x);
                                    a d - b c
(%o1)                sin(%pi x) + ---------
                                       2
                                      h
(%i2) dpart (dpart (expr, 1, 1), 2, 2);
                          """""""    a d - b c
(%o2)                sin("%pi x") + ---------
                          """""""      """"
                                       " 2"
                                       "h "
                                       """"
(%i3) expr2: lpart (BAR, lpart (FOO, %, 1), 2);
                  FOO""""""""""""   BAR"""""""""
                  "    """""""" "   "a d - b c"
(%o3)             "sin("%pi x")" + "---------"
                  "    """""""" "   "  """"   "
                  """""""""""""""   "  " 2"   "
                                    "  "h "   "
                                    "  """"   "
                                    """""""""""
```

```
(%i4) rembox (expr2, unlabelled);
                                    BAR"""""""""
                     FOO""""""""""  "a d - b c"
(%o4)                "sin(%pi x)" + "---------"
                     """""""""""""  "    2    "
                                    "   h     "
                                    """""""""""
(%i5) rembox (expr2, FOO);
                                    BAR"""""""""
                         ********   "a d - b c"
(%o5)                sin("%pi x") + "---------"
                         ********   "  """"   "
                                    "  " 2"   "
                                    "  "h "   "
                                    "  """"   "
                                    """""""""""
(%i6) rembox (expr2, BAR);
                     FOO"""""""""""
                     "    ******** "   a d - b c
(%o6)                "sin("%pi x")" + ---------
                     "    ******** "     """"
                     """""""""""""""     " 2"
                                         "h "
                                         """"
(%i7) rembox (expr2);
                                    a d - b c
(%o7)                 sin(%pi x) + ---------
                                        2
                                       h
```

`reveal` (*expr*, *depth*) [Function]

Replaces parts of *expr* at the specified integer *depth* with descriptive summaries.

- Sums and differences are replaced by `Sum(n)` where *n* is the number of operands of the sum.
- Products are replaced by `Product(n)` where *n* is the number of operands of the product.
- Exponentials are replaced by `Expt`.
- Quotients are replaced by `Quotient`.
- Unary negation is replaced by `Negterm`.
- Lists are replaced by `List(n)` where *n* is the number of elements of the list.

When *depth* is greater than or equal to the maximum depth of *expr*, `reveal (expr, depth)` returns *expr* unmodified.

`reveal` evaluates its arguments. `reveal` returns the summarized expression.

Example:

```
(%i1) e: expand ((a - b)^2)/expand ((exp(a) + exp(b))^2);
                          2           2
                         b  - 2 a b + a
(%o1)                -------------------------
                         b + a     2 b     2 a
                     2 %e      + %e    + %e
(%i2) reveal (e, 1);
(%o2)                         Quotient
(%i3) reveal (e, 2);
                              Sum(3)
(%o3)                         ------
                              Sum(3)
(%i4) reveal (e, 3);
                       Expt + Negterm + Expt
(%o4)                  ------------------------
                       Product(2) + Expt + Expt
(%i5) reveal (e, 4);
                        2                  2
                       b  - Product(3) + a
(%o5)          ------------------------------------
                          Product(2)     Product(2)
               2 Expt + %e           + %e
(%i6) reveal (e, 5);
                          2           2
                         b  - 2 a b + a
(%o6)                -------------------------
                         Sum(2)     2 b     2 a
                     2 %e       + %e    + %e
(%i7) reveal (e, 6);
                          2           2
                         b  - 2 a b + a
(%o7)                -------------------------
                         b + a     2 b     2 a
                     2 %e      + %e    + %e
```

`sqrtdenest` (*expr*) [Function]

Denests `sqrt` of simple, numerical, binomial surds, where possible. E.g.

```
(%i1) sqrt(sqrt(3)/2+1)/sqrt(11*sqrt(2)-12);
                              sqrt(3)
                          sqrt(------- + 1)
                                 2
(%o1)                   ---------------------
                        sqrt(11 sqrt(2) - 12)
```

```
(%i2) sqrtdenest(%);
                                 sqrt(3)   1
                                 ------- + -
                                    2      2
(%o2)                           -------------
                                   1/4    3/4
                                3 2    - 2
```

Sometimes it helps to apply `sqrtdenest` more than once, on such as `(19601-13860 sqrt(2))^(7/4)`.

`sublis` (*list*, *expr*) [Function]

Makes multiple parallel substitutions into an expression. *list* is a list of equations. The left hand side of the equations must be an atom.

The variable `sublis_apply_lambda` controls simplification after `sublis`.

See also `psubst` for making parallel substitutions.

Example:

```
(%i1) sublis ([a=b, b=a], sin(a) + cos(b));
(%o1)                     sin(b) + cos(a)
```

`sublis_apply_lambda` [Option variable]

Default value: `true`

Controls whether `lambda`'s substituted are applied in simplification after `sublis` is used or whether you have to do an `ev` to get things to apply. `true` means do the application.

`subnumsimp` [Option variable]

Default value: `false`

If `true` then the functions `subst` and `psubst` can substitute a subscripted variable `f[x]` with a number, when only the symbol `f` is given.

See also `subst`.

subst: cannot substitute 100 for operator g in expression g x – an error. To debug this try: debugmode(true);

```
(%i1) subst(100,g,g[x]+2);
(%i2) subst(100,g,g[x]+2),subnumsimp:true;
(%o2)                            102
```

`subst` (`a`, *b*, *c*) [Function]

Substitutes *a* for *b* in *c*. *b* must be an atom or a complete subexpression of *c*. For example, `x+y+z` is a complete subexpression of `2*(x+y+z)/w` while `x+y` is not. When *b* does not have these characteristics, one may sometimes use `substpart` or `ratsubst` (see below). Alternatively, if *b* is of the form `e/f` then one could use `subst (a*f, e, c)` while if *b* is of the form `e^(1/f)` then one could use `subst (a^f, e, c)`. The `subst` command also discerns the `x^y` in `x^-y` so that `subst (a, sqrt(x), 1/sqrt(x))` yields `1/a`. *a* and *b* may also be operators of an expression enclosed in double-quotes `"` or they may be function names. If one wishes to substitute for the

independent variable in derivative forms then the `at` function (see below) should be used.

`subst` is an alias for `substitute`.

The commands `subst (`*eq_1*`,` *expr*`)` or `subst ([`*eq_1*`, ..., `*eq_k*`],` *expr*`)` are other permissible forms. The *eq_i* are equations indicating substitutions to be made. For each equation, the right side will be substituted for the left in the expression *expr*. The equations are substituted in serial from left to right in *expr*. See the functions `sublis` and `psubst` for making parallel substitutions.

`exptsubst` if `true` permits substitutions like `y` for `%e^x` in `%e^(a*x)` to take place.

When `opsubst` is `false`, `subst` will not attempt to substitute into the operator of an expression. E.g. `(opsubst: false, subst (x^2, r, r+r[0]))` will work.

See also `at`, `ev` and `psubst`, as well as `let` and `letsimp`.

Examples:

```
(%i1) subst (a, x+y, x + (x+y)^2 + y);
                                        2
(%o1)                        y + x + a
(%i2) subst (-%i, %i, a + b*%i);
(%o2)                         a - %i b
```

The substitution is done in serial for a list of equations. Compare this with a parallel substitution:

```
(%i1) subst([a=b, b=c], a+b);
(%o1)                             2 c
(%i2) sublis([a=b, b=c], a+b);
(%o2)                            c + b
```

Single-character Operators like `+` and `-` have to be quoted in order to be replaced by subst. It is to note, though, that `a+b-c` might be expressed as `a+b+(-1*c)` internally.

```
(%i1) subst(["+"="-"],a+b-c);
(%o1)                         c - b + a
```

The difference between `subst` and `at` can be seen in the following example:

```
(%i1) g1:y(t)=a*x(t)+b*diff(x(t),t);
                               d
(%o1)              y(t) = b (-- (x(t))) + a x(t)
                               dt
(%i2) subst('diff(x(t),t)=1,g1);
(%o2)                      y(t) = a x(t) + b
(%i3) at(g1,'diff(x(t),t)=1);
                                 |
                         d       |
(%o3)       y(t) = b (-- (x(t))) |             ) + a x(t)
                         dt      |d
                                 |-- (x(t)) = 1
                                  dt
```

For further examples, do `example (subst)`.

`substinpart` (*x*, *expr*, *n_1*, ..., *n_k*) [Function]

Similar to `substpart`, but `substinpart` works on the internal representation of *expr*.

Examples:

```
(%i1) x . 'diff (f(x), x, 2);
                                2
                               d
(%o1)                   x . (--- (f(x)))
                                 2
                               dx
(%i2) substinpart (d^2, %, 2);
                                    2
(%o2)                           x . d
(%i3) substinpart (f1, f[1](x + 1), 0);
(%o3)                         f1(x + 1)
```

If the last argument to a `part` function is a list of indices then several subexpressions are picked out, each one corresponding to an index of the list. Thus

```
(%i1) part (x + y + z, [1, 3]);
(%o1)                           z + x
```

`piece` holds the value of the last expression selected when using the `part` functions. It is set during the execution of the function and thus may be referred to in the function itself as shown below. If `partswitch` is set to `true` then `end` is returned when a selected part of an expression doesn't exist, otherwise an error message is given.

```
(%i1) expr: 27*y^3 + 54*x*y^2 + 36*x^2*y + y + 8*x^3 + x + 1;
               3         2       2            3
(%o1)     27 y  + 54 x y  + 36 x  y + y + 8 x  + x + 1
(%i2) part (expr, 2, [1, 3]);
                                  2
(%o2)                          54 y
(%i3) sqrt (piece/54);
                               abs(y)
(%i4) substpart (factor (piece), expr, [1, 2, 3, 5]);
                                 3
(%o4)                (3 y + 2 x)  + y + x + 1
(%i5) expr: 1/x + y/x - 1/z;
                               1   y   1
(%o5)                        - - + - + -
                               z   x   x
(%i6) substpart (xthru (piece), expr, [2, 3]);
                              y + 1   1
(%o6)                         ----- - -
                                x     z
```

Also, setting the option `inflag` to `true` and calling `part` or `substpart` is the same as calling `inpart` or `substinpart`.

`substpart` (*x*, *expr*, *n_1*, ..., *n_k*) [Function]

Substitutes *x* for the subexpression picked out by the rest of the arguments as in `part`. It returns the new value of *expr*. *x* may be some operator to be substituted for an operator of *expr*. In some cases *x* needs to be enclosed in double-quotes `"` (e.g. `substpart ("+", a*b, 0)` yields `b + a`).

Example:

```
(%i1) 1/(x^2 + 2);
                                 1
(%o1)                          ------
                                2
                               x  + 2
(%i2) substpart (3/2, %, 2, 1, 2);
                                   1
(%o2)                          --------
                                3/2
                               x    + 2
(%i3) a*x + f(b, y);
(%o3)                       a x + f(b, y)
(%i4) substpart ("+", %, 1, 0);
(%o4)                     x + f(b, y) + a
```

Also, setting the option `inflag` to `true` and calling `part` or `substpart` is the same as calling `inpart` or `substinpart`.

`symbolp` (*expr*) [Function]

Returns `true` if *expr* is a symbol, else `false`.

See also Section 8.3 [Identifiers], page 110.

`unorder` () [Function]

Disables the aliasing created by the last use of the ordering commands `ordergreat` and `orderless`. `ordergreat` and `orderless` may not be used more than one time each without calling `unorder`. `unorder` does not substitute back in expressions the original symbols for the aliases introduced by `ordergreat` and `orderless`. Therefore, after execution of `unorder` the aliases appear in previous expressions.

See also `ordergreat` and `orderless`.

Examples:

`ordergreat(a)` introduces an alias for the symbol `a`. Therefore, the difference of `%o2` and `%o4` does not vanish. `unorder` does not substitute back the symbol `a` and the alias appears in the output `%o7`.

```
(%i1) unorder();
(%o1)                            []
(%i2) b*x + a^2;
                                     2
(%o2)                          b x + a
(%i3) ordergreat (a);
(%o3)                           done
```

```
(%i4) b*x + a^2;
 %th(1) - %th(3);
                                2
(%o4)                          a  + b x
(%i5) unorder();
                                 2    2
(%o5)                            a  - a
(%i6) %th(2);
(%o6)                              [a]
```

`verbify` (*f*) [Function]

Returns the verb form of the function name *f*. See also `verb`, `noun`, and `nounify`.

Examples:

```
(%i1) verbify ('foo);
(%o1)                              foo
(%i2) :lisp $%
$FOO
(%i2) nounify (foo);
(%o2)                              foo
(%i3) :lisp $%
%FOO
```

9 Operators

9.1 Introduction to operators

It is possible to define new operators with specified precedence, to undefine existing operators, or to redefine the precedence of existing operators. An operator may be unary prefix or unary postfix, binary infix, n-ary infix, matchfix, or nofix. "Matchfix" means a pair of symbols which enclose their argument or arguments, and "nofix" means an operator which takes no arguments. As examples of the different types of operators, there are the following.

unary prefix
: negation `- a`

unary postfix
: factorial `a!`

binary infix
: exponentiation `a^b`

n-ary infix
: addition `a + b`

matchfix
: list construction `[a, b]`

(There are no built-in nofix operators; for an example of such an operator, see `nofix`.)

The mechanism to define a new operator is straightforward. It is only necessary to declare a function as an operator; the operator function might or might not be defined.

An example of user-defined operators is the following. Note that the explicit function call `"dd" (a)` is equivalent to `dd a`, likewise `"<-" (a, b)` is equivalent to `a <- b`. Note also that the functions `"dd"` and `"<-"` are undefined in this example.

```
(%i1) prefix ("dd");
(%o1)                           dd
(%i2) dd a;
(%o2)                          dd a
(%i3) "dd" (a);
(%o3)                          dd a
(%i4) infix ("<-");
(%o4)                           <-
(%i5) a <- dd b;
(%o5)                       a <- dd b
(%i6) "<-" (a, "dd" (b));
(%o6)                       a <- dd b
```

The Maxima functions which define new operators are summarized in this table, stating the default left and right binding powers (lbp and rbp, respectively). (Binding power determines operator precedence. However, since left and right binding powers can differ, binding power is somewhat more complicated than precedence.) Some of the operation definition functions take additional arguments; see the function descriptions for details.

`prefix`
: rbp=180

`postfix`
: lbp=180

`infix` lbp=180, rbp=180

`nary` lbp=180, rbp=180

`matchfix` (binding power not applicable)

`nofix` (binding power not applicable)

For comparison, here are some built-in operators and their left and right binding powers.

Operator	lbp	rbp
:	180	20
::	180	20
:=	180	20
::=	180	20
!	160	
!!	160	
^	140	139
.	130	129
*	120	
/	120	120
+	100	100
-	100	134
=	80	80
#	80	80
>	80	80
>=	80	80
<	80	80
<=	80	80
not		70
and	65	
or	60	
,	10	
$	-1	
;	-1	

`remove` and `kill` remove operator properties from an atom. `remove ("a", op)` removes only the operator properties of *a*. `kill ("a")` removes all properties of *a*, including the operator properties. Note that the name of the operator must be enclosed in quotation marks.

incorrect syntax: # is not a prefix operator 5 ## ^ incorrect syntax: # is not a prefix operator 5 ## ^

```
(%i1) infix ("##");
(%o1)                           ##
(%i2) "##" (a, b) := a^b;
                                          b
(%o2)                    (a ## b) := a
(%i3) 5 ## 3;
(%o3)                           125
```

```
(%i4) remove ("##", op);
(%o4)                          done
(%i5) 5 ## 3;
(%i5) "##" (5, 3);
(%o5)                           125
(%i6) infix ("##");
(%o6)                           ##
(%i7) 5 ## 3;
(%o7)                           125
(%i8) kill ("##");
(%o8)                          done
(%i9) 5 ## 3;
(%i9) "##" (5, 3);
(%o9)                       ##(5, 3)
```

9.2 Arithmetic operators

`+` [Operator]
`-` [Operator]
`*` [Operator]
`/` [Operator]
`^` [Operator]

The symbols `+ * / ^` represent addition, multiplication, division, and exponentiation, respectively. The names of these operators are `"+" "*" "/"` and `"^"`, which may appear where the name of a function or operator is required.

The symbols `+` and `-` represent unary addition and negation, respectively, and the names of these operators are `"+"` and `"-"`, respectively.

Subtraction `a - b` is represented within Maxima as addition, `a + (- b)`. Expressions such as `a + (- b)` are displayed as subtraction. Maxima recognizes `"-"` only as the name of the unary negation operator, and not as the name of the binary subtraction operator.

Division `a / b` is represented within Maxima as multiplication, `a * b^(- 1)`. Expressions such as `a * b^(- 1)` are displayed as division. Maxima recognizes `"/"` as the name of the division operator.

Addition and multiplication are n-ary, commutative operators. Division and exponentiation are binary, noncommutative operators.

Maxima sorts the operands of commutative operators to construct a canonical representation. For internal storage, the ordering is determined by `orderlessp`. For display, the ordering for addition is determined by `ordergreatp`, and for multiplication, it is the same as the internal ordering.

Arithmetic computations are carried out on literal numbers (integers, rationals, ordinary floats, and bigfloats). Except for exponentiation, all arithmetic operations on numbers are simplified to numbers. Exponentiation is simplified to a number if either operand is an ordinary float or bigfloat or if the result is an exact integer or rational; otherwise an exponentiation may be simplified to `sqrt` or another exponentiation or left unchanged.

Floating-point contagion applies to arithmetic computations: if any operand is a bigfloat, the result is a bigfloat; otherwise, if any operand is an ordinary float, the result is an ordinary float; otherwise, the operands are rationals or integers and the result is a rational or integer.

Arithmetic computations are a simplification, not an evaluation. Thus arithmetic is carried out in quoted (but simplified) expressions.

Arithmetic operations are applied element-by-element to lists when the global flag `listarith` is `true`, and always applied element-by-element to matrices. When one operand is a list or matrix and another is an operand of some other type, the other operand is combined with each of the elements of the list or matrix.

Examples:

Addition and multiplication are n-ary, commutative operators. Maxima sorts the operands to construct a canonical representation. The names of these operators are `"+"` and `"*"`.

```
(%i1) c + g + d + a + b + e + f;
(%o1)               g + f + e + d + c + b + a
(%i2) [op (%), args (%)];
(%o2)              [+, [g, f, e, d, c, b, a]]
(%i3) c * g * d * a * b * e * f;
(%o3)                       a b c d e f g
(%i4) [op (%), args (%)];
(%o4)              [*, [a, b, c, d, e, f, g]]
(%i5) apply ("+", [a, 8, x, 2, 9, x, x, a]);
(%o5)                     3 x + 2 a + 19
(%i6) apply ("*", [a, 8, x, 2, 9, x, x, a]);
                                    2  3
(%o6)                        144 a  x
```

Division and exponentiation are binary, noncommutative operators. The names of these operators are `"/"` and `"^"`.

```
(%i1) [a / b, a ^ b];
                                 a   b
(%o1)                           [-, a ]
                                 b
(%i2) [map (op, %), map (args, %)];
(%o2)              [[/, ^], [[a, b], [a, b]]]
(%i3) [apply ("/", [a, b]), apply ("^", [a, b])];
                                 a   b
(%o3)                           [-, a ]
                                 b
```

Subtraction and division are represented internally in terms of addition and multiplication, respectively.

```
(%i1) [inpart (a - b, 0), inpart (a - b, 1), inpart (a - b, 2)];
(%o1)                        [+, a, - b]
```

```
(%i2) [inpart (a / b, 0), inpart (a / b, 1), inpart (a / b, 2)];
                                      1
(%o2)                          [*, a, -]
                                      b
```

Computations are carried out on literal numbers. Floating-point contagion applies.

```
(%i1) 17 + b - (1/2)*29 + 11^(2/4);
                                          5
(%o1)                    b + sqrt(11) + -
                                          2
(%i2) [17 + 29, 17 + 29.0, 17 + 29b0];
(%o2)                    [46, 46.0, 4.6b1]
```

Arithmetic computations are a simplification, not an evaluation.

```
(%i1) simp : false;
(%o1)                             false
(%i2) '(17 + 29*11/7 - 5^3);
                                  29 11    3
(%o2)                       17 + ----- - 5
                                   7
(%i3) simp : true;
(%o3)                             true
(%i4) '(17 + 29*11/7 - 5^3);
                                   437
(%o4)                             - ---
                                     7
```

Arithmetic is carried out element-by-element for lists (depending on `listarith`) and matrices.

```
(%i1) matrix ([a, x], [h, u]) - matrix ([1, 2], [3, 4]);
                           [ a - 1  x - 2 ]
(%o1)                      [              ]
                           [ h - 3  u - 4 ]
(%i2) 5 * matrix ([a, x], [h, u]);
                             [ 5 a  5 x ]
(%o2)                        [          ]
                             [ 5 h  5 u ]
(%i3) listarith : false;
(%o3)                             false
(%i4) [a, c, m, t] / [1, 7, 2, 9];
                             [a, c, m, t]
(%o4)                        ------------
                             [1, 7, 2, 9]
(%i5) [a, c, m, t] ^ x;
                                        x
(%o5)                        [a, c, m, t]
(%i6) listarith : true;
(%o6)                             true
```

```
(%i7) [a, c, m, t] / [1, 7, 2, 9];
                                c  m  t
(%o7)                      [a, -, -, -]
                                7  2  9
(%i8) [a, c, m, t] ^ x;
                               x   x   x   x
(%o8)                     [a , c , m , t ]
```

** [Operator]

Exponentiation operator. Maxima recognizes ** as the same operator as ^ in input, and it is displayed as ^ in 1-dimensional output, or by placing the exponent as a superscript in 2-dimensional output.

The fortran function displays the exponentiation operator as **, whether it was input as ** or ^.

Examples:

```
(%i1) is (a**b = a^b);
(%o1)                           true
(%i2) x**y + x^z;
                                z    y
(%o2)                          x  + x
(%i3) string (x**y + x^z);
(%o3)                          x^z+x^y
(%i4) fortran (x**y + x^z);
      x**z+x**y
(%o4)                           done
```

^^ [Operator]

Noncommutative exponentiation operator. ^^ is the exponentiation operator corresponding to noncommutative multiplication ., just as the ordinary exponentiation operator ^ corresponds to commutative multiplication *.

Noncommutative exponentiation is displayed by ^^ in 1-dimensional output, and by placing the exponent as a superscript within angle brackets < > in 2-dimensional output.

Examples:

```
(%i1) a . a . b . b . b + a * a * a * b * b;
                         3  2    <2>    <3>
(%o1)                   a  b  + a    . b
(%i2) string (a . a . b . b . b + a * a * a * b * b);
(%o2)                   a^3*b^2+a^^2 . b^^3
```

. [Operator]

The dot operator, for matrix (non-commutative) multiplication. When "." is used in this way, spaces should be left on both sides of it, e.g. A . B This distinguishes it plainly from a decimal point in a floating point number.

See also Section 25.1.1 Dot, dot0nscsimp, dot0simp, dot1simp, dotassoc, dotconstrules, dotdistrib, dotexptsimp, dotident, and dotscrules.

9.3 Relational operators

< [Operator]
<= [Operator]
>= [Operator]
> [Operator]

The symbols `<` `<=` `>=` and `>` represent less than, less than or equal, greater than or equal, and greater than, respectively. The names of these operators are `"<"` `"<="` `">="` and `">"`, which may appear where the name of a function or operator is required.

These relational operators are all binary operators; constructs such as `a < b < c` are not recognized by Maxima.

Relational expressions are evaluated to Boolean values by the functions `is` and `maybe`, and the programming constructs `if`, `while`, and `unless`. Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated (when evaluation is not otherwise prevented by quotation).

When a relational expression cannot be evaluated to `true` or `false`, the behavior of `is` and `if` are governed by the global flag `prederror`. When `prederror` is `true`, `is` and `if` trigger an error. When `prederror` is `false`, `is` returns `unknown`, and `if` returns a partially-evaluated conditional expression.

`maybe` always behaves as if `prederror` were `false`, and `while` and `unless` always behave as if `prederror` were `true`.

Relational operators do not distribute over lists or other aggregates.

See also `=`, `#`, `equal`, and `notequal`.

Examples:

Relational expressions are evaluated to Boolean values by some functions and programming constructs.

```
(%i1) [x, y, z] : [123, 456, 789];
(%o1)                     [123, 456, 789]
(%i2) is (x < y);
(%o2)                          true
(%i3) maybe (y > z);
(%o3)                          false
(%i4) if x >= z then 1 else 0;
(%o4)                            0
(%i5) block ([S], S : 0, for i:1 while i <= 100 do S : S + i,
       return (S));
(%o5)                          5050
```

Relational expressions are not otherwise evaluated or simplified to Boolean values, although the arguments of relational expressions are evaluated.

```
(%i1) [x, y, z] : [123, 456, 789];
(%o1)                     [123, 456, 789]
(%i2) [x < y, y <= z, z >= y, y > z];
(%o2)    [123 < 456, 456 <= 789, 789 >= 456, 456 > 789]
```

```
(%i3) map (is, %);
(%o3)                [true, true, true, false]
```

9.4 Logical operators

`and` [Operator]

The logical conjunction operator. `and` is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.

`and` forces evaluation (like `is`) of one or more operands, and may force evaluation of all operands.

Operands are evaluated in the order in which they appear. `and` evaluates only as many of its operands as necessary to determine the result. If any operand is `false`, the result is `false` and no further operands are evaluated.

The global flag `prederror` governs the behavior of `and` when an evaluated operand cannot be determined to be `true` or `false`. `and` prints an error message when `prederror` is `true`. Otherwise, operands which do not evaluate to `true` or `false` are accepted, and the result is a Boolean expression.

`and` is not commutative: `a and b` might not be equal to `b and a` due to the treatment of indeterminate operands.

`not` [Operator]

The logical negation operator. `not` is a prefix operator; its operand is a Boolean expression, and its result is a Boolean value.

`not` forces evaluation (like `is`) of its operand.

The global flag `prederror` governs the behavior of `not` when its operand cannot be determined to be `true` or `false`. `not` prints an error message when `prederror` is `true`. Otherwise, operands which do not evaluate to `true` or `false` are accepted, and the result is a Boolean expression.

`or` [Operator]

The logical disjunction operator. `or` is an n-ary infix operator; its operands are Boolean expressions, and its result is a Boolean value.

`or` forces evaluation (like `is`) of one or more operands, and may force evaluation of all operands.

Operands are evaluated in the order in which they appear. `or` evaluates only as many of its operands as necessary to determine the result. If any operand is `true`, the result is `true` and no further operands are evaluated.

The global flag `prederror` governs the behavior of `or` when an evaluated operand cannot be determined to be `true` or `false`. `or` prints an error message when `prederror` is `true`. Otherwise, operands which do not evaluate to `true` or `false` are accepted, and the result is a Boolean expression.

`or` is not commutative: `a or b` might not be equal to `b or a` due to the treatment of indeterminate operands.

9.5 Operators for Equations

[Operator]

Represents the negation of syntactic equality =.

Note that because of the rules for evaluation of predicate expressions (in particular because not *expr* causes evaluation of *expr*), not a = b is equivalent to is(a # b), instead of a # b.

Examples:

```
(%i1) a = b;
(%o1)                          a = b
(%i2) is (a = b);
(%o2)                          false
(%i3) a # b;
(%o3)                          a # b
(%i4) not a = b;
(%o4)                          true
(%i5) is (a # b);
(%o5)                          true
(%i6) is (not a = b);
(%o6)                          true
```

= [Operator]

The equation operator.

An expression a = b, by itself, represents an unevaluated equation, which might or might not hold. Unevaluated equations may appear as arguments to solve and algsys or some other functions.

The function is evaluates = to a Boolean value. is(a = b) evaluates a = b to true when a and b are identical. That is, a and b are atoms which are identical, or they are not atoms and their operators are identical and their arguments are identical. Otherwise, is(a = b) evaluates to false; it never evaluates to unknown. When is(a = b) is true, a and b are said to be syntactically equal, in contrast to equivalent expressions, for which is(equal(a, b)) is true. Expressions can be equivalent and not syntactically equal.

The negation of = is represented by #. As with =, an expression a # b, by itself, is not evaluated. is(a # b) evaluates a # b to true or false.

In addition to is, some other operators evaluate = and # to true or false, namely if, and, or, and not.

Note that because of the rules for evaluation of predicate expressions (in particular because not *expr* causes evaluation of *expr*), not a = b is equivalent to is(a # b), instead of a # b.

rhs and lhs return the right-hand and left-hand sides, respectively, of an equation or inequation.

See also equal and notequal.

Examples:

An expression `a = b`, by itself, represents an unevaluated equation, which might or might not hold.

```
(%i1) eq_1 : a * x - 5 * y = 17;
(%o1)                     a x - 5 y = 17
(%i2) eq_2 : b * x + 3 * y = 29;
(%o2)                     3 y + b x = 29
(%i3) solve ([eq_1, eq_2], [x, y]);
                         196         29 a - 17 b
(%o3)           [[x = ---------, y = -----------]]
                      5 b + 3 a       5 b + 3 a
(%i4) subst (%, [eq_1, eq_2]);
         196 a     5 (29 a - 17 b)
(%o4) [--------- - --------------- = 17,
       5 b + 3 a      5 b + 3 a
                                  196 b     3 (29 a - 17 b)
                                --------- + --------------- = 29]
                                5 b + 3 a      5 b + 3 a
(%i5) ratsimp (%);
(%o5)                    [17 = 17, 29 = 29]
```

`is(a = b)` evaluates `a = b` to `true` when *a* and *b* are syntactically equal (that is, identical). Expressions can be equivalent and not syntactically equal.

```
(%i1) a : (x + 1) * (x - 1);
(%o1)                      (x - 1) (x + 1)
(%i2) b : x^2 - 1;
                                2
(%o2)                          x  - 1
(%i3) [is (a = b), is (a # b)];
(%o3)                       [false, true]
(%i4) [is (equal (a, b)), is (notequal (a, b))];
(%o4)                       [true, false]
```

Some operators evaluate `=` and `#` to `true` or `false`.

```
(%i1) if expand ((x + y)^2) = x^2 + 2 * x * y + y^2 then FOO else
      BAR;
(%o1)                           FOO
(%i2) eq_3 : 2 * x = 3 * x;
(%o2)                         2 x = 3 x
(%i3) eq_4 : exp (2) = %e^2;
                                2     2
(%o3)                          %e  = %e
(%i4) [eq_3 and eq_4, eq_3 or eq_4, not eq_3];
(%o4)                    [false, true, true]
```

Because `not` *expr* causes evaluation of *expr*, `not a = b` is equivalent to `is(a # b)`.

```
(%i1) [2 * x # 3 * x, not (2 * x = 3 * x)];
(%o1)                      [2 x # 3 x, true]
(%i2) is (2 * x # 3 * x);
(%o2)                           true
```

9.6 Assignment operators

: [Operator]

Assignment operator.

When the left-hand side is a simple variable (not subscripted), : evaluates its right-hand side and associates that value with the left-hand side.

When the left-hand side is a subscripted element of a list, matrix, declared Maxima array, or Lisp array, the right-hand side is assigned to that element. The subscript must name an existing element; such objects cannot be extended by naming nonexistent elements.

When the left-hand side is a subscripted element of a hashed array, the right-hand side is assigned to that element, if it already exists, or a new element is allocated, if it does not already exist.

When the left-hand side is a list of simple and/or subscripted variables, the right-hand side must evaluate to a list, and the elements of the right-hand side are assigned to the elements of the left-hand side, in parallel.

See also kill and remvalue, which undo the association between the left-hand side and its value.

Examples:

Assignment to a simple variable.

```
(%i1) a;
(%o1)                           a
(%i2) a : 123;
(%o2)                          123
(%i3) a;
(%o3)                          123
```

Assignment to an element of a list.

```
(%i1) b : [1, 2, 3];
(%o1)                       [1, 2, 3]
(%i2) b[3] : 456;
(%o2)                          456
(%i3) b;
(%o3)                      [1, 2, 456]
```

Assignment to a variable that neither is the name of a list nor of an array creates a hashed array.

```
(%i1) c[99] : 789;
(%o1)                          789
(%i2) c[99];
(%o2)                          789
(%i3) c;
(%o3)                           c
(%i4) arrayinfo (c);
(%o4)                  [hashed, 1, [99]]
(%i5) listarray (c);
(%o5)                        [789]
```

Multiple assignment.

```
(%i1) [a, b, c] : [45, 67, 89];
(%o1)                      [45, 67, 89]
(%i2) a;
(%o2)                           45
(%i3) b;
(%o3)                           67
(%i4) c;
(%o4)                           89
```

Multiple assignment is carried out in parallel. The values of `a` and `b` are exchanged in this example.

```
(%i1) [a, b] : [33, 55];
(%o1)                        [33, 55]
(%i2) [a, b] : [b, a];
(%o2)                        [55, 33]
(%i3) a;
(%o3)                           55
(%i4) b;
(%o4)                           33
```

`::` [Operator]

Assignment operator.

`::` is the same as `:` (which see) except that `::` evaluates its left-hand side as well as its right-hand side.

Examples:

```
(%i1) x : 'foo;
(%o1)                           foo
(%i2) x :: 123;
(%o2)                           123
(%i3) foo;
(%o3)                           123
(%i4) x : '[a, b, c];
(%o4)                        [a, b, c]
(%i5) x :: [11, 22, 33];
(%o5)                      [11, 22, 33]
(%i6) a;
(%o6)                           11
(%i7) b;
(%o7)                           22
(%i8) c;
(%o8)                           33
```

`::=` [Operator]

Macro function definition operator. `::=` defines a function (called a "macro" for historical reasons) which quotes its arguments, and the expression which it returns

(called the "macro expansion") is evaluated in the context from which the macro was called. A macro function is otherwise the same as an ordinary function.

`macroexpand` returns a macro expansion (without evaluating it). `macroexpand (foo (x))` followed by `''%` is equivalent to `foo (x)` when `foo` is a macro function.

`::=` puts the name of the new macro function onto the global list `macros`. `kill`, `remove`, and `remfunction` unbind macro function definitions and remove names from `macros`.

`fundef` or `dispfun` return a macro function definition or assign it to a label, respectively.

Macro functions commonly contain `buildq` and `splice` expressions to construct an expression, which is then evaluated.

Examples

A macro function quotes its arguments, so message (1) shows `y - z`, not the value of `y - z`. The macro expansion (the quoted expression `'(print ("(2) x is equal to", x))`) is evaluated in the context from which the macro was called, printing message (2).

```
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printq1 (x) ::= block (print ("(1) x is equal to", x),
                                '(print ("(2) x is equal to", x)))$
(%i5) printq1 (y - z);
(1) x is equal to y - z
(2) x is equal to %pi
(%o5)                           %pi
```

An ordinary function evaluates its arguments, so message (1) shows the value of `y - z`. The return value is not evaluated, so message (2) is not printed until the explicit evaluation `''%`.

```
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
(%i4) printe1 (x) := block (print ("(1) x is equal to", x),
      '(print ("(2) x is equal to", x)))$
(%i5) printe1 (y - z);
(1) x is equal to 1234 - 1729 w
(%o5)              print((2) x is equal to, x)
(%i6) ''%;
(2) x is equal to %pi
(%o6)                           %pi
```

`macroexpand` returns a macro expansion. `macroexpand (foo (x))` followed by `''%` is equivalent to `foo (x)` when `foo` is a macro function.

```
(%i1) x: %pi$
(%i2) y: 1234$
(%i3) z: 1729 * w$
```

```
(%i4) g (x) ::= buildq ([x], print ("x is equal to", x))$
(%i5) macroexpand (g (y - z));
(%o5)              print(x is equal to, y - z)
(%i6) ''%;
x is equal to 1234 - 1729 w
(%o6)                      1234 - 1729 w
(%i7) g (y - z);
x is equal to 1234 - 1729 w
(%o7)                      1234 - 1729 w
```

`:=` [Operator]

The function definition operator.

f`(`*x_1*`, ..., `*x_n*`) :=` *expr* defines a function named *f* with arguments *x_1*, ..., *x_n* and function body *expr*. `:=` never evaluates the function body (unless explicitly evaluated by quote-quote `''`). The function body is evaluated every time the function is called.

f`[`*x_1*`, ..., `*x_n*`] :=` *expr* defines a so-called memoizing function. Its function body is evaluated just once for each distinct value of its arguments, and that value is returned, without evaluating the function body, whenever the arguments have those values again. (A function of this kind is also known as a "array function".)

f`[`*x_1*`, ..., `*x_n*`](`*y_1*`, ..., `*y_m*`) :=` *expr* is a special case of a memoizing function. *f*`[`*x_1*`, ..., `*x_n*`]` is a memoizing function which returns a lambda expression with arguments *y_1*, ..., *y_m*. The function body is evaluated once for each distinct value of *x_1*, ..., *x_n*, and the body of the lambda expression is that value.

When the last or only function argument *x_n* is a list of one element, the function defined by `:=` accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments *x_1*, ..., *x_(n - 1)*, and any further actual arguments, if present, are assigned to *x_n* as a list.

All function definitions appear in the same namespace; defining a function `f` within another function `g` does not automatically limit the scope of `f` to `g`. However, `local(f)` makes the definition of function `f` effective only within the block or other compound expression in which `local` appears.

If some formal argument *x_k* is a quoted symbol, the function defined by `:=` does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also `define` and `::=`.

Examples:

`:=` never evaluates the function body (unless explicitly evaluated by quote-quote).

```
(%i1) expr : cos(y) - sin(x);
(%o1)                      cos(y) - sin(x)
(%i2) F1 (x, y) := expr;
(%o2)                    F1(x, y) := expr
(%i3) F1 (a, b);
(%o3)                      cos(y) - sin(x)
```

```
(%i4) F2 (x, y) := ''expr;
(%o4)              F2(x, y) := cos(y) - sin(x)
(%i5) F2 (a, b);
(%o5)                     cos(b) - sin(a)
```

`f(x_1, ..., x_n) := ...` defines an ordinary function.

```
(%i1) G1(x, y) := (print ("Evaluating G1 for x=", x, "and y=", y),
 x.y - y.x);
(%o1) G1(x, y) := (print("Evaluating G1 for x=", x, "and y=",
                                                  y), x . y - y . x)
(%i2) G1([1, a], [2, b]);
Evaluating G1 for x= [1, a] and y= [2, b]
(%o2)                             0
(%i3) G1([1, a], [2, b]);
Evaluating G1 for x= [1, a] and y= [2, b]
(%o3)                             0
```

`f[x_1, ..., x_n] := ...` defines a memoizing function.

```
(%i1) G2[a] := (print ("Evaluating G2 for a=", a), a^2);
                                                          2
(%o1)     G2  := (print("Evaluating G2 for a=", a), a )
            a
(%i2) G2[1234];
Evaluating G2 for a= 1234
(%o2)                          1522756
(%i3) G2[1234];
(%o3)                          1522756
(%i4) G2[2345];
Evaluating G2 for a= 2345
(%o4)                          5499025
(%i5) arrayinfo (G2);
(%o5)              [hashed, 1, [1234], [2345]]
(%i6) listarray (G2);
(%o6)                  [1522756, 5499025]
```

f`[x_1, ..., x_n](y_1, ..., y_m) :=` *expr* is a special case of a memoizing function.

```
(%i1) G3[n](x) := (print ("Evaluating G3 for n=", n), diff (sin(x)^2,
 x, n));
(%o1) G3 (x) := (print("Evaluating G3 for n=", n),
        n
                                                          2
                                                 diff(sin (x), x, n))
(%i2) G3[2];
Evaluating G3 for n= 2
                                  2           2
(%o2)          lambda([x], 2 cos (x) - 2 sin (x))
```

```
(%i3) G3[2];
                                  2           2
(%o3)          lambda([x], 2 cos (x) - 2 sin (x))
(%i4) G3[2](1);
                             2           2
(%o4)                 2 cos (1) - 2 sin (1)
(%i5) arrayinfo (G3);
(%o5)                    [hashed, 1, [2]]
(%i6) listarray (G3);
                                  2           2
(%o6)         [lambda([x], 2 cos (x) - 2 sin (x))]
```

When the last or only function argument *x_n* is a list of one element, the function defined by `:=` accepts a variable number of arguments.

```
(%i1) H ([L]) := apply ("+", L);
(%o1)                H([L]) := apply("+", L)
(%i2) H (a, b, c);
(%o2)                        c + b + a
```

`local` makes a local function definition.

```
(%i1) foo (x) := 1 - x;
(%o1)                     foo(x) := 1 - x
(%i2) foo (100);
(%o2)                           - 99
(%i3) block (local (foo), foo (x) := 2 * x, foo (100));
(%o3)                            200
(%i4) foo (100);
(%o4)                           - 99
```

9.7 User defined operators

`infix` [Function]

`infix` (*op*)
`infix` (*op*, *lbp*, *rbp*)
`infix` (*op*, *lbp*, *rbp*, *lpos*, *rpos*, *pos*)

Declares *op* to be an infix operator. An infix operator is a function of two arguments, with the name of the function written between the arguments. For example, the subtraction operator `-` is an infix operator.

`infix` (*op*) declares *op* to be an infix operator with default binding powers (left and right both equal to 180) and parts of speech (left and right both equal to `any`).

`infix` (*op*, *lbp*, *rbp*) declares *op* to be an infix operator with stated left and right binding powers and default parts of speech (left and right both equal to `any`).

`infix` (*op*, *lbp*, *rbp*, *lpos*, *rpos*, *pos*) declares *op* to be an infix operator with stated left and right binding powers and parts of speech *lpos*, *rpos*, and *pos* for the left operand, the right operand, and the operator result, respectively.

"Part of speech", in reference to operator declarations, means expression type. Three types are recognized: `expr`, `clause`, and `any`, indicating an algebraic expression, a

Boolean expression, or any kind of expression, respectively. Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

The precedence of *op* with respect to other operators derives from the left and right binding powers of the operators in question. If the left and right binding powers of *op* are both greater the left and right binding powers of some other operator, then *op* takes precedence over the other operator. If the binding powers are not both greater or less, some more complicated relation holds.

The associativity of *op* depends on its binding powers. Greater left binding power (*lbp*) implies an instance of *op* is evaluated before other operators to its left in an expression, while greater right binding power (*rbp*) implies an instance of *op* is evaluated before other operators to its right in an expression. Thus greater *lbp* makes *op* right-associative, while greater *rbp* makes *op* left-associative. If *lbp* is equal to *rbp*, *op* is left-associative.

See also Section 9.1 [Introduction to operators], page 137.

Examples:

If the left and right binding powers of *op* are both greater the left and right binding powers of some other operator, then *op* takes precedence over the other operator.

```
(%i1) :lisp (get '$+ 'lbp)
100
(%i1) :lisp (get '$+ 'rbp)
134
(%i1) infix ("##", 101, 101);
(%o1)                           ##
(%i2) "##"(a, b) := sconcat("(", a, ",", b, ")");
(%o2)       (a ## b) := sconcat("(", a, ",", b, ")")
(%i3) 1 + a ## b + 2;
(%o3)                        (a,b) + 3
(%i4) infix ("##", 99, 99);
(%o4)                           ##
(%i5) 1 + a ## b + 2;
(%o5)                        (a+1,b+2)
```

Greater *lbp* makes *op* right-associative, while greater *rbp* makes *op* left-associative.

```
(%i1) infix ("##", 100, 99);
(%o1)                           ##
(%i2) "##"(a, b) := sconcat("(", a, ",", b, ")")$
(%i3) foo ## bar ## baz;
(%o3)                     (foo,(bar,baz))
(%i4) infix ("##", 100, 101);
(%o4)                           ##
(%i5) foo ## bar ## baz;
(%o5)                     ((foo,bar),baz)
```

Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

incorrect syntax: Found ALGEBRAIC expression where LOGICAL expression expected if x ## y then ^

```
(%i1) infix ("##", 100, 99, expr, expr, expr);
(%o1)                           ##
(%i2) if x ## y then 1 else 0;
(%i2) infix ("##", 100, 99, expr, expr, clause);
(%o2)                           ##
(%i3) if x ## y then 1 else 0;
(%o3)                if x ## y then 1 else 0
```

matchfix [Function]

matchfix (*ldelimiter*, *rdelimiter*)
matchfix (*ldelimiter*, *rdelimiter*, *arg_pos*, *pos*)

Declares a matchfix operator with left and right delimiters *ldelimiter* and *rdelimiter*. The delimiters are specified as strings.

A "matchfix" operator is a function of any number of arguments, such that the arguments occur between matching left and right delimiters. The delimiters may be any strings, so long as the parser can distinguish the delimiters from the operands and other expressions and operators. In practice this rules out unparseable delimiters such as `%`, `,`, `$` and `;`, and may require isolating the delimiters with white space. The right delimiter can be the same or different from the left delimiter.

A left delimiter can be associated with only one right delimiter; two different matchfix operators cannot have the same left delimiter.

An existing operator may be redeclared as a matchfix operator without changing its other properties. In particular, built-in operators such as addition `+` can be declared matchfix, but operator functions cannot be defined for built-in operators.

The command `matchfix (`*ldelimiter*`, `*rdelimiter*`, `*arg_pos*`, `*pos*`)` declares the argument part-of-speech *arg_pos* and result part-of-speech *pos*, and the delimiters *ldelimiter* and *rdelimiter*.

"Part of speech", in reference to operator declarations, means expression type. Three types are recognized: `expr`, `clause`, and `any`, indicating an algebraic expression, a Boolean expression, or any kind of expression, respectively. Maxima can detect some syntax errors by comparing the declared part of speech to an actual expression.

The function to carry out a matchfix operation is an ordinary user-defined function. The operator function is defined in the usual way with the function definition operator `:=` or `define`. The arguments may be written between the delimiters, or with the left delimiter as a quoted string and the arguments following in parentheses. `dispfun (`*ldelimiter*`)` displays the function definition.

The only built-in matchfix operator is the list constructor `[ ]`. Parentheses `( )` and double-quotes `" "` act like matchfix operators, but are not treated as such by the Maxima parser.

`matchfix` evaluates its arguments. `matchfix` returns its first argument, *ldelimiter*.

Examples:

Delimiters may be almost any strings.

```
(%i1) matchfix ("@@", "~");
(%o1)                           @@
```

```
(%i2) @@ a, b, c ~;
(%o2)                        @@a, b, c~
(%i3) matchfix (">>", "<<");
(%o3)                             >>
(%i4) >> a, b, c <<;
(%o4)                        >>a, b, c<<
(%i5) matchfix ("foo", "oof");
(%o5)                             foo
(%i6) foo a, b, c oof;
(%o6)                       fooa, b, coof
(%i7) >> w + foo x, y oof + z << / @@ p, q ~;
                       >>z + foox, yoof + w<<
(%o7)                  ----------------------
                              @@p, q~
```

Matchfix operators are ordinary user-defined functions.

```
(%i1) matchfix ("!-", "-!");
(%o1)                             "!-"
(%i2) !- x, y -! := x/y - y/x;
                                      x   y
(%o2)                     !-x, y-! := - - -
                                      y   x
(%i3) define (!-x, y-!, x/y - y/x);
                                      x   y
(%o3)                     !-x, y-! := - - -
                                      y   x
(%i4) define ("!-" (x, y), x/y - y/x);
                                      x   y
(%o4)                     !-x, y-! := - - -
                                      y   x
(%i5) dispfun ("!-");
                                      x   y
(%t5)                     !-x, y-! := - - -
                                      y   x

(%o5)                            done
(%i6) !-3, 5-!;
                                   16
(%o6)                            - --
                                   15
(%i7) "!-" (3, 5);
                                   16
(%o7)                            - --
                                   15
```

nary [Function]

nary (*op*)
nary (*op*, *bp*, *arg_pos*, *pos*)

An `nary` operator is used to denote a function of any number of arguments, each of which is separated by an occurrence of the operator, e.g. A+B or A+B+C. The `nary("x")` function is a syntax extension function to declare `x` to be an `nary` operator. Functions may be declared to be `nary`. If `declare(j,nary);` is done, this tells the simplifier to simplify, e.g. `j(j(a,b),j(c,d))` to `j(a, b, c, d)`.

See also Section 9.1 [Introduction to operators], page 137.

nofix [Function]

nofix (*op*)
nofix (*op*, *pos*)

`nofix` operators are used to denote functions of no arguments. The mere presence of such an operator in a command will cause the corresponding function to be evaluated. For example, when one types `"exit;"` to exit from a Maxima break, `"exit"` is behaving similar to a `nofix` operator. The function `nofix("x")` is a syntax extension function which declares `x` to be a `nofix` operator.

See also Section 9.1 [Introduction to operators], page 137.

postfix [Function]

postfix (*op*)
postfix (*op*, *lbp*, *lpos*, *pos*)

`postfix` operators like the `prefix` variety denote functions of a single argument, but in this case the argument immediately precedes an occurrence of the operator in the input string, e.g. 3!. The `postfix("x")` function is a syntax extension function to declare `x` to be a `postfix` operator.

See also Section 9.1 [Introduction to operators], page 137.

prefix [Function]

prefix (*op*)
prefix (*op*, *rbp*, *rpos*, *pos*)

A `prefix` operator is one which signifies a function of one argument, which argument immediately follows an occurrence of the operator. `prefix("x")` is a syntax extension function to declare `x` to be a `prefix` operator.

See also Section 9.1 [Introduction to operators], page 137.

10 Evaluation

10.1 Functions and Variables for Evaluation

' [Operator]

The single quote operator ' prevents evaluation.

Applied to a symbol, the single quote prevents evaluation of the symbol.

Applied to a function call, the single quote prevents evaluation of the function call, although the arguments of the function are still evaluated (if evaluation is not otherwise prevented). The result is the noun form of the function call.

Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression. E.g., `'(f(x))` means do not evaluate the expression `f(x)`. `'f(x)` (with the single quote applied to `f` instead of `f(x)`) means return the noun form of `f` applied to `[x]`.

The single quote does not prevent simplification.

When the global flag `noundisp` is `true`, nouns display with a single quote. This switch is always `true` when displaying function definitions.

See also the quote-quote operator [quote-quote], page 159 and `nouns`.

Examples:

Applied to a symbol, the single quote prevents evaluation of the symbol.

```
(%i1) aa: 1024;
(%o1)                          1024
(%i2) aa^2;
(%o2)                         1048576
(%i3) 'aa^2;
                                 2
(%o3)                           aa
(%i4) ''%;
(%o4)                         1048576
```

Applied to a function call, the single quote prevents evaluation of the function call. The result is the noun form of the function call.

```
(%i1) x0: 5;
(%o1)                            5
(%i2) x1: 7;
(%o2)                            7
(%i3) integrate (x^2, x, x0, x1);
                                218
(%o3)                           ---
                                 3
(%i4) 'integrate (x^2, x, x0, x1);
```

```
                                7
                               /
                               [   2
(%o4)                          I  x  dx
                               ]
                               /
                                5
(%i5) %, nouns;
                                  218
(%o5)                             ---
                                   3
```

Applied to a parenthesized expression, the single quote prevents evaluation of all symbols and function calls in the expression.

```
(%i1) aa: 1024;
(%o1)                           1024
(%i2) bb: 19;
(%o2)                            19
(%i3) sqrt(aa) + bb;
(%o3)                            51
(%i4) '(sqrt(aa) + bb);
(%o4)                      bb + sqrt(aa)
(%i5) ''%;
(%o5)                            51
```

The single quote does not prevent simplification.

```
(%i1) sin (17 * %pi) + cos (17 * %pi);
(%o1)                           - 1
(%i2) '(sin (17 * %pi) + cos (17 * %pi));
(%o2)                           - 1
```

Maxima considers floating point operations by its in-built mathematical functions to be a simplification.

```
(%i1) sin(1.0);
(%o1)                          .8414709848078965
(%i2) '(sin(1.0));
(%o2)                          .8414709848078965
```

When the global flag `noundisp` is `true`, nouns display with a single quote.

```
(%i1) x:%pi;
(%o1)                                 %pi
(%i2) bfloat(x);
(%o2)                         3.141592653589793b0
(%i3) sin(x);
(%o3)                                  0
(%i4) noundisp;
(%o4)                                false
(%i5) 'bfloat(x);
(%o5)                            bfloat(%pi)
```

```
(%i6) bfloat('x);
(%o6)                                  x
(%i7) 'sin(x);
(%o7)                                  0
(%i8) sin('x);
(%o8)                                sin(x)
(%i9) noundisp : not noundisp;
(%o9)                                 true
(%i10) 'bfloat(x);
(%o10)                            'bfloat(%pi)
(%i11) bfloat('x);
(%o11)                                 x
(%i12) 'sin(x);
(%o12)                                 0
(%i13) sin('x);
(%o13)                               sin(x)
(%i14)
```

'' [Operator]

The quote-quote operator '' (two single quote marks) modifies evaluation in input expressions.

Applied to a general expression *expr*, quote-quote causes the value of *expr* to be substituted for *expr* in the input expression.

Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression. The quote-quote operator is always applied as soon as it is parsed, and cannot be quoted. Thus quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote '.

Quote-quote is recognized by batch and load.

See also ev, the single-quote operator [quote], page 157 and nouns.

Examples:

Applied to a general expression *expr*, quote-quote causes the value of *expr* to be substituted for *expr* in the input expression.

```
(%i1) expand ((a + b)^3);
                     3        2      2      3
(%o1)               b  + 3 a b  + 3 a  b + a
(%i2) [_, ''_];
                          3    3        2      2      3
(%o2)     [expand((b + a) ), b  + 3 a b  + 3 a  b + a ]
(%i3) [%i1, ''%i1];
                          3    3        2      2      3
(%o3)     [expand((b + a) ), b  + 3 a b  + 3 a  b + a ]
(%i4) [aa : cc, bb : dd, cc : 17, dd : 29];
```

```
(%o4)                    [cc, dd, 17, 29]
(%i5) foo_1 (x) := aa - bb * x;
(%o5)                 foo_1(x) := aa - bb x
(%i6) foo_1 (10);
(%o6)                       cc - 10 dd
(%i7) ''%;
(%o7)                          - 273
(%i8) ''(foo_1 (10));
(%o8)                          - 273
(%i9) foo_2 (x) := ''aa - ''bb * x;
(%o9)                 foo_2(x) := cc - dd x
(%i10) foo_2 (10);
(%o10)                         - 273
(%i11) [x0 : x1, x1 : x2, x2 : x3];
(%o11)                     [x1, x2, x3]
(%i12) x0;
(%o12)                          x1
(%i13) ''x0;
(%o13)                          x2
(%i14) '' ''x0;
(%o14)                          x3
```

Applied to the operator of an expression, quote-quote changes the operator from a noun to a verb (if it is not already a verb).

```
(%i1) declare (foo, noun);
(%o1)                          done
(%i2) foo (x) := x - 1729;
(%o2)                 ''foo(x) := x - 1729
(%i3) foo (100);
(%o3)                        foo(100)
(%i4) ''foo (100);
(%o4)                         - 1629
```

The quote-quote operator is applied by the input parser; it is not stored as part of a parsed input expression.

```
(%i1) [aa : bb, cc : dd, bb : 1234, dd : 5678];
(%o1)                 [bb, dd, 1234, 5678]
(%i2) aa + cc;
(%o2)                          dd + bb
(%i3) display (_, op (_), args (_));
                             _ = cc + aa

                          op(cc + aa) = +

                       args(cc + aa) = [cc, aa]

(%o3)                          done
(%i4) ''(aa + cc);
```

```
(%o4)                          6912
(%i5) display (_, op (_), args (_));
                             _ = dd + bb

                          op(dd + bb) = +

                       args(dd + bb) = [dd, bb]

(%o5)                          done
```

Quote-quote causes evaluation when evaluation is otherwise suppressed, such as in function definitions, lambda expressions, and expressions quoted by single quote '.

```
(%i1) foo_1a (x) := ''(integrate (log (x), x));
(%o1)               foo_1a(x) := x log(x) - x
(%i2) foo_1b (x) := integrate (log (x), x);
(%o2)           foo_1b(x) := integrate(log(x), x)
(%i3) dispfun (foo_1a, foo_1b);
(%t3)               foo_1a(x) := x log(x) - x

(%t4)           foo_1b(x) := integrate(log(x), x)

(%o4)                         [%t3, %t4]
(%i5) integrate (log (x), x);
(%o5)                        x log(x) - x
(%i6) foo_2a (x) := ''%;
(%o6)               foo_2a(x) := x log(x) - x
(%i7) foo_2b (x) := %;
(%o7)                    foo_2b(x) := %
(%i8) dispfun (foo_2a, foo_2b);
(%t8)               foo_2a(x) := x log(x) - x

(%t9)                    foo_2b(x) := %

(%o9)                         [%t7, %t8]
(%i10) F : lambda ([u], diff (sin (u), u));
(%o10)             lambda([u], diff(sin(u), u))
(%i11) G : lambda ([u], ''(diff (sin (u), u)));
(%o11)                   lambda([u], cos(u))
(%i12) '(sum (a[k], k, 1, 3) + sum (b[k], k, 1, 3));
(%o12)         sum(b , k, 1, 3) + sum(a , k, 1, 3)
                     k                   k
(%i13) '(''(sum (a[k], k, 1, 3)) + ''(sum (b[k], k, 1, 3)));
(%o13)             b  + a  + b  + a  + b  + a
                    3    3    2    2    1    1
```

ev (*expr*, *arg_1*, ..., *arg_n*) [Function]

Evaluates the expression *expr* in the environment specified by the arguments *arg_1*, ..., *arg_n*. The arguments are switches (Boolean flags), assignments, equations, and functions. `ev` returns the result (another expression) of the evaluation.

The evaluation is carried out in steps, as follows.

1. First the environment is set up by scanning the arguments which may be any or all of the following.
   - `simp` causes *expr* to be simplified regardless of the setting of the switch `simp` which inhibits simplification if `false`.
   - `noeval` suppresses the evaluation phase of `ev` (see step (4) below). This is useful in conjunction with the other switches and in causing *expr* to be resimplified without being reevaluated.
   - `nouns` causes the evaluation of noun forms (typically unevaluated functions such as `'integrate` or `'diff`) in *expr*.
   - `expand` causes expansion.
   - `expand (m, n)` causes expansion, setting the values of `maxposex` and `maxnegex` to *m* and *n* respectively.
   - `detout` causes any matrix inverses computed in *expr* to have their determinant kept outside of the inverse rather than dividing through each element.
   - `diff` causes all differentiations indicated in *expr* to be performed.
   - `derivlist (x, y, z, ...)` causes only differentiations with respect to the indicated variables. See also `derivlist`.
   - `risch` causes integrals in *expr* to be evaluated using the Risch algorithm. See `risch`. The standard integration routine is invoked when using the special symbol `nouns`.
   - `float` causes non-integral rational numbers to be converted to floating point.
   - `numer` causes some mathematical functions (including exponentiation) with numerical arguments to be evaluated in floating point. It causes variables in *expr* which have been given numervals to be replaced by their values. It also sets the `float` switch on.
   - `pred` causes predicates (expressions which evaluate to `true` or `false`) to be evaluated.
   - `eval` causes an extra post-evaluation of *expr* to occur. (See step (5) below.) `eval` may occur multiple times. For each instance of `eval`, the expression is evaluated again.
   - `A` where `A` is an atom declared to be an evaluation flag `evflag` causes `A` to be bound to `true` during the evaluation of *expr*.
   - `V: expression` (or alternately `V=expression`) causes `V` to be bound to the value of `expression` during the evaluation of *expr*. Note that if `V` is a Maxima option, then `expression` is used for its value during the evaluation of *expr*. If more than one argument to `ev` is of this type then the binding is done in parallel. If `V` is a non-atomic expression then a substitution rather than a binding is performed.

- `F` where `F`, a function name, has been declared to be an evaluation function `evfun` causes `F` to be applied to *expr*.
- Any other function names, e.g. `sum`, cause evaluation of occurrences of those names in *expr* as though they were verbs.
- In addition a function occurring in *expr* (say `F(x)`) may be defined locally for the purpose of this evaluation of *expr* by giving `F(x) := expression` as an argument to `ev`.
- If an atom not mentioned above or a subscripted variable or subscripted expression was given as an argument, it is evaluated and if the result is an equation or assignment then the indicated binding or substitution is performed. If the result is a list then the members of the list are treated as if they were additional arguments given to `ev`. This permits a list of equations to be given (e.g. `[X=1, Y=A**2]`) or a list of names of equations (e.g., `[%t1, %t2]` where `%t1` and `%t2` are equations) such as that returned by `solve`.

The arguments of `ev` may be given in any order with the exception of substitution equations which are handled in sequence, left to right, and evaluation functions which are composed, e.g., `ev (`*`expr`*`, ratsimp, realpart)` is handled as `realpart (ratsimp (`*`expr`*`))`.

The `simp`, `numer`, and `float` switches may also be set locally in a block, or globally in Maxima so that they will remain in effect until being reset.

If *expr* is a canonical rational expression (CRE), then the expression returned by `ev` is also a CRE, provided the `numer` and `float` switches are not both `true`.

2. During step (1), a list is made of the non-subscripted variables appearing on the left side of equations in the arguments or in the value of some arguments if the value is an equation. The variables (subscripted variables which do not have associated `memoizing functions` as well as non-subscripted variables) in the expression *expr* are replaced by their global values, except for those appearing in this list. Usually, *expr* is just a label or `%` (as in `%i2` in the example below), so this step simply retrieves the expression named by the label, so that `ev` may work on it.
3. If any substitutions are indicated by the arguments, they are carried out now.
4. The resulting expression is then re-evaluated (unless one of the arguments was `noeval`) and simplified according to the arguments. Note that any function calls in *expr* will be carried out after the variables in it are evaluated and that `ev(F(x))` thus may behave like `F(ev(x))`.
5. For each instance of `eval` in the arguments, steps (3) and (4) are repeated.

See also `[quote-quote], page 159`, `at` and `subst`.

Examples:

```
(%i1) sin(x) + cos(y) + (w+1)^2 + 'diff (sin(w), w);
                                      d                     2
(%o1)              cos(y) + sin(x) + -- (sin(w)) + (w + 1)
                                      dw
(%i2) ev (%, numer, expand, diff, x=2, y=1);
                                    2
```

```
(%o2)                cos(w) + w  + 2 w + 2.449599732693821
```

An alternate top level syntax has been provided for `ev`, whereby one may just type in its arguments, without the `ev()`. That is, one may write simply

`expr, arg_1, ..., arg_n`

This is not permitted as part of another expression, e.g., in functions, blocks, etc.

Notice the parallel binding process in the following example.

```
(%i3) programmode: false;
(%o3)                                false
(%i4) x+y, x: a+y, y: 2;
(%o4)                              y + a + 2
(%i5) 2*x - 3*y = 3$
(%i6) -3*x + 2*y = -4$
(%i7) solve ([%o5, %o6]);
Solution

                                          1
(%t7)                               y = - -
                                          5

                                        6
(%t8)                                 x = -
                                        5
(%o8)                            [[%t7, %t8]]
(%i8) %o6, %o8;
(%o8)                              - 4 = - 4
(%i9) x + 1/x > gamma (1/2);
                                        1
(%o9)                           x + - > sqrt(%pi)
                                        x
(%i10) %, numer, x=1/2;
(%o10)                     2.5 > 1.772453850905516
(%i11) %, pred;
(%o11)                               true
```

`eval` [Special symbol]

As an argument in a call to `ev (`*`expr`*`)`, `eval` causes an extra evaluation of *expr*. See `ev`.

Example:

```
(%i1) [a:b,b:c,c:d,d:e];
(%o1)                            [b, c, d, e]
(%i2) a;
(%o2)                                  b
(%i3) ev(a);
(%o3)                                  c
(%i4) ev(a),eval;
(%o4)                                  e
```

```
(%i5) a,eval,eval;
(%o5)                                  e
```

evflag [Property]

When a symbol *x* has the `evflag` property, the expressions `ev(`*expr*`, `*x*`)` and *expr*`, `*x* (at the interactive prompt) are equivalent to `ev(`*expr*`, `*x*` = true)`. That is, *x* is bound to `true` while *expr* is evaluated.

The expression `declare(x, evflag)` gives the `evflag` property to the variable *x*.

The flags which have the `evflag` property by default are the following:

```
algebraic          cauchysum       demoivre
dotscrules         %emode          %enumer
exponentialize     exptisolate     factorflag
float              halfangles      infeval
isolate_wrt_times  keepfloat       letrat
listarith          logabs          logarc
logexpand          lognegint
m1pbranch          numer_pbranch   programmode
radexpand          ratalgdenom     ratfac
ratmx              ratsimpexpons   simp
simpproduct        simpsum         sumexpand
trigexpand
```

Examples:

```
(%i1) sin (1/2);
                                   1
(%o1)                         sin(-)
                                   2
(%i2) sin (1/2), float;
(%o2)                    0.479425538604203
(%i3) sin (1/2), float=true;
(%o3)                    0.479425538604203
(%i4) simp : false;
(%o4)                           false
(%i5) 1 + 1;
(%o5)                           1 + 1
(%i6) 1 + 1, simp;
(%o6)                             2
(%i7) simp : true;
(%o7)                           true
(%i8) sum (1/k^2, k, 1, inf);
                               inf
                               ====
                               \     1
(%o8)                           >    --
                               /      2
                               ====  k
                               k = 1
```

```
(%i9) sum (1/k^2, k, 1, inf), simpsum;
                                  2
                                 %pi
(%o9)                            ----
                                  6
(%i10) declare (aa, evflag);
(%o10)                          done
(%i11) if aa = true then YES else NO;
(%o11)                           NO
(%i12) if aa = true then YES else NO, aa;
(%o12)                           YES
```

`evfun` [Property]

When a function F has the `evfun` property, the expressions `ev(`*`expr`*`, `*`F`*`)` and *`expr`*`, `*`F`* (at the interactive prompt) are equivalent to *`F`*`(ev(`*`expr`*`))`.

If two or more `evfun` functions F, G, etc., are specified, the functions are applied in the order that they are specified.

The expression `declare(`*`F`*`, evfun)` gives the `evfun` property to the function F. The functions which have the `evfun` property by default are the following:

```
bfloat          factor       fullratsimp
logcontract     polarform    radcan
ratexpand       ratsimp      rectform
rootscontract   trigexpand   trigreduce
```

Examples:

```
(%i1) x^3 - 1;
                                3
(%o1)                          x  - 1
(%i2) x^3 - 1, factor;
                                  2
(%o2)                  (x - 1) (x  + x + 1)
(%i3) factor (x^3 - 1);
                                  2
(%o3)                  (x - 1) (x  + x + 1)
(%i4) cos(4 * x) / sin(x)^4;
                              cos(4 x)
(%o4)                         --------
                                 4
                              sin (x)
(%i5) cos(4 * x) / sin(x)^4, trigexpand;
                 4           2      2         4
               sin (x) - 6 cos (x) sin (x) + cos (x)
(%o5)          ------------------------------------
                                 4
                              sin (x)
(%i6) cos(4 * x) / sin(x)^4, trigexpand, ratexpand;
                             2         4
```

```
                        6 cos (x)   cos (x)
(%o6)                 - --------- + ------- + 1
                             2          4
                          sin (x)    sin (x)
(%i7) ratexpand (trigexpand (cos(4 * x) / sin(x)^4));
                             2          4
                        6 cos (x)   cos (x)
(%o7)                 - --------- + ------- + 1
                             2          4
                          sin (x)    sin (x)
(%i8) declare ([F, G], evfun);
(%o8)                          done
(%i9) (aa : bb, bb : cc, cc : dd);
(%o9)                           dd
(%i10) aa;
(%o10)                          bb
(%i11) aa, F;
(%o11)                         F(cc)
(%i12) F (aa);
(%o12)                         F(bb)
(%i13) F (ev (aa));
(%o13)                         F(cc)
(%i14) aa, F, G;
(%o14)                       G(F(cc))
(%i15) G (F (ev (aa)));
(%o15)                       G(F(cc))
```

`infeval` [Option variable]

Enables "infinite evaluation" mode. `ev` repeatedly evaluates an expression until it stops changing. To prevent a variable, say `X`, from being evaluated away in this mode, simply include `X='X` as an argument to `ev`. Of course expressions such as `ev (X, X=X+1, infeval)` will generate an infinite loop.

`noeval` [Special symbol]

`noeval` suppresses the evaluation phase of `ev`. This is useful in conjunction with other switches and in causing expressions to be resimplified without being reevaluated.

`nouns` [Special symbol]

`nouns` is an `evflag`. When used as an option to the `ev` command, `nouns` converts all "noun" forms occurring in the expression being `ev`'d to "verbs", i.e., evaluates them. See also `noun`, `nounify`, `verb`, and `verbify`.

`pred` [Special symbol]

As an argument in a call to `ev (expr)`, `pred` causes predicates (expressions which evaluate to `true` or `false`) to be evaluated. See `ev`.

Example:

```
(%i1) 1<2;
(%o1)                                1 < 2
```

```
(%i2) 1<2,pred;
(%o2)                                true
```

11 Simplification

11.1 Introduction to Simplification

Maxima interacts with the user through a cycle of actions called the read-eval-print loop (REPL). This consists of three steps: reading and parsing, evaluating and simplifying, and outputting. Parsing converts a syntactically valid sequence of typed characters into a internal data structure. Evaluation replaces variable and function names with their values and simplification rewrites expressions to be easier for the user or other programs to understand. Output displays results in a variety of different formats and notations.

Evaluation and simplification sometimes appear to have similar functionality, and Maxima uses simplification in many cases where other systems use evaluation. For example, arithmetic both on numbers and on symbolic expressions is simplification, not evaluation: `2+2` simplifies to `4`, `2+x+x` simplifies to `2+2*x`, and `sqrt(7)^4` simplifies to `49`. Evaluation and simplification are interleaved. For example, `factor(integrate(x+1,x))` first calls the built-in function `integrate`, giving `x+x*x*2^-1`; that simplifies to `x+(1/2)*x^2`; this in turn is passed to the `factor` function, which returns `(x*(x+2))/2`.

Evaluation is what makes Maxima a programming language: it implements functions, subroutines, variables, values, loops, assignments and so on. Evaluation replaces built-in or user-defined function names by their definitions and variables by their values. This is largely the same as activities of a conventional programming language, but extended to work with symbolic mathematical data. The system has various optional "flags" which the user can set to control the details of evaluation. See Section 10.1 [Functions and Variables for Evaluation], page 157.

Simplification maintains the value of an expression while re-formulating its form to be smaller, simpler to understand, or to conform to a particular specification (like expanded). For example, `sin(%pi/2)` to `1`, and `x+x` to `2*x`. There are many flags which control simplification. For example, with `triginverses:true`, `atan(tan(x))` does not simplify to `x`, but with `triginverses:all`, it does.

Simplification can be provided in three ways:

- The internal, built-in automated simplifier,
- User-written pattern-matching transformations, linked to the simplifier by using "tellsimp" or "tellsimpafter" and called automatically,
- User-written simplification routines adding using the `simplifying` subsystem.

The internal simplifier belongs to the heart of Maxima. It is a large and complicated collection of programs, and it has been refined over many years and by thousands of users. Nevertheless, especially if you are trying out novel ideas or unconventional notation, you may find it helpful to make small (or large) changes to the program yourself. For details see for example the paper at the end of https://people.eecs.berkeley.edu/~fateman/papers/intro5.txt.

Maxima internally represents expressions as "trees" with operators or "roots" like `+`, `*` , `=` and operands ("leaves") which are variables like x, y, z, functions or sub-trees, like `x*y`. Each operator has a simplification program associated with it. `+` (which also covers binary `-` since `a-b = a+(-1)*b)` and `*` (which also covers `/` since `a/b = a*b^(-1)`) have rather elaborate

simplification programs. These simplification programs (simplus, simptimes, simpexpt, etc.) are called whenever the simplifier encounters the respective arithmetic operators in an expression tree to be analyzed.

The structure of the simplifier dates back to 1965, and many hands have worked on it through the years. It is data-directed, or object-oriented in the sense that it dispatches to the appropriate routine depending on the root of some sub-tree of the expression, recursively. This general approach means that modifications to simplification are generally localized. In many cases it is straightforward to add an operator and its simplification routine without disturbing existing code.

Maxima also provides a variety of transformation routines that can change the form of an expression, including `factor` (polynomial factorization), `horner` (reorganize a polynomial using Horner's rule), `partfrac` (rewrite a rational function as partial fractions), `trigexpand` (apply the sum formulas for trigonometric functions), and so on.

Users can also write routines that change the form of an expression.

Besides this general simplifier operating on algebraic expression trees, there are several other representations of expressions in Maxima which have separate methods. For example, the `rat` function converts polynomials to vectors of coefficients to assist in rapid manipulation of such forms. Other representations include Taylor series and the (rarely used) Poisson series.

All operators introduced by the user initially have no simplification programs associated with them. Maxima does not know anything about function `"f"` and so typing `f(a,b)` will result in simplifying a,b, but not `f`. Even some built-in operators have no simplifications. For example, `=` does not `"simplify"` – it is a place-holder with no simplification semantics other than to simplify its two arguments, in this case referred to as the left and right sides. Other parts of Maxima such as the solve program take special note of equations, that is, trees with `=` as the root. (Note – in Maxima, the assignment operation is `:` . That is, `q: 4` sets the value of the symbol q to `4`. Function definition is done with `:=`.)

The general simplifier returns results with an internal flag indicating the expression and each sub-expression has been simplified. This does not guarantee that it is unique over all possible equivalent expressions. That's too hard (theoretically, not possible given the generality of what can be expressed in Maxima). However, some aspects of the expression, such as the ordering of terms in a sum or product, are made uniform. This is important for the other programs to work properly.

A number of option variables control simplification. Indeed, simplification can be turned off entirely using `simp:false`. However, many internal routines will not operate correctly with `simp:false`. (About the only time it seems plausible to turn off the simplifier is in the rare case that you want to over-ride a built-in simplification. In that case you might temporarily disable the simplifier, put in the new transformation via `tellsimp`, and then re-enable the simplifier by `simp:true`.)

It is more plausible for you to associate user-defined symbolic function names or operators with properties (`additive`, `lassociative`, `oddfun`, `antisymmetric`, `linear`, `outative`, `commutative`, `multiplicative`, `rassociative`, `evenfun`, `nary` and `symmetric`). These options steer the simplifier processing in systematic directions.

For example, `declare(f,oddfun)` specifies that `f` is an odd function. Maxima will simplify `f(-x)` to `-f(x)`. In the case of an even function, that is `declare(g,evenfun)`,

Maxima will simplify `g(-x)` to `g(x)`. You can also associate a programming function with a name such as `h(x):=x^2+1`. In that case the evaluator will immediately replace `h(3)` by `10`, and `h(a+1)` by `(a+1)^2+1`, so any properties of `h` will be ignored.

In addition to these directly related properties set up by the user, facts and properties from the actual context may have an impact on the simplifier's behavior, too. See Section 13.1 [Introduction to Maxima's Database], page 219.

Example: `sin(n*%pi)` is simplified to zero, if *n* is an integer.

```
(%i1) sin(n*%pi);
(%o1)                        sin(%pi n)
(%i2) declare(n, integer);
(%o2)                           done
(%i3) sin(n*%pi);
(%o3)                             0
```

If automated simplification is not sufficient, you can consider a variety of built-in, but explicitly called simplification functions (`ratsimp`, `expand`, `factor`, `radcan` and others). There are also flags that will push simplification into one or another direction. Given `demoivre:true` the simplifier rewrites complex exponentials as trigonometric forms. Given `exponentialize:true` the simplifier tries to do the reverse: rewrite trigonometric forms as complex exponentials.

As everywhere in Maxima, by writing your own functions (be it in the Maxima user language or in the implementation language Lisp) and explicitly calling them at selected places in the program, you can respond to your individual simplification needs. Lisp gives you a handle on all the internal mechanisms, but you rarely need this full generality. "Tellsimp" is designed to generate much of the Lisp internal interface into the simplifier automatically. See See Chapter 30 [Rules and Patterns], page 549.

Over the years (Maxima/Macsyma's origins date back to about 1966!) users have contributed numerous application packages and tools to extend or alter its functional behavior. Various non-standard and "share" packages exist to modify or extend simplification as well. You are invited to look into this more experimental material where work is still in progress See Chapter 92 [Package simplification], page 1245.

The following appended material is optional on a first reading, and reading it is not necessary for productive use of Maxima. It is for the curious user who wants to understand what is going on, or the ambitious programmer who might wish to change the (open-source) code. Experimentation with redefining Maxima Lisp code is easily possible: to change the definition of a Lisp program (say the one that simplifies `cos()`, named `simp%cos`), you simply load into Maxima a text file that will overwrite the `simp%cos` function from the maxima package.

11.2 Functions and Variables for Simplification

`additive` [Property]

If `declare(f,additive)` has been executed, then:

(1) If `f` is univariate, whenever the simplifier encounters `f` applied to a sum, `f` will be distributed over that sum. I.e. `f(y+x)` will simplify to `f(y)+f(x)`.

(2) If `f` is a function of 2 or more arguments, additivity is defined as additivity in the first argument to `f`, as in the case of `sum` or `integrate`, i.e. `f(h(x)+g(x),x)` will simplify to `f(h(x),x)+f(g(x),x)`. This simplification does not occur when `f` is applied to expressions of the form `sum(x[i],i,lower-limit,upper-limit)`.

Example:

```
(%i1) F3 (a + b + c);
(%o1)                      F3(c + b + a)
(%i2) declare (F3, additive);
(%o2)                           done
(%i3) F3 (a + b + c);
(%o3)                 F3(c) + F3(b) + F3(a)
```

`antisymmetric` [Property]

If `declare(h,antisymmetric)` is done, this tells the simplifier that `h` is antisymmetric. E.g. `h(x,z,y)` will simplify to `- h(x, y, z)`. That is, it will give (-1)^n times the result given by `symmetric` or `commutative`, where n is the number of interchanges of two arguments necessary to convert it to that form.

Examples:

```
(%i1) S (b, a);
(%o1)                          S(b, a)
(%i2) declare (S, symmetric);
(%o2)                           done
(%i3) S (b, a);
(%o3)                          S(a, b)
(%i4) S (a, c, e, d, b);
(%o4)                    S(a, b, c, d, e)
(%i5) T (b, a);
(%o5)                          T(b, a)
(%i6) declare (T, antisymmetric);
(%o6)                           done
(%i7) T (b, a);
(%o7)                        - T(a, b)
(%i8) T (a, c, e, d, b);
(%o8)                    T(a, b, c, d, e)
```

`combine` (*expr*) [Function]

Simplifies the sum *expr* by combining terms with the same denominator into a single term.

See also: `rncombine`.

Example:

```
(%i1) 1*f/2*b + 2*c/3*a + 3*f/4*b +c/5*b*a;
                      5 b f   a b c   2 a c
(%o1)                 ----- + ----- + -----
                        4       5       3
```

```
(%i2) combine (%);
                    75 b f + 4 (3 a b c + 10 a c)
(%o2)               -----------------------------
                                  60
```

`commutative` [Property]

If `declare(h, commutative)` is done, this tells the simplifier that `h` is a commutative function. E.g. `h(x, z, y)` will simplify to `h(x, y, z)`. This is the same as `symmetric`.

Example:

```
(%i1) S (b, a);
(%o1)                          S(b, a)
(%i2) S (a, b) + S (b, a);
(%o2)                    S(b, a) + S(a, b)
(%i3) declare (S, commutative);
(%o3)                           done
(%i4) S (b, a);
(%o4)                          S(a, b)
(%i5) S (a, b) + S (b, a);
(%o5)                         2 S(a, b)
(%i6) S (a, c, e, d, b);
(%o6)                    S(a, b, c, d, e)
```

`demoivre` (*expr*) [Function]

`demoivre` [Option variable]

The function `demoivre (expr)` converts one expression without setting the global variable `demoivre`.

When the variable `demoivre` is `true`, complex exponentials are converted into equivalent expressions in terms of circular functions: `exp (a + b*%i)` simplifies to `%e^a * (cos(b) + %i*sin(b))` if `b` is free of `%i`. `a` and `b` are not expanded.

The default value of `demoivre` is `false`.

`exponentialize` converts circular and hyperbolic functions to exponential form. `demoivre` and `exponentialize` cannot both be true at the same time.

`distrib` (*expr*) [Function]

Distributes sums over products. It differs from `expand` in that it works at only the top level of an expression, i.e., it doesn't recurse and it is faster than `expand`. It differs from `multthru` in that it expands all sums at that level.

Examples:

```
(%i1) distrib ((a+b) * (c+d));
(%o1)                  b d + a d + b c + a c
(%i2) multthru ((a+b) * (c+d));
(%o2)                  (b + a) d + (b + a) c
(%i3) distrib (1/((a+b) * (c+d)));
                                 1
(%o3)                    ---------------
                         (b + a) (d + c)
```

```
(%i4) expand (1/((a+b) * (c+d)), 1, 0);
                                  1
(%o4)                 ---------------------
                      b d + a d + b c + a c
```

distribute_over [Option variable]

Default value: true

distribute_over controls the mapping of functions over bags like lists, matrices, and equations. At this time not all Maxima functions have this property. It is possible to look up this property with the command properties..

The mapping of functions is switched off, when setting distribute_over to the value false.

Examples:

The sin function maps over a list:

```
(%i1) sin([x,1,1.0]);
(%o1)         [sin(x), sin(1), 0.8414709848078965]
```

mod is a function with two arguments which maps over lists. Mapping over nested lists is possible too:

```
(%i1) mod([x,11,2*a],10);
(%o1)             [mod(x, 10), 1, 2 mod(a, 5)]
(%i2) mod([[x,y,z],11,2*a],10);
(%o2) [[mod(x, 10), mod(y, 10), mod(z, 10)], 1, 2 mod(a, 5)]
```

Mapping of the floor function over a matrix and an equation:

```
(%i1) floor(matrix([a,b],[c,d]));
                      [ floor(a)  floor(b) ]
(%o1)                 [                    ]
                      [ floor(c)  floor(d) ]
(%i2) floor(a=b);
(%o2)                  floor(a) = floor(b)
```

Functions with more than one argument map over any of the arguments or all arguments:

```
(%i1) expintegral_e([1,2],[x,y]);
(%o1) [[expintegral_e(1, x), expintegral_e(1, y)],
                         [expintegral_e(2, x), expintegral_e(2, y)]]
```

Check if a function has the property distribute_over:

```
(%i1) properties(abs);
(%o1) [limit function, integral, rule, distributes over bags,
                                            noun, gradef, transfun]
```

The mapping of functions is switched off, when setting distribute_over to the value false.

```
(%i1) distribute_over;
(%o1)                          true
(%i2) sin([x,1,1.0]);
(%o2)         [sin(x), sin(1), 0.8414709848078965]
```

```
(%i3) distribute_over : not distribute_over;
(%o3)                          false
(%i4) sin([x,1,1.0]);
(%o4)                   sin([x, 1, 1.0])
```

`domain` [Option variable]

Default value: `real`

When `domain` is set to `complex`, `sqrt (x^2)` will remain `sqrt (x^2)` instead of returning `abs(x)`.

`evenfun` [Property]
`oddfun` [Property]

`declare(f, evenfun)` or `declare(f, oddfun)` tells Maxima to recognize the function `f` as an even or odd function.

Examples:

```
(%i1) o (- x) + o (x);
(%o1)                       o(x) + o(- x)
(%i2) declare (o, oddfun);
(%o2)                          done
(%i3) o (- x) + o (x);
(%o3)                            0
(%i4) e (- x) - e (x);
(%o4)                       e(- x) - e(x)
(%i5) declare (e, evenfun);
(%o5)                          done
(%i6) e (- x) - e (x);
(%o6)                            0
```

`expand` [Function]

`expand` (*expr*)
`expand` (*expr*, *p*, *n*)

Expand expression *expr*. Products of sums and exponentiated sums are multiplied out, numerators of rational expressions which are sums are split into their respective terms, and multiplication (commutative and non-commutative) are distributed over addition at all levels of *expr*.

For polynomials one should usually use `ratexpand` which uses a more efficient algorithm.

`maxnegex` and `maxposex` control the maximum negative and positive exponents, respectively, which will expand.

`expand (`*expr*`, `*p*`, `*n*`)` expands *expr*, using *p* for `maxposex` and *n* for `maxnegex`. This is useful in order to expand part but not all of an expression.

`expon` - the exponent of the largest negative power which is automatically expanded (independent of calls to `expand`). For example if `expon` is 4 then `(x+1)^(-5)` will not be automatically expanded.

`expop` - the highest positive exponent which is automatically expanded. Thus `(x+1)^3`, when typed, will be automatically expanded only if `expop` is greater than

or equal to 3. If it is desired to have `(x+1)^n` expanded where `n` is greater than `expop` then executing `expand ((x+1)^n)` will work only if `maxposex` is not less than `n`.

`expand(expr, 0, 0)` causes a resimplification of `expr`. `expr` is not reevaluated. In distinction from `ev(expr, noeval)` a special representation (e. g. a CRE form) is removed. See also `resimplify` and `ev`.

The `expand` flag used with `ev` causes expansion.

The file `share/simplification/facexp.mac` contains several related functions (in particular `facsum`, `factorfacsum` and `collectterms`, which are autoloaded) and variables (`nextlayerfactor` and `facsum_combine`) that provide the user with the ability to structure expressions by controlled expansion. Brief function descriptions are available in `simplification/facexp.usg`. A demo is available by doing `demo("facexp")`.

Examples:

```
(%i1) expr:(x+1)^2*(y+1)^3;
                                2        3
(%o1)                    (x + 1)  (y + 1)
(%i2) expand(expr);
       2  3        3    3      2  2        2      2      2
(%o2) x  y  + 2 x y  + y  + 3 x  y  + 6 x y  + 3 y  + 3 x  y
                                                          2
                                      + 6 x y + 3 y + x  + 2 x + 1
(%i3) expand(expr,2);
                2        3              3         3
(%o3)         x  (y + 1)  + 2 x (y + 1)  + (y + 1)
(%i4) expr:(x+1)^-2*(y+1)^3;
                                    3
                             (y + 1)
(%o4)                        --------
                                    2
                             (x + 1)
(%i5) expand(expr);
             3               2
            y             3 y            3 y             1
(%o5) ------------ + ------------ + ------------ + ------------
       2              2              2              2
      x  + 2 x + 1   x  + 2 x + 1   x  + 2 x + 1   x  + 2 x + 1
(%i6) expand(expr,2,2);
                                    3
                               (y + 1)
(%o6)                        ------------
                              2
                             x  + 2 x + 1
```

Resimplify an expression without expansion:

```
(%i1) expr:(1+x)^2*sin(x);
                                 2
(%o1)                     (x + 1)  sin(x)
```

```
(%i2) exponentialize:true;
(%o2)                          true
(%i3) expand(expr,0,0);
                       %i x     - %i x            2
                    (%e     - %e      ) %i (x + 1)
(%o3)           - ------------------------------
                                 2
```

expandwrt (*expr*, *x_1*, ..., *x_n*) [Function]

Expands expression `expr` with respect to the variables *x_1*, ..., *x_n*. All products involving the variables appear explicitly. The form returned will be free of products of sums of expressions that are not free of the variables. *x_1*, ..., *x_n* may be variables, operators, or expressions.

By default, denominators are not expanded, but this can be controlled by means of the switch `expandwrt_denom`.

This function is autoloaded from `simplification/stopex.mac`.

expandwrt_denom [Option variable]

Default value: `false`

`expandwrt_denom` controls the treatment of rational expressions by `expandwrt`. If `true`, then both the numerator and denominator of the expression will be expanded according to the arguments of `expandwrt`, but if `expandwrt_denom` is `false`, then only the numerator will be expanded in that way.

expandwrt_factored (*expr*, *x_1*, ..., *x_n*) [Function]

is similar to `expandwrt`, but treats expressions that are products somewhat differently. `expandwrt_factored` expands only on those factors of `expr` that contain the variables *x_1*, ..., *x_n*.

This function is autoloaded from `simplification/stopex.mac`.

expon [Option variable]

Default value: 0

`expon` is the exponent of the largest negative power which is automatically expanded (independent of calls to `expand`). For example, if `expon` is 4 then `(x+1)^(-5)` will not be automatically expanded.

exponentialize (*expr*) [Function]

exponentialize [Option variable]

The function `exponentialize (expr)` converts circular and hyperbolic functions in *expr* to exponentials, without setting the global variable `exponentialize`.

When the variable `exponentialize` is `true`, all circular and hyperbolic functions are converted to exponential form. The default value is `false`.

`demoivre` converts complex exponentials into circular functions. `exponentialize` and `demoivre` cannot both be true at the same time.

expop [Option variable]

Default value: 0

`expop` is the highest positive exponent which is automatically expanded. Thus `(x + 1)^3`, when typed, will be automatically expanded only if `expop` is greater than or equal to 3. If it is desired to have `(x + 1)^n` expanded where `n` is greater than `expop` then executing `expand ((x + 1)^n)` will work only if `maxposex` is not less than n.

`lassociative` [Property]

`declare (g, lassociative)` tells the Maxima simplifier that `g` is left-associative. E.g., `g (g (a, b), g (c, d))` will simplify to `g (g (g (a, b), c), d)`.

See also `rassociative`.

`linear` [Property]

One of Maxima's operator properties. For univariate `f` so declared, "expansion" `f(x + y)` yields `f(x) + f(y)`, `f(a*x)` yields `a*f(x)` takes place where `a` is a "constant". For functions of two or more arguments, "linearity" is defined to be as in the case of `sum` or `integrate`, i.e., `f (a*x + b, x)` yields `a*f(x,x) + b*f(1,x)` for `a` and `b` free of `x`.

Example:

```
(%i1) declare (f, linear);
(%o1)                           done
(%i2) f(x+y);
(%o2)                       f(y) + f(x)
(%i3) declare (a, constant);
(%o3)                           done
(%i4) f(a*x);
(%o4)                         a f(x)
```

`linear` is equivalent to `additive` and `outative`. See also `opproperties`.

Example:

```
(%i1) 'sum (F(k) + G(k), k, 1, inf);
                        inf
                        ____
                        \
(%o1)                    >    (G(k) + F(k))
                        /
                        ____
                        k = 1
(%i2) declare (nounify (sum), linear);
(%o2)                           done
(%i3) 'sum (F(k) + G(k), k, 1, inf);
                      inf          inf
                      ____         ____
                      \            \
(%o3)                  >    G(k) +  >    F(k)
                      /            /
                      ____         ____
                      k = 1        k = 1
```

`maxnegex` [Option variable]

Default value: 1000

`maxnegex` is the largest negative exponent which will be expanded by the `expand` command, see also `maxposex`.

`maxposex` [Option variable]

Default value: 1000

`maxposex` is the largest exponent which will be expanded with the `expand` command, see also `maxnegex`.

`multiplicative` [Property]

`declare(f, multiplicative)` tells the Maxima simplifier that `f` is multiplicative.

1. If `f` is univariate, whenever the simplifier encounters `f` applied to a product, `f` distributes over that product. E.g., `f(x*y)` simplifies to `f(x)*f(y)`. This simplification is not applied to expressions of the form `f('product(...))`.
2. If `f` is a function of 2 or more arguments, multiplicativity is defined as multiplicativity in the first argument to `f`, e.g., `f (g(x) * h(x), x)` simplifies to `f (g(x) ,x) * f (h(x), x)`.

`declare(nounify(product), multiplicative)` tells Maxima to simplify symbolic products.

Example:

```
(%i1) F2 (a * b * c);
(%o1)                          F2(a b c)
(%i2) declare (F2, multiplicative);
(%o2)                            done
(%i3) F2 (a * b * c);
(%o3)                    F2(a) F2(b) F2(c)
```

`declare(nounify(product), multiplicative)` tells Maxima to simplify symbolic products.

```
(%i1) product (a[i] * b[i], i, 1, n);
                                n
                              -----
                              |   |
(%o1)                         |   | a  b
                              |   |  i  i
                              i = 1
(%i2) declare (nounify (product), multiplicative);
(%o2)                            done
(%i3) product (a[i] * b[i], i, 1, n);
                             n         n
                           -----     -----
                           |   |     |   |
(%o3)                     (|   | a ) |   | b
                           |   |  i  |   |  i
                           i = 1     i = 1
```

multthru [Function]

multthru (*expr*)
multthru (*expr_1*, *expr_2*)

Multiplies a factor (which should be a sum) of *expr* by the other factors of *expr*. That is, *expr* is *f_1 f_2 ... f_n* where at least one factor, say *f_i*, is a sum of terms. Each term in that sum is multiplied by the other factors in the product. (Namely all the factors except *f_i*). multthru does not expand exponentiated sums. This function is the fastest way to distribute products (commutative or noncommutative) over sums. Since quotients are represented as products multthru can be used to divide sums by products as well.

multthru (*expr_1*, *expr_2*) multiplies each term in *expr_2* (which should be a sum or an equation) by *expr_1*. If *expr_1* is not itself a sum then this form is equivalent to multthru (*expr_1**expr_2*).

```
(%i1) x/(x-y)^2 - 1/(x-y) - f(x)/(x-y)^3;
                        1        x         f(x)
(%o1)               - ----- + -------- - --------
                      x - y          2          3
                                (x - y)    (x - y)
(%i2) multthru ((x-y)^3, %);
                                          2
(%o2)                x (x - y) - (x - y)  - f(x)
(%i3) ratexpand (%);
                               2
(%o3)                        - y  + x y - f(x)
(%i4) ((a+b)^10*s^2 + 2*a*b*s + (a*b)^2)/(a*b*s^2);
                           10  2              2  2
                  (b + a)   s  + 2 a b s + a  b
(%o4)             -----------------------------
                                    2
                                 a b s
(%i5) multthru (%);  /* note that this does not expand (b+a)^10 */
                                          10
                        2   a b   (b + a)
(%o5)                   - + --- + ---------
                        s    2       a b
                             s
(%o6)             a . f + a . c . (e + d) + a . b
(%i7) multthru (a.(b+c.(d+e)+f));
(%o7)          a . f + a . c . e + a . c . d + a . b
(%i8) expand (a.(b+c.(d+e)+f));
```

nary [Property]

declare(f, nary) tells Maxima to recognize the function f as an n-ary function.

The nary declaration is not the same as calling the [function_nary], page 156 function. The sole effect of declare(f, nary) is to instruct the Maxima simplifier to flatten nested expressions, for example, to simplify foo(x, foo(y, z)) to foo(x, y, z). See also declare.

Example:

```
(%i1) H (H (a, b), H (c, H (d, e)));
(%o1)               H(H(a, b), H(c, H(d, e)))
(%i2) declare (H, nary);
(%o2)                          done
(%i3) H (H (a, b), H (c, H (d, e)));
(%o3)                    H(a, b, c, d, e)
```

`negdistrib` [Option variable]

Default value: `true`

When `negdistrib` is `true`, -1 distributes over an expression. E.g., `-(x + y)` becomes `- y - x`. Setting it to `false` will allow `- (x + y)` to be displayed like that. This is sometimes useful but be very careful: like the `simp` flag, this is one flag you do not want to set to `false` as a matter of course or necessarily for other than local use in your Maxima.

Example:

```
(%i1) negdistrib;
(%o1)                           true
(%i2) -(x+y);
(%o2)                         - y - x
(%i3) negdistrib : not negdistrib ;
(%o3)                           false
(%i4) -(x+y);
(%o4)                        - (y + x)
```

`oppropertie`s [System variable]

`oppropertie`s is the list of the special operator properties recognized by the Maxima simplifier.

Items are added to the `oppropertie`s list by the function `define_oppropert`y.

Example:

```
(%i1) oppropertie;
(%o1) [linear, additive, multiplicative, outative, evenfun,
oddfun, commutative, symmetric, antisymmetric, nary,
lassociative, rassociative]
```

`define_oppropert`y (*property_name*, *simplifier_fn*) [Function]

Declares the symbol *property_name* to be an operator property, which is simplified by *simplifier_fn*, which may be the name of a Maxima or Lisp function or a lambda expression. After `define_oppropert`y is called, functions and operators may be declared to have the *property_name* property, and *simplifier_fn* is called to simplify them.

simplifier_fn must be a function of one argument, which is an expression in which the main operator is declared to have the *property_name* property.

simplifier_fn is called with the global flag `simp` disabled. Therefore *simplifier_fn* must be able to carry out its simplification without making use of the general simplifier.

`define_oppropert`y appends *property_name* to the global list `oppropertie`s.

`define_opproperty` returns `done`.

Example:

Declare a new property, `identity`, which is simplified by `simplify_identity`. Declare that `f` and `g` have the new property.

```
(%i1) define_opproperty (identity, simplify_identity);
(%o1)                          done
(%i2) simplify_identity(e) := first(e);
(%o2)           simplify_identity(e) := first(e)
(%i3) declare ([f, g], identity);
(%o3)                          done
(%i4) f(10 + t);
(%o4)                         t + 10
(%i5) g(3*u) - f(2*u);
(%o5)                            u
```

`outative` [Property]

`declare(f, outative)` tells the Maxima simplifier that constant factors in the argument of `f` can be pulled out.

1. If `f` is univariate, whenever the simplifier encounters `f` applied to a product, that product will be partitioned into factors that are constant and factors that are not and the constant factors will be pulled out. E.g., `f(a*x)` will simplify to `a*f(x)` where `a` is a constant. Non-atomic constant factors will not be pulled out.
2. If `f` is a function of 2 or more arguments, outativity is defined as in the case of `sum` or `integrate`, i.e., `f (a*g(x), x)` will simplify to `a * f(g(x), x)` for `a` free of `x`.

`sum`, `integrate`, and `limit` are all `outative`.

Example:

```
(%i1) F1 (100 * x);
(%o1)                        F1(100 x)
(%i2) declare (F1, outative);
(%o2)                          done
(%i3) F1 (100 * x);
(%o3)                        100 F1(x)
(%i4) declare (zz, constant);
(%o4)                          done
(%i5) F1 (zz * y);
(%o5)                        zz F1(y)
```

`radcan` (*expr*) [Function]

Simplifies *expr*, which can contain logs, exponentials, and radicals, by converting it into a form which is canonical over a large class of expressions and a given ordering of variables; that is, all functionally equivalent forms are mapped into a unique form. For a somewhat larger class of expressions, `radcan` produces a regular form. Two equivalent expressions in this class do not necessarily have the same appearance, but their difference can be simplified by `radcan` to zero.

For some expressions `radcan` is quite time consuming. This is the cost of exploring certain relationships among the components of the expression for simplifications based on factoring and partial-fraction expansions of exponents.

Examples:

```
(%i1) radcan((log(x+x^2)-log(x))^a/log(1+x)^(a/2));
                                        a/2
(%o1)                       log(x + 1)
(%i2) radcan((log(1+2*a^x+a^(2*x))/log(1+a^x)));
(%o2)                              2
(%i3) radcan((%e^x-1)/(1+%e^(x/2)));
                                 x/2
(%o3)                           %e    - 1
```

`radexpand` [Option variable]

Default value: `true`

`radexpand` controls some simplifications of radicals.

When `radexpand` is `all`, causes nth roots of factors of a product which are powers of n to be pulled outside of the radical. E.g. if `radexpand` is `all`, `sqrt (16*x^2)` simplifies to `4*x`.

More particularly, consider `sqrt (x^2)`.

- If `radexpand` is `all` or `assume (x > 0)` has been executed, `sqrt(x^2)` simplifies to `x`.
- If `radexpand` is `true` and `domain` is `real` (its default), `sqrt(x^2)` simplifies to `abs(x)`.
- If `radexpand` is `false`, or `radexpand` is `true` and `domain` is `complex`, `sqrt(x^2)` is not simplified.

Note that `domain` only matters when `radexpand` is `true`.

`rassociative` [Property]

`declare (g, rassociative)` tells the Maxima simplifier that `g` is right-associative. E.g., `g(g(a, b), g(c, d))` simplifies to `g(a, g(b, g(c, d)))`.

See also `lassociative`.

`resimplify (`*expr*`)` [Function]

Resimplifies the expression *expr* based on the current environment. This function is useful when the fact database, option variables, or tellsimp rules have changed since the expression was last simplified.

Example:

```
(%i1) expr : sin(x)^2 + cos(x)^2;
                              2         2
(%o1)                    sin (x) + cos (x)
(%i2) exponentialize : true;
(%o2)                            true
(%i3) expr;
                              2         2
(%o3)                    sin (x) + cos (x)
```

```
(%i4) resimplify(%);
                %i x      - %i x 2       %i x      - %i x 2
             (%e     + %e       )    (%e     - %e       )
(%o4)        -------------------- - --------------------
                      4                       4
(%i5) ratsimp(%);
(%o5)                              1
```

scsimp (*expr*, *rule_1*, ..., *rule_n*) [Function]

Sequential Comparative Simplification (method due to Stoute). `scsimp` attempts to simplify *expr* according to the rules *rule_1*, ..., *rule_n*. If a smaller expression is obtained, the process repeats. Otherwise after all simplifications are tried, it returns the original answer.

`example (scsimp)` displays some examples.

simp [Option variable]

Default value: `true`

`simp` enables simplification. This is the default. `simp` is also an `evflag`, which is recognized by the function `ev`. See ev.

When `simp` is used as an `evflag` with a value `false`, the simplification is suppressed only during the evaluation phase of an expression. The flag does not suppress the simplification which follows the evaluation phase.

Many Maxima functions and operations require simplification to be enabled to work normally. When simplification is disabled, many results will be incomplete, and in addition there may be incorrect results or program errors.

Examples:

The simplification is switched off globally. The expression `sin(1.0)` is not simplified to its numerical value. The `simp`-flag switches the simplification on.

```
(%i1) simp:false;
(%o1)                           false
(%i2) sin(1.0);
(%o2)                         sin(1.0)
(%i3) sin(1.0),simp;
(%o3)                  0.8414709848078965
```

The simplification is switched on again. The `simp`-flag cannot suppress the simplification completely. The output shows a simplified expression, but the variable `x` has an unsimplified expression as a value, because the assignment has occurred during the evaluation phase of the expression.

```
(%i1) simp:true;
(%o1)                           true
(%i2) x:sin(1.0),simp:false;
(%o2)                  0.8414709848078965
(%i3) :lisp $x
((%SIN) 1.0)
```

`symmetric` [Property]

`declare (h, symmetric)` tells the Maxima simplifier that `h` is a symmetric function. E.g., `h (x, z, y)` simplifies to `h (x, y, z)`.

`commutative` is synonymous with `symmetric`.

`xthru (`*expr*`)` [Function]

Combines all terms of *expr* (which should be a sum) over a common denominator without expanding products and exponentiated sums as `ratsimp` does. `xthru` cancels common factors in the numerator and denominator of rational expressions but only if the factors are explicit.

Sometimes it is better to use `xthru` before `ratsimp`ing an expression in order to cause explicit factors of the gcd of the numerator and denominator to be canceled thus simplifying the expression to be `ratsimp`ed.

Examples:

```
(%i1) ((x+2)^20 - 2*y)/(x+y)^20 + (x+y)^(-19) - x/(x+y)^20;
                                  20
                 1       (x + 2)   - 2 y       x
(%o1)        --------- + --------------- - ---------
                   19             20             20
             (y + x)        (y + x)        (y + x)
(%i2) xthru (%);
                                 20
                          (x + 2)   - y
(%o2)                     -------------
                                 20
                            (y + x)
```

12 Elementary Functions

12.1 Functions for Numbers

abs (*z*) [Function]

The abs function represents the mathematical absolute value function and works for both numerical and symbolic values. If the argument, *z*, is a real or complex number, abs returns the absolute value of *z*. If possible, symbolic expressions using the absolute value function are also simplified.

Maxima can differentiate, integrate and calculate limits for expressions containing abs. The abs_integrate package further extends Maxima's ability to calculate integrals involving the abs function. See (%i12) in the examples below.

When applied to a list or matrix, abs automatically distributes over the terms. Similarly, it distributes over both sides of an equation. To alter this behaviour, see the variable distribute_over.

See also cabs.

Examples:

Calculation of abs for real and complex numbers, including numerical constants and various infinities. The first example shows how abs distributes over the elements of a list.

```
(%i1) abs([-4, 0, 1, 1+%i]);
(%o1)                    [4, 0, 1, sqrt(2)]
(%i2) abs((1+%i)*(1-%i));
(%o2)                               2
(%i3) abs(%e+%i);
                                     2
(%o3)                         sqrt(%e  + 1)
(%i4) abs([inf, infinity, minf]);
(%o4)                       [inf, inf, inf]
```

Simplification of expressions containing abs:

```
(%i1) abs(x^2);
                                    2
(%o1)                               x
(%i2) abs(x^3);
                                 2
(%o2)                           x  abs(x)
(%i3) abs(abs(x));
(%o3)                            abs(x)
(%i4) abs(conjugate(x));
(%o4)                            abs(x)
```

Integrating and differentiating with the abs function. Note that more integrals involving the abs function can be performed, if the abs_integrate package is loaded. The last example shows the Laplace transform of abs: see laplace.

```
(%i1) diff(x*abs(x),x),expand;
(%o1)                          2 abs(x)
```

```
(%i2) integrate(abs(x),x);
                              x abs(x)
(%o2)                         --------
                                 2
(%i3) integrate(x*abs(x),x);
                             /
                             |
(%o3)                        | x abs(x) dx
                             |
                             /
(%i4) load("abs_integrate")$
(%i5) integrate(x*abs(x),x);
                              3
                             x  signum(x)
(%o5)                        ------------
                                 3
(%i6) integrate(abs(x),x,-2,%pi);
                                 2
                               %pi
(%o6)                          ---- + 2
                                2
(%i7) laplace(abs(x),x,s);
                                 1
(%o7)                            --
                                  2
                                 s
```

`ceiling` (*x*) [Function]

When *x* is a real number, return the least integer that is greater than or equal to *x*.

If *x* is a constant expression (`10 * %pi`, for example), `ceiling` evaluates *x* using big floating point numbers, and applies `ceiling` to the resulting big float. Because `ceiling` uses floating point evaluation, it's possible, although unlikely, that `ceiling` could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for `fpprec`.

For non-constant inputs, `ceiling` tries to return a simplified value. Here are examples of the simplifications that `ceiling` knows about:

```
(%i1) ceiling (ceiling (x));
(%o1)                        ceiling(x)
(%i2) ceiling (floor (x));
(%o2)                         floor(x)
(%i3) declare (n, integer)$
(%i4) [ceiling (n), ceiling (abs (n)), ceiling (max (n, 6))];
(%o4)                  [n, abs(n), max(6, n)]
(%i5) assume (x > 0, x < 1)$
(%i6) ceiling (x);
(%o6)                             1
```

```
(%i7) tex (ceiling (a));
$$\left \lceil a \right \rceil$$
(%o7)                          false
```

The `ceiling` function distributes over lists, matrices and equations. See `distribute_over`.

Finally, for all inputs that are manifestly complex, `ceiling` returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be `integervalued`. Both the `ceiling` and `floor` functions can use this information; for example:

```
(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
(%o2)                           f(x)
(%i3) ceiling (f(x) - 1);
(%o3)                        f(x) - 1
```

Example use:

```
(%i1) unitfrac(r) := block([uf : [], q],
    if not(ratnump(r)) then
       error("unitfrac: argument must be a rational number"),
    while r # 0 do (
        uf : cons(q : 1/ceiling(1/r), uf),
        r : r - q),
    reverse(uf));
(%o1) unitfrac(r) := block([uf : [], q],
if not ratnump(r) then error("unitfrac: argument must be a rational number"
                                        1
), while r # 0 do (uf : cons(q : ----------, uf), r : r - q),
                                        1
                                   ceiling(-)
                                        r

reverse(uf))
(%i2) unitfrac (9/10);
                             1  1  1
(%o2)                       [-, -, --]
                             2  3  15
(%i3) apply ("+", %);
                                9
(%o3)                           --
                                10
(%i4) unitfrac (-9/10);
                                   1
(%o4)                       [- 1, --]
                                   10
(%i5) apply ("+", %);
                                 9
(%o5)                          - --
                                 10
```

```
(%i6) unitfrac (36/37);
                          1  1  1  1    1
(%o6)                    [-, -, -, --, ----]
                          2  3  8  69  6808
(%i7) apply ("+", %);
                                36
(%o7)                           --
                                37
```

`entier` (*x*) [Function]

Returns the largest integer less than or equal to *x* where *x* is numeric. `fix` (as in `fixnum`) is a synonym for this, so `fix(x)` is precisely the same.

`floor` (*x*) [Function]

When *x* is a real number, return the largest integer that is less than or equal to *x*.

If *x* is a constant expression (`10 * %pi`, for example), `floor` evaluates *x* using big floating point numbers, and applies `floor` to the resulting big float. Because `floor` uses floating point evaluation, it's possible, although unlikely, that `floor` could return an erroneous value for constant inputs. To guard against errors, the floating point evaluation is done using three values for `fpprec`.

For non-constant inputs, `floor` tries to return a simplified value. Here are examples of the simplifications that `floor` knows about:

```
(%i1) floor (ceiling (x));
(%o1)                       ceiling(x)
(%i2) floor (floor (x));
(%o2)                        floor(x)
(%i3) declare (n, integer)$
(%i4) [floor (n), floor (abs (n)), floor (min (n, 6))];
(%o4)                 [n, abs(n), min(6, n)]
(%i5) assume (x > 0, x < 1)$
(%i6) floor (x);
(%o6)                            0
(%i7) tex (floor (a));
$$\left \lfloor a \right \rfloor$$
(%o7)                          false
```

The `floor` function distributes over lists, matrices and equations. See `distribute_over`.

Finally, for all inputs that are manifestly complex, `floor` returns a noun form.

If the range of a function is a subset of the integers, it can be declared to be `integervalued`. Both the `ceiling` and `floor` functions can use this information; for example:

```
(%i1) declare (f, integervalued)$
(%i2) floor (f(x));
(%o2)                          f(x)
(%i3) ceiling (f(x) - 1);
(%o3)                       f(x) - 1
```

`fix (x)` [Function]
A synonym for `entier (x)`.

`hstep (x)` [Function]
The Heaviside unit step function, equal to 0 if x is negative, equal to 1 if x is positive and equal to $1/2$ if x is equal to zero.

If you want a unit step function that takes on the value of 0 at x equal to zero, use `unit_step`.

`lmax (L)` [Function]
When L is a list or a set, return `apply ('max, args (L))`. When L is not a list or a set, signal an error. See also `lmin` and `max`.

`lmin (L)` [Function]
When L is a list or a set, return `apply ('min, args (L))`. When L is not a list or a set, signal an error. See also `lmax` and `min`.

`max (x_1, ..., x_n)` [Function]
Return a simplified value for the numerical maximum of the expressions *x_1* through *x_n*. For an empty argument list, `max` yields `minf`.

The option variable `maxmin_effort` controls which simplification methods are applied. Using the default value of *twelve* for `maxmin_effort`, `max` uses *all* available simplification methods. To to inhibit all simplifications, set `maxmin_effort` to zero.

When `maxmin_effort` is one or more, for an explicit list of real numbers, `max` returns a number.

Unless `max` needs to simplify a lengthy list of expressions, we suggest using the default value of `maxmin_effort`. Setting `maxmin_effort` to zero (no simplifications), will cause problems for some Maxima functions; accordingly, generally `maxmin_effort` should be nonzero.

See also `min`, `lmax`., and `lmin`..

Examples:

In the first example, setting `maxmin_effort` to zero suppresses simplifications.

```
(%i1) block([maxmin_effort : 0], max(1,2,x,x, max(a,b)));
(%o1) max(1,2,max(a,b),x,x)

(%i2) block([maxmin_effort : 1], max(1,2,x,x, max(a,b)));
(%o2) max(2,a,b,x)
```

When `maxmin_effort` is two or more, `max` compares pairs of members:

```
(%i1) block([maxmin_effort : 1], max(x,x+1,x+3));
(%o1) max(x,x+1,x+3)

(%i2) block([maxmin_effort : 2], max(x,x+1,x+3));
(%o2) x+3
```

Finally, when `maxmin_effort` is three or more, `max` compares triples members and excludes those that are in between; for example

```
(%i1) block([maxmin_effort : 4], max(x, 2*x, 3*x, 4*x));
(%o1) max(x,4*x)
```

`min` (*x_1*, ..., *x_n*) [Function]

Return a simplified value for the numerical minimum of the expressions *x_1* through *x_n*. For an empty argument list, `minf` yields `inf`.

The option variable `maxmin_effort` controls which simplification methods are applied. Using the default value of *twelve* for `maxmin_effort`, `max` uses *all* available simplification methods. To to inhibit all simplifications, set `maxmin_effort` to zero.

When `maxmin_effort` is one or more, for an explicit list of real numbers, `min` returns a number.

Unless `min` needs to simplify a lengthy list of expressions, we suggest using the default value of `maxmin_effort`. Setting `maxmin_effort` to zero (no simplifications), will cause problems for some Maxima functions; accordingly, generally `maxmin_effort` should be nonzero.

See also `max`, `lmax`., and `lmin`..

Examples:

In the first example, setting `maxmin_effort` to zero suppresses simplifications.

```
(%i1) block([maxmin_effort : 0], min(1,2,x,x, min(a,b)));
(%o1) min(1,2,a,b,x,x)

(%i2) block([maxmin_effort : 1], min(1,2,x,x, min(a,b)));
(%o2) min(1,a,b,x)
```

When `maxmin_effort` is two or more, `min` compares pairs of members:

```
(%i1) block([maxmin_effort : 1], min(x,x+1,x+3));
(%o1) min(x,x+1,x+3)

(%i2) block([maxmin_effort : 2], min(x,x+1,x+3));
(%o2) x
```

Finally, when `maxmin_effort` is three or more, `min` compares triples members and excludes those that are in between; for example

```
(%i1) block([maxmin_effort : 4], min(x, 2*x, 3*x, 4*x));
(%o1) max(x,4*x)
```

`round` (*x*) [Function]

When *x* is a real number, returns the closest integer to *x*. Multiples of 1/2 are rounded to the nearest even integer. Evaluation of *x* is similar to `floor` and `ceiling`.

The `round` function distributes over lists, matrices and equations. See `distribute_over`.

`signum` (*x*) [Function]

For either real or complex numbers *x*, the signum function returns 0 if *x* is zero; for a nonzero numeric input *x*, the signum function returns `x/abs(x)`.

For non-numeric inputs, Maxima attempts to determine the sign of the input. When the sign is negative, zero, or positive, `signum` returns -1,0, 1, respectively. For all other values for the sign, `signum` a simplified but equivalent form. The simplifications include reflection (`signum(-x)` gives `-signum(x)`) and multiplicative identity (`signum(x*y)` gives `signum(x) * signum(y)`).

The `signum` function distributes over a list, a matrix, or an equation. See `sign` and `distribute_over`.

`truncate` (*x*) [Function]

When *x* is a real number, return the closest integer to *x* not greater in absolute value than *x*. Evaluation of *x* is similar to `floor` and `ceiling`.

The `truncate` function distributes over lists, matrices and equations. See `distribute_over`.

12.2 Functions for Complex Numbers

`cabs` (*expr*) [Function]

Calculates the absolute value of an expression representing a complex number. Unlike the function `abs`, the `cabs` function always decomposes its argument into a real and an imaginary part. If `x` and `y` represent real variables or expressions, the `cabs` function calculates the absolute value of `x + %i*y` as

```
(%i1) cabs (1);
(%o1)                           1
(%i2) cabs (1 + %i);
(%o2)                        sqrt(2)
(%i3) cabs (exp (%i));
(%o3)                           1
(%i4) cabs (exp (%pi * %i));
(%o4)                           1
(%i5) cabs (exp (3/2 * %pi * %i));
(%o5)                           1
(%i6) cabs (17 * exp (2 * %i));
(%o6)                          17
```

If `cabs` returns a noun form this most commonly is caused by some properties of the variables involved not being known:

```
(%i1) cabs (a+%i*b);
                                  2    2
(%o1)                       sqrt(b  + a )
(%i2) declare(a,real,b,real);
(%o2)                          done
(%i3) cabs (a+%i*b);
                                  2    2
(%o3)                       sqrt(b  + a )
(%i4) assume(a>0,b>0);
(%o4)                      [a > 0, b > 0]
(%i5) cabs (a+%i*b);
                                  2    2
(%o5)                       sqrt(b  + a )
```

The `cabs` function can use known properties like symmetry properties of complex functions to help it calculate the absolute value of an expression. If such identities exist, they can be advertised to `cabs` using function properties. The symmetries that

`cabs` understands are: mirror symmetry, conjugate function and complex characteristic.

`cabs` is a verb function and is not suitable for symbolic calculations. For such calculations (including integration, differentiation and taking limits of expressions containing absolute values), use `abs`.

The result of `cabs` can include the absolute value function, `abs`, and the arc tangent, `atan2`.

When applied to a list or matrix, `cabs` automatically distributes over the terms. Similarly, it distributes over both sides of an equation.

For further ways to compute with complex numbers, see the functions `rectform`, `realpart`, `imagpart`, `carg`, `conjugate` and `polarform`.

Examples:

Examples with `sqrt` and `sin`.

```
(%i1) cabs(sqrt(1+%i*x));
                                2     1/4
(%o1)                         (x  + 1)
(%i2) cabs(sin(x+%i*y));
                     2         2         2         2
(%o2)       sqrt(cos (x) sinh (y) + sin (x) cosh (y))
```

The error function, `erf`, has mirror symmetry, which is used here in the calculation of the absolute value with a complex argument:

```
(%i1) cabs(erf(x+%i*y));
                                                 2
           (erf(%i y + x) - erf(%i y - x))
(%o1) sqrt(-------------------------------
                          4
                                                              2
                              (- erf(%i y + x) - erf(%i y - x))
                            - --------------------------------)
                                              4
```

Maxima knows complex identities for the Bessel functions, which allow it to compute the absolute value for complex arguments. Here is an example for `bessel_j`.

```
(%i1) cabs(bessel_j(1,%i));
(%o1)                    bessel_i(1, 1)
```

`carg` (*z*) [Function]

Returns the complex argument of *z*. The complex argument is an angle `theta` in `(-%pi, %pi]` such that `r exp (theta %i) = z` where `r` is the magnitude of *z*.

`carg` is a computational function, not a simplifying function.

See also `abs` (complex magnitude), `polarform`, `rectform`, `realpart`, and `imagpart`.

Examples:

```
(%i1) carg (1);
(%o1)                           0
```

```
(%i2) carg (1 + %i);
                                %pi
(%o2)                           ---
                                 4
(%i3) carg (exp (%i));
                                sin(1)
(%o3)                     atan(------)
                                cos(1)
(%i4) carg (exp (%pi * %i));
(%o4)                           %pi
(%i5) carg (exp (3/2 * %pi * %i));
                                 %pi
(%o5)                          - ---
                                  2
(%i6) carg (17 * exp (2 * %i));
                               sin(2)
(%o6)                  atan(------) + %pi
                               cos(2)
```

If `carg` returns a noun form this most commonly is caused by some properties of the variables involved not being known:

```
(%i1) carg (a+%i*b);
(%o1)                       atan2(b, a)
(%i2) declare(a,real,b,real);
(%o2)                          done
(%i3) carg (a+%i*b);
(%o3)                       atan2(b, a)
(%i4) assume(a>0,b>0);
(%o4)                     [a > 0, b > 0]
(%i5) carg (a+%i*b);
                                  b
(%o5)                        atan(-)
                                  a
```

`conjugate` (*x*) [Function]

Returns the complex conjugate of *x*.

```
(%i1) declare ([aa, bb], real, cc, complex, ii, imaginary);
(%o1)                          done
(%i2) conjugate (aa + bb*%i);
(%o2)                       aa - %i bb
(%i3) conjugate (cc);
(%o3)                      conjugate(cc)
(%i4) conjugate (ii);
(%o4)                          - ii
(%i5) conjugate (xx + yy);
(%o5)                        yy + xx
```

`imagpart` (*expr*) [Function]

Returns the imaginary part of the expression *expr*.

`imagpart` is a computational function, not a simplifying function.

See also `abs`, `carg`, `polarform`, `rectform`, and `realpart`.

Example:

```
(%i1) imagpart (a+b*%i);
(%o1)                           b
(%i2) imagpart (1+sqrt(2)*%i);
(%o2)                         sqrt(2)
(%i3) imagpart (1);
(%o3)                           0
(%i4) imagpart (sqrt(2)*%i);
(%o4)                         sqrt(2)
```

`polarform` (*expr*) [Function]

Returns an expression `r %e^(%i theta)` equivalent to *expr*, such that `r` and `theta` are purely real.

Example:

```
(%i1) polarform(a+b*%i);
                     %i atan2(b, a)       2    2
(%o1)             %e                  sqrt(b  + a )
(%i2) polarform(1+%i);
                                       %i %pi
                                       ------
                                         4
(%o2)                     sqrt(2) %e
(%i3) polarform(1+2*%i);
                                       %i atan(2)
(%o3)                  sqrt(5) %e
```

`realpart` (*expr*) [Function]

Returns the real part of *expr*. `realpart` and `imagpart` will work on expressions involving trigonometric and hyperbolic functions, as well as square root, logarithm, and exponentiation.

Example:

```
(%i1) realpart (a+b*%i);
(%o1)                           a
(%i2) realpart (1+sqrt(2)*%i);
(%o2)                           1
(%i3) realpart (sqrt(2)*%i);
(%o3)                           0
(%i4) realpart (1);
(%o4)                           1
```

`rectform` (*expr*) [Function]

Returns an expression `a + b %i` equivalent to *expr*, such that *a* and *b* are purely real.

Example:

```
(%i1) rectform(sqrt(2)*%e^(%i*%pi/4));
(%o1)                          %i + 1
(%i2) rectform(sqrt(b^2+a^2)*%e^(%i*atan2(b, a)));
(%o2)                         %i b + a
(%i3) rectform(sqrt(5)*%e^(%i*atan(2)));
(%o3)                         2 %i + 1
```

12.3 Combinatorial Functions

`!!` [Operator]

The double factorial operator.

For an integer, float, or rational number `n`, `n!!` evaluates to the product `n (n-2) (n-4) (n-6) ... (n - 2 (k-1))` where `k` is equal to `entier (n/2)`, that is, the largest integer less than or equal to `n/2`. Note that this definition does not coincide with other published definitions for arguments which are not integers.

For an even (or odd) integer `n`, `n!!` evaluates to the product of all the consecutive even (or odd) integers from 2 (or 1) through `n` inclusive.

For an argument `n` which is not an integer, float, or rational, `n!!` yields a noun form `genfact (n, n/2, 2)`.

`binomial` (*x*, *y*) [Function]

The binomial coefficient `x!/(y! (x - y)!)`. If *x* and *y* are integers, then the numerical value of the binomial coefficient is computed. If *y*, or *x* - *y*, is an integer, the binomial coefficient is expressed as a polynomial.

Examples:

```
(%i1) binomial (11, 7);
(%o1)                           330
(%i2) 11! / 7! / (11 - 7)!;
(%o2)                           330
(%i3) binomial (x, 7);
        (x - 6) (x - 5) (x - 4) (x - 3) (x - 2) (x - 1) x
(%o3)   ------------------------------------------------
                              5040
(%i4) binomial (x + 7, x);
      (x + 1) (x + 2) (x + 3) (x + 4) (x + 5) (x + 6) (x + 7)
(%o4) -------------------------------------------------------
                              5040
(%i5) binomial (11, y);
(%o5)                    binomial(11, y)
```

`factcomb` (*expr*) [Function]

Tries to combine the coefficients of factorials in *expr* with the factorials themselves by converting, for example, `(n + 1)*n!` into `(n + 1)!`.

`sumsplitfact` if set to `false` will cause `minfactorial` to be applied after a `factcomb`.

Example:

```
(%i1) sumsplitfact;
(%o1)                          true
(%i2) (n + 1)*(n + 1)*n!;
                                      2
(%o2)                        (n + 1)  n!
(%i3) factcomb (%);
(%o3)                   (n + 2)! - (n + 1)!
(%i4) sumsplitfact: not sumsplitfact;
(%o4)                          false
(%i5) (n + 1)*(n + 1)*n!;
                                      2
(%o5)                        (n + 1)  n!
(%i6) factcomb (%);
(%o6)                 n (n + 1)! + (n + 1)!
```

`factorial` [Function]
`!` [Operator]

Represents the factorial function. Maxima treats `factorial (x)` the same as `x!`.

For any complex number `x`, except for negative integers, `x!` is defined as `gamma(x+1)`.

For an integer `x`, `x!` simplifies to the product of the integers from 1 to `x` inclusive. `0!` simplifies to 1. For a real or complex number in float or bigfloat precision `x`, `x!` simplifies to the value of `gamma (x+1)`. For `x` equal to `n/2` where `n` is an odd integer, `x!` simplifies to a rational factor times `sqrt (%pi)` (since `gamma (1/2)` is equal to `sqrt (%pi)`).

The option variables `factlim` and `gammalim` control the numerical evaluation of factorials for integer and rational arguments. The functions `minfactorial` and `factcomb` simplifies expressions containing factorials.

The functions `gamma`, `bffac`, and `cbffac` are varieties of the gamma function. `bffac` and `cbffac` are called internally by `gamma` to evaluate the gamma function for real and complex numbers in bigfloat precision.

`makegamma` substitutes `gamma` for factorials and related functions.

Maxima knows the derivative of the factorial function and the limits for specific values like negative integers.

The option variable `factorial_expand` controls the simplification of expressions like `(n+x)!`, where `n` is an integer.

See also `binomial`.

The factorial of an integer is simplified to an exact number unless the operand is greater than `factlim`. The factorial for real and complex numbers is evaluated in float or bigfloat precision.

```
(%i1) factlim : 10;
(%o1)                           10
(%i2) [0!, (7/2)!, 8!, 20!];
                          105 sqrt(%pi)
(%o2)              [1, -------------, 40320, 20!]
                               16
```

```
(%i3) [4,77!, (1.0+%i)!];
(%o3) [4, 77!, 0.3430658398165453 %i + 0.6529654964201667]
(%i4) [2.86b0!, (1.0b0+%i)!];
(%o4) [5.046635586910012b0, 3.430658398165454b-1 %i
                                          + 6.529654964201667b-1]
```

The factorial of a known constant, or general expression is not simplified. Even so it may be possible to simplify the factorial after evaluating the operand.

```
(%i1) [(%i + 1)!, %pi!, %e!, (cos(1) + sin(1))!];
(%o1)      [(%i + 1)!, %pi!, %e!, (sin(1) + cos(1))!]
(%i2) ev (%, numer, %enumer);
(%o2) [0.3430658398165453 %i + 0.6529654964201667,
         7.188082728976031, 4.260820476357003, 1.227580202486819]
```

Factorials are simplified, not evaluated. Thus `x!` may be replaced even in a quoted expression.

```
(%i1) '([0!, (7/2)!, 4.77!, 8!, 20!]);
           105 sqrt(%pi)
(%o1) [1, -------------, 81.44668037931197, 40320,
                 16
                                              2432902008176640000]
```

Maxima knows the derivative of the factorial function.

```
(%i1) diff(x!,x);
(%o1)                     x! psi (x + 1)
                                 0
```

The option variable `factorial_expand` controls expansion and simplification of expressions with the factorial function.

```
(%i1) (n+1)!/n!,factorial_expand:true;
(%o1)                          n + 1
```

`factlim` [Option variable]

Default value: 100000

`factlim` specifies the highest factorial which is automatically expanded. If it is -1 then all integers are expanded.

`factorial_expand` [Option variable]

Default value: false

The option variable `factorial_expand` controls the simplification of expressions like `(x+n)!`, where `n` is an integer. See `factorial` for an example.

`genfact` (*x*, *y*, *z*) [Function]

Returns the generalized factorial, defined as `x (x-z) (x - 2 z) ... (x - (y - 1) z)`. Thus, when *x* is an integer, `genfact (x, x, 1) = x!` and `genfact (x, x/2, 2) = x!!`.

`minfactorial` (*expr*) [Function]

Examines *expr* for occurrences of two factorials which differ by an integer. `minfactorial` then turns one into a polynomial times the other.

```
(%i1) n!/(n+2)!;
                                   n!
(%o1)                          --------
                               (n + 2)!
(%i2) minfactorial (%);
                                    1
(%o2)                     ---------------
                          (n + 1) (n + 2)
```

sumsplitfact [Option variable]

Default value: true

When sumsplitfact is false, minfactorial is applied after a factcomb.

```
(%i1) sumsplitfact;
(%o1)                           true
(%i2) n!/(n+2)!;
                                   n!
(%o2)                          --------
                               (n + 2)!
(%i3) factcomb(%);
                                   n!
(%o3)                          --------
                               (n + 2)!
(%i4) sumsplitfact: not sumsplitfact ;
(%o4)                           false
(%i5) n!/(n+2)!;
                                   n!
(%o5)                          --------
                               (n + 2)!
(%i6) factcomb(%);
                                    1
(%o6)                     ---------------
                          (n + 1) (n + 2)
```

12.4 Root, Exponential and Logarithmic Functions

%e_to_numlog [Option variable]

Default value: false

When true, r some rational number, and x some expression, %e^(r*log(x)) will be simplified into x^r . It should be noted that the radcan command also does this transformation, and more complicated transformations of this ilk as well. The logcontract command "contracts" expressions containing log.

%emode [Option variable]

Default value: true

When %emode is true, %e^(%pi %i x) is simplified as follows.

%e^(%pi %i x) simplifies to cos (%pi x) + %i sin (%pi x) if x is a floating point number, an integer, or a multiple of 1/2, 1/3, 1/4, or 1/6, and then further simplified.

For other numerical `x`, `%e^(%pi %i x)` simplifies to `%e^(%pi %i y)` where `y` is `x - 2 k` for some integer `k` such that `abs(y) < 1`.

When `%emode` is `false`, no special simplification of `%e^(%pi %i x)` is carried out.

```
(%i1) %emode;
(%o1)                          true
(%i2) %e^(%pi*%i*1);
(%o2)                           - 1
(%i3) %e^(%pi*%i*216/144);
(%o3)                          - %i
(%i4) %e^(%pi*%i*192/144);
                            sqrt(3) %i   1
(%o4)                     - ---------- - -
                                2        2
(%i5) %e^(%pi*%i*180/144);
                             %i         1
(%o5)                    - ------- - -------
                           sqrt(2)   sqrt(2)
(%i6) %e^(%pi*%i*120/144);
                            %i   sqrt(3)
(%o6)                       -- - -------
                            2       2
(%i7) %e^(%pi*%i*121/144);
                              121 %i %pi
                              ----------
                                 144
(%o7)                        %e
```

`%enumer` [Option variable]

Default value: `false`

When `%enumer` is `true`, `%e` is replaced by its numeric value 2.718... whenever `numer` is `true`.

When `%enumer` is `false`, this substitution is carried out only if the exponent in `%e^x` evaluates to a number.

See also `ev` and `numer`.

```
(%i1) %enumer;
(%o1)                          false
(%i2) numer;
(%o2)                          false
(%i3) 2*%e;
(%o3)                          2 %e
(%i4) %enumer: not %enumer;
(%o4)                          true
(%i5) 2*%e;
(%o5)                          2 %e
(%i6) numer: not numer;
(%o6)                          true
```

```
(%i7) 2*%e;
(%o7)                    5.43656365691809
(%i8) 2*%e^1;
(%o8)                    5.43656365691809
(%i9) 2*%e^x;
                                          x
(%o9)                 2 2.718281828459045
```

`exp` (*x*) [Function]

Represents the exponential function. Instances of `exp (x)` in input are simplified to `%e^x`; `exp` does not appear in simplified expressions.

`demoivre` if `true` causes `%e^(a + b %i)` to simplify to `%e^(a (cos(b) + %i sin(b)))` if `b` is free of `%i`. See `demoivre`.

`%emode`, when `true`, causes `%e^(%pi %i x)` to be simplified. See `%emode`.

`%enumer`, when `true` causes `%e` to be replaced by 2.718... whenever `numer` is `true`. See `%enumer`.

```
(%i1) demoivre;
(%o1)                          false
(%i2) %e^(a + b*%i);
                                 %i b + a
(%o2)                          %e
(%i3) demoivre: not demoivre;
(%o3)                          true
(%i4) %e^(a + b*%i);
                           a
(%o4)                    %e  (%i sin(b) + cos(b))
```

`li` [*s*] (*z*) [Function]

Represents the polylogarithm function of order s and argument z, defined by the infinite series

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

`li[1](z)` is $-\log(1-z)$. `li[2]` and `li[3]` are the dilogarithm and trilogarithm functions, respectively.

When the order is 1, the polylogarithm simplifies to `- log (1 - z)`, which in turn simplifies to a numerical value if z is a real or complex floating point number or the `numer` evaluation flag is present.

When the order is 2 or 3, the polylogarithm simplifies to a numerical value if z is a real floating point number or the `numer` evaluation flag is present.

Examples:

RETRIEVE: End of file encountered. – an error. To debug this try: debugmode(true);

```
(%i1) assume (x > 0);
(%o1)                         [x > 0]
```

```
(%i2) integrate ((log (1 - t)) / t, t, 0, x);
Is x - 1 positive, negative or zero?

Is x - 1 positive, negative or zero?
li[4](1);
Is x - 1 positive, negative or zero?
li[5](1);
Is x - 1 positive, negative or zero?
li[2](1/2);
Is x - 1 positive, negative or zero?
li[2](%i);
Is x - 1 positive, negative or zero?
li[2](1+%i);
Is x - 1 positive, negative or zero?
li [2] (7);
Is x - 1 positive, negative or zero?
li [2] (7), numer;
Is x - 1 positive, negative or zero?
li [3] (7);
Is x - 1 positive, negative or zero?
li [2] (7), numer;
Is x - 1 positive, negative or zero?
L : makelist (i / 4.0, i, 0, 8);
Is x - 1 positive, negative or zero?
map (lambda ([x], li [2] (x)), L);
Is x - 1 positive, negative or zero?
map (lambda ([x], li [3] (x)), L);
```

`log` (*x*) [Function]

Represents the natural (base *e*) logarithm of *x*.

Maxima does not have a built-in function for the base 10 logarithm or other bases. `log10(x) := log(x) / log(10)` is a useful definition.

Simplification and evaluation of logarithms is governed by several global flags:

`logexpand`
: causes `log(a^b)` to become `b*log(a)`. If it is set to `all`, `log(a*b)` will also simplify to `log(a)+log(b)`. If it is set to `super`, then `log(a/b)` will also simplify to `log(a)-log(b)` for rational numbers `a/b`, `a#1`. (`log(1/b)`, for `b` integer, always simplifies.) If it is set to `false`, all of these simplifications will be turned off.

`logsimp`
: if `false` then no simplification of `%e` to a power containing `log`'s is done.

`lognegint`
: if `true` implements the rule `log(-n) -> log(n)+%i*%pi` for `n` a positive integer.

`%e_to_numlog`
: when `true`, `r` some rational number, and `x` some expression, the expression `%e^(r*log(x))` will be simplified into `x^r`. It should be noted that

the `radcan` command also does this transformation, and more complicated transformations of this as well. The `logcontract` command "contracts" expressions containing `log`.

`logabs` [Option variable]

Default value: `false`

When doing indefinite integration where logs are generated, e.g. `integrate(1/x,x)`, the answer is given in terms of `log(abs(...))` if `logabs` is `true`, but in terms of `log(...)` if `logabs` is `false`. For definite integration, the `logabs:true` setting is used, because here "evaluation" of the indefinite integral at the endpoints is often needed.

`logarc` (*expr*) [Function]

The function `logarc(`*expr*`)` carries out the replacement of inverse circular and hyperbolic functions with equivalent logarithmic functions for an expression *expr* without setting the global variable `logarc`.

`logarc` [Option variable]

When the global variable `logarc` is `true`, inverse circular and hyperbolic functions are replaced by equivalent logarithmic functions. The default value of `logarc` is `false`.

`logconcoeffp` [Option variable]

Default value: `false`

Controls which coefficients are contracted when using `logcontract`. It may be set to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do `logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$` . Then `logcontract(1/2*log(x));` will give `log(sqrt(x))`.

`logcontract` (*expr*) [Function]

Recursively scans the expression *expr*, transforming subexpressions of the form `a1*log(b1) + a2*log(b2) + c` into `log(ratsimp(b1^a1 * b2^a2)) + c`

```
(%i1) 2*(a*log(x) + 2*a*log(y))$
(%i2) logcontract(%);
                                    2  4
(%o2)                        a log(x  y )
```

The declaration `declare(n,integer)` causes `logcontract(2*a*n*log(x))` to simplify to `a*log(x^(2*n))`. The coefficients that "contract" in this manner are those such as the 2 and the `n` here which satisfy `featurep(coeff,integer)`. The user can control which coefficients are contracted by setting the option `logconcoeffp` to the name of a predicate function of one argument. E.g. if you like to generate SQRTs, you can do `logconcoeffp:'logconfun$ logconfun(m):=featurep(m,integer) or ratnump(m)$` . Then `logcontract(1/2*log(x));` will give `log(sqrt(x))`.

`logexpand` [Option variable]

Default value: `true`

If `true`, that is the default value, causes `log(a^b)` to become `b*log(a)`. If it is set to `all`, `log(a*b)` will also simplify to `log(a)+log(b)`. If it is set to `super`, then `log(a/b)` will also simplify to `log(a)-log(b)` for rational numbers `a/b`, `a#1`.

(`log(1/b)`, for integer `b`, always simplifies.) If it is set to `false`, all of these simplifications will be turned off.

When `logexpand` is set to `all` or `super`, the logarithm of a product expression simplifies to a summation of logarithms.

Examples:

When `logexpand` is `true`, `log(a^b)` simplifies to `b*log(a)`.

```
(%i1) log(n^2), logexpand=true;
(%o1)                         2 log(n)
```

When `logexpand` is `all`, `log(a*b)` simplifies to `log(a)+log(b)`.

```
(%i1) log(10*x), logexpand=all;
(%o1)                     log(x) + log(10)
```

When `logexpand` is `super`, `log(a/b)` simplifies to `log(a)-log(b)` for rational numbers `a/b` with `a#1`.

```
(%i1) log(a/(n + 1)), logexpand=super;
(%o1)                    log(a) - log(n + 1)
```

When `logexpand` is set to `all` or `super`, the logarithm of a product expression simplifies to a summation of logarithms.

```
(%i1) my_product : product (X(i), i, 1, n);
                               n
                             -----
                             |   |
(%o1)                        |   | X(i)
                             |   |
                             i = 1
(%i2) log(my_product), logexpand=all;
                              n
                             ____
                             \
(%o2)                         >    log(X(i))
                             /
                             ----
                             i = 1
(%i3) log(my_product), logexpand=super;
                              n
                             ____
                             \
(%o3)                         >    log(X(i))
                             /
                             ----
                             i = 1
```

When `logexpand` is `false`, these simplifications are disabled.

```
(%i1) logexpand : false $
(%i2) log(n^2);
                                    2
(%o2)                         log(n )
```

```
(%i3) log(10*x);
(%o3)                          log(10 x)
(%i4) log(a/(n + 1));
                                    a
(%o4)                         log(-----)
                                  n + 1
(%i5) log ('product (X(i), i, 1, n));
                                   n
                                 -----
                                 |   |
(%o5)                      log(  |   | X(i))
                                 |   |
                                 i = 1
```

`lognegint` [Option variable]

Default value: `false`

If `true` implements the rule `log(-n) -> log(n)+%i*%pi` for `n` a positive integer.

`logsimp` [Option variable]

Default value: `true`

If `false` then no simplification of `%e` to a power containing `log`'s is done.

`plog` (*x*) [Function]

Represents the principal branch of the complex-valued natural logarithm with `-%pi < carg(x) <= +%pi` .

`sqrt` (*x*) [Function]

The square root of *x*. It is represented internally by `x^(1/2)`. See also `rootscontract` and `radexpand`.

12.5 Trigonometric Functions

12.5.1 Introduction to Trigonometric

Maxima has many trigonometric functions defined. Not all trigonometric identities are programmed, but it is possible for the user to add many of them using the pattern matching capabilities of the system. The trigonometric functions defined in Maxima are: `acos`, `acosh`, `acot`, `acoth`, `acsc`, `acsch`, `asec`, `asech`, `asin`, `asinh`, `atan`, `atanh`, `cos`, `cosh`, `cot`, `coth`, `csc`, `csch`, `sec`, `sech`, `sin`, `sinh`, `tan`, and `tanh`. There are a number of commands especially for handling trigonometric functions, see `trigexpand`, `trigreduce`, and the switch `trigsign`. Three share packages extend the simplification rules built into Maxima, `ntrig`, `atrig1` and Section 99.1 [Introduction to trigtools], page 1345. Do `describe(`*`command`*`)` for details.

12.5.2 Functions and Variables for Trigonometric

12.5.2.1 Trigonometric and Hyperbolic Functions

`acos` (*x*) [Function]
– Arc Cosine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`acosh` (*x*) [Function]
– Hyperbolic Arc Cosine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`acot` (*x*) [Function]
– Arc Cotangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`acoth` (*x*) [Function]
– Hyperbolic Arc Cotangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`acsc` (*x*) [Function]
– Arc Cosecant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`acsch` (*x*) [Function]
– Hyperbolic Arc Cosecant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`asec` (*x*) [Function]
– Arc Secant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`asech` (*x*) [Function]
– Hyperbolic Arc Secant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`asin` (*x*) [Function]
– Arc Sine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`asinh` (*x*) [Function]
– Hyperbolic Arc Sine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`atan` (*x*) [Function]
– Arc Tangent.

See also `atan2`.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`atan2` (*y*, *x*) [Function]
– yields the value of $\tan^{-1}(y/x)$ in the interval $-\pi$ to π taking into consideration the quadrant of the point (x, y).

Along the branch cut with $y = 0$ and $x < 0$, `atan2` is continuous with the second quadrant. See also `atan`.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`atanh` (*x*) [Function]
– Hyperbolic Arc Tangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`cos` (*x*) [Function]
– Cosine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`cosh` (*x*) [Function]
– Hyperbolic Cosine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`cot` (*x*) [Function]
– Cotangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`coth` (*x*) [Function]
– Hyperbolic Cotangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`csc` (*x*) [Function]
– Cosecant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`csch` (*x*) [Function]
– Hyperbolic Cosecant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`sec` (*x*) [Function]
– Secant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`sech` (*x*) [Function]
– Hyperbolic Secant.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`sin` (*x*) [Function]
– Sine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`sinh` (*x*) [Function]
– Hyperbolic Sine.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`tan` (*x*) [Function]
– Tangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

`tanh` (*x*) [Function]
– Hyperbolic Tangent.

For variables that control simplification see [%piargs], page 210, [%iargs], page 210, [halfangles], page 211, [triginverses], page 213, and [trigsign], page 212.

12.5.2.2 Options Controlling Simplification

`%piargs` [Option variable]

Default value: `true`

When `%piargs` is `true`, trigonometric functions are simplified to algebraic constants when the argument is an integer multiple of π, $\pi/2$, $\pi/4$, or $\pi/6$.

Maxima knows some identities which can be applied when π, etc., are multiplied by an integer variable (that is, a symbol declared to be integer).

Examples:

```
(%i1) %piargs : false$
(%i2) [sin (%pi), sin (%pi/2), sin (%pi/3)];
                                  %pi       %pi
(%o2)              [sin(%pi), sin(---), sin(---)]
                                   2         3
(%i3) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];
                        %pi       %pi       %pi
(%o3)              [sin(---), sin(---), sin(---)]
                         4         5         6
(%i4) %piargs : true$
(%i5) [sin (%pi), sin (%pi/2), sin (%pi/3)];
                                  sqrt(3)
(%o5)                      [0, 1, -------]
                                    2
(%i6) [sin (%pi/4), sin (%pi/5), sin (%pi/6)];
                            1         %pi   1
(%o6)                  [-------, sin(---), -]
                        sqrt(2)       5    2
(%i7) [cos (%pi/3), cos (10*%pi/3), tan (10*%pi/3),
       cos (sqrt(2)*%pi/3)];
                  1    1                sqrt(2) %pi
(%o7)            [-, - -, sqrt(3), cos(-----------)]
                  2    2                     3
```

Some identities are applied when π and $\pi/2$ are multiplied by an integer variable.

```
(%i1) declare (n, integer, m, even)$
(%i2) [sin (%pi * n), cos (%pi * m), sin (%pi/2 * m),
       cos (%pi/2 * m)];
                                        m/2
(%o2)                   [0, 1, 0, (- 1)   ]
```

`%iargs` [Option variable]

Default value: `true`

When `%iargs` is `true`, trigonometric functions are simplified to hyperbolic functions when the argument is apparently a multiple of the imaginary unit i.

Even when the argument is demonstrably real, the simplification is applied; Maxima considers only whether the argument is a literal multiple of i.

Examples:

```
(%i1) %iargs : false$
(%i2) [sin (%i * x), cos (%i * x), tan (%i * x)];
(%o2)           [sin(%i x), cos(%i x), tan(%i x)]
(%i3) %iargs : true$
(%i4) [sin (%i * x), cos (%i * x), tan (%i * x)];
(%o4)           [%i sinh(x), cosh(x), %i tanh(x)]
```

Even when the argument is demonstrably real, the simplification is applied.

```
(%i1) declare (x, imaginary)$
(%i2) [featurep (x, imaginary), featurep (x, real)];
(%o2)                      [true, false]
(%i3) sin (%i * x);
(%o3)                      %i sinh(x)
```

halfangles [Option variable]

Default value: `false`

When `halfangles` is `true`, trigonometric functions of arguments *expr*/2 are simplified to functions of *expr*.

For a real argument x in the interval $0 \le x < 2\pi$, $\sin \frac{x}{2}$ simplifies to a simple formula:

$$\frac{\sqrt{1-\cos x}}{\sqrt{2}}$$

A complicated factor is needed to make this formula correct for all complex arguments $z = x + iy$:

$$(-1)^{\lfloor x/(2\pi)\rfloor}\left[1 - \text{unit\_step}(-\text{y})\left(1 + (-1)^{\lfloor \text{x}/(2\pi)\rfloor - \lceil \text{x}/(2\pi)\rceil}\right)\right]$$

Maxima knows this factor and similar factors for the functions `sin`, `cos`, `sinh`, and `cosh`. For special values of the argument z these factors simplify accordingly.

Examples:

```
(%i1) halfangles : false$
(%i2) sin (x / 2);
                                  x
(%o2)                         sin(-)
                                  2
(%i3) halfangles : true$
(%i4) sin (x / 2);
                              x
                     floor(-----)
                            2 %pi
                (- 1)             sqrt(1 - cos(x))
(%o4)           ----------------------------------
                              sqrt(2)
(%i5) assume(x>0, x<2*%pi)$
(%i6) sin(x / 2);
                         sqrt(1 - cos(x))
(%o6)                    ----------------
                              sqrt(2)
```

`trigsign` [Option variable]

Default value: `true`

When `trigsign` is `true`, it permits simplification of negative arguments to trigonometric functions. E.g., $\sin(-x)$ will become $-\sin x$ only if `trigsign` is `true`.

12.5.2.3 Explicit Simplifications Using Identities

`trigexpand` (*expr*) [Function]

Expands trigonometric and hyperbolic functions of sums of angles and of multiple angles occurring in *expr*. For best results, *expr* should be expanded. To enhance user control of simplification, this function expands only one level at a time, expanding sums of angles or multiple angles. To obtain full expansion into sines and cosines immediately, set the switch `trigexpand: true`.

`trigexpand` is governed by the following global flags:

`trigexpand`
: If `true` causes expansion of all expressions containing sin's and cos's occurring subsequently.

`halfangles`
: If `true` causes half-angles to be simplified away.

`trigexpandplus`
: Controls the "sum" rule for `trigexpand`, expansion of sums (e.g. `sin(x + y)`) will take place only if `trigexpandplus` is `true`.

`trigexpandtimes`
: Controls the "product" rule for `trigexpand`, expansion of products (e.g. `sin(2 x)`) will take place only if `trigexpandtimes` is `true`.

Examples:

```
(%i1) x+sin(3*x)/sin(x),trigexpand=true,expand;
                            2           2
(%o1)                - sin (x) + 3 cos (x) + x
(%i2) trigexpand(sin(10*x+y));
(%o2)          cos(10 x) sin(y) + sin(10 x) cos(y)
```

`trigexpandplus` [Option variable]

Default value: `true`

`trigexpandplus` controls the "sum" rule for `trigexpand`. Thus, when the `trigexpand` command is used or the `trigexpand` switch set to `true`, expansion of sums (e.g. `sin(x+y)`) will take place only if `trigexpandplus` is `true`.

`trigexpandtimes` [Option variable]

Default value: `true`

`trigexpandtimes` controls the "product" rule for `trigexpand`. Thus, when the `trigexpand` command is used or the `trigexpand` switch set to `true`, expansion of products (e.g. `sin(2*x)`) will take place only if `trigexpandtimes` is `true`.

`triginverses` [Option variable]

Default value: `true`

`triginverses` controls the simplification of the composition of trigonometric and hyperbolic functions with their inverse functions.

If `all`, both e.g. `atan(tan(x))` and `tan(atan(x))` simplify to x.

If `true`, the *`arcfun`*`(`*`fun`*`(x))` simplification is turned off.

If `false`, both the *`arcfun`*`(`*`fun`*`(x))` and *`fun`*`(`*`arcfun`*`(x))` simplifications are turned off.

`trigreduce` [Function]

`trigreduce (`*`expr`*`, `*`x`*`)`
`trigreduce (`*`expr`*`)`

Combines products and powers of trigonometric and hyperbolic sin's and cos's of x into those of multiples of x. It also tries to eliminate these functions when they occur in denominators. If x is omitted then all variables in *expr* are used.

See also `poissimp`.

```
(%i1) trigreduce(-sin(x)^2+3*cos(x)^2+x);
                 cos(2 x)      cos(2 x)   1        1
(%o1)            -------- + 3 (-------- + -) + x - -
                    2             2       2        2
```

`trigsimp (`*`expr`*`)` [Function]

Employs the identities $\sin(x)^2 + \cos(x)^2 = 1$ and $\cosh(x)^2 - \sinh(x)^2 = 1$ to simplify expressions containing `tan`, `sec`, etc., to `sin`, `cos`, `sinh`, `cosh`.

`trigreduce`, `ratsimp`, and `radcan` may be able to further simplify the result.

`demo ("trgsmp.dem")` displays some examples of `trigsimp`.

`trigrat (`*`expr`*`)` [Function]

Gives a canonical simplified quasilinear form of a trigonometrical expression; *expr* is a rational fraction of several `sin`, `cos` or `tan`, the arguments of them are linear forms in some variables (or kernels) and `%pi/`*`n`* (n integer) with integer coefficients. The result is a simplified fraction with numerator and denominator linear in `sin` and `cos`. Thus `trigrat` linearize always when it is possible.

```
(%i1) trigrat(sin(3*a)/sin(a+%pi/3));
(%o1)            sqrt(3) sin(2 a) + cos(2 a) - 1
```

The following example is taken from Davenport, Siret, and Tournier, *Calcul Formel*, Masson (or in English, Addison-Wesley), section 1.5.5, Morley theorem.

```
(%i1) c : %pi/3 - a - b$
(%i2) bc : sin(a)*sin(3*c)/sin(a+b);
                                          %pi
                    sin(a) sin(3 (- b - a + ---))
                                           3
(%o2)               ------------------------------
                              sin(b + a)
```

```
(%i3) ba : bc, c=a, a=c;
                                          %pi
                    sin(3 a) sin(b + a - ---)
                                           3
(%o3)               -------------------------
                                   %pi
                          sin(a - ---)
                                    3
(%i4) ac2 : ba^2 + bc^2 - 2*bc*ba*cos(b);
         2         2         %pi
      sin (3 a) sin (b + a - ---)
                              3
(%o4) ---------------------------
                2     %pi
             sin (a - ---)
                       3
                                        %pi
 - (2 sin(a) sin(3 a) sin(3 (- b - a + ---)) cos(b)
                                         3
             %pi            %pi
 sin(b + a - ---))/(sin(a - ---) sin(b + a))
              3              3
      2       2              %pi
   sin (a) sin (3 (- b - a + ---))
                              3
 + -------------------------------
                2
             sin (b + a)
(%i5) trigrat (ac2);
(%o5) - (sqrt(3) sin(4 b + 4 a) - cos(4 b + 4 a)
 - 2 sqrt(3) sin(4 b + 2 a) + 2 cos(4 b + 2 a)
 - 2 sqrt(3) sin(2 b + 4 a) + 2 cos(2 b + 4 a)
 + 4 sqrt(3) sin(2 b + 2 a) - 8 cos(2 b + 2 a) - 4 cos(2 b - 2 a)
 + sqrt(3) sin(4 b) - cos(4 b) - 2 sqrt(3) sin(2 b) + 10 cos(2 b)
 + sqrt(3) sin(4 a) - cos(4 a) - 2 sqrt(3) sin(2 a) + 10 cos(2 a)
 - 9)/4
```

12.5.2.4 Additional Functions

`atrig1` [Package]

The `atrig1` package contains several additional simplification rules for inverse trigonometric functions. Together with rules already known to Maxima, the following angles are fully implemented: 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. Corresponding angles in the other three quadrants are also available. Do `load("atrig1");` to use them.

`ntrig` [Package]

The `ntrig` package contains a set of simplification rules that are used to simplify trigonometric function whose arguments are of the form `f(n %pi/10)` where f is any of the functions `sin`, `cos`, `tan`, `csc`, `sec` and `cot`.

12.6 Random Numbers

`make_random_state` [Function]

`make_random_state (n)`
`make_random_state (s)`
`make_random_state (true)`
`make_random_state (false)`

A random state object represents the state of the random number generator. The state comprises 627 32-bit words.

`make_random_state (n)` returns a new random state object created from an integer seed value equal to *n* modulo 2^32. *n* may be negative.

`make_random_state (s)` returns a copy of the random state *s*.

`make_random_state (true)` returns a new random state object, using the current computer clock time as the seed.

`make_random_state (false)` returns a copy of the current state of the random number generator.

`set_random_state (s)` [Function]

Copies *s* to the random number generator state.

`set_random_state` always returns `done`.

`random (x)` [Function]

Returns a pseudorandom number. If *x* is an integer, `random (x)` returns an integer from 0 through `x - 1` inclusive. If *x* is a floating point number, `random (x)` returns a nonnegative floating point number less than *x*. `random` complains with an error if *x* is neither an integer nor a float, or if *x* is not positive.

The functions `make_random_state` and `set_random_state` maintain the state of the random number generator.

The Maxima random number generator is an implementation of the Mersenne twister MT 19937.

Examples:

```
(%i1) s1: make_random_state (654321)$
(%i2) set_random_state (s1);
(%o2)                          done
(%i3) random (1000);
(%o3)                           768
(%i4) random (9573684);
(%o4)                         7657880
(%i5) random (2^75);
(%o5)                 1180449161503683163639 0
(%i6) s2: make_random_state (false)$
(%i7) random (1.0);
(%o7)                  0.2310127244107132
(%i8) random (10.0);
(%o8)                  4.3945536458708245
```

```
(%i9) random (100.0);
(%o9)                    32.28666704056853
(%i10) set_random_state (s2);
(%o10)                         done
(%i11) random (1.0);
(%o11)                  0.2310127244107132
(%i12) random (10.0);
(%o12)                  4.3945536458708245
(%i13) random (100.0);
(%o13)                   32.28666704056853
```

13 Maxima's Database

13.1 Introduction to Maxima's Database

13.2 Functions and Variables for Properties

`alphabetic` [Property]

`alphabetic` is a property type recognized by `declare`. The expression `declare(s, alphabetic)` tells Maxima to recognize as alphabetic all of the characters in *s*, which must be a string.

See also Section 8.3 [Identifiers], page 110.

Example:

```
(%i1) xx\~yy\‘\@ : 1729;
(%o1)                          1729
(%i2) declare ("~‘@", alphabetic);
(%o2)                          done
(%i3) xx~yy‘@ + @yy‘xx + ‘xx@@yy~;
(%o3)               ‘xx@@yy~ + @yy‘xx + 1729
(%i4) listofvars (%);
(%o4)                   [@yy‘xx, ‘xx@@yy~]
```

`bindtest` [Property]

The command `declare(x, bindtest)` tells Maxima to trigger an error when the symbol *x* is evaluated unbound.

evaluation: unbound variable aa – an error. To debug this try: debugmode(true);

```
(%i1) aa + bb;
(%o1)                         bb + aa
(%i2) declare (aa, bindtest);
(%o2)                          done
(%i3) aa + bb;
(%i4) aa : 1234;
(%o4)                          1234
(%i5) aa + bb;
(%o5)                       bb + 1234
```

`constant` [Property]

`declare(a, constant)` declares *a* to be a constant. The declaration of a symbol to be constant does not prevent the assignment of a nonconstant value to the symbol.

See `constantp` and `declare`.

Example:

```
(%i1) declare(c, constant);
(%o1)                          done
(%i2) constantp(c);
(%o2)                          true
```

```
(%i3) c : x;
(%o3)                           x
(%i4) constantp(c);
(%o4)                         false
```

`constantp` (*expr*) [Function]

Returns `true` if *expr* is a constant expression, otherwise returns `false`.

An expression is considered a constant expression if its arguments are numbers (including rational numbers, as displayed with `/R/`), symbolic constants such as `%pi`, `%e`, and `%i`, variables bound to a constant or declared constant by `declare`, or functions whose arguments are constant.

`constantp` evaluates its arguments.

See the property `constant` which declares a symbol to be constant.

Examples:

```
(%i1) constantp (7 * sin(2));
(%o1)                         true
(%i2) constantp (rat (17/29));
(%o2)                         true
(%i3) constantp (%pi * sin(%e));
(%o3)                         true
(%i4) constantp (exp (x));
(%o4)                         false
(%i5) declare (x, constant);
(%o5)                         done
(%i6) constantp (exp (x));
(%o6)                         true
(%i7) constantp (foo (x) + bar (%e) + baz (2));
(%o7)                         false
```

`declare` (*a_1*, *p_1*, *a_2*, *p_2*, ...) [Function]

Assigns the atom or list of atoms *a_i* the property or list of properties *p_i*. When *a_i* and/or *p_i* are lists, each of the atoms gets all of the properties.

`declare` quotes its arguments. `declare` always returns `done`.

As noted in the description for each declaration flag, for some flags `featurep(`*object*`,` *feature*`)` returns `true` if *object* has been declared to have *feature*.

For more information about the features system, see `features`. To remove a property from an atom, use `remove`.

`declare` recognizes the following properties:

`additive` Tells Maxima to simplify *a_i* expressions by the substitution `a_i(x + y + z + ...) --> a_i(x) + a_i(y) + a_i(z) + ...`. The substitution is carried out on the first argument only.

`alphabetic`
Tells Maxima to recognize all characters in *a_i* (which must be a string) as alphabetic characters.

`antisymmetric`, `commutative`, `symmetric`
: Tells Maxima to recognize *a_i* as a symmetric or antisymmetric function. `commutative` is the same as `symmetric`.

`bindtest`
: Tells Maxima to trigger an error when *a_i* is evaluated unbound.

`constant`
: Tells Maxima to consider *a_i* a symbolic constant.

`even`, `odd`
: Tells Maxima to recognize *a_i* as an even or odd integer variable.

`evenfun`, `oddfun`
: Tells Maxima to recognize *a_i* as an odd or even function.

`evflag`
: Makes *a_i* known to the `ev` function so that *a_i* is bound to `true` during the execution of `ev` when *a_i* appears as a flag argument of `ev`.

`evfun`
: Makes *a_i* known to `ev` so that the function named by *a_i* is applied when *a_i* appears as a flag argument of `ev`.

`feature`
: Tells Maxima to recognize *a_i* as the name of a feature. Other atoms may then be declared to have the *a_i* property.

`increasing`, `decreasing`
: Tells Maxima to recognize *a_i* as an increasing or decreasing function.

`integer`, `noninteger`
: Tells Maxima to recognize *a_i* as an integer or noninteger variable.

`integervalued`
: Tells Maxima to recognize *a_i* as an integer-valued function.

`lassociative`, `rassociative`
: Tells Maxima to recognize *a_i* as a right-associative or left-associative function.

`linear`
: Equivalent to declaring *a_i* both `outative` and `additive`.

`mainvar`
: Tells Maxima to consider *a_i* a "main variable". A main variable succeeds all other constants and variables in the canonical ordering of Maxima expressions, as determined by `ordergreatp`.

`multiplicative`
: Tells Maxima to simplify *a_i* expressions by the substitution `a_i(x * y * z * ...) --> a_i(x) * a_i(y) * a_i(z) * ...`. The substitution is carried out on the first argument only.

`nary`
: Tells Maxima to recognize *a_i* as an n-ary function.

  The `nary` declaration is not the same as calling the `nary` function. The sole effect of `declare(foo, nary)` is to instruct the Maxima simplifier to flatten nested expressions, for example, to simplify `foo(x, foo(y, z))` to `foo(x, y, z)`.

`nonarray`
: Tells Maxima to consider *a_i* not an array. This declaration prevents multiple evaluation of a subscripted variable name.

`nonscalar`
: Tells Maxima to consider *a_i* a nonscalar variable. The usual application is to declare a variable as a symbolic vector or matrix.

`noun`
: Tells Maxima to parse *a_i* as a noun. The effect of this is to replace instances of *a_i* with `'a_i` or `nounify(a_i)`, depending on the context.

`outative`
: Tells Maxima to simplify *a_i* expressions by pulling constant factors out of the first argument.

  When *a_i* has one argument, a factor is considered constant if it is a literal or declared constant.

  When *a_i* has two or more arguments, a factor is considered constant if the second argument is a symbol and the factor is free of the second argument.

`posfun`
: Tells Maxima to recognize *a_i* as a positive function.

`rational`, `irrational`
: Tells Maxima to recognize *a_i* as a rational or irrational real variable.

`real`, `imaginary`, `complex`
: Tells Maxima to recognize *a_i* as a real, pure imaginary, or complex variable.

`scalar`
: Tells Maxima to consider *a_i* a scalar variable.

Examples of the usage of the properties are available in the documentation for each separate description of a property.

`decreasing` [Property]
`increasing` [Property]

The commands `declare(f, decreasing)` or `declare(f, increasing)` tell Maxima to recognize the function *f* as an decreasing or increasing function.

See also `declare` for more properties.

Example:

```
(%i1) assume(a > b);
(%o1)                          [a > b]
(%i2) is(f(a) > f(b));
(%o2)                          unknown
(%i3) declare(f, increasing);
(%o3)                           done
(%i4) is(f(a) > f(b));
(%o4)                           true
```

`even` [Property]
`odd` [Property]

`declare(a, even)` or `declare(a, odd)` tells Maxima to recognize the symbol *a* as an even or odd integer variable. The properties `even` and `odd` are not recognized by the functions `evenp`, `oddp`, and `integerp`.

See also `declare` and `askinteger`.

Example:

```
(%i1) declare(n, even);
(%o1)                         done
(%i2) askinteger(n, even);
(%o2)                          yes
(%i3) askinteger(n);
(%o3)                          yes
(%i4) evenp(n);
(%o4)                         false
```

`feature` [Property]

Maxima understands two distinct types of features, system features and features which apply to mathematical expressions. See also `status` for information about system features. See also `features` and `featurep` for information about mathematical features.

`feature` itself is not the name of a function or variable.

`featurep` (*a*, *f*) [Function]

Attempts to determine whether the object *a* has the feature *f* on the basis of the facts in the current database. If so, it returns `true`, else `false`.

Note that `featurep` returns `false` when neither *f* nor the negation of *f* can be established.

`featurep` evaluates its argument.

See also `declare` and `features`.

```
(%i1) declare (j, even)$
(%i2) featurep (j, integer);
(%o2)                         true
```

`features` [Declaration]

Maxima recognizes certain mathematical properties of functions and variables. These are called "features".

`declare (`*x*`, `*foo*`)` gives the property *foo* to the function or variable *x*.

`declare (`*foo*`, feature)` declares a new feature *foo*. For example, `declare ([red, green, blue], feature)` declares three new features, `red`, `green`, and `blue`.

The predicate `featurep (`*x*`, `*foo*`)` returns `true` if *x* has the *foo* property, and `false` otherwise.

The infolist `features` is a list of known features. These are

```
integer        noninteger      even
odd            rational        irrational
real           imaginary       complex
analytic       increasing      decreasing
oddfun         evenfun         posfun
constant       commutative     lassociative
rassociative   symmetric       antisymmetric
integervalued
```

plus any user-defined features.

`features` is a list of mathematical features. There is also a list of non-mathematical, system-dependent features. See `status`.

Example:

```
(%i1) declare (FOO, feature);
(%o1)                         done
(%i2) declare (x, FOO);
(%o2)                         done
(%i3) featurep (x, FOO);
(%o3)                         true
```

`get` (`a`, *i*) [Function]

Retrieves the user property indicated by *i* associated with atom *a* or returns `false` if *a* doesn't have property *i*.

`get` evaluates its arguments.

See also `put` and `qput`.

```
(%i1) put (%e, 'transcendental, 'type);
(%o1)                     transcendental
(%i2) put (%pi, 'transcendental, 'type)$
(%i3) put (%i, 'algebraic, 'type)$
(%i4) typeof (expr) := block ([q],
        if numberp (expr)
        then return ('algebraic),
        if not atom (expr)
        then return (maplist ('typeof, expr)),
        q: get (expr, 'type),
        if q=false
        then errcatch (error(expr,"is not numeric.")) else q)$
(%i5) typeof (2*%e + x*%pi);
x is not numeric.
(%o5)  [[transcendental, []], [algebraic, transcendental]]
(%i6) typeof (2*%e + %pi);
(%o6)     [transcendental, [algebraic, transcendental]]
```

`integer` [Property]

`noninteger` [Property]

`declare(a, integer)` or `declare(a, noninteger)` tells Maxima to recognize *a* as an integer or noninteger variable.

See also `declare`.

Example:

```
(%i1) declare(n, integer, x, noninteger);
(%o1)                         done
(%i2) askinteger(n);
(%o2)                          yes
(%i3) askinteger(x);
(%o3)                          no
```

integervalued [Property]

declare(f, integervalued) tells Maxima to recognize f as an integer-valued function.

See also declare.

Example:

```
(%i1) exp(%i)^f(x);
                                 %i f(x)
(%o1)                           %e
(%i2) declare(f, integervalued);
(%o2)                            done
(%i3) exp(%i)^f(x);
                                 %i f(x)
(%o3)                           %e
```

nonarray [Property]

The command declare(a, nonarray) tells Maxima to consider a not an array. This declaration prevents multiple evaluation, if a is a subscripted variable.

See also declare.

Example:

```
(%i1) a:'b$ b:'c$ c:'d$
(%i4) a[x];
(%o4)                             d
                                   x
(%i5) declare(a, nonarray);
(%o5)                            done
(%i6) a[x];
(%o6)                             a
                                   x
```

nonscalar [Property]

Makes atoms behave as does a list or matrix with respect to the dot operator.

See also declare.

nonscalarp (*expr*) [Function]

Returns true if *expr* is a non-scalar, i.e., it contains atoms declared as non-scalars, lists, or matrices.

See also the predicate function scalarp and declare.

posfun [Property]

declare (f, posfun) declares f to be a positive function. is (f(x) > 0) yields true.

See also declare.

printprops [Function]

printprops (a, *i*)
printprops ([*a_1*, ..., *a_n*], *i*)
printprops (*all*, *i*)

Displays the property with the indicator i associated with the atom a. a may also be a list of atoms or the atom all in which case all of the atoms with the given

property will be used. For example, `printprops ([f, g], atvalue)`. `printprops` is for properties that cannot otherwise be displayed, i.e. for `atvalue`, `atomgrad`, `gradef`, and `matchdeclare`.

`properties (a)` [Function]
Returns a list of the names of all the properties associated with the atom *a*.

`props` [System variable]
Default value: `[]`

`props` are atoms which have any property other than those explicitly mentioned in `infolists`, such as specified by `atvalue`, `matchdeclare`, etc., as well as properties specified in the `declare` function.

`propvars (prop)` [Function]
Returns a list of those atoms on the `props` list which have the property indicated by *prop*. Thus `propvars (atvalue)` returns a list of atoms which have atvalues.

`put (atom, value, indicator)` [Function]
Assigns *value* to the property (specified by *indicator*) of *atom*. *indicator* may be the name of any property, not just a system-defined property.

`rem` reverses the effect of `put`.

`put` evaluates its arguments. `put` returns *value*.

See also `qput` and `get`.

Examples:

```
(%i1) put (foo, (a+b)^5, expr);
                                        5
(%o1)                          (b + a)
(%i2) put (foo, "Hello", str);
(%o2)                            Hello
(%i3) properties (foo);
(%o3)              [[user properties, str, expr]]
(%i4) get (foo, expr);
                                        5
(%o4)                          (b + a)
(%i5) get (foo, str);
(%o5)                            Hello
```

`qput (atom, value, indicator)` [Function]
Assigns *value* to the property (specified by *indicator*) of *atom*. This is the same as `put`, except that the arguments are quoted.

See also `get`.

Example:

```
(%i1) foo: aa$
(%i2) bar: bb$
(%i3) baz: cc$
(%i4) put (foo, bar, baz);
(%o4)                             bb
```

```
(%i5) properties (aa);
(%o5)                 [[user properties, cc]]
(%i6) get (aa, cc);
(%o6)                             bb
(%i7) qput (foo, bar, baz);
(%o7)                             bar
(%i8) properties (foo);
(%o8)            [value, [user properties, baz]]
(%i9) get ('foo, 'baz);
(%o9)                             bar
```

`rational` [Property]
`irrational` [Property]

`declare(a, rational)` or `declare(a, irrational)` tells Maxima to recognize *a* as a rational or irrational real variable.

See also `declare`.

`real` [Property]
`imaginary` [Property]
`complex` [Property]

`declare(a, real)`, `declare(a, imaginary)`, or `declare(a, complex)` tells Maxima to recognize *a* as a real, pure imaginary, or complex variable.

See also `declare`.

`rem` (*atom*, *indicator*) [Function]

Removes the property indicated by *indicator* from *atom*. `rem` reverses the effect of `put`.

`rem` returns `done` if *atom* had an *indicator* property when `rem` was called, or `false` if it had no such property.

`remove` [Function]

`remove` (*a_1*, *p_1*, ..., *a_n*, *p_n*)
`remove` ([*a_1*, ..., *a_m*], [*p_1*, ..., *p_n*], ...)
`remove` ("a", *operator*)
`remove` (a, *transfun*)
`remove` (*all*, *p*)

Removes properties associated with atoms.

`remove (a_1, p_1, ..., a_n, p_n)` removes property `p_k` from atom `a_k`.

`remove ([a_1, ..., a_m], [p_1, ..., p_n], ...)` removes properties *p_1*, ..., *p_n* from atoms *a_1*, ..., *a_m*. There may be more than one pair of lists.

`remove (all, p)` removes the property *p* from all atoms which have it.

The removed properties may be system-defined properties such as `function`, `macro`, or `mode_declare`. `remove` does not remove properties defined by `put`.

A property may be `transfun` to remove the translated Lisp version of a function. After executing this, the Maxima version of the function is executed rather than the translated version.

`remove ("a", operator)` or, equivalently, `remove ("a", op)` removes from *a* the operator properties declared by `prefix`, `infix`, `[function_nary], page 156`, `postfix`, `matchfix`, or `nofix`. Note that the name of the operator must be written as a quoted string.

`remove` always returns `done` whether or not an atom has a specified property. This behavior is unlike the more specific remove functions `remvalue`, `remarray`, `remfunction`, and `remrule`.

`remove` quotes its arguments.

`scalar` [Property]

`declare(a, scalar)` tells Maxima to consider *a* a scalar variable.

See also `declare`.

`scalarp` (*expr*) [Function]

Returns `true` if *expr* is a number, constant, or variable declared `scalar` with `declare`, or composed entirely of numbers, constants, and such variables, but not containing matrices or lists.

See also the predicate function `nonscalarp`.

13.3 Functions and Variables for Facts

`activate` (*context_1*, ..., *context_n*) [Function]

Activates the contexts *context_1*, ..., *context_n*. The facts in these contexts are then available to make deductions and retrieve information. The facts in these contexts are not listed by `facts ()`.

The variable `activecontexts` is the list of contexts which are active by way of the `activate` function.

`activecontexts` [System variable]

Default value: `[]`

`activecontexts` is a list of the contexts which are active by way of the `activate` function, as opposed to being active because they are subcontexts of the current context.

`askequal` [Function]

`askequal` (*expr1*, *expr2*)

`askequal(expr1, expr2)` attempts to determine from the `assume` database whether *expr1* is equal to *expr2*, and prompts the user if it cannot tell.

If the user provides the answer, the answer is stored in the `assume` database for the duration of the evaluation of the expression currently in progress. When the evaluation is completed, the user-provided answer is removed from the database.

`askequal` returns `yes` or `no`, whether the answer was determined from the `assume` database or provided by the user.

See also `equal`.

askinteger [Function]

askinteger (*expr*, *integer*)
askinteger (*expr*)
askinteger (*expr*, *even*)
askinteger (*expr*, *odd*)

`askinteger (expr, integer)` attempts to determine from the `assume` database whether *expr* is an integer. `askinteger` prompts the user if it cannot tell otherwise, and attempt to install the information in the database if possible. `askinteger (expr)` is equivalent to `askinteger (expr, integer)`.

`askinteger (expr, even)` and `askinteger (expr, odd)` likewise attempt to determine if *expr* is an even integer or odd integer, respectively.

asksign (*expr*) [Function]

First attempts to determine whether the specified expression is positive, negative, or zero. If it cannot, it asks the user the necessary questions to complete its deduction. The user's answer is recorded in the data base for the duration of the current computation. The return value of `asksign` is one of `pos`, `neg`, or `zero`.

assume (*pred_1*, ..., *pred_n*) [Function]

Adds predicates *pred_1*, ..., *pred_n* to the current context. If a predicate is inconsistent or redundant with the predicates in the current context, it is not added to the context. The context accumulates predicates from each call to `assume`.

`assume` returns a list whose elements are the predicates added to the context or the atoms `redundant` or `inconsistent` where applicable.

The predicates *pred_1*, ..., *pred_n* can only be expressions with the relational operators `<` `<=` `equal` `notequal` `>=` and `>`. Predicates cannot be literal equality `=` or literal inequality `#` expressions, nor can they be predicate functions such as `integerp`.

Compound predicates of the form *pred_1* `and` ... `and` *pred_n* are recognized, but not *pred_1* `or` ... `or` *pred_n*. `not` *pred_k* is recognized if *pred_k* is a relational predicate. Expressions of the form `not (`*pred_1* `and` *pred_2*`)` and `not (`*pred_1* `or` *pred_2*`)` are not recognized.

Maxima's deduction mechanism is not very strong; there are many obvious consequences which cannot be determined by `is`. This is a known weakness.

`assume` does not handle predicates with complex numbers. If a predicate contains a complex number `assume` returns `inconsistent` or `redundant`.

`assume` evaluates its arguments.

See also `is`, `facts`, `forget`, `context`, and `declare`.

Examples:

```
(%i1) assume (xx > 0, yy < -1, zz >= 0);
(%o1)              [xx > 0, yy < - 1, zz >= 0]
(%i2) assume (aa < bb and bb < cc);
(%o2)                   [bb > aa, cc > bb]
(%i3) facts ();
(%o3)     [xx > 0, - 1 > yy, zz >= 0, bb > aa, cc > bb]
(%i4) is (xx > yy);
(%o4)                          true
```

```
(%i5) is (yy < -yy);
(%o5)                          true
(%i6) is (sinh (bb - aa) > 0);
(%o6)                          true
(%i7) forget (bb > aa);
(%o7)                        [bb > aa]
(%i8) prederror : false;
(%o8)                          false
(%i9) is (sinh (bb - aa) > 0);
(%o9)                         unknown
(%i10) is (bb^2 < cc^2);
(%o10)                        unknown
```

`assumescalar` [Option variable]

Default value: `true`

`assumescalar` helps govern whether expressions `expr` for which `nonscalarp (expr)` is `false` are assumed to behave like scalars for certain transformations.

Let `expr` represent any expression other than a list or a matrix, and let `[1, 2, 3]` represent any list or matrix. Then `expr . [1, 2, 3]` yields `[expr, 2 expr, 3 expr]` if `assumescalar` is `true`, or `scalarp (expr)` is `true`, or `constantp (expr)` is `true`.

If `assumescalar` is `true`, such expressions will behave like scalars only for commutative operators, but not for noncommutative multiplication `..`

When `assumescalar` is `false`, such expressions will behave like non-scalars.

When `assumescalar` is `all`, such expressions will behave like scalars for all the operators listed above.

`assume_pos` [Option variable]

Default value: `false`

When `assume_pos` is `true` and the sign of a parameter x cannot be determined from the current context or other considerations, `sign` and `asksign (x)` return `true`. This may forestall some automatically-generated `asksign` queries, such as may arise from `integrate` or other computations.

By default, a parameter is x such that `symbolp (x)` or `subvarp (x)`. The class of expressions considered parameters can be modified to some extent via the variable `assume_pos_pred`.

`sign` and `asksign` attempt to deduce the sign of expressions from the sign of operands within the expression. For example, if `a` and `b` are both positive, then `a + b` is also positive.

However, there is no way to bypass all `asksign` queries. In particular, when the `asksign` argument is a difference `x - y` or a logarithm `log(x)`, `asksign` always requests an input from the user, even when `assume_pos` is `true` and `assume_pos_pred` is a function which returns `true` for all arguments.

`assume_pos_pred` [Option variable]

Default value: `false`

When `assume_pos_pred` is assigned the name of a function or a lambda expression of one argument *x*, that function is called to determine whether *x* is considered a parameter for the purpose of `assume_pos`. `assume_pos_pred` is ignored when `assume_pos` is `false`.

The `assume_pos_pred` function is called by `sign` and `asksign` with an argument *x* which is either an atom, a subscripted variable, or a function call expression. If the `assume_pos_pred` function returns `true`, *x* is considered a parameter for the purpose of `assume_pos`.

By default, a parameter is *x* such that `symbolp (x)` or `subvarp (x)`.

See also `assume` and `assume_pos`.

Examples:

```
(%i1) assume_pos: true$
(%i2) assume_pos_pred: symbolp$
(%i3) sign (a);
(%o3)                          pos
(%i4) sign (a[1]);
(%o4)                          pnz
(%i5) assume_pos_pred: lambda ([x], display (x), true)$
(%i6) asksign (a);
                               x = a

(%o6)                          pos
(%i7) asksign (a[1]);
                              x = a
                                   1

(%o7)                          pos
(%i8) asksign (foo (a));
                            x = foo(a)

(%o8)                          pos
(%i9) asksign (foo (a) + bar (b));
                            x = foo(a)

                            x = bar(b)

(%o9)                          pos
```

```
(%i10) asksign (log (a));
                                x = a

                                x = a

                                x = a

                                x = a

                                x = a

                                x = a

                                x = a

                                x = a

                                x = a

Is a - 1 positive, negative or zero?
p;
(%o10)                           pos
(%i11) asksign (a - b);
                                x = a

                                x = b

                                x = a

                                x = b

Is b - a positive, negative or zero?
p;
(%o11)                           neg
```

`context` [Option variable]

Default value: `initial`

`context` names the collection of facts maintained by `assume` and `forget`. `assume` adds facts to the collection named by `context`, while `forget` removes facts.

Binding `context` to a name *foo* changes the current context to *foo*. If the specified context *foo* does not yet exist, it is created automatically by a call to `newcontext`. The specified context is activated automatically.

See `contexts` for a general description of the context mechanism.

`contexts` [Option variable]

Default value: `[initial, global]`

`contexts` is a list of the contexts which currently exist, including the currently active context.

The context mechanism makes it possible for a user to bind together and name a collection of facts, called a context. Once this is done, the user can have Maxima assume or forget large numbers of facts merely by activating or deactivating their context.

Any symbolic atom can be a context, and the facts contained in that context will be retained in storage until destroyed one by one by calling `forget` or destroyed as a whole by calling `kill` to destroy the context to which they belong.

Contexts exist in a hierarchy, with the root always being the context `global`, which contains information about Maxima that some functions need. When in a given context, all the facts in that context are "active" (meaning that they are used in deductions and retrievals) as are all the facts in any context which is a subcontext of the active context.

When a fresh Maxima is started up, the user is in a context called `initial`, which has `global` as a subcontext.

See also `facts`, `newcontext`, `supcontext`, `killcontext`, `activate`, `deactivate`, `assume`, and `forget`.

`csign` (*expr*) [Function]

Attempts to determine the sign of *expr* on the basis of the facts in the current data base without assuming that *expr* is real-valued. It returns one of the following answers: `pos` (positive), `neg` (negative), `zero`, `pz` (positive or zero), `nz` (negative or zero), `pn` (positive or negative), `pnz` (positive, negative, or zero), `imaginary` (purely imaginary), or `complex`, (complex, i.e. nothing known).

Note that while this function does not assume that *expr* is real-valued, it still assumes that variables are real-valued unless declared otherwise. This means that `csign(z)` will return `pnz` unless `declare(z,complex)` or `declare(z,imaginary)` has been evaluated beforehand.

See also `sign`.

`deactivate` (*context_1*, ..., *context_n*) [Function]

Deactivates the specified contexts *context_1*, ..., *context_n*.

`facts` [Function]

`facts` (*item*)
`facts` ()

If *item* is the name of a context, `facts` (*item*) returns a list of the facts in the specified context.

If *item* is not the name of a context, `facts` (*item*) returns a list of the facts known about *item* in the current context. Facts that are active, but in a different context, are not listed.

`facts` () (i.e., without an argument) lists the current context.

forget [Function]

forget (*pred_1*, ..., *pred_n*)
forget (*L*)

Removes predicates established by assume. The predicates may be expressions equivalent to (but not necessarily identical to) those previously assumed.

forget (*L*), where *L* is a list of predicates, forgets each item on the list.

is (*expr*) [Function]

Attempts to determine whether the predicate *expr* is provable from the facts in the assume database.

If the predicate is provably true or false, is returns true or false, respectively. Otherwise, the return value is governed by the global flag prederror. When prederror is true, is complains with an error message. Otherwise, is returns unknown.

ev(*expr*, pred) (which can be written *expr*, pred at the interactive prompt) is equivalent to is(*expr*).

See also assume, facts, and maybe.

Examples:

is causes evaluation of predicates.

```
(%i1) %pi > %e;
(%o1)                         %pi > %e
(%i2) is (%pi > %e);
(%o2)                           true
```

is attempts to derive predicates from the assume database.

```
(%i1) assume (a > b);
(%o1)                         [a > b]
(%i2) assume (b > c);
(%o2)                         [b > c]
(%i3) is (a < b);
(%o3)                          false
(%i4) is (a > c);
(%o4)                          true
(%i5) is (equal (a, c));
(%o5)                          false
```

If is can neither prove nor disprove a predicate from the assume database, the global flag prederror governs the behavior of is.

Unable to evaluate predicate a > 0 – an error. To debug this try: debugmode(true);

```
(%i1) assume (a > b);
(%o1)                         [a > b]
(%i2) prederror: true$
(%i3) is (a > 0);
(%i4) prederror: false$
(%i5) is (a > 0);
(%o5)                        unknown
```

killcontext (*context_1*, ..., *context_n*) [Function]

Kills the contexts *context_1*, ..., *context_n*.

If one of the contexts is the current context, the new current context will become the first available subcontext of the current context which has not been killed. If the first available unkilled context is `global` then `initial` is used instead. If the `initial` context is killed, a new, empty `initial` context is created.

`killcontext` refuses to kill a context which is currently active, either because it is a subcontext of the current context, or by use of the function `activate`.

`killcontext` evaluates its arguments. `killcontext` returns `done`.

maybe (*expr*) [Function]

Attempts to determine whether the predicate *expr* is provable from the facts in the `assume` database.

If the predicate is provably `true` or `false`, `maybe` returns `true` or `false`, respectively. Otherwise, `maybe` returns `unknown`.

`maybe` is functionally equivalent to `is` with `prederror: false`, but the result is computed without actually assigning a value to `prederror`.

See also `assume`, `facts`, and `is`.

Examples:

```
(%i1) maybe (x > 0);
(%o1)                         unknown
(%i2) assume (x > 1);
(%o2)                         [x > 1]
(%i3) maybe (x > 0);
(%o3)                          true
```

newcontext [Function]

`newcontext` (*name*)
`newcontext` ()

Creates a new, empty context, called *name*, which has `global` as its only subcontext. The newly-created context becomes the currently active context.

If *name* is not specified, a new name is created (via `gensym`) and returned.

`newcontext` evaluates its argument. `newcontext` returns *name* (if specified) or the new context name.

sign (*expr*) [Function]

Attempts to determine the sign of *expr* on the basis of the facts in the current data base. It returns one of the following answers: `pos` (positive), `neg` (negative), `zero`, `pz` (positive or zero), `nz` (negative or zero), `pn` (positive or negative), or `pnz` (positive, negative, or zero, i.e. nothing known).

Note that this function assumes that *expr* is a real-valued expression, such that for example `sign(sqrt(x))` will yield `pz` even though `sqrt(x)` may return a complex-valued result for `x<0`.

See also `signum`.

supcontext [Function]

supcontext (*name*, *context*)
supcontext (*name*)
supcontext ()

Creates a new context, called *name*, which has *context* as a subcontext. *context* must exist.

If *context* is not specified, the current context is assumed.

If *name* is not specified, a new name is created (via gensym) and returned.

supcontext evaluates its argument. supcontext returns *name* (if specified) or the new context name.

13.4 Functions and Variables for Predicates

charfun (*p*) [Function]

Return 0 when the predicate *p* evaluates to false; return 1 when the predicate evaluates to true. When the predicate evaluates to something other than true or false (unknown), return a noun form.

Examples:

```
(%i1) charfun (x < 1);
(%o1)                      charfun(x < 1)
(%i2) subst (x = -1, %);
(%o2)                              1
(%i3) e : charfun ('"and" (-1 < x, x < 1))$
(%i4) [subst (x = -1, e), subst (x = 0, e), subst (x = 1, e)];
(%o4) [charfun((- 1 < - 1) and (- 1 < 1)),
  charfun((- 1 < 0) and (0 < 1)), charfun((- 1 < 1) and (1 < 1))]
```

compare (*x*, *y*) [Function]

Return a comparison operator *op* (<, <=, >, >=, =, or #) such that is (*x op y*) evaluates to true; when either *x* or *y* depends on %i and *x* # *y*, return notcomparable; when there is no such operator or Maxima isn't able to determine the operator, return unknown.

Examples:

```
(%i1) compare (1, 2);
(%o1)                              <
(%i2) compare (1, x);
(%o2)                           unknown
(%i3) compare (%i, %i);
(%o3)                              =
(%i4) compare (%i, %i + 1);
(%o4)                         notcomparable
(%i5) compare (1/x, 0);
(%o5)                              #
(%i6) compare (x, abs(x));
(%o6)                             <=
```

The function `compare` doesn't try to determine whether the real domains of its arguments are nonempty; thus

```
(%i1) compare (acos (x^2 + 1), acos (x^2 + 1) + 1);
(%o1)                          <
```

The real domain of `acos (x^2 + 1)` is empty.

`equal (a, b)` [Function]

Represents equivalence, that is, equal value.

By itself, `equal` does not evaluate or simplify. The function `is` attempts to evaluate `equal` to a Boolean value. `is(equal(a, b))` returns `true` (or `false`) if and only if *a* and *b* are equal (or not equal) for all possible values of their variables, as determined by evaluating `ratsimp(a - b)`; if `ratsimp` returns 0, the two expressions are considered equivalent. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

When `is` fails to reduce `equal` to `true` or `false`, the result is governed by the global flag `prederror`. When `prederror` is `true`, `is` complains with an error message. Otherwise, `is` returns `unknown`.

In addition to `is`, some other operators evaluate `equal` and `notequal` to `true` or `false`, namely `if`, `and`, `or`, and `not`.

The negation of `equal` is `notequal`.

Examples:

By itself, `equal` does not evaluate or simplify.

```
(%i1) equal (x^2 - 1, (x + 1) * (x - 1));
                          2
(%o1)            equal(x  - 1, (x - 1) (x + 1))
(%i2) equal (x, x + 1);
(%o2)                     equal(x, x + 1)
(%i3) equal (x, y);
(%o3)                       equal(x, y)
```

The function `is` attempts to evaluate `equal` to a Boolean value. `is(equal(a, b))` returns `true` when `ratsimp(a - b)` returns 0. Two expressions may be equivalent even if they are not syntactically equal (i.e., identical).

```
(%i1) ratsimp (x^2 - 1 - (x + 1) * (x - 1));
(%o1)                           0
(%i2) is (equal (x^2 - 1, (x + 1) * (x - 1)));
(%o2)                          true
(%i3) is (x^2 - 1 = (x + 1) * (x - 1));
(%o3)                          false
(%i4) ratsimp (x - (x + 1));
(%o4)                           - 1
(%i5) is (equal (x, x + 1));
(%o5)                          false
(%i6) is (x = x + 1);
(%o6)                          false
```

```
(%i7) ratsimp (x - y);
(%o7)                          x - y
(%i8) is (equal (x, y));
(%o8)                         unknown
(%i9) is (x = y);
(%o9)                          false
```

When `is` fails to reduce `equal` to `true` or `false`, the result is governed by the global flag `prederror`.

2 2 Unable to evaluate predicate equal(x + 2 x + 1, x - 2 x - 1) – an error. To debug this try: debugmode(true);

```
(%i1) [aa : x^2 + 2*x + 1, bb : x^2 - 2*x - 1];
                     2             2
(%o1)              [x  + 2 x + 1, x  - 2 x - 1]
(%i2) ratsimp (aa - bb);
(%o2)                         4 x + 2
(%i3) prederror : true;
(%o3)                          true
(%i4) is (equal (aa, bb));
(%i5) prederror : false;
(%o5)                          false
(%i6) is (equal (aa, bb));
(%o6)                         unknown
```

Some operators evaluate `equal` and `notequal` to `true` or `false`.

```
(%i1) if equal (y, y - 1) then FOO else BAR;
(%o1)                           BAR
(%i2) eq_1 : equal (x, x + 1);
(%o2)                      equal(x, x + 1)
(%i3) eq_2 : equal (y^2 + 2*y + 1, (y + 1)^2);
                           2                   2
(%o3)              equal(y  + 2 y + 1, (y + 1) )
(%i4) [eq_1 and eq_2, eq_1 or eq_2, not eq_1];
(%o4)                   [false, true, true]
```

Because `not` *expr* causes evaluation of *expr*, `not equal(a, b)` is equivalent to `is(notequal(a, b))`.

```
(%i1) [notequal (2*z, 2*z - 1), not equal (2*z, 2*z - 1)];
(%o1)             [notequal(2 z, 2 z - 1), true]
(%i2) is (notequal (2*z, 2*z - 1));
(%o2)                          true
```

`notequal (a, b)` [Function]

Represents the negation of `equal(a, b)`.

Examples:

```
(%i1) equal (a, b);
(%o1)                       equal(a, b)
(%i2) maybe (equal (a, b));
(%o2)                         unknown
```

```
(%i3) notequal (a, b);
(%o3)                     notequal(a, b)
(%i4) not equal (a, b);
(%o4)                     notequal(a, b)
(%i5) maybe (notequal (a, b));
(%o5)                        unknown
(%i6) assume (a > b);
(%o6)                        [a > b]
(%i7) equal (a, b);
(%o7)                      equal(a, b)
(%i8) maybe (equal (a, b));
(%o8)                         false
(%i9) notequal (a, b);
(%o9)                     notequal(a, b)
(%i10) maybe (notequal (a, b));
(%o10)                        true
```

`unknown` (*expr*) [Function]

Returns `true` if and only if *expr* contains an operator or function not recognized by the Maxima simplifier.

`zeroequiv` (*expr*, *v*) [Function]

Tests whether the expression *expr* in the variable *v* is equivalent to zero, returning `true`, `false`, or `dontknow`.

`zeroequiv` has these restrictions:

1. Do not use functions that Maxima does not know how to differentiate and evaluate.
2. If the expression has poles on the real line, there may be errors in the result (but this is unlikely to occur).
3. If the expression contains functions which are not solutions to first order differential equations (e.g. Bessel functions) there may be incorrect results.
4. The algorithm uses evaluation at randomly chosen points for carefully selected subexpressions. This is always a somewhat hazardous business, although the algorithm tries to minimize the potential for error.

For example `zeroequiv (sin(2 * x) - 2 * sin(x) * cos(x), x)` returns `true` and `zeroequiv (%e^x + x, x)` returns `false`. On the other hand `zeroequiv (log(a * b) - log(a) - log(b), a)` returns `dontknow` because of the presence of an extra parameter `b`.

14 Plotting

14.1 Introduction to Plotting

To make the plots, Maxima can use an external plotting package or its own graphical interface Xmaxima (see the section on Section 14.2 Plotting Formats). The plotting functions calculate a set of points and pass them to the plotting package together with a set of commands specific to that graphic program. In some cases those commands and data are saved in a file and the graphic program is executed giving it the name of that file to be parsed.

If the format used is either `gnuplot`, `xmaxima` or `geomview`, the commands used to create the plot are placed in a file named `random-string.format`, where `random-string` is a 16-character random string and `format` is the name of the plotting format being used (`gnuplot`, `xmaxima` or `geomview`). That file is then saved in the temporary directory and should be deleted after the maxima session ends.

The temporary directory is specified by the system variable `maxima_tempdir`. That location can be changed by changing the value of that variable (or using the environment variable `MAXIMA_TEMPDIR`). Its value must be a string that represents a valid directory where Maxima can create new files. In the case of the `gnuplot` format, option `gnuplot_out_file` can be used to change the default name `random-string.gnuplot` for the output file. If the name given includes a directory name, it will saved in that directory instead of in the temporary directory, and it will not be removed after the Maxima session ends. For example, the plotting option:

```
[gnuplot_out_file, "./my_gnuplotfile.txt"]
```

will save the commands to produce the plot in a file `my_gnuplotfile.txt` in the current directory.

There are commands to save the plot in a graphic format file, rather than showing it in the screen. The default name for that graphic file is `random-string.extension`, where `extension` is the extension normally used for the kind of graphic file selected, but that name can also be specified by the user.

When a plotting command fails the file where the plotting commands were saved can be inspected to diagnose the problem. Since the `gnuplot_pipes` format does not create any file, if a plotting command fails while using that format it is recommended to repeat the same plotting command adding the `gnuplot` option and ending the command with semicolon. That way, you can see the name of the file where the commands sent to Gnuplot were saved, which can be inspected to diagnose the problem (before it is deleted when the Maxima session ends or use `gnuplot_out_file` as explained above to create a permanent file):

The additional package [draw], page 849, provides functions similar to the ones described in this section with some extra features, but it only works with `gnuplot`. Note that some plotting options have the same name in both plotting packages, but their syntax and behavior is different. To view the documentation for a graphic option `opt`, type `?? opt` in order to choose the information for either of those two packages.

14.2 Plotting Formats

Maxima can use either Gnuplot, Xmaxima or Geomview as graphics program. Gnuplot and Geomview are external programs which must be installed separately, while Xmaxima is distributed with Maxima. To see which plotting format you are currently using, use the command `get_plot_option(plot_format);` and to change to another format, you can use `set_plot_option([plot_format, <format>])`, where `<format>` is the name of one of the formats described below. Those two commands show and change the global plot format, while each individual plotting command can use its own format, if it includes an option `[plot_format, <format>]` (see `get_plot_option` and `set_plot_option`).

The plotting formats are the following:

- **gnuplot**

  Used to launch the external program gnuplot, which must be installed in your system. All plotting commands and data are saved into a file `random-string.gnuplot` in the temporary directory, where `random-string` is a 16-character random string.

- **gnuplot_pipes** (default value)

  It is similar to the `gnuplot` format except that the commands and plot data are sent directly to `gnuplot` without creating any files. A single gnuplot process is kept open, with a single graphic window, and subsequent plot commands will be sent to the same process, replacing previous plots in that same window. Even if the graphic window is closed, the `gnuplot` process is still running until the end of the session or until it is killed with `gnuplot_close`.. The function `gnuplot_replot` can be used to modify a plot that has already been displayed on the screen or to open again the graphic window after it was closed.

  This format is only used to plot to the screen; whenever graphic files are to be created, the format is silently switched to `gnuplot` and the commands needed to create the graphic file are saved with the data into a file `random-string.gnuplot` in the temporary directory, where `random-string` is a 16-character random string.

- **mgnuplot**

  Mgnuplot is a Tk-based wrapper around gnuplot. It is an old interface still included in the Maxima distribution, but it is currently disabled because it does not have most of the features introduced by recent versions of the plotting commands. Mgnuplot requires an external gnuplot installation and, in Unix systems, the Tcl/Tk system.

- **xmaxima**

  Xmaxima is a Tcl/Tk graphical interface for Maxima that can also be used to display plots created when Maxima is run from the console or from other graphical interfaces. To use this format, the xmaxima program, which is distributed together with Maxima, must be installed; in some Linux distributions Xmaxima is distributed in a package separate from other parts of Maxima. If Maxima is being run from the Xmaxima console, the data and commands are passed to xmaxima through the same socket used for the communication between Maxima and the Xmaxima console. When used from a terminal or from graphical interfaces different from Xmaxima, the commands and data are saved in a file `random-string.xmaxima` in the temporary directory, where `random-string` is a 16-character random string. The program xmaxima is run with the name of that file as argument.

- **geomview**

  Geomview, a Motif based interactive 3D viewing program for Unix. It can only be used to display plots created with `plot3d`. To use this format, the geomview program must be installed.

14.3 Functions and Variables for Plotting

`geomview_command` [System variable]

This variable stores the name of the command used to run the geomview program when the plot format is `geomview`. Its default value is `"geomview"`. If the geomview program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,

```
(%i1) geomview_command: "/usr/local/bin/my_geomview"$
```

This variable must contain a string.

See also `gnuplot_command` and `xmaxima_plot_command`.

`get_plot_option` (*keyword*, *index*) [Function]

Returns the current default value of the option named *keyword*, which is a list. The optional argument *index* must be a positive integer which can be used to extract only one element from the list (element 1 is the name of the option).

See also `set_plot_option`, `remove_plot_option` and the section on `Section 14.4 Plotting Options`.

`gnuplot_command` [System variable]

This variable stores the name of the command used to run the gnuplot program when the plot format is `gnuplot` or `gnuplot_pipes`. Its default value is `"gnuplot"`. If the gnuplot program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,

```
(%i1) gnuplot_command: "/usr/local/bin/my_gnuplot"$
```

This variable must contain a string.

See also `geomview_command` and `xmaxima_plot_command`.

`gnuplot_file_args` [System variable]

When a graphic file is going to be created using `gnuplot`, this variable is used to specify the format used to print the file name given to gnuplot. Its default value is `"~a"` in SBCL and Openmcl, and `"~s"` in other lisp versions, which means that the name of the file will be passed without quotes if SBCL or Openmcl are used and within quotes if other Lisp versions are used. The contents of this variable can be changed in order to add options for the gnuplot program, adding those options before the format directive `"~s"`.

`gnuplot_view_args` [System variable]

This variable is the format used to parse the argument that will be passed to the gnuplot program when the plot format is `gnuplot`. Its default value is `"-persist ~a"` when SBCL or Openmcl are used, and `"-persist ~s"` with other Lisp variants, where `"~a"` or `"~s"` will be replaced with the name of the file where the gnuplot commands

have been written (usually "random-string.gnuplot"). The option `-persist` tells gnuplot to exit after the commands in the file have been executed, without closing the window that displays the plot.

Those familiar with gnuplot, might want to change the value of this variable. For example, by changing it to:

```
(%i1) gnuplot_view_args: "~s -"$
```

gnuplot will not be closed after the commands in the file have been executed; thus, the window with the plot will remain, as well as the gnuplot interactive shell where other commands can be issued in order to modify the plot.

In Windows versions of Gnuplot older than 4.6.3 the behavior of "~s -" and "-persist ~s" were the opposite; namely, "-persist ~s" made the plot window and the gnuplot interactive shell remain, while "~s -" closed the gnuplot shell keeping the plot window. Therefore, when older gnuplot versions are used in Windows, it might be necessary to adjust the value of `gnuplot_view_args`.

`julia` (*x*, *y*, *...options...*) [Function]

Creates a graphic representation of the Julia set for the complex number (*x* + i *y*). The two mandatory parameters *x* and *y* must be real. This program is part of the additional package `dynamics`, but that package does not have to be loaded; the first time julia is used, it will be loaded automatically.

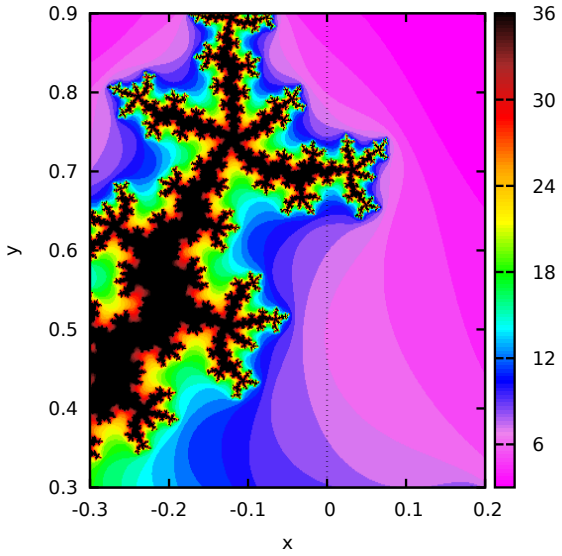
Each pixel in the grid is given a color corresponding to the number of iterations it takes the sequence that starts at that point to move out of the convergence circle of radius 2 centered at the origin. The number of pixels in the grid is controlled by the `grid` plot option (default 30 by 30). The maximum number of iterations is set with the option `iterations`. The program sets its own default palette: magenta, violet, blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by adding an explicit `palette` option in the command.

The default domain used goes from -2 to 2 in both axes and can be changed with the `x` and `y` options. By default, the two axes are shown with the same scale, unless the option `yx_ratio` is used or the option `same_xy` is disabled. Other general plot options are also accepted.

The following example shows a region of the Julia set for the number -0.55 + i0.6. The option `color_bar_tics` is used to prevent Gnuplot from adjusting the color box up to 40, in which case the points corresponding the maximum 36 iterations would not be black.

Warning: line 394 - example input lines must begin with ’

```
(%i1) julia (-0.55, 0.6, [iterations, 36], [x, -0.3, 0.2],
  [y, 0.3, 0.9], [grid, 400, 400], [color_bar_tics, 0, 6, 36])$
```



`make_transform` ([*var1*, *var2*, *var3*], *fx*, *fy*, *fz*) [Function]

Returns a function suitable to be used in the option `transform_xy` of plot3d. The three variables *var1*, *var2*, *var3* are three dummy variable names, which represent the 3 variables given by the plot3d command (first the two independent variables and then the function that depends on those two variables). The three functions *fx*, *fy*, *fz* must depend only on those 3 variables, and will give the corresponding x, y and z coordinates that should be plotted. There are two transformations defined by default: `polar_to_xy` and `spherical_to_xyz`. See the documentation for those two transformations.

`mandelbrot` (*options*) [Function]

Creates a graphic representation of the Mandelbrot set. This program is part of the additional package `dynamics`, but that package does not have to be loaded; the first time mandelbrot is used, the package will be loaded automatically.

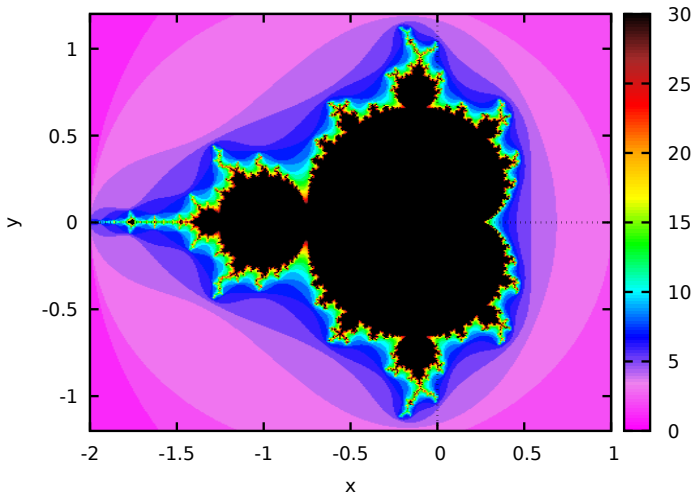
This program can be called without any arguments, in which case it will use a default value of 9 iterations per point, a grid with dimensions set by the `grid` plot option (default 30 by 30) and a region that extends from -2 to 2 in both axes. The options are all the same that plot2d accepts, plus an option `iterations` to change the number of iterations.

Each pixel in the grid is given a color corresponding to the number of iterations it takes the sequence starting at zero to move out of the convergence circle of radius 2, centered at the origin. The maximum number of iterations is set by the option `iterations`. The program uses its own default palette: magenta,violet, blue, cyan, green, yellow, orange, red, brown and black, but it can be changed by adding an explicit `palette` option in the command. By default, the two axes are shown with the same scale, unless the option `yx_ratio` is used or the option `same_xy` is disabled.

Example:

[grid,400,400])$ incorrect syntax: end of file while scanning expression.
^

```
(%i1) mandelbrot ([iterations, 30], [x, -2, 1], [y, -1.2, 1.2],
```



`polar_to_xy` [System function]

It can be given as value for the `transform_xy` option of plot3d. Its effect will be to interpret the two independent variables in plot3d as the distance from the z axis and the azimuthal angle (polar coordinates), and transform them into x and y coordinates.

`plot2d` [Function]

`plot2d` (*expr*, *range_x*, *options*)
`plot2d` (*expr_1*=*expr_2*, *range_x*, *range_y*, *options*)
`plot2d` ([*parametric*, *expr_x*, *expr_y*, *range*], *options*)
`plot2d` ([*discrete*, *points*], *options*)
`plot2d` ([*contour*, *expr*], *range_x*, *range_y*, *options*)
`plot2d` ([*type_1*, ..., *type_n*], *options*)

There are 5 types of plots that can be plotted by `plot2d`:

1. **Explicit functions**. `plot2d` (*expr*, *range_x*, *options*), where *expr* is an expression that depends on only one variable, or the name of a function with one input parameter and numerical results. *range_x* is a list with three elements, the first one being the name of the variable that will be shown on the horizontal axis of the plot, and the other two elements should be two numbers, the first one smaller than the second, that define the minimum and maximum values to be shown on the horizontal axis. The name of the variable used in *range_x* must be the same variable on which *expr* depends. The result will show in the vertical axis the corresponding values of the expression or function for each value of the variable in the horizontal axis.
2. **Implicit functions**. `plot2d` (*expr_1*=*expr_2*, *range_x*, *range_y*, *options*), where *expr_1* and *expr_2* are two expressions that can depend on one or two variables. *range_x* and *range_y* must be two lists of three elements that define the ranges for the variables in the two axes of the plot; the first element of each list is the name of the corresponding variable, and the other two elements are the minimum and maximum values for that variable. The two variables on which *expr_1* and *expr_2* can depend are those specified by *range_x* and *range_y*. The result will be a curve or a set of curves where the equation *expr_1*=*expr_2* is true.

3. **Parametric functions**. `plot2d` ([parametric, *expr_x*, *expr_y*, *range*], *options*), where *expr_x* and *expr_y* are two expressions that depend on a single parameter. *range* must be a three-element list; the first element must be the name of the parameter on which *expr_x* and *expr_y* depend, and the other two elements must be the minimum and maximum values for that parameter. The result will be a curve in which the horizontal and vertical coordinates of each point are the values of *expr_x* and *expr_y* for a value of the parameter within the range given.

4. **Set of points**. `plot2d` ([discrete, *points*], *options*), displays a list of points, joined by segments by default. The horizontal and vertical coordinates of each of those points can be specified in three different ways: With two lists of the same length, in which the elements of the first list are the horizontal coordinates of the points and the second list are the vertical coordinates, or with a list of two-element lists, each one corresponding to the two coordinates of one of the points, or with a single list that defines the vertical coordinates of the points; in this last case, the horizontal coordinates of the n points will be assumed to be the first n natural numbers.

5. **Contour lines**. `plot2d` ([contour, *expr*], *range_x*, *range_y*, *options*), where *expr* is an expression that depends on two variables. *range_x* and *range_y* will be lists whose first elements are the names of those two variables, followed by two numbers that set the minimum and maximum values for them. The first variable will be represented along the horizontal axis and the second along the vertical axis. The result will be a set of curves along which the given expression has certain values. If those values are not specified with the option `levels`, plotd2d will try to choose, at the most, 8 values of the form d*10^n, where d is either 1, 2 or 5, all of them within the minimum and maximum values of *expr* within the given ranges.

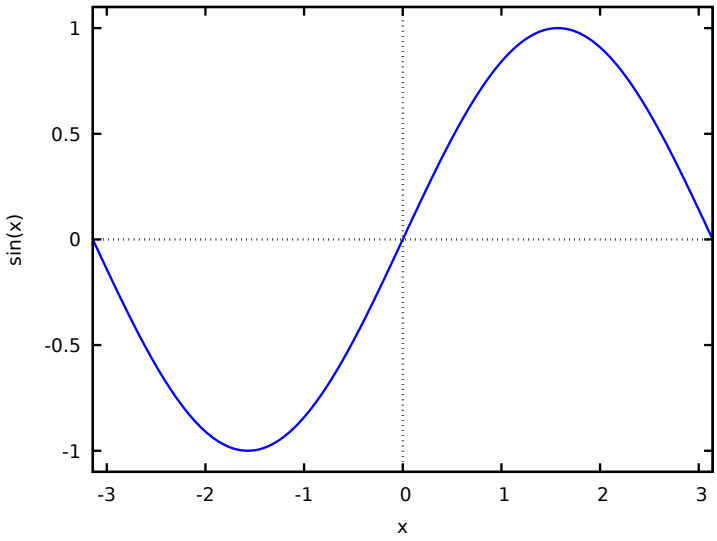
At the end of a plot2d command several of the options described in Section 14.4 Plotting Options can be used. Many instances of the 5 types described above can be combined into a single plot, by putting them inside a list: `plot2d` ([*type_1*, ..., *type_n*], *options*). If one of the types included in the list require *range_x* or *range_y*, those ranges should come immediately after the list.

If there are several plots to be plotted, a legend will be written to identity each of the expressions. The labels that should be used in that legend can be given with the option `legend`. If that option is not used, Maxima will create labels from the expressions or function names.

Examples:

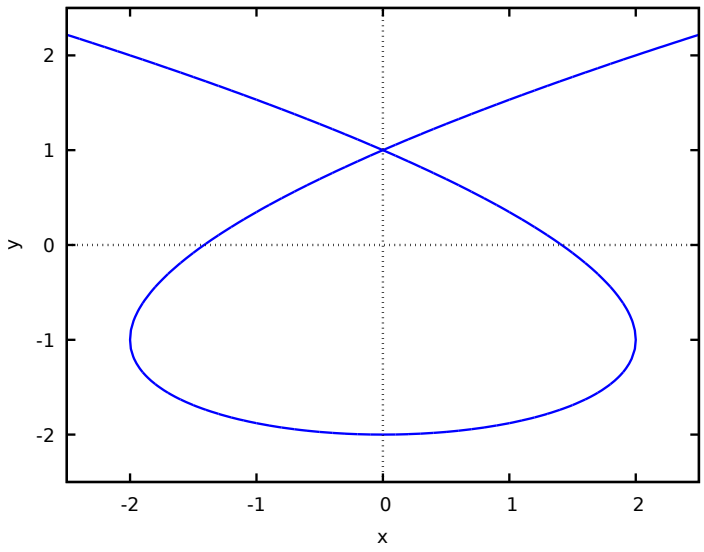
1. Explicit function.

```
(%i1) plot2d (sin(x), [x, -%pi, %pi])$
```



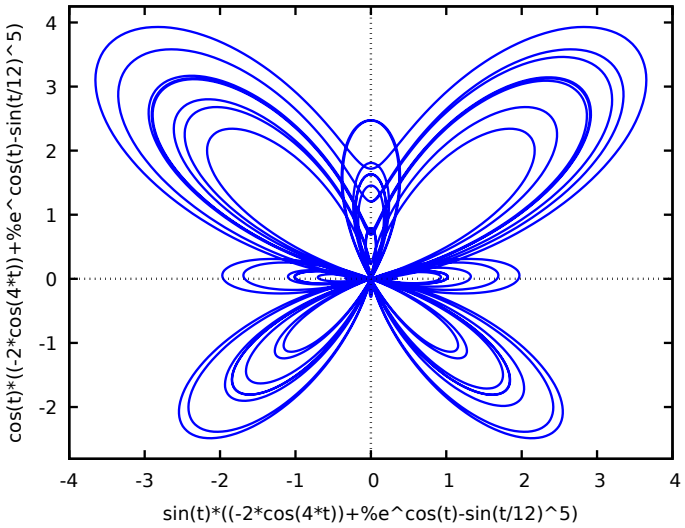
2. Implicit function.

```
(%i1) plot2d (x^2-y^3+3*y=2, [x,-2.5,2.5], [y,-2.5,2.5])$
```



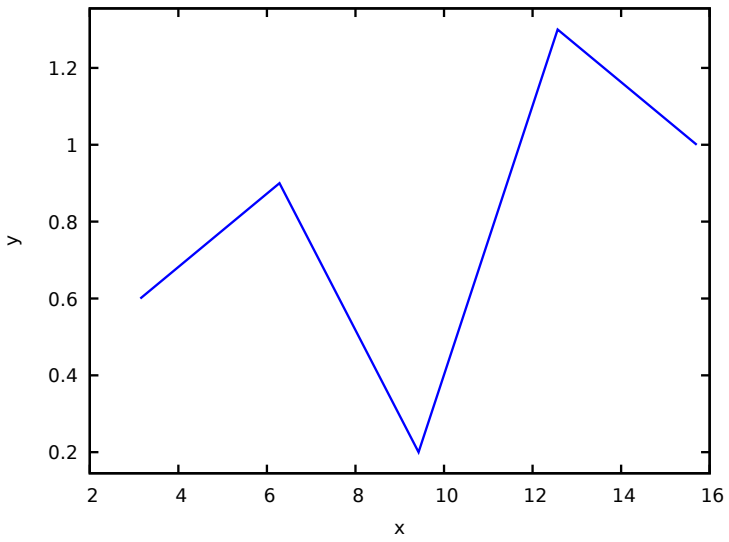
3. Parametric function.

```
(%i1) r: (exp(cos(t))-2*cos(4*t)-sin(t/12)^5)$
(%i2) plot2d([parametric, r*sin(t), r*cos(t), [t,-8*%pi,8*%pi]])$
```



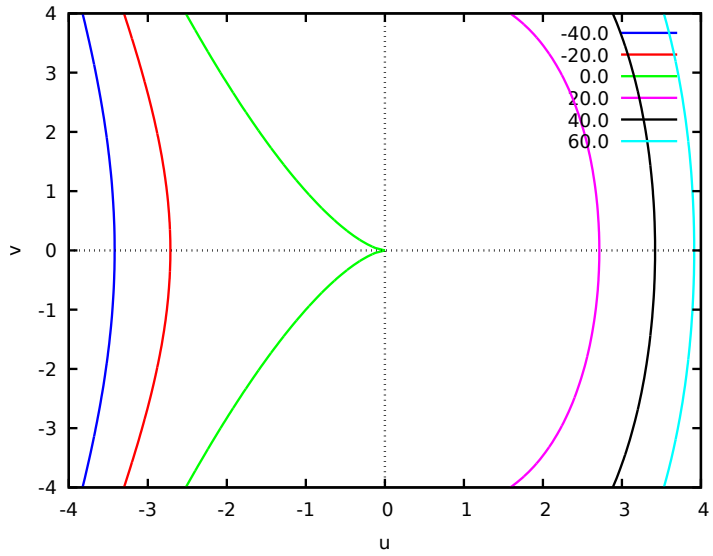
4. Set of points.

```
(%i1) plot2d ([discrete, makelist(i*%pi, i, 1, 5),
                                [0.6, 0.9, 0.2, 1.3, 1]])$
```



5. Contour lines.

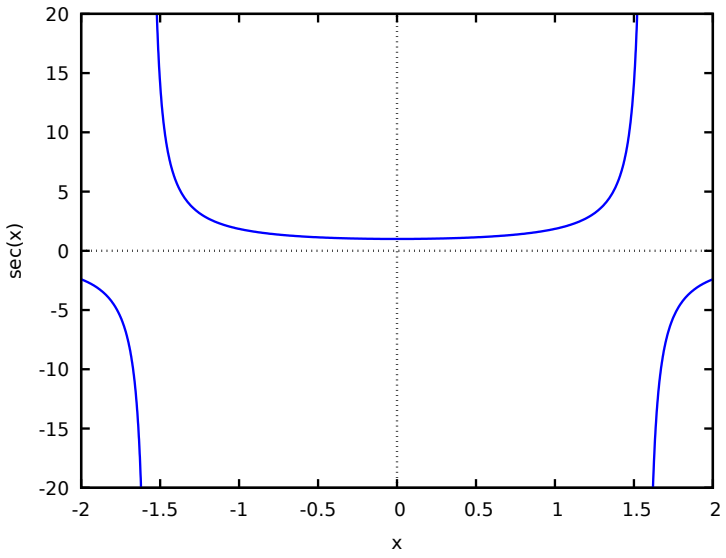
```
(%i1) plot2d ([contour, u^3 + v^2], [u, -4, 4], [v, -4, 4])$
```



Examples using options.

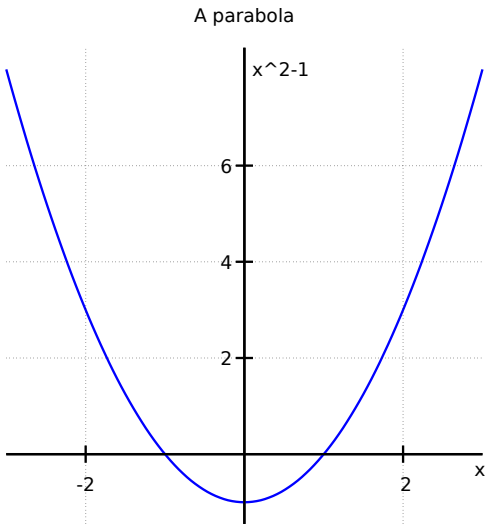
If an explicit function grows too fast, the `y` option can be used to limit the values in the vertical axis:

```
(%i1) plot2d (sec(x), [x, -2, 2], [y, -20, 20])$
plot2d: some values will be clipped.
```



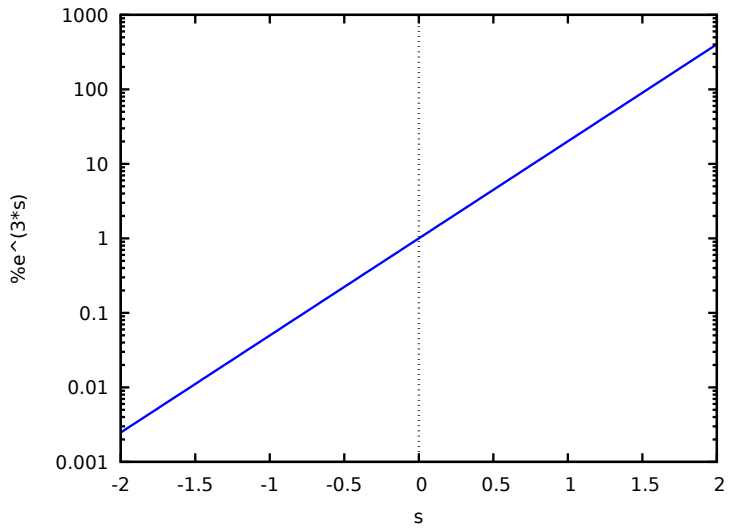
When the plot box is disabled, no labels are created for the axes. In that case, instead of using `xlabel` and `ylabel` to set the names of the axes, it is better to use option `label`, which allows more flexibility. Option `yx_ratio` is used to change the default rectangular shape of the plot; in this example the plot will fill a square.

```
(%i1) plot2d ( x^2 - 1, [x, -3, 3], nobox, grid2d,
     [yx_ratio, 1], [axes, solid], [xtics, -2, 4, 2],
     [ytics, 2, 2, 6], [label, ["x", 2.9, -0.3],
     ["x^2-1", 0.1, 8]], [title, "A parabola"])$
```



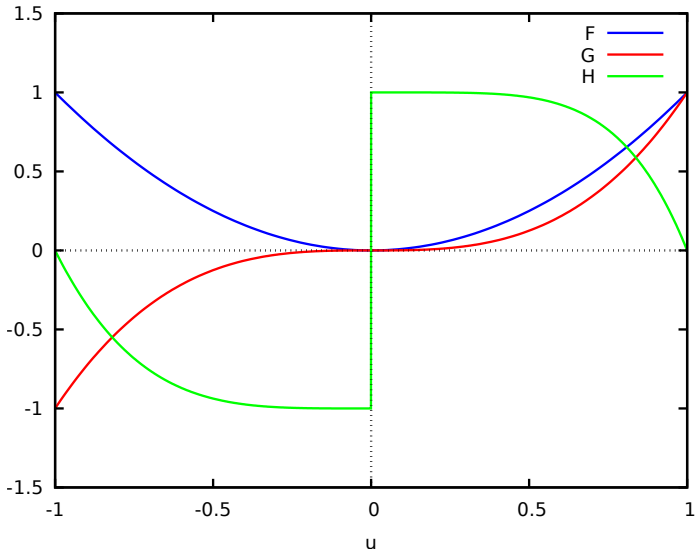
A plot with a logarithmic scale in the vertical axis:

```
(%i1) plot2d (exp(3*s), [s, -2, 2], logy)$
```



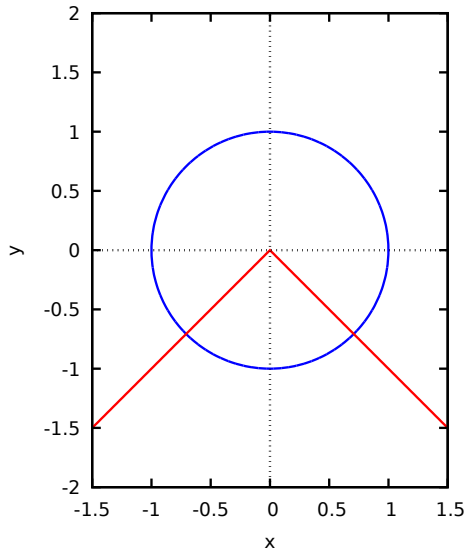
Plotting functions by name:

```
(%i1) F(x) := x^2 $
(%i2) :lisp (defun |$g| (x) (m* x x x))
$g
(%i2) H(x) := if x < 0 then x^4 - 1 else 1 - x^5 $
(%i3) plot2d ([F, G, H], [u, -1, 1], [y, -1.5, 1.5])$
```



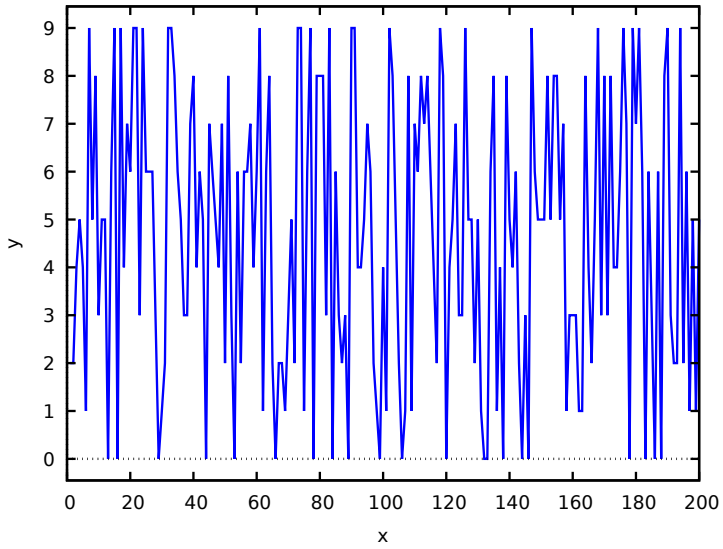
Plot of a circumference of radius 1 centered at the origin, together with the function `y=-|x|`. To prevent the circumference to look like an ellipse, option `same_xy` is used to scale the two axes at the same rate. Option `nolegend` is also used to remove the legend identifying the two functions.

```
(%i1) plot2d([x^2+y^2=1, -abs(x)], [x, -1.5, 1.5], [y, -2, 2], same_xy,
        nolegend);
(%o1)                           false
```



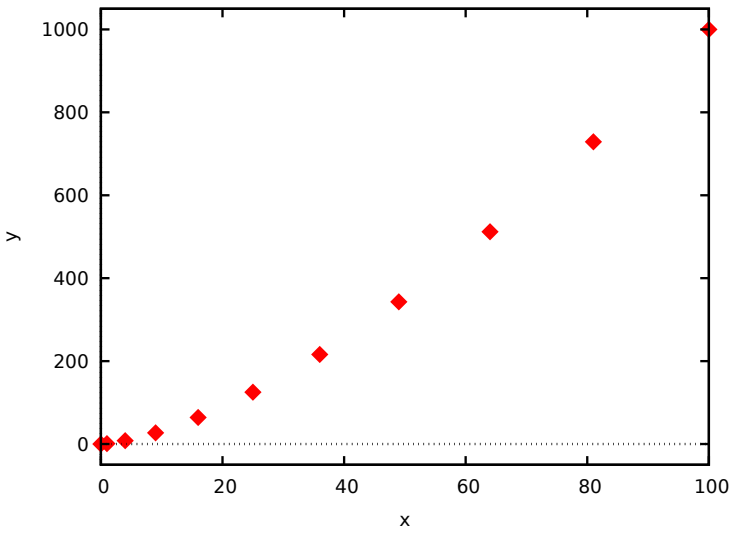
A plot of 200 random numbers between 0 and 9:

```
(%i1) plot2d ([discrete, makelist ( random(10), 200)])$
```



In the next example a table with three columns is saved in a file "data.txt" which is then read and the second and third column are plotted on the two axes:

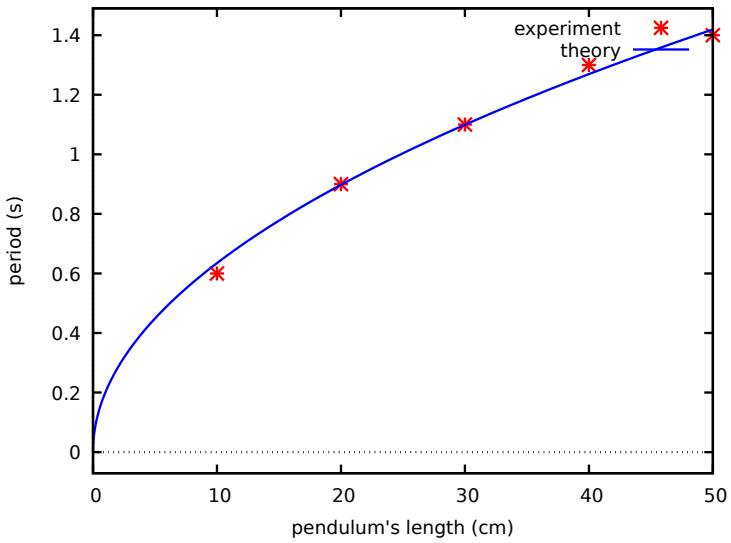
```
(%i1) display2d:false$
(%i2) with_stdout ("data.txt", for x:0 thru 10 do
                               print (x, x^2, x^3))$
(%i3) data: read_matrix ("data.txt")$
(%i4) plot2d ([discrete, transpose(data)[2], transpose(data)[3]],
  [style,points], [point_type,diamond], [color,red])$
```



A plot of discrete data points together with a continuous function:

```
(%i1) xy: [[10, .6], [20, .9], [30, 1.1], [40, 1.3], [50, 1.4]]$
```

```
(%i2) plot2d([[discrete, xy], 2*%pi*sqrt(l/980)], [l,0,50],
        [style, points, lines], [color, red, blue],
        [point_type, asterisk],
        [legend, "experiment", "theory"],
        [xlabel, "pendulum's length (cm)"],
        [ylabel, "period (s)"])$
```



See also the Section 14.4 Plotting Options section.

plot3d [Function]

plot3d (*expr*, *x_range*, *y_range*, ..., *options*, ...)
plot3d ([*expr_1*, ..., *expr_n*], *x_range*, *y_range*, ..., *options*, ...)

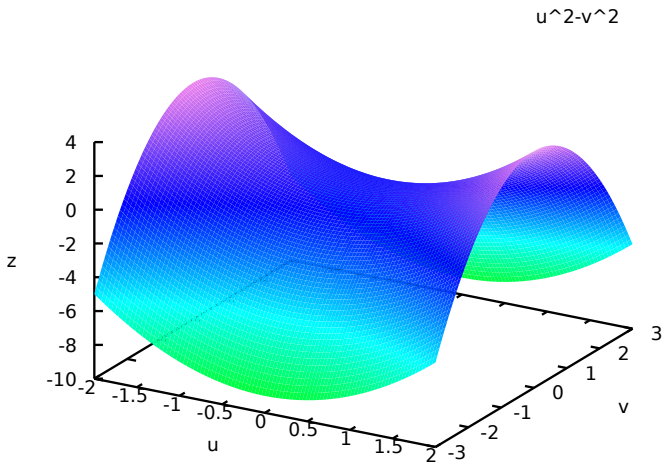
Displays a plot of one or more surfaces defined as functions of two variables or in parametric form.

The functions to be plotted may be specified as expressions or function names. The mouse can be used to rotate the plot looking at the surface from different sides.

Examples.

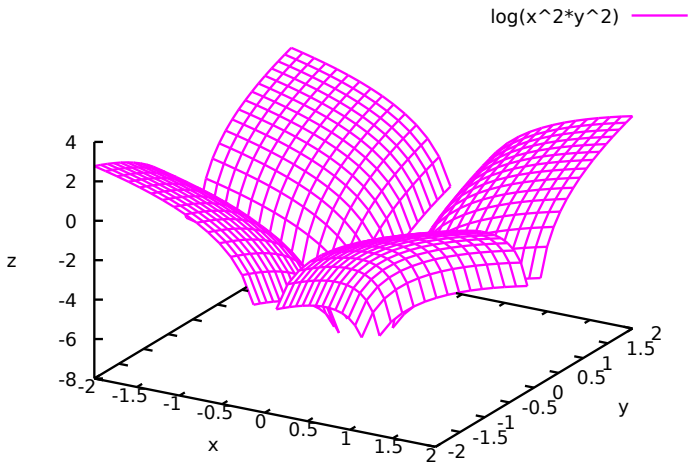
Plot of a function of two variables:

```
(%i1) plot3d (u^2 - v^2, [u, -2, 2], [v, -3, 3], [grid, 100, 100],
       nomesh_lines)$
```



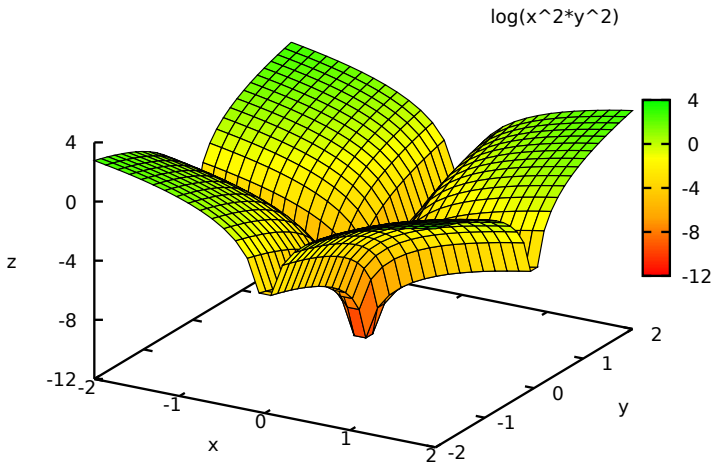
Use of the `z` option to limit a function that goes to infinity (in this case the function is minus infinity on the x and y axes); this also shows how to plot with only lines and no shading:

```
(%i1) plot3d ( log ( x^2*y^2 ), [x, -2, 2], [y, -2, 2], [z, -8, 4],
         nopalette, [color, magenta])$
log: encountered log(0).
```



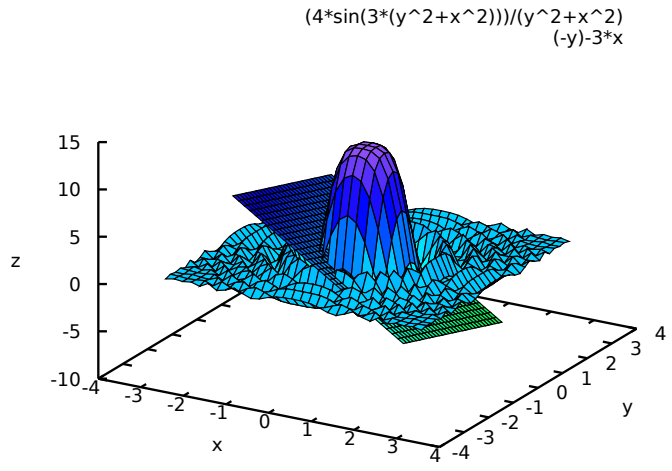
The infinite values of z can also be avoided by choosing a grid that does not fall on any points where the function is undefined, as in the next example, which also shows how to change the palette and how to include a color bar that relates colors to values of the z variable:

```
(%i1) plot3d (log (x^2*y^2), [x, -2, 2], [y, -2, 2],[grid, 29, 29],
      [palette, [gradient, red, orange, yellow, green]],
      color_bar, [xtics, 1], [ytics, 1], [ztics, 4],
      [color_bar_tics, 4])$
log: encountered log(0).
```



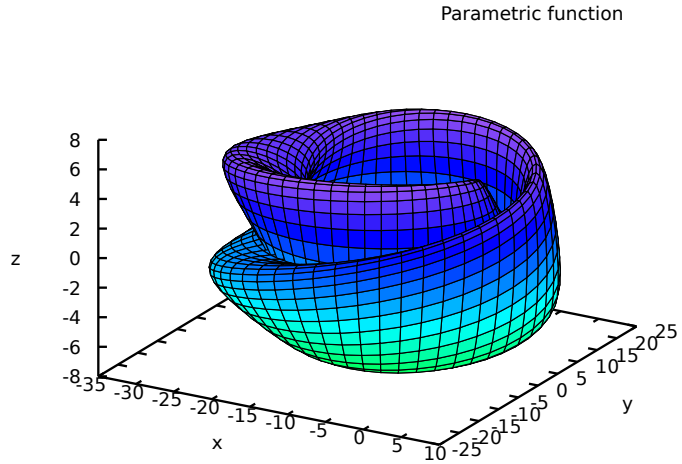
Two surfaces in the same plot. Ranges specific to one of the surfaces can be given by placing each expression and its ranges in a separate list; global ranges for the complete plot are also given after the function definitions.

```
(%i1) plot3d ([[-3*x - y, [x, -2, 2], [y, -2, 2]],
   4*sin(3*(x^2 + y^2))/(x^2 + y^2), [x, -3, 3], [y, -3, 3]],
   [x, -4, 4], [y, -4, 4])$
expt: undefined: 0 to a negative exponent.
```



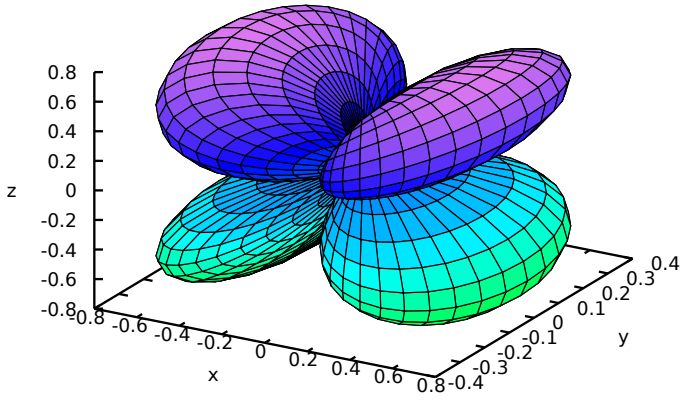
Plot of a Klein bottle, defined parametrically:

```
(%i1) expr_1: 5*cos(x)*(cos(x/2)*cos(y)+sin(x/2)*sin(2*y)+3)-10$
(%i2) expr_2: -5*sin(x)*(cos(x/2)*cos(y)+sin(x/2)*sin(2*y)+3)$
(%i3) expr_3: 5*(-sin(x/2)*cos(y)+cos(x/2)*sin(2*y))$
(%i4) plot3d ([expr_1, expr_2, expr_3], [x, -%pi, %pi],
        [y, -%pi, %pi], [grid, 50, 50])$
```



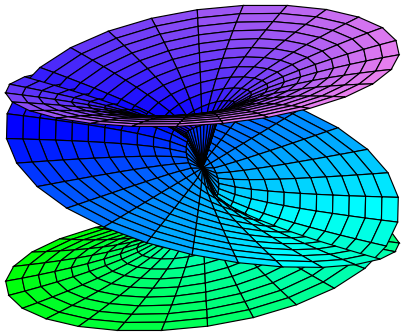
Plot of a "spherical harmonic" function, using the predefined transformation, `spherical_to_xyz` to transform from spherical coordinates to rectangular coordinates. See the documentation for `spherical_to_xyz`.

```
(%i1) plot3d (sin(2*theta)*cos(phi), [theta,0,%pi], [phi,0,2*%pi],
   [transform_xy, spherical_to_xyz], [grid, 30, 60], nolegend)$
```



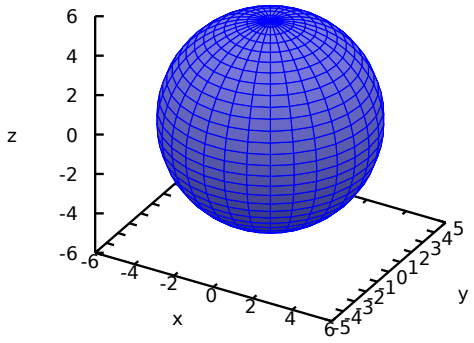
Use of the pre-defined function `polar_to_xy` to transform from cylindrical to rectangular coordinates. See the documentation for `polar_to_xy`.

```
(%i1) plot3d (r^.33*cos(th/3), [r,0,1], [th,0,6*%pi], nobox,
    nolegend, [grid, 12, 80], [transform_xy, polar_to_xy])$
```



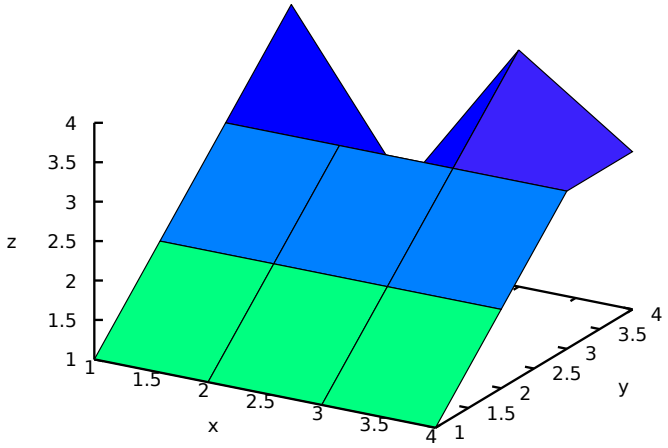
Plot of a sphere using the transformation from spherical to rectangular coordinates. Option `same_xyz` is used to get the three axes scaled in the same proportion. When transformations are used, it is not convenient to eliminate the mesh lines, because Gnuplot will not show the surface correctly.

```
(%i1) plot3d ( 5, [theta,0,%pi], [phi,0,2*%pi], same_xyz, nolegend,
  [transform_xy, spherical_to_xyz], [mesh_lines_color,blue],
  [palette,[gradient,"#1b1b4e", "#8c8cf8"]])$
```



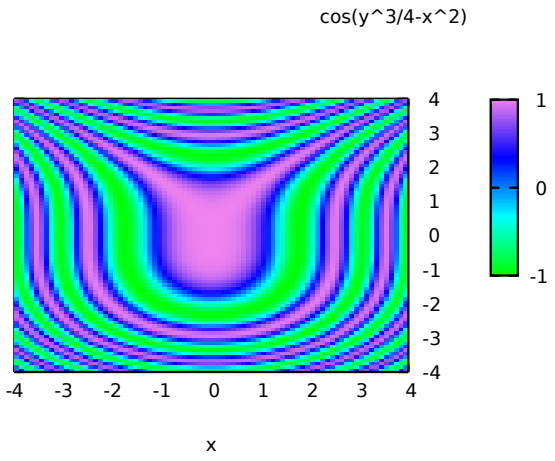
Definition of a function of two-variables using a matrix. Notice the single quote in the definition of the function, to prevent `plot3d` from failing when it realizes that the matrix will require integer indices.

```
(%i1) M: matrix([1,2,3,4], [1,2,3,2], [1,2,3,4], [1,2,3,3])$
(%i2) f(x, y) := float('M [round(x), round(y)])$
(%i3) plot3d (f(x,y), [x,1,4], [y,1,4], [grid,3,3], nolegend)$
```



By setting the elevation equal to zero, a surface can be seen as a map in which each color represents a different level.

```
(%i1) plot3d (cos (-x^2 + y^3/4), [x,-4,4], [y,-4,4], [zlabel,""],
       [mesh_lines_color,false], [elevation,0], [azimuth,0],
       color_bar, [grid,80,80], noztics, [color_bar_tics,1])$
```



See also Section 14.4 Plotting Options.

`plot_options` [System variable]

This variable is being kept for compatibility with older versions of Wxmaxima, but its use is deprecated. To set global plotting options, see their current values or remove options, use `set_plot_option`, `get_plot_option` `remove_plot_option`, and `reset_plot_options`.

`remove_plot_option` (*name*) [Function]

Removes the global value of an option. The name of the option must be given.

See also `set_plot_option`, `get_plot_option` and Section 14.4 Plotting Options.

`reset_plot_options` () [Function]

Sets the default global values of the plotting options. After changing the global values of some plotting options, this function is used to recovered the same values as when Maxima was started.

`set_plot_option` (*option*) [Function]

Accepts any of the options listed in the section Section 14.4 Plotting Options, and saves them for use in plotting commands. The values of the options set in each plotting command will have precedence, but if those options are not given, the default values set with this function will be used.

`set_plot_option` evaluates its argument and returns the complete list of options (after modifying the option given). If called without any arguments, it will simply show the list of current default options.

See also `remove_plot_option`, `get_plot_option` and the section on Section 14.4 Plotting Options.

Example:

Modification of the `grid` values.

```
(%i1) set_plot_option ([grid, 30, 40]);
(%o1) [[plot_format, gnuplot_pipes], [grid, 30, 40],
[run_viewer, true], [axes, true], [nticks, 29],
[adapt_depth, 5], [color, blue, red, green, magenta, black,
cyan], [point_type, bullet, box, triangle, plus, times,
asterisk], [palette, [hue, 0.33333333, 0.7, 1, 0.5],
[hue, 0.8, 0.7, 1, 0.4]], [gnuplot_svg_background, white],
[gnuplot_preamble, ], [gnuplot_term, default]]
```

spherical_to_xyz [System function]

It can be given as value for the transform_xy option of plot3d. Its effect will be to interpret the two independent variables and the function in plot3d as the spherical coordinates of a point (first, the angle with the z axis, then the angle of the xy projection with the x axis and finally the distance from the origin) and transform them into x, y and z coordinates.

xmaxima_plot_command [System variable]

This variable stores the name of the command used to run the xmaxima program when the plot format is xmaxima. Its default value is "xmaxima". If the xmaxima program is not found unless you give its complete path or if you want to try a different version of it, you may change the value of this variable. For instance,

```
(%i1) xmaxima_plot_command: "/usr/local/bin/my_xmaxima"$
```

This variable must contain a string.

See also gnuplot_command and geomview_command.

14.4 Plotting Options

There is a variable plot_options which is kept for backward compatibility with old versions of Wxmaxima but it is not used by the plotting programs. The global options used by the plotting programs are in a list which can only be accessed or modified in Maxima using functions set_plot_option, get_plot_option, remove_plot_option, and reset_plot_options.

Plotting options can also be included inside a plotting command, in which case they have effect only for that plotting command, without changing the global values of those options.

Plotting options consist of a list starting with one of the keywords in this section, followed by one or more values. The options that accept among their possible values true or false, can also be set to true by simply writing their names, and false by writing their names with the prefix no. For instance, typing logx as an option is equivalent to writing [logx, true] and nobox is equivalent to [box, false]. When just the name of the option is used for an option which cannot have a value true, it means that any value previously assigned to that option will be removed, leaving it to the graphic program to decide what to do. There are also some abbreviations for some options as will be shown below.

Some of the options may have different effects in different plotting commands as it will be pointed out for those particular options.

`adapt_depth` [*adapt_depth*, *integer*] [Plot option]

Default value: `5`

The maximum number of splittings used by the adaptive plotting routine of `plot2d`; *integer* must be a non-negative integer. A value of zero means that adaptive plotting will not be used and the resulting plot will have 1+4**nticks* points (see option `nticks`). To have more control on the number of points and their positions, a list of points can be created and then plotted using the `discrete` method of `plot2d`.

`axes` [*axes*, *symbol*], *axes*, *noaxes* [Plot option]

Default value: `true`

Where *symbol* can be either `true`, `false`, `x`, `y` or `solid`. If `false`, no axes are shown; if equal to `x` or `y` only the x or y axis will be shown; if it is equal to `true`, both axes will be shown and `solid` will show the two axes with a solid line, rather than the default broken line. This option does not have any effect in the 3 dimensional plots. The single keywords `axes` and `noaxes` can be used as synonyms for `[axes, true]` and `[axes, false]`.

`azimuth` [*azimuth*, *number*] [Plot option]

Default value: `30`

A plot3d plot can be thought of as starting with the x and y axis in the horizontal and vertical axis, as in plot2d, and the z axis coming out of the screen. The z axis is then rotated around the x axis through an angle equal to `elevation` and then the new xy plane is rotated around the new z axis through an angle `azimuth`. This option sets the value for the azimuth, in degrees.

See also `elevation`.

`box` [*box*, *symbol*], *box*, *nobox* [Plot option]

Default value: `true`

If set to `true`, a bounding box will be drawn for the plot; if set to `false`, no box will be drawn. The single keywords `box` and `nobox` can be used as synonyms for `[box, true]` and `[box, false]`.

`color` [*color*, *color_1*, ..., *color_n*] [Plot option]

In 2d plots it defines the color (or colors) for the various curves. In `plot3d`, it defines the colors used for the mesh lines of the surfaces, when no palette is being used.

If there are more curves or surfaces than colors, the colors will be repeated in sequence. The valid colors are `red`, `green`, `blue`, `magenta`, `cyan`, `yellow`, `orange`, `violet`, `brown`, `gray`, `black`, `white`, or a string starting with the character # and followed by six hexadecimal digits: two for the red component, two for green component and two for the blue component. If the name of a given color is unknown color, black will be used instead.

`color_bar` [*color_bar*, *symbol*], *color_bar*, *nocolor_bar* [Plot option]

Default value: `false` in plot3d, `true` in mandelbrot and julia

Where *symbol* can be either `true` or `false`. If `true`, whenever `plot3d`, `mandelbrot` or `julia` use a palette to represent different values, a box will be shown on the right, showing the corresponding between colors and values. The single keywords `color_bar`

and `nocolor_bar` can be used as synonyms for `[color_bar, true]` and `[color_bar, false]`.

`color_bar_tics` [*color_bar_tics*, *x1*, *x2*, *x3*], *color_bar_tics*, *nocolor_bar_tics* [Plot option]

Defines the values at which a mark and a number will be placed in the color bar. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted. When only one number is given, it will be used as the increment from an initial value that will be chosen automatically. The single keyword `color_bar_tics` removes a value given previously, making the graphic program use its default for the values of the tics and `nocolor_bar_tics` will not show any tics on the color bar.

`elevation` [*elevation*, *number*] [Plot option]

Default value: `60`

A plot3d plot can be thought of as starting with the x and y axis in the horizontal and vertical axis, as in plot2d, and the z axis coming out of the screen. The z axis is then rotated around the x axis through an angle equal to `elevation` and then the new xy plane is rotated around the new z axis through an angle `azimuth`. This option sets the value for the elevation, in degrees.

See also `azimuth`.

`gnuplot` [Plot option]

This is an abbreviation for `[plot_format, gnuplot]`. See `plot_format`.

`gnuplot_pipes` [Plot option]

This is an abbreviation for `[plot_format, gnuplot_pipes]`. See `plot_format`.

`grid` [*grid*, *integer*, *integer*] [Plot option]

Default value: `30`, `30`

Sets the number of grid points to use in the x- and y-directions for three-dimensional plotting or for the `julia` and `mandelbrot` programs.

For a way to actually draw a grid See `grid2d`.

`grid2d` [*grid2d*, *value*], *grid2d*, *nogrid2d* [Plot option]

Default value: `false`

Shows a grid of lines on the xy plane. The points where the grid lines are placed are the same points where tics are marked in the x and y axes, which can be controlled with the `xtics` and `ytics` options. The single keywords `grid2d` and `nogrid2d` can be used as synonyms for `[grid2d, true]` and `[grid2d, false]`.

See also `grid`.

`iterations` [*iterations*, *value*] [Plot option]

Default value: `9`

Number of iterations made by the programs mandelbrot and julia.

`label` [*label*, [*string*, *x*, *y*], ...] [Plot option]

Writes one or several labels in the points with *x*, *y* coordinates indicated after each label.

`legend` [*legend*, *string_1*, ..., *string_n*], *legend*, *nolegend* [Plot option]

It specifies the labels for the plots when various plots are shown. If there are more plots than the number of labels given, they will be repeated. If given the value `false`, no legends will be shown. By default, the names of the expressions or functions will be used, or the words discrete1, discrete2, ..., for discrete sets of points. The single keyword `legend` removes any previously defined legends, leaving it to the plotting program to set up a legend. The keyword `nolegend` is a synonym for `[legend, false]`.

`levels` [*levels*, *number*, ...] [Plot option]

This option is used by `plot2d` to do contour plots. If only one number is given after the keyword `levels`, it must be a natural number; `plot2d` will try to plot that number of contours with values between the minimum and maximum value of the expression given, with the form d*10^n, where d is either 1, 2 or 5. Since not always it will be possible to find that number of levels in that interval, the number of contour lines show will be smaller than the number specified by this option.

If more than one number are given after the keyword `levels`, `plot2d`. will show the contour lines corresponding to those values of the expression plotted, if they exist within the domain used.

`logx` [*logx*, *value*], *logx*, *nologx* [Plot option]

Default value: `false`

Makes the horizontal axes to be scaled logarithmically. It can be either true or false. The single keywords `logx` and `nologx` can be used as synonyms for `[logx, true]` and `[logx, false]`.

`logy` [*logy*, *value*], *logy*, *nology* [Plot option]

Default value: `false`

Makes the vertical axes to be scaled logarithmically. It can be either true or false. The single keywords `logy` and `nology` can be used as synonyms for `[logy, true]` and `[logy, false]`.

`mesh_lines_color` [*mesh_lines_color*, *color*], *mesh_lines_color*, *nomesh_lines* [Plot option]

Default value: `black`

It sets the color used by plot3d to draw the mesh lines, when a palette is being used. It accepts the same colors as for the option `color` (see the list of allowed colors in `color`). It can also be given a value `false` to eliminate completely the mesh lines. The single keyword `mesh_lines_color` removes any previously defined colors, leaving it to the graphic program to decide what color to use. The keyword `nomesh_lines` is a synonym for `[mesh_lines_color, false]`

`nticks` [*nticks*, *integer*] [Plot option]

Default value: `29`

When plotting functions with `plot2d`, it is gives the initial number of points used by the adaptive plotting routine for plotting functions. When plotting parametric functions with `plot3d`, it sets the number of points that will be shown for the plot.

`palette` [*palette*, [`palette_1`], ..., [`palette_n`]], *palette*, *nopalette* [Plot option]

It can consist of one palette or a list of several palettes. Each palette is a list with a keyword followed by values. If the keyword is gradient, it should be followed by a list of valid colors.

If the keyword is hue, saturation or value, it must be followed by 4 numbers. The first three numbers, which must be between 0 and 1, define the hue, saturation and value of a basic color to be assigned to the minimum value of z. The keyword specifies which of the three attributes (hue, saturation or value) will be increased according to the values of z. The last number indicates the increase corresponding to the maximum value of z. That last number can be bigger than 1 or negative; the corresponding values of the modified attribute will be rounded modulo 1.

Gnuplot only uses the first palette in the list; xmaxima will use the palettes in the list sequentially, when several surfaces are plotted together; if the number of palettes is exhausted, they will be repeated sequentially.

The color of the mesh lines will be given by the option `mesh_lines_color`. If `palette` is given the value `false`, the surfaces will not be shaded but represented with a mesh of curves only. In that case, the colors of the lines will be determined by the option `color`.

The single keyword `palette` removes any palette previously defined, leaving it to the graphic program to decide the palette to use and `nopalette` is a synonym for `[palette, false]`.

`plotepsilon` [*plotepsilon*, *value*] [Plot option]

Default value: 1e-6

This value is used by `plot2d` when plotting implicit functions or contour lines. When plotting an explicit function `expr_1=expr_2`, it is converted into `expr_1-expr_2` and the points where that equals zero are searched. When a contour line for `expr` equal to some value is going to be plotted, the points where `expr` minus that value are equal to zero are searched. The search is done by computing those expressions at a grid of points (see `sample`). If at one of the points in that grid the absolute value of the expression is less than the value of `plotepsilon`, it will be assumed to be zero, and therefore, as being part of the curve to be plotted.

`plot_format` [*plot_format*, *format*], *gnuplot*, *gnuplot_pipes*, *xmaxima* [Plot option]

Default value: `gnuplot`, in Windows systems, or `gnuplot_pipes` in other systems.

Where *format* is one of the following: gnuplot, xmaxima, mgnuplot, gnuplot_pipes or geomview. The three possibilities: `[plot_format, gnuplot]`, `[plot_format, gnuplot_pipes]` and `[plot_format, xmaxima]`, can be abbreviated by the keywords `gnuplot`, `gnuplot_pipes` and `xmaxima`.

It sets the format to be used for plotting as explained in Section 14.2 Plotting Formats.

`plot_realpart` [*plot_realpart*, *symbol*], *plot_realpart*, *noplot_realpart* [Plot option]

Default value: `false`

If set to `true`, the functions to be plotted will be considered as complex functions whose real value should be plotted; this is equivalent to plotting `realpart(`*`function`*`)`. If set to `false`, nothing will be plotted when the function does not give a real value. For instance, when `x` is negative, `log(x)` gives a complex value, with real value equal to `log(abs(x))`; if `plot_realpart` were `true`, `log(-5)` would be plotted as `log(5)`, while nothing would be plotted if `plot_realpart` were `false`. The single keyword `plot_realpart` can be used as a synonym for `[plot_realpart, true]` and `noplot_realpart` is a synonym for `[plot_realpart, false]`.

`point_type` [*point_type*, *type_1*, ..., *type_n*] [Plot option]
In gnuplot, each set of points to be plotted with the style "points" or "linespoints" will be represented with objects taken from this list, in sequential order. If there are more sets of points than objects in this list, they will be repeated sequentially. The possible objects that can be used are: `bullet`, `circle`, `plus`, `times`, `asterisk`, `box`, `square`, `triangle`, `delta`, `wedge`, `nabla`, `diamond`, `lozenge`.

`pdf_file` [*pdf_file*, *file_name*] [Plot option]
Saves the plot into a PDF file with name equal to *file_name*, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable `maxima_tempdir`, unless *file_name* contains the character "/", in which case it will be assumed to contain the complete path where the file should be created. The value of `maxima_tempdir` can be changed to save the file in a different directory. When the option `gnuplot_pdf_term_command` is also given, it will be used to set up Gnuplot's PDF terminal; otherwise, Gnuplot's pdfcairo terminal will be used with solid colored lines of width 3, plot size of 17.2 cm by 12.9 cm and font of 18 points.

`png_file` [*png_file*, *file_name*] [Plot option]
Saves the plot into a PNG graphics file with name equal to *file_name*, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable `maxima_tempdir`, unless *file_name* contains the character "/", in which case it will be assumed to contain the complete path where the file should be created. The value of `maxima_tempdir` can be changed to save the file in a different directory. When the option `gnuplot_png_term_command` is also given, it will be used to set up Gnuplot's PNG terminal; otherwise, Gnuplot's pngcairo terminal will be used, with a font of size 12.

`ps_file` [*ps_file*, *file_name*] [Plot option]
Saves the plot into a Postscript file with name equal to *file_name*, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable `maxima_tempdir`, unless *file_name* contains the character "/", in which case it will be assumed to contain the complete path where the file should be created. The value of `maxima_tempdir` can be changed to save the file in a different directory. When the option `gnuplot_ps_term_command` is also given, it will be used to set up Gnuplot's Postscript terminal; otherwise, Gnuplot's postscript terminal will be used with the EPS option, solid colored lines of width 2, plot size of 16.4 cm by 12.3 cm and font of 24 points.

`run_viewer` [*run_viewer*, *symbol*], *run_viewer*, *norun_viewer* [Plot option]
Default value: `true`

This option is only used when the plot format is `gnuplot` and the terminal is `default` or when the Gnuplot terminal is set to `dumb` (see `gnuplot_term`) and can have a true or false value.

If the terminal is `default`, a file `random-string.gnuplot` (or other name specified with `gnuplot_out_file`) is created with the gnuplot commands necessary to generate the plot. Option `run_viewer` controls whether or not Gnuplot will be launched to execute those commands and show the plot.

If the terminal is `default`, gnuplot is run to execute the commands in `random-string.gnuplot`, producing another file `maxplot.txt` (or other name specified with `gnuplot_out_file`). Option `run_viewer` controls whether or not that file, with an ASCII representation of the plot, will be shown in the Maxima or Xmaxima console.

Its default value, true, makes the plots appear in either the console or a separate graphics window. `run_viewer` and `norun_viewer` are synonyms for `[run_viewer, true]` and `[run_viewer, false]`.

`same_xy` [*same_xy* , *value*], *same_xy*, *nosame_xy* [Plot option]
It can be either true or false. If true, the scales used in the x and y axes will be the same, in either 2d or 3d plots. See also `yx_ratio`. `same_xy` and `nosame_xy` are synonyms for `[same_xy, true]` and `[same_xy, false]`.

`same_xyz` [*same_xyz* , *value*], *same_xyz*, *nosame_xyz* [Plot option]
It can be either true or false. If true, the scales used in the 3 axes of a 3d plot will be the same. `same_xyz` and `nosame_xyz` are synonyms for `[same_xyz, true]` and `[same_xyz, false]`.

`sample` [*sample*, *nx*, *ny*] [Plot option]
Default value: `[sample, 47, 47]`

nx and *ny* must be two natural numbers that will be used by `plot2d` to look for the points that make part of the plot of an implicit function or a contour line. The domain is divided into *nx* intervals in the horizontal axis and *ny* intervals in the vertical axis and the numerical value of the expression is computed at the borders of those intervals. Higher values of *nx* and *ny* will give smoother curves, but will increase the time needed to trace the plot. When there are critical points in the plot where the curve changes direction, to get better results it is more important to try to make those points to be at the border of the intervals, rather than increasing *nx* and *ny*.

`style` [*style*, *type_1*, ..., *type_n*], [*style*, [*style_1*], ..., [*style_n*]] [Plot option]
The styles that will be used for the various functions or sets of data in a 2d plot. The word *style* must be followed by one or more styles. If there are more functions and data sets than the styles given, the styles will be repeated. Each style can be either *lines* for line segments, *points* for isolated points, *linespoints* for segments and points, or *dots* for small isolated dots. Gnuplot accepts also an *impulses* style.

Each of the styles can be enclosed inside a list with some additional parameters. *lines* accepts one or two numbers: the width of the line and an integer that identifies a color. The default color codes are: 1: blue, 2: red, 3: magenta, 4: orange, 5: brown, 6: lime and 7: aqua. If you use Gnuplot with a terminal different than X11, those colors might be different; for example, if you use the option [*gnuplot_term*, *ps*], color index 4 will correspond to black, instead of orange.

points accepts one, two, or three parameters; the first parameter is the radius of the points, the second parameter is an integer that selects the color, using the same code used for *lines* and the third parameter is currently used only by Gnuplot and it corresponds to several objects instead of points. The default types of objects are: 1: filled circles, 2: open circles, 3: plus signs, 4: x, 5: *, 6: filled squares, 7: open squares, 8: filled triangles, 9: open triangles, 10: filled inverted triangles, 11: open inverted triangles, 12: filled lozenges and 13: open lozenges.

linespoints accepts up to five parameters: line width, points radius, color, type of object to replace the points, and the gnuplot pointinterval option to control space between points.

See also `color` and `point_type`.

`svg_file` [*svg_file*, *file_name*] [Plot option]

Saves the plot into an SVG file with name equal to *file_name*, rather than showing it in the screen. By default, the file will be created in the directory defined by the variable `maxima_tempdir`, unless *file_name* contains the character "/", in which case it will be assumed to contain the complete path where the file should be created. The value of `maxima_tempdir` can be changed to save the file in a different directory. When the option `gnuplot_svg_term_command` is also given, it will be used to set up Gnuplot's SVG terminal; otherwise, Gnuplot's svg terminal will be used with font of 14 points.

`t` [*t*, *min*, *max*] [Plot option]

Default range for parametric plots.

`title` [*title*, *text*] [Plot option]

Defines a title that will be written at the top of the plot.

`transform_xy` [*transform_xy*, *symbol*], *notransform_xy* [Plot option]

Where *symbol* is either `false` or the result obtained by using the function `transform_xy`. If different from `false`, it will be used to transform the 3 coordinates in plot3d. `notransform_xy` removes any transformation function previously defined.

See `make_transform`, `polar_to_xy` and `spherical_to_xyz`.

`window` [*window*, *n*] [Plot option]

Opens the plot in window number *n*, instead of the default window 0. If window number *n* is already opened, the plot in that window will be replaced by the current plot.

`x` [*x*, *min*, *max*] [Plot option]

When used as the first option in a `plot2d` command (or any of the first two in `plot3d`), it indicates that the first independent variable is x and it sets its range. It can also

be used again after the first option (or after the second option in plot3d) to define the effective horizontal domain that will be shown in the plot.

`xlabel` [*xlabel*, *string*], *noxlabel* [Plot option]
Specifies the *string* that will label the first axis; if this option is not used, that label will be the name of the independent variable, when plotting functions with `plot2d` the name of the first variable, when plotting surfaces with `plot3d`, or the first expression in the case of a parametric plot. `noxlabel` is equivalent to `[xlabel, ""]`, which does not print any label on the first axis.

`xmaxima` [Plot option]
This is an abbreviation for `[plot_format, xmaxima]`. See `plot_format`.

`xtics` [*xtics*, *x1*, *x2*, *x3*], [*xtics*, `false`], *xtics*, *noxtics* [Plot option]
Defines the values at which a mark and a number will be placed in the x axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in which case the first number is the increment from an initial value that will be chosen by the graphic program. If `[xtics, false]` is used, no marks or numbers will be shown along the x axis.

The single keyword `xtics` removes any values previously defined, leaving it to the graphic program to decide the values to use and `noxtics` is a synonym for `[xtics, false]`

`xy_scale` [*xy_scale*, *sx*, *sy*] [Plot option]
In a 2d plot, it defines the ratio of the total size of the Window to the size that will be used for the plot. The two numbers given as arguments are the scale factors for the x and y axes.

This option does not change the size of the graphic window or the placement of the graph in the window. If you want to change the aspect ratio of the plot, it is better to use option `yx_ratio`. For instance, `[yx_ratio, 10]` instead of `[xy_scale, 0.1, 1]`.

`y` [*y*, *min*, *max*] [Plot option]
When used as one of the first two options in `plot3d`, it indicates that one of the independent variables is y and it sets its range. Otherwise, it defines the effective domain of the second variable that will be shown in the plot.

`ylabel` [*ylabel*, *string*], *noylabel* [Plot option]
Specifies the *string* that will label the second axis; if this option is not used, that label will be "y", when plotting explicit functions with `plot2d`, or the name of the second variable, when plotting surfaces with `plot3d`, or the second expression in the case of a parametric plot. `noylabel` is equivalent to `[ylabel, ""]`, which does not print any label on the second axis.

`ytics` [*ytics*, *y1*, *y2*, *y3*], [*ytics*, `false`], *ytics*, *noytics* [Plot option]
Defines the values at which a mark and a number will be placed in the y axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in

which case the first number is the increment from an initial value that will be chosen by the graphic program. If `[ytics, false]` is used, no marks or numbers will be shown along the y axis.

The single keyword `ytics` removes any values previously defined, leaving it to the graphic program to decide the values to use and `noytics` is a synonym for `[ytics, false]`

`yx_ratio` [*yx_ratio*, `r`] [Plot option]
In a 2d plot, the ratio between the vertical and the horizontal sides of the rectangle used to make the plot. See also `same_xy`.

`z` [*z*, `min`, `max`] [Plot option]
Used in `plot3d` to set the effective range of values of z that will be shown in the plot.

`zlabel` [*zlabel*, `string`], *nozlabel* [Plot option]
Specifies the *string* that will label the third axis, when using `plot3d`. If this option is not used, that label will be "z", when plotting surfaces, or the third expression in the case of a parametric plot. It can not be used with `set_plot_option` and it will be ignored by `plot2d`. `nozlabel` is equivalent to `[zlabel, ""]`, which does not print any label on the third axis.

`zmin` [*zmin*, `z`], *zmin* [Plot option]
In 3d plots, the value of z that will be at the bottom of the plot box.

The single keyword `zmin` removes any value previously defined, leaving it to the graphic program to decide the value to use.

`ztics` [*ztics*, `z1`, `z2`, `z3`], [*ztics*, `false`], *ztics*, *noztics* [Plot option]
Defines the values at which a mark and a number will be placed in the z axis. The first number is the initial value, the second the increments and the third is the last value where a mark is placed. The second and third numbers can be omitted, in which case the first number is the increment from an initial value that will be chosen by the graphic program. If `[ztics, false]` is used, no marks or numbers will be shown along the z axis.

The single keyword `ztics` removes any values previously defined, leaving it to the graphic program to decide the values to use and `noztics` is a synonym for `[ztics, false]`

14.5 Gnuplot Options

There are several plot options specific to gnuplot. All of them consist of a keyword (the name of the option), followed by a string that should be a valid gnuplot command, to be passed directly to gnuplot. In most cases, there exist a corresponding plotting option that will produce a similar result and whose use is more recommended than the gnuplot specific option.

`gnuplot_term` [*gnuplot_term*, `terminal_name`] [Plot option]
Sets the output terminal type for gnuplot. The argument *terminal_name* can be a string or one of the following 3 special symbols

- **default** (default value)

  Gnuplot output is displayed in a separate graphical window and the gnuplot terminal used will be specified by the value of the option gnuplot_default_term_command.

- **dumb**

  Gnuplot output is saved to a file `random-string.txt` using "ASCII art" approximation to graphics. If the option gnuplot_out_file is set to *filename*, the plot will be saved there, instead of the default `random-string.gnuplot`. The settings for the "dumb" terminal of Gnuplot are given by the value of option gnuplot_dumb_term_command. If option run_viewer is set to true and the plot_format is `gnuplot` that ASCII representation will also be shown in the Maxima or Xmaxima console.

- **ps**

  Gnuplot generates commands in the PostScript page description language. If the option gnuplot_out_file is set to *filename*, gnuplot writes the PostScript commands to *filename*. Otherwise, it is saved as `maxplot.ps` file. The settings for this terminal are given by the value of the option gnuplot_dumb_term_command.

- A string representing any valid gnuplot term specification

  Gnuplot can generate output in many other graphical formats such as png, jpeg, svg etc. To use those formats, option `gnuplot_term` can be set to any supported gnuplot term name (which must be a symbol) or even a full gnuplot term specification with any valid options (which must be a string). For example `[gnuplot_term, png]` creates output in PNG (Portable Network Graphics) format while `[gnuplot_term, "png size 1000,1000"]` creates PNG of 1000 x 1000 pixels size. If the option gnuplot_out_file is set to *filename*, gnuplot writes the output to *filename*. Otherwise, it is saved as `maxplot.`*`term`* file, where *term* is gnuplot terminal name.

`gnuplot_out_file` [*gnuplot_out_file*, *`file_name`*] [Plot option]

It can be used to replace the default name for the file that contains the commands that will interpreted by gnuplot, when the terminal is set to `default`, or to replace the default name of the graphic file that gnuplot creates, when the terminal is different from `default`. If it contains one or more slashes, "/", the name of the file will be left as it is; otherwise, it will be appended to the path of the temporary directory. The complete name of the files created by the plotting commands is always sent as output of those commands so they can be seen if the command is ended by semi-colon.

`gnuplot_script_file` [*gnuplot_script_file*, *`file_name_or_function`*] [Plot option]

Creates a plot with `plot2d`, `plot3d`, `mandelbrot` or `julia` using the `gnuplot` plot_format and saving the script sent to Gnuplot in the file specified by *file_name_or_function*. The value *file_name_or_function* can be a string or a Maxima function of one variable that returns a string. If that string corresponds to a complete file path (directory and file name), the script will be created in that file and will not be deleted after Maxima is closed; otherwise, the string will give the

name of a file to be created in the temporary directory and deleted when Maxima is closed.

In this example, the script file name is set to "sin.gnuplot", in the current directory.

```
(%i1) plot2d( sin(x), [x,0,2*%pi], [gnuplot_script_file, "./sin.gnuplot"]);
(%o1)                      [./sin.gnuplot]
```

In this example, `gnuplot_prt(file)` is a function that takes the default file name, *file*. It constructs a full file name for the data file by interpolating a random 8-digit integer with a pad of zeros into the default file name (a 16-character random string followed by ".gnuplot"). The temporary directory is determined by `maxima_tempdir` (it is "/tmp" in this example).

```
(%i1) gnuplot_prt(file) := block([beg,end],
       [beg,end] : split(file,"."),
       printf(false,"~a_~8,'0d.~a",beg,random(10^8-1),end)) $
(%i2) plot2d( sin(x), [x,0,2*%pi], [gnuplot_script_file, gnuplot_prt]);
(%o2)       [/tmp/nxuo4x28s6wocvjw_99211646.gnuplot]
```

By default, the script would have been saved in a file named `random-string.gnuplot` (random-string = nxuo4x28s6wocvjw_99211646 in this example) in the temporary directory. If the print statement in function `gnuplot_prt` above included a directory path, the file would have been saved in that directory rather than in the temporary directory.

`gnuplot_pm3d` [*gnuplot_pm3d*, *value*] [Plot option]

With a value of `false`, it can be used to disable the use of PM3D mode, which is enabled by default.

`gnuplot_preamble` [*gnuplot_preamble*, *string*] [Plot option]

This option inserts gnuplot commands before any other commands sent to Gnuplot. Any valid gnuplot commands may be used. Multiple commands should be separated with a semi-colon. See also `gnuplot_postamble`.

`gnuplot_postamble` [*gnuplot_postamble*, *string*] [Plot option]

This option inserts gnuplot commands after other commands sent to Gnuplot and right before the plot command is sent. Any valid gnuplot commands may be used. Multiple commands should be separated with a semi-colon. See also `gnuplot_preamble`.

`gnuplot_default_term_command` [Plot option]

[gnuplot_default_term_command, *command*]

The gnuplot command to set the terminal type for the default terminal. It this option is not set, the command used will be: `"set term wxt size 640,480 font \",12\"; set term pop"`.

`gnuplot_dumb_term_command` [Plot option]

[gnuplot_dumb_term_command, *command*]

The gnuplot command to set the terminal type for the dumb terminal. It this option is not set, the command used will be: `"set term dumb 79 22"`, which makes the text output 79 characters by 22 characters.

`gnuplot_pdf_term_command` [*gnuplot_pdf_term_command*, *command*] [Plot option]

The gnuplot command to set the terminal type for the PDF terminal. If this option is not set, the command used will be: `"set term pdfcairo color solid lw 3 size 17.2 cm, 12.9 cm font \",18\""`. See the gnuplot documentation for more information.

`gnuplot_png_term_command` [*gnuplot_png_term_command*, *command*] [Plot option]

The gnuplot command to set the terminal type for the PNG terminal. If this option is not set, the command used will be: `"set term pngcairo font \",12\""`. See the gnuplot documentation for more information.

`gnuplot_ps_term_command` [*gnuplot_ps_term_command*, *command*] [Plot option]

The gnuplot command to set the terminal type for the PostScript terminal. If this option is not set, the command used will be: `"set term postscript eps color solid lw 2 size 16.4 cm, 12.3 cm font \",24\""`. See the gnuplot documentation for `set term postscript` for more information.

`gnuplot_strings` [*gnuplot_strings*, *value*] [Plot option]

With a value of `true`, all strings used in labels and titles will be interpreted by gnuplot as "enhanced" text, if the terminal being used supports it. In enhanced mode, some characters such as `^` and `_` are not printed, but interpreted as formatting characters. For a list of the formatting characters and their meaning, see the documentation for enhanced in Gnuplot. The default value for this option is `false`, which will not treat any characters as formatting characters.

`gnuplot_svg_background` [Plot option]

[gnuplot_svg_background, *color*]
nognuplot_svg_background

Specify the background color for SVG image output. *color* must be a string which specifies a color name recognized by Gnuplot, or an RGB triple of the form `"#xxxxxx"` where `x` is a hexadecimal digit. The default value is `"white"`.

When the value of `gnuplot_svg_background` is `false`, the background is the default determined by Gnuplot. At present (April 2023), the Gnuplot default is to specify the background in SVG output as `"none"`.

`nognuplot_svg_background`, specified by itself without a value, is equivalent to `[gnuplot_svg_background, false]`.

`gnuplot_svg_term_command` [*gnuplot_svg_term_command*, *command*] [Plot option]

The gnuplot command to set the terminal type for the SVG terminal. If this option is not set, the command used will be: `"set term svg font \",14\""`. See the gnuplot documentation for more information.

`gnuplot_curve_titles` [Plot option]

This is an obsolete option that has been replaced by `legend` described above.

`gnuplot_curve_styles` [Plot option]

This is an obsolete option that has been replaced by `style`.

14.6 Gnuplot_pipes Format Functions

`gnuplot_start` () [Function]

Opens the pipe to gnuplot used for plotting with the `gnuplot_pipes` format. Is not necessary to manually open the pipe before plotting.

`gnuplot_close` () [Function]

Closes the pipe to gnuplot which is used with the `gnuplot_pipes` format.

`gnuplot_send` (*s*) [Function]

Sends the command *s* to the gnuplot pipe. If that pipe is not currently opened, it will be opened before sending the command. *s* must be a string.

`gnuplot_restart` () [Function]

Closes the pipe to gnuplot which is used with the `gnuplot_pipes` format and opens a new pipe.

`gnuplot_replot` [Function]

`gnuplot_replot` ()
`gnuplot_replot` (*s*)

Updates the gnuplot window. If `gnuplot_replot` is called with a gnuplot command in a string *s*, then `s` is sent to gnuplot before reploting the window.

`gnuplot_reset` () [Function]

Resets the state of gnuplot used with the `gnuplot_pipes` format. To update the gnuplot window call `gnuplot_replot` after `gnuplot_reset`.

15 File Input and Output

15.1 Comments

A comment in Maxima input is any text between `/*` and `*/`.

The Maxima parser treats a comment as whitespace for the purpose of finding tokens in the input stream; a token always ends at a comment. An input such as `a/* foo */b` contains two tokens, `a` and `b`, and not a single token `ab`. Comments are otherwise ignored by Maxima; neither the content nor the location of comments is stored in parsed input expressions.

Comments can be nested to arbitrary depth. The `/*` and `*/` delimiters form matching pairs. There must be the same number of `/*` as there are `*/`.

Examples:

```
(%i1) /* aa is a variable of interest */  aa : 1234;
(%o1)                          1234
(%i2) /* Value of bb depends on aa */  bb : aa^2;
(%o2)                         1522756
(%i3) /* User-defined infix operator */  infix ("b");
(%o3)                            b
(%i4) /* Parses same as a b c, not abc */  a/* foo */b/* bar */c;
(%o4)                          a b c
(%i5) /* Comments /* can be nested /* to any depth */ */ */  1 + xyz;
(%o5)                         xyz + 1
```

15.2 Files

A file is simply an area on a particular storage device which contains data or text. Files on the disks are figuratively grouped into "directories". A directory is just a list of files. Commands which deal with files are:

appendfile	batch	batchload
closefile	file_output_append	filename_merge
file_search	file_search_cache	file_search_maxima
file_search_lisp	file_search_demo	file_search_usage
file_search_tests	file_type	file_type_lisp
file_type_maxima	load	load_pathname
loadfile	loadprint	pathname_directory
pathname_name	pathname_type	printfile
save	stringout	with_stdout
writefile		

When a file name is passed to functions like `plot2d`, `save`, or `writefile` and the file name does not include a path, Maxima stores the file in the current working directory. The current working directory depends on the system like Windows or Linux and on the installation.

15.3 Functions and Variables for File Input and Output

appendfile (*filename*) [Function]

Appends a console transcript to *filename*. `appendfile` is the same as `writefile`, except that the transcript file, if it exists, is always appended.

`closefile` closes the transcript file opened by `appendfile` or `writefile`.

batch [Function]

`batch` (*filename*)
`batch` (*filename*, *option*)
`batch` (*S*)
`batch` (*S*, *option*)

`batch(`*filename*`)` reads Maxima expressions from *filename* and evaluates them. `batch` searches for *filename* in the list `file_search_maxima`. See also `file_search`.

`batch(`*S*`)` reads Maxima expressions from the input stream *S* as created by `openr`. The behavior of `batch` in this case is the same as if the input were a file name, and in the remainder of this description, what is said about input files applies to input streams as well, except that the comments about searching for files do not apply to streams.

`batch(`*filename*`, demo)` is like `demo(`*filename*`)`. In this case `batch` searches for *filename* in the list `file_search_demo`. See `demo`.

`batch(`*filename*`, test)` is like `run_testsuite` with the option `display_all=true`. For this case `batch` searches *filename* in the list `file_search_maxima` and not in the list `file_search_tests` like `run_testsuite`. Furthermore, `run_testsuite` runs tests which are in the list `testsuite_files`. With `batch` it is possible to run any file in a test mode, which can be found in the list `file_search_maxima`. This is useful, when writing a test file.

filename comprises a sequence of Maxima expressions, each terminated with `;` or `$`. The special variable `%` and the function `%th` refer to previous results within the file. The file may include `:lisp` constructs. Spaces, tabs, and newlines in the file are ignored. A suitable input file may be created by a text editor or by the `stringout` function.

`batch` reads each input expression from *filename*, displays the input to the console, computes the corresponding output expression, and displays the output expression. Input labels are assigned to the input expressions and output labels are assigned to the output expressions. `batch` evaluates every input expression in the file unless there is an error. If user input is requested (by `asksign` or `askinteger`, for example) `batch` pauses to collect the requisite input and then continue; if `batch_answers_from_file` is `true`, the input is read from the file itself. See also `batch_answers_from_file`.

It may be possible to halt `batch` by typing `control-C` at the console. The effect of `control-C` depends on the underlying Lisp implementation.

`batch` has several uses, such as to provide a reservoir for working command lines, to give error-free demonstrations, or to help organize one's thinking in solving complex problems.

`batch` evaluates its arguments.

When called with no second argument or with the option `demo`, `batch` returns the path of *filename*, if the argument is a file name, or the path of the file for which the input stream was opened, if the argument is a file input stream. If the argument is a string input stream, a representation of the input stream is returned.

When called with the option `test`, the return value is a an empty list `[]` or a list with *filename* and the numbers of the tests which have failed.

See also `load`, `batchload`, `batch_answers_from_file`, and `demo`.

`batchload` [Function]

`batchload (`*filename*`)`
`batchload (`*S*`)`

Reads Maxima expressions from input file *filename* or input stream *S* and evaluates them, without displaying the input or output expressions and without assigning labels to output expressions. Printed output (such as produced by `print` or `describe`)) is displayed, however.

The special variable `%` and the function `%th` refer to previous results from the interactive interpreter, not results within the file. The file cannot include `:lisp` constructs.

`batchload` evaluates its argument.

`batchload` returns the path of *filename*, if the argument is a file name, or the path of the file for which the input stream was opened, if the argument is a file input stream. If the argument is a string input stream, a representation of the input stream is returned.

See also `batch`, and `load`.

`closefile ()` [Function]

Closes the transcript file opened by `writefile` or `appendfile`.

`batch_answers_from_file` [Option variable]

Default value: `false`

If `true`, then `batch` reads answers to interactive questions from its input file or stream.

Example: Maxima's interactive testsuite includes something like following.

```
[asksign (foo), sign (foo), sign (foo)];
p;
[pos, pos, pos];
```

The first line makes Maxima ask a question; when `batch_answers_from_file` is `true`, the second line is read as the answer to the question; and the third line provides the expected result.

The command-line option `--batch-string` binds `batch_answers_from_file` to `true`. The `run_testsuite` function, as a default, also binds `batch_answers_from_file` to `true`. See also `command_line_options` and `run_testsuite`.

`file_output_append` [Option variable]

Default value: `false`

`file_output_append` governs whether file output functions append or truncate their output file. When `file_output_append` is `true`, such functions append to their output file. Otherwise, the output file is truncated.

save, stringout, and with_stdout respect file_output_append. Other functions which write output files do not respect file_output_append. In particular, plotting and translation functions always truncate their output file, and tex and appendfile always append.

filename_merge (*path*, *filename*) [Function]

Constructs a modified path from *path* and *filename*. If the final component of *path* is of the form ###.*something*, the component is replaced with *filename*.*something*. Otherwise, the final component is simply replaced by *filename*.

The result is a Lisp pathname object.

file_search [Function]

file_search (*filename*)
file_search (*filename*, *pathlist*)

file_search searches for the file *filename* and returns the path to the file (as a string) if it can be found; otherwise file_search returns false. file_search (*filename*) searches in the default search directories, which are specified by the file_search_maxima, file_search_lisp, and file_search_demo variables.

file_search first checks if the actual name passed exists, before attempting to match it to "wildcard" file search patterns. See file_search_maxima concerning file search patterns.

The argument *filename* can be a path and file name, or just a file name, or, if a file search directory includes a file search pattern, just the base of the file name (without an extension). For example,

```
file_search ("/home/wfs/special/zeta.mac");
file_search ("zeta.mac");
file_search ("zeta");
```

all find the same file, assuming the file exists and /home/wfs/special/###.mac is in file_search_maxima.

file_search (*filename*, *pathlist*) searches only in the directories specified by *pathlist*, which is a list of strings. The argument *pathlist* supersedes the default search directories, so if the path list is given, file_search searches only the ones specified, and not any of the default search directories. Even if there is only one directory in *pathlist*, it must still be given as an one-element list.

The user may modify the default search directories. See file_search_maxima.

file_search is invoked by load with file_search_maxima and file_search_lisp as the search directories.

See also file_search_cache.

file_search_cache [Option variable]

Default value: auto

file_search_cache controls the use of a cache by file_search in order to speed up the search. Currently, it can only speed up searches where the filename part does not contain wildcards (but the directory part may).

When file_search_cache is auto (the default value), the cache will be activated if a one-time check determines that the system (Lisp implementation, file system) fulfills

the requirements. This can be overridden by setting `file_search_cache` to `true`, which is not recommended. The value `false` prevents the use of the cache.

See also `file_search`.

`file_search_maxima` [Option variable]
`file_search_lisp` [Option variable]
`file_search_demo` [Option variable]
`file_search_usage` [Option variable]
`file_search_tests` [Option variable]

These variables specify lists of directories to be searched by `load`, `demo`, and some other Maxima functions. The default values of these variables name various directories in the Maxima installation.

The user can modify these variables, either to replace the default values or to append additional directories. For example,

```
file_search_maxima: ["/usr/local/foo/*.mac",
    "/usr/local/bar/*.mac"]$
```

replaces the default value of `file_search_maxima`, while

```
file_search_maxima: append (file_search_maxima,
    ["/usr/local/foo/*.mac", "/usr/local/bar/*.mac"])$
```

appends two additional directories. It may be convenient to put such an expression in the file `maxima-init.mac` so that the file search path is assigned automatically when Maxima starts. See also Section 3.1 [Introduction for Runtime Environment], page 9.

Each element of the search list is a Common Lisp wildcard pathname. Briefly, a wildcard filename looks like `"*.lisp"`, which matches any filename with an extension of `"lisp"`. A directory component of `*` matches any directory in the current directory, and `**` matches any directory and subdirectories in the current directory.

So, `file_search_maxima` includes `"/home/username/.maxima/**/*.mac"`. This means look in all subdirectories of `"/home/username/.maxima/"` for files with extension `"mac"`. This includes subdirectories of subdirectories. Thus, `load("file")` will find `"/home/username/.maxima/dir1/subdir1/file.mac"`.

To only search for a single level of subdirectories, use `"/home/username/.maxima/*/*.mac"`. This means Maxima will not find the file `"/home/username/.maxima/dir1/subdir1/file.mac"` when Maxima tries to, say, `load("file")`.

Further information about Common Lisp pathnames maybe be found in CLHS Section 19.2: Pathnames.

`file_type` (*filename*) [Function]

Returns a guess about the content of *filename*, based on the filename extension. *filename* need not refer to an actual file; no attempt is made to open the file and inspect the content.

The return value is a symbol, either `object`, `lisp`, or `maxima`. If the extension is matches one of the values in `file_type_maxima`, `file_type` returns `maxima`. If the extension matches one of the values in `file_type_lisp`, `file_type` returns `lisp`. If none of the above, `file_type` returns `object`.

See also `pathname_type`.

See file_type_maxima and file_type_lisp for the default values.

Examples:

```
(%i1) map('file_type,
    ["test.lisp", "test.mac", "test.dem", "test.txt"]);
(%o1)            [lisp, maxima, maxima, object]
```

file_type_lisp [Option variable]

Default value: [l, lsp, lisp]

file_type_lisp is a list of file extensions that maxima recognizes as denoting a Lisp source file.

See also file_type.

file_type_maxima [Option variable]

Default value: [mac, mc, demo, dem, dm1, dm2, dm3, dmt, wxm]

file_type_maxima is a list of file extensions that maxima recognizes as denoting a Maxima source file.

See also file_type.

load (*filename*) [Function]

Evaluates expressions in *filename*, thus bringing variables, functions, and other objects into Maxima. The binding of any existing object is clobbered by the binding recovered from *filename*.

filename must be a string, symbol, or Lisp pathname (as created by filename_merge). To find the file, load calls file_search with file_search_maxima and file_search_lisp as the search directories. If load succeeds, it returns the name of the file. Otherwise load prints an error message.

load works equally well for Lisp code and Maxima code. Files created by save, translate_file, and compile_file, which create Lisp code, and stringout, which creates Maxima code, can all be processed by load. load calls loadfile to load Lisp files and batchload to load Maxima files.

load does not recognize :lisp constructs in Maxima files, and while processing *filename*, the global variables _, __, %, and %th have whatever bindings they had when load was called.

Note also that structures will only be read back as structures if they have been defined by defstruct before the load command is called.

See also loadfile, for Lisp files; and batch, batchload, and demo. for Maxima files.

See file_search for more detail about the file search mechanism. The numericalio chapter describes many functions for loading csv and other data files.

During Maxima file loading, the variable load_pathname is bound to the pathname of the file being loaded.

load evaluates its argument.

load_pathname [System variable]

Default value: false

When a file is loaded with the functions load, loadfile or batchload the system variable load_pathname is bound to the pathname of the file which is processed.

The variable load_pathname can be accessed from the file during the loading.

Example:

Suppose we have a batchfile test.mac in the directory

"/home/dieter/workspace/mymaxima/temp/" with the following commands

```
print("The value of load_pathname is: ", load_pathname)$
print("End of batchfile")$
```

then we get the following output

```
(%i1) load("/home/dieter/workspace/mymaxima/temp/test.mac")$
The value of load_pathname is:
                    /home/dieter/workspace/mymaxima/temp/test.mac
End of batchfile
```

loadfile (*filename*) [Function]

Evaluates Lisp expressions in *filename*. loadfile does not invoke file_search, so filename must include the file extension and as much of the path as needed to find the file.

loadfile can process files created by save, translate_file, and compile_file. The user may find it more convenient to use load instead of loadfile.

loadprint [Option variable]

Default value: true

loadprint tells whether to print a message when a file is loaded.

- When loadprint is true, always print a message.
- When loadprint is 'loadfile, print a message only if a file is loaded by the function loadfile.
- When loadprint is 'autoload, print a message only if a file is automatically loaded. See setup_autoload.
- When loadprint is false, never print a message.

directory (*path*) [Function]

Returns a list of the files and directories found in *path* in the file system.

path may contain wildcard characters (i.e., characters which represent unspecified parts of the path), which include at least the asterisk on most systems, and possibly other characters, depending on the system.

directory relies on the Lisp function DIRECTORY, which may have implementation-specific behavior.

pathname_directory (*pathname*) [Function]
pathname_name (*pathname*) [Function]
pathname_type (*pathname*) [Function]

These functions return the components of *pathname*.

Examples:

```
(%i1) pathname_directory("/home/dieter/maxima/changelog.txt");
(%o1)                  /home/dieter/maxima/
(%i2) pathname_name("/home/dieter/maxima/changelog.txt");
(%o2)                         changelog
(%i3) pathname_type("/home/dieter/maxima/changelog.txt");
(%o3)                             txt
```

`printfile` (*path*) [Function]

Prints the file named by *path* to the console. *path* may be a string or a symbol; if it is a symbol, it is converted to a string.

If *path* names a file which is accessible from the current working directory, that file is printed to the console. Otherwise, `printfile` attempts to locate the file by appending *path* to each of the elements of `file_search_usage` via `filename_merge`.

`printfile` returns *path* if it names an existing file, or otherwise the result of a successful filename merge.

`save` [Function]

`save` (*filename*, *name_1*, *name_2*, *name_3*, ...)
`save` (*filename*, *values*, *functions*, *labels*, ...)
`save` (*filename*, [*m*, *n*])
`save` (*filename*, *name_1*=*expr_1*, ...)
`save` (*filename*, *all*)
`save` (*filename*, *name_1*=*expr_1*, *name_2*=*expr_2*, ...)

Stores the current values of *name_1*, *name_2*, *name_3*, ..., in *filename*. The arguments are the names of variables, functions, or other objects. If a name has no value or function associated with it, it is ignored. `save` returns *filename*.

`save` stores data in the form of Lisp expressions. If *filename* ends in `.lisp` the data stored by `save` may be recovered by `load (`*filename*`)`. See `load`.

The global flag `file_output_append` governs whether `save` appends or truncates the output file. When `file_output_append` is `true`, `save` appends to the output file. Otherwise, `save` truncates the output file. In either case, `save` creates the file if it does not yet exist.

The special form `save (`*filename*`, values, functions, labels, ...)` stores the items named by `values`, `functions`, `labels`, etc. The names may be any specified by the variable `infolists`. `values` comprises all user-defined variables.

The special form `save (`*filename*`, [`*m*`, `*n*`])` stores the values of input and output labels *m* through *n*. Note that *m* and *n* must be literal integers. Input and output labels may also be stored one by one, e.g., `save ("foo.1", %i42, %o42)`. `save (`*filename*`, labels)` stores all input and output labels. When the stored labels are recovered, they clobber existing labels.

The special form `save (`*filename*`, `*name_1*`=`*expr_1*`, `*name_2*`=`*expr_2*`, ...)` stores the values of *expr_1*, *expr_2*, ..., with names *name_1*, *name_2*, ... It is useful to apply this form to input and output labels, e.g., `save ("foo.1", aa=%o88)`. The right-hand side of the equality in this form may be any expression, which is

evaluated. This form does not introduce the new names into the current Maxima environment, but only stores them in *filename*.

These special forms and the general form of `save` may be mixed at will. For example, `save (filename, aa, bb, cc=42, functions, [11, 17])`.

The special form `save (filename, all)` stores the current state of Maxima. This includes all user-defined variables, functions, arrays, etc., as well as some automatically defined items. The saved items include system variables, such as `file_search_maxima` or `showtime`, if they have been assigned new values by the user; see `myoptions`.

`save` evaluates *filename* and quotes all other arguments.

`stringout` [Function]

`stringout (filename, expr_1, expr_2, expr_3, ...)`
`stringout (filename, [m, n])`
`stringout (filename, input)`
`stringout (filename, functions)`
`stringout (filename, values)`

`stringout` writes expressions to a file in the same form the expressions would be typed for input. The file can then be used as input for the `batch` or `demo` commands, and it may be edited for any purpose. `stringout` can be executed while `writefile` is in progress.

The global flag `file_output_append` governs whether `stringout` appends or truncates the output file. When `file_output_append` is `true`, `stringout` appends to the output file. Otherwise, `stringout` truncates the output file. In either case, `stringout` creates the file if it does not yet exist.

The general form of `stringout` writes the values of one or more expressions to the output file. Note that if an expression is a variable, only the value of the variable is written and not the name of the variable. As a useful special case, the expressions may be input labels (`%i1`, `%i2`, `%i3`, ...) or output labels (`%o1`, `%o2`, `%o3`, ...).

If `grind` is `true`, `stringout` formats the output using the `grind` format. Otherwise the `string` format is used. See `grind` and `string`.

The special form `stringout (filename, [m, n])` writes the values of input labels m through n, inclusive.

The special form `stringout (filename, input)` writes all input labels to the file.

The special form `stringout (filename, functions)` writes all user-defined functions (named by the global list `functions`)) to the file.

The special form `stringout (filename, values)` writes all user-assigned variables (named by the global list `values`)) to the file. Each variable is printed as an assignment statement, with the name of the variable, a colon, and its value. Note that the general form of `stringout` does not print variables as assignment statements.

`with_stdout` [Function]

`with_stdout (`*f*`, `*expr_1*`, `*expr_2*`, `*expr_3*`, ...)`
`with_stdout (`*s*`, `*expr_1*`, `*expr_2*`, `*expr_3*`, ...)`

Evaluates *expr_1*, *expr_2*, *expr_3*, ... and writes any output thus generated to a file *f* or output stream *s*. The evaluated expressions are not written to the output. Output may be generated by `print`, `display`, `grind`, among other functions.

The global flag `file_output_append` governs whether `with_stdout` appends or truncates the output file *f*. When `file_output_append` is `true`, `with_stdout` appends to the output file. Otherwise, `with_stdout` truncates the output file. In either case, `with_stdout` creates the file if it does not yet exist.

`with_stdout` returns the value of its final argument.

See also `writefile` and `display2d`.

```
(%i1) with_stdout ("tmp.out", for i:5 thru 10 do
      print (i, "! yields", i!))$
(%i2) printfile ("tmp.out")$
5 ! yields 120
6 ! yields 720
7 ! yields 5040
8 ! yields 40320
9 ! yields 362880
10 ! yields 3628800
```

`writefile (`*filename*`)` [Function]

Begins writing a transcript of the Maxima session to *filename*. All interaction between the user and Maxima is then recorded in this file, just as it appears on the console.

As the transcript is printed in the console output format, it cannot be reloaded into Maxima. To make a file containing expressions which can be reloaded, see `save` and `stringout`. `save` stores expressions in Lisp form, while `stringout` stores expressions in Maxima form.

The effect of executing `writefile` when *filename* already exists depends on the underlying Lisp implementation; the transcript file may be clobbered, or the file may be appended. `appendfile` always appends to the transcript file.

It may be convenient to execute `playback` after `writefile` to save the display of previous interactions. As `playback` displays only the input and output variables (`%i1`, `%o1`, etc.), any output generated by a print statement in a function (as opposed to a return value) is not displayed by `playback`.

`closefile` closes the transcript file opened by `writefile` or `appendfile`.

15.4 Functions and Variables for TeX Output

Note that the built-in TeX output functionality of wxMaxima makes no use of the functions described here but uses its own implementation instead.

`tex` [Function]

`tex` (*expr*)
`tex` (*expr*, *destination*)
`tex` (*expr*, *false*)
`tex` (*label*)
`tex` (*label*, *destination*)
`tex` (*label*, *false*)

Prints a representation of an expression suitable for the TeX document preparation system. The result is a fragment of a document, which can be copied into a larger document but not processed by itself.

`tex` (*expr*) prints a TeX representation of *expr* on the console.

`tex` (*label*) prints a TeX representation of the expression named by *label* and assigns it an equation label (to be displayed to the left of the expression). The TeX equation label is the same as the Maxima label.

destination may be an output stream or file name. When *destination* is a file name, `tex` appends its output to the file. The functions `openw` and `opena` create output streams.

`tex` (*expr*, `false`) and `tex` (*label*, `false`) return their TeX output as a string.

`tex` evaluates its first argument after testing it to see if it is a label. Quote-quote `''` forces evaluation of the argument, thereby defeating the test and preventing the label.

See also `tex1` and `texput`.

Examples:

```
(%i1) integrate (1/(1+x^3), x);
                                    2 x - 1
                2            atan(-------)
           log(x  - x + 1)        sqrt(3)    log(x + 1)
(%o1)    - --------------- + ------------- + ----------
                 6             sqrt(3)          3
(%i2) tex (%o1);
$$-\left({{\log \left(x^2-x+1\right)}\over{6}}\right)+{{\arctan
 \left({{2\,x-1}\over{\sqrt{3}}}\right)}\over{\sqrt{3}}}+{{\log
 \left(x+1\right)}\over{3}}\leqno{\tt (\%o1)}$$
(%o2)                        (\%o1)
(%i3) tex (integrate (sin(x), x));
$$-\cos x$$
(%o3)                         false
(%i4) tex (%o1, "foo.tex");
(%o4)                        (\%o1)
```

`tex` (*expr*, `false`) returns its TeX output as a string.

```
(%i1) S : tex (x * y * z, false);
(%o1) $$x\,y\,z$$
(%i2) S;
(%o2) $$x\,y\,z$$
```

`tex1 (e)` [Function]

Returns a string which represents the TeX output for the expressions *e*. The TeX output is not enclosed in delimiters for an equation or any other environment.

See also `tex` and `texput`.

Examples:

```
(%i1) tex1 (sin(x) + cos(x));
(%o1)                      \sin x+\cos x
```

`texput` [Function]

`texput (a, s)`
`texput (a, f)`
`texput (a, s, operator_type)`
`texput (a, [s_1, s_2], matchfix)`
`texput (a, [s_1, s_2, s_3], matchfix)`

Assign the TeX output for the atom *a*, which can be a symbol or the name of an operator.

`texput (a, s)` causes the `tex` function to interpolate the string *s* into the TeX output in place of *a*.

`texput (a, f)` causes the `tex` function to call the function *f* to generate TeX output. *f* must accept one argument, which is an expression which has operator *a*, and must return either a string (the TeX output) or `false`, indicating that the TeX function in effect when `texput` is called should handle the expression. *f* may call `tex1` to generate TeX output for the arguments of the input expression.

`texput (a, s, operator_type)`, where *operator_type* is `prefix`, `infix`, `postfix`, `nary`, or `nofix`, causes the `tex` function to interpolate *s* into the TeX output in place of *a*, and to place the interpolated text in the appropriate position.

`texput (a, [s_1, s_2], matchfix)` causes the `tex` function to interpolate *s_1* and *s_2* into the TeX output on either side of the arguments of *a*. The arguments (if more than one) are separated by commas.

`texput (a, [s_1, s_2, s_3], matchfix)` causes the `tex` function to interpolate *s_1* and *s_2* into the TeX output on either side of the arguments of *a*, with *s_3* separating the arguments.

See also `tex` and `tex1`.

Examples:

Assign TeX output for a variable.

```
(%i1) texput (me,"\\mu_e");
(%o1)                          \mu_e
(%i2) tex (me);
$$\mu_e$$
(%o2)                          false
```

Assign TeX output for an ordinary function (not an operator).

```
(%i1) texput (lcm, "\\mathrm{lcm}");
(%o1)                      \mathrm{lcm}
```

```
(%i2) tex (lcm (a, b));
$$\mathrm{lcm}\left(a , b\right)$$
(%o2)                          false
```

Call a function to generate TeX output.

```
(%i1) texfoo (e) := block ([a, b], [a, b] : args (e),
  concat ("\\left[\\stackrel{",tex1(b),"}{",tex1(a),"}\\right]"))$
(%i2) texput (foo, texfoo);
(%o2)                         texfoo
(%i3) tex (foo (2^x, %pi));
$$\left[\stackrel{\pi}{2^{x}}\right]$$
(%o3)                          false
```

Assign TeX output for a prefix operator.

```
(%i1) prefix ("grad");
(%o1)                          grad
(%i2) texput ("grad", " \\nabla ", prefix);
(%o2)                         \nabla
(%i3) tex (grad f);
$$ \nabla f$$
(%o3)                          false
```

Assign TeX output for an infix operator.

```
(%i1) infix ("~");
(%o1)                            ~
(%i2) texput ("~", " \\times ", infix);
(%o2)                         \times
(%i3) tex (a ~ b);
$$a \times b$$
(%o3)                          false
```

Assign TeX output for a postfix operator.

```
(%i1) postfix ("##");
(%o1)                           ##
(%i2) texput ("##", "!!", postfix);
(%o2)                           !!
(%i3) tex (x ##);
$$x!!$$
(%o3)                          false
```

Assign TeX output for a nary operator.

```
(%i1) nary ("@@");
(%o1)                           @@
(%i2) texput ("@@", " \\circ ", nary);
(%o2)                           \circ
(%i3) tex (a @@ b @@ c @@ d);
$$a \circ b \circ c \circ d$$
(%o3)                          false
```

Assign TeX output for a nofix operator.

```
(%i1) nofix ("foo");
(%o1)                           foo
(%i2) texput ("foo", "\\mathsc{foo}", nofix);
(%o2)                       \mathsc{foo}
(%i3) tex (foo);
$$\mathsc{foo}$$
(%o3)                           false
```

Assign TeX output for a matchfix operator.

```
(%i1) matchfix ("<<", ">>");
(%o1)                            <<
(%i2) texput ("<<", [" \\langle ", " \\rangle "], matchfix);
(%o2)                 [ \langle ,  \rangle ]
(%i3) tex (<<a>>);
$$ \langle a \rangle $$
(%o3)                           false
(%i4) tex (<<a, b>>);
$$ \langle a , b \rangle $$
(%o4)                           false
(%i5) texput ("<<", [" \\langle ", " \\rangle ", " \\, | \\,"],
      matchfix);
(%o5)           [ \langle ,  \rangle ,  \, | \,]
(%i6) tex (<<a>>);
$$ \langle a \rangle $$
(%o6)                           false
(%i7) tex (<<a, b>>);
$$ \langle a \, | \,b \rangle $$
(%o7)                           false
```

`get_tex_environment` (*op*) [Function]
`set_tex_environment` (*op*, *before*, *after*) [Function]

Customize the TeX environment output by `tex`. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

Only the TeX environment of the top-level operator in an expression is output; TeX environments associated with other operators are ignored.

`get_tex_environment` returns the TeX environment which is applied to the operator *op*; returns the default if no other environment has been assigned.

`set_tex_environment` assigns the TeX environment for the operator *op*.

Examples:

```
(%i1) get_tex_environment (":=");
(%o1) [
\begin{verbatim}
, ;
\end{verbatim}
]
```

```
(%i2) tex (f (x) := 1 - x);
\begin{verbatim}
f(x):=1-x;
\end{verbatim}

(%o2)                          false
(%i3) set_tex_environment (":=", "$$", "$$");
(%o3)                        [$$, $$]
(%i4) tex (f (x) := 1 - x);
$$f(x):=1-x$$
(%o4)                          false
```

`get_tex_environment_default` () [Function]
`set_tex_environment_default` (*before*, *after*) [Function]

Customize the TeX environment output by `tex`. As maintained by these functions, the TeX environment comprises two strings: one is printed before any other TeX output, and the other is printed after.

`get_tex_environment_default` returns the TeX environment which is applied to expressions for which the top-level operator has no specific TeX environment (as assigned by `set_tex_environment`).

`set_tex_environment_default` assigns the default TeX environment.

Examples:

```
(%i1) get_tex_environment_default ();
(%o1)                        [$$, $$]
(%i2) tex (f(x) + g(x));
$$g\left(x\right)+f\left(x\right)$$
(%o2)                          false
(%i3) set_tex_environment_default ("\\begin{equation}
(%o3) [\begin{equation}
,
\end{equation}]
(%i4) ", "
\begin{equation}
g\left(x\right)+f\left(x\right)
\end{equation}
(%o4)                          false
(%i5) \\end{equation}");
```

15.5 Functions and Variables for Fortran Output

`fortindent` [Option variable]

Default value: `0`

`fortindent` controls the left margin indentation of expressions printed out by the `fortran` command. `0` gives normal printout (i.e., 6 spaces), and positive values will causes the expressions to be printed farther to the right.

`fortran` (*expr*) [Function]

Prints *expr* as a Fortran statement. The output line is indented with spaces. If the line is too long, `fortran` prints continuation lines. `fortran` prints the exponentiation operator `^` as `**`, and prints a complex number `a + b %i` in the form `(a,b)`.

expr may be an equation. If so, `fortran` prints an assignment statement, assigning the right-hand side of the equation to the left-hand side. In particular, if the right-hand side of *expr* is the name of a matrix, then `fortran` prints an assignment statement for each element of the matrix.

If *expr* is not something recognized by `fortran`, the expression is printed in `grind` format without complaint. `fortran` does not know about lists, arrays, or functions.

`fortindent` controls the left margin of the printed lines. `0` is the normal margin (i.e., indented 6 spaces). Increasing `fortindent` causes expressions to be printed further to the right.

When `fortspaces` is `true`, `fortran` fills out each printed line with spaces to 80 columns.

`fortran` evaluates its arguments; quoting an argument defeats evaluation. `fortran` always returns `done`.

See also the function [function_f90], page 1003 for printing one or more expressions as a Fortran 90 program.

Examples:

```
(%i1) expr: (a + b)^12$
(%i2) fortran (expr);
      (b+a)**12
(%o2)                          done
(%i3) fortran ('x=expr);
      x = (b+a)**12
(%o3)                          done
(%i4) fortran ('x=expand (expr));
      x = b**12+12*a*b**11+66*a**2*b**10+220*a**3*b**9+495*a**4*b**8+792
     1   *a**5*b**7+924*a**6*b**6+792*a**7*b**5+495*a**8*b**4+220*a**9*b
     2   **3+66*a**10*b**2+12*a**11*b+a**12
(%o4)                          done
(%i5) fortran ('x=7+5*%i);
      x = (7,5)
(%o5)                          done
(%i6) fortran ('x=[1,2,3,4]);
      x(1) = 1
      x(2) = 2
      x(3) = 3
      x(4) = 4
(%o6)                          done
(%i7) f(x) := x^2$
(%i8) fortran (f);
      f
(%o8)                          done
```

`fortspaces` [Option variable]

Default value: `false`

When `fortspaces` is `true`, `fortran` fills out each printed line with spaces to 80 columns.

16 Polynomials

16.1 Introduction to Polynomials

Polynomials are stored in Maxima either in General Form or as Canonical Rational Expressions (CRE) form. The latter is a standard form, and is used internally by operations such as `factor`, `ratsimp`, and so on.

Canonical Rational Expressions constitute a kind of representation which is especially suitable for expanded polynomials and rational functions (as well as for partially factored polynomials and rational functions when `ratfac` is set to `true`). In this CRE form an ordering of variables (from most to least main) is assumed for each expression.

Polynomials are represented recursively by a list consisting of the main variable followed by a series of pairs of expressions, one for each term of the polynomial. The first member of each pair is the exponent of the main variable in that term and the second member is the coefficient of that term which could be a number or a polynomial in another variable again represented in this form. Thus the principal part of the CRE form of `3*x^2-1` is `(X 2 3 0 -1)` and that of `2*x*y+x-3` is `(Y 1 (X 1 2) 0 (X 1 1 0 -3))` assuming `y` is the main variable, and is `(X 1 (Y 1 2 0 1) 0 -3)` assuming `x` is the main variable. "Main"-ness is usually determined by reverse alphabetical order.

The "variables" of a CRE expression needn't be atomic. In fact any subexpression whose main operator is not `+`, `-`, `*`, `/` or `^` with integer power will be considered a "variable" of the expression (in CRE form) in which it occurs. For example the CRE variables of the expression `x+sin(x+1)+2*sqrt(x)+1` are `x`, `sqrt(x)`, and `sin(x+1)`. If the user does not specify an ordering of variables by using the `ratvars` function Maxima will choose an alphabetic one.

In general, CRE's represent rational expressions, that is, ratios of polynomials, where the numerator and denominator have no common factors, and the denominator is positive. The internal form is essentially a pair of polynomials (the numerator and denominator) preceded by the variable ordering list. If an expression to be displayed is in CRE form or if it contains any subexpressions in CRE form, the symbol `/R/` will follow the line label.

See the `rat` function for converting an expression to CRE form.

An extended CRE form is used for the representation of Taylor series. The notion of a rational expression is extended so that the exponents of the variables can be positive or negative rational numbers rather than just positive integers and the coefficients can themselves be rational expressions as described above rather than just polynomials. These are represented internally by a recursive polynomial form which is similar to and is a generalization of CRE form, but carries additional information such as the degree of truncation. As with CRE form, the symbol `/T/` follows the line label of such expressions.

16.2 Functions and Variables for Polynomials

`algebraic` [Option variable]

Default value: `false`

`algebraic` must be set to `true` in order for the simplification of algebraic integers to take effect.

`berlefact` [Option variable]

Default value: `true`

When `berlefact` is `false` then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

`bezout` (*p1*, *p2*, *x*) [Function]

an alternative to the `resultant` command. It returns a matrix. `determinant` of this matrix is the desired resultant.

Examples:

```
(%i1) bezout(a*x+b, c*x^2+d, x);
                            [ b c  - a d ]
(%o1)                       [            ]
                            [  a     b   ]
(%i2) determinant(%);
                              2      2
(%o2)                        a  d + b  c
(%i3) resultant(a*x+b, c*x^2+d, x);
                              2      2
(%o3)                        a  d + b  c
```

`bothcoef` (*expr*, *x*) [Function]

Returns a list whose first member is the coefficient of *x* in *expr* (as found by `ratcoef` if *expr* is in CRE form otherwise by `coeff`) and whose second member is the remaining part of *expr*. That is, `[A, B]` where *expr* = `A*x + B`.

Example:

```
(%i1) islinear (expr, x) := block ([c],
        c: bothcoef (rat (expr, x), x),
        is (freeof (x, c) and c[1] # 0))$
(%i2) islinear ((r^2 - (x - r)^2)/x, x);
(%o2)                          true
```

`coeff` [Function]

`coeff` (*expr*, *x*, *n*)
`coeff` (*expr*, *x*)

Returns the coefficient of *x*`^`*n* in *expr*, where *expr* is a polynomial or a monomial term in *x*. Other than `ratcoef` `coeff` is a strictly syntactical operation and will only find literal instances of *x*`^`*n* in the internal representation of *expr*.

`coeff(`*expr*`,` *x*`^`*n*`)` is equivalent to `coeff(`*expr*`,` *x*`,` *n*`)`. `coeff(`*expr*`,` *x*`,` `0`) returns the remainder of *expr* which is free of *x*. If omitted, *n* is assumed to be 1.

x may be a simple variable or a subscripted variable, or a subexpression of *expr* which comprises an operator and all of its arguments.

It may be possible to compute coefficients of expressions which are equivalent to *expr* by applying `expand` or `factor`. `coeff` itself does not apply `expand` or `factor` or any other function.

`coeff` distributes over lists, matrices, and equations.

See also `ratcoef`.

Examples:

`coeff` returns the coefficient *x^n* in *expr*.

```
(%i1) coeff (b^3*a^3 + b^2*a^2 + b*a + 1, a^3);
                                3
(%o1)                           b
```

`coeff(`*expr*`,` *x^n*`)` is equivalent to `coeff(`*expr*`,` *x*`,` *n*`)`.

```
(%i1) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z, 3);
(%o1)                          - c
                                 3
(%i2) coeff (c[4]*z^4 - c[3]*z^3 - c[2]*z^2 + c[1]*z, z^3);
(%o2)                          - c
                                 3
```

`coeff(`*expr*`,` *x*`,` `0)` returns the remainder of *expr* which is free of *x*.

```
(%i1) coeff (a*u + b^2*u^2 + c^3*u^3, b, 0);
                             3  3
(%o1)                       c  u  + a u
```

x may be a simple variable or a subscripted variable, or a subexpression of *expr* which comprises an operator and all of its arguments.

```
(%i1) coeff (h^4 - 2*%pi*h^2 + 1, h, 2);
(%o1)                        - 2 %pi
(%i2) coeff (v[1]^4 - 2*%pi*v[1]^2 + 1, v[1], 2);
(%o2)                        - 2 %pi
(%i3) coeff (sin(1+x)*sin(x) + sin(1+x)^3*sin(x)^3, sin(1+x)^3);
                                 3
(%o3)                         sin (x)
(%i4) coeff ((d - a)^2*(b + c)^3 + (a + b)^4*(c - d), a + b, 4);
(%o4)                          c - d
```

`coeff` itself does not apply `expand` or `factor` or any other function.

```
(%i1) coeff (c*(a + b)^3, a);
(%o1)                             0
(%i2) expand (c*(a + b)^3);
                3          2        2        3
(%o2)          b  c + 3 a b  c + 3 a  b c + a  c
(%i3) coeff (%, a);
                                  2
(%o3)                          3 b  c
(%i4) coeff (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c, (a + b)^3);
(%o4)                             0
(%i5) factor (b^3*c + 3*a*b^2*c + 3*a^2*b*c + a^3*c);
                                    3
(%o5)                        (b + a)  c
(%i6) coeff (%, (a + b)^3);
(%o6)                             c
```

`coeff` distributes over lists, matrices, and equations.

```
(%i1) coeff ([4*a, -3*a, 2*a], a);
(%o1)                        [4, - 3, 2]
(%i2) coeff (matrix ([a*x, b*x], [-c*x, -d*x]), x);
                             [  a    b  ]
(%o2)                        [          ]
                             [ - c  - d ]
(%i3) coeff (a*u - b*v = 7*u + 3*v, u);
(%o3)                            a = 7
```

`content` (*p_1*, *x_1*, ..., *x_n*) [Function]

Returns a list whose first element is the greatest common divisor of the coefficients of the terms of the polynomial *p_1* in the variable *x_n* (this is the content) and whose second element is the polynomial *p_1* divided by the content.

Examples:

```
(%i1) content (2*x*y + 4*x^2*y^2, y);
                                        2
(%o1)                    [2 x, 2 x y  + y]
```

`denom` (*expr*) [Function]

Returns the denominator of the rational expression *expr*.

See also `num`

```
(%i1) g1:(x+2)*(x+1)/((x+3)^2);
                           (x + 1) (x + 2)
(%o1)                      ---------------
                                    2
                             (x + 3)
(%i2) denom(g1);
                                    2
(%o2)                        (x + 3)
(%i3) g2:sin(x)/10*cos(x)/y;
                            cos(x) sin(x)
(%o3)                       -------------
                                10 y
(%i4) denom(g2);
(%o4)                           10 y
```

`divide` (*p_1*, *p_2*, *x_1*, ..., *x_n*) [Function]

computes the quotient and remainder of the polynomial *p_1* divided by the polynomial *p_2*, in a main polynomial variable, *x_n*. The other variables are as in the `ratvars` function. The result is a list whose first element is the quotient and whose second element is the remainder.

Examples:

```
(%i1) divide (x + y, x - y, x);
(%o1)                         [1, 2 y]
(%i2) divide (x + y, x - y);
(%o2)                        [- 1, 2 x]
```

Note that `y` is the main variable in the second example.

`eliminate` (*[eqn_1, ..., eqn_n]*, *[x_1, ..., x_k]*) [Function]

Eliminates variables from equations (or expressions assumed equal to zero) by taking successive resultants. This returns a list of `n - k` expressions with the *k* variables *x_1*, ..., *x_k* eliminated. First *x_1* is eliminated yielding `n - 1` expressions, then `x_2` is eliminated, etc. If `k = n` then a single expression in a list is returned free of the variables *x_1*, ..., *x_k*. In this case `solve` is called to solve the last resultant for the last variable.

Example:

```
(%i1) expr1: 2*x^2 + y*x + z;
                                        2
(%o1)                     z + x y + 2 x
(%i2) expr2: 3*x + 5*y - z - 1;
(%o2)                   - z + 5 y + 3 x - 1
(%i3) expr3: z^2 + x - y^2 + 5;
                          2    2
(%o3)                     z  - y  + x + 5
(%i4) eliminate ([expr3, expr2, expr1], [y, z]);
              8         7          6          5          4
(%o4) [7425 x  - 1170 x  + 1299 x  + 12076 x  + 22887 x
                                     3         2
                               - 5154 x  - 1291 x  + 7688 x + 15376]
```

`ezgcd` (*p_1*, *p_2*, *p_3*, ...) [Function]

Returns a list whose first element is the greatest common divisor of the polynomials *p_1*, *p_2*, *p_3*, ... and whose remaining elements are the polynomials divided by the greatest common divisor. This always uses the `ezgcd` algorithm.

See also `gcd`, `gcdex`, `gcdivide`, and `poly_gcd`.

Examples:

The three polynomials have the greatest common divisor `2*x-3`. The gcd is first calculated with the function `gcd` and then with the function `ezgcd`.

```
(%i1) p1 : 6*x^3-17*x^2+14*x-3;
                          3       2
(%o1)                6 x  - 17 x  + 14 x - 3
(%i2) p2 : 4*x^4-14*x^3+12*x^2+2*x-3;
                     4       3       2
(%o2)            4 x  - 14 x  + 12 x  + 2 x - 3
(%i3) p3 : -8*x^3+14*x^2-x-3;
                           3       2
(%o3)                - 8 x  + 14 x  - x - 3
(%i4) gcd(p1, gcd(p2, p3));
(%o4)                          2 x - 3
(%i5) ezgcd(p1, p2, p3);
                     2                3      2           2
(%o5) [2 x - 3, 3 x  - 4 x + 1, 2 x  - 4 x  + 1, - 4 x  + x + 1]
```

`facexpand` [Option variable]

Default value: `true`

`facexpand` controls whether the irreducible factors returned by `factor` are in expanded (the default) or recursive (normal CRE) form.

`factor` [Function]

`factor` (*expr*)
`factor` (*expr*, *p*)

Factors the expression *expr*, containing any number of variables or functions, into factors irreducible over the integers. `factor (expr, p)` factors *expr* over the field of rationals with an element adjoined whose minimum polynomial is *p*.

`factor` uses `ifactors` function for factoring integers.

`factorflag` if `false` suppresses the factoring of integer factors of rational expressions.

`dontfactor` may be set to a list of variables with respect to which factoring is not to occur. (It is initially empty). Factoring also will not take place with respect to any variables which are less important (using the variable ordering assumed for CRE form) than those on the `dontfactor` list.

`savefactors` if `true` causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

`berlefact` if `false` then the Kronecker factoring algorithm will be used otherwise the Berlekamp algorithm, which is the default, will be used.

`intfaclim` if `true` maxima will give up factorization of integers if no factor is found after trial divisions and Pollard's rho method. If set to `false` (this is the case when the user calls `factor` explicitly), complete factorization of the integer will be attempted. The user's setting of `intfaclim` is used for internal calls to `factor`. Thus, `intfaclim` may be reset to prevent Maxima from taking an inordinately long time factoring large integers.

`factor_max_degree` if set to a positive integer `n` will prevent certain polynomials from being factored if their degree in any variable exceeds `n`.

See also `collectterms` and `sqfr`

Examples:

```
(%i1) factor (2^63 - 1);
                      2
(%o1)              7  73 127 337 92737 649657
(%i2) factor (-8*y - 4*x + z^2*(2*y + x));
(%o2)                (2 y + x) (z - 2) (z + 2)
(%i3) -1 - 2*x - x^2 + y^2 + 2*x*y^2 + x^2*y^2;
                  2  2        2    2    2
(%o3)           x  y  + 2 x y  + y  - x  - 2 x - 1
(%i4) block ([dontfactor: [x]], factor (%/36/(1 + 2*y + y^2)));
                        2
                       (x  + 2 x + 1) (y - 1)
(%o4)                  ----------------------
                              36 (y + 1)
(%i5) factor (1 + %e^(3*x));
                         x         2 x     x
(%o5)               (%e  + 1) (%e    - %e  + 1)
```

```
(%i6) factor (1 + x^4, a^2 - 2);
                    2              2
(%o6)             (x  - a x + 1) (x  + a x + 1)
(%i7) factor (-y^2*z^2 - x*z^2 + x^2*y^2 + x^3);
                       2
(%o7)              - (y  + x) (z - x) (z + x)
(%i8) (2 + x)/(3 + x)/(b + x)/(c + x)^2;
                             x + 2
(%o8)               ------------------------
                                          2
                    (x + 3) (x + b) (x + c)
(%i9) ratsimp (%);
                4                  3
(%o9) (x + 2)/(x  + (2 c + b + 3) x
     2                      2             2                   2
 + (c  + (2 b + 6) c + 3 b) x  + ((b + 3) c  + 6 b c) x + 3 b c )
(%i10) partfrac (%, x);
           2                  4                3
(%o10) - (c  - 4 c - b + 6)/((c  + (- 2 b - 6) c
     2              2         2                2
 + (b  + 12 b + 9) c  + (- 6 b  - 18 b) c + 9 b ) (x + c))
                 c - 2
 - ---------------------------------
     2                             2
   (c  + (- b - 3) c + 3 b) (x + c)
                         b - 2
 + ---------------------------------------------------
             2             2       3      2
   ((b - 3) c  + (6 b - 2 b ) c + b  - 3 b ) (x + b)
                         1
 - ----------------------------------------------
             2
   ((b - 3) c  + (18 - 6 b) c + 9 b - 27) (x + 3)
(%i11) map ('factor, %);
              2
             c  - 4 c - b + 6                 c - 2
(%o11) - ------------------------- - ------------------------
                2        2                                2
         (c - 3)  (c - b)  (x + c)   (c - 3) (c - b) (x + c)
                       b - 2                        1
            + ------------------------ - ------------------------
                             2                          2
              (b - 3) (c - b)  (x + b)   (b - 3) (c - 3)  (x + 3)
(%i12) ratsimp ((x^5 - 1)/(x - 1));
                       4    3    2
(%o12)                x  + x  + x  + x + 1
```

```
(%i13) subst (a, x, %);
                         4    3    2
(%o13)                  a  + a  + a  + a + 1
(%i14) factor (%th(2), %);
                         2        3        3    2
(%o14)   (x - a) (x - a ) (x - a ) (x + a  + a  + a + 1)
(%i15) factor (1 + x^12);
                         4        8    4
(%o15)                (x  + 1) (x  - x  + 1)
(%i16) factor (1 + x^99);
                  2            6    3
(%o16) (x + 1) (x  - x + 1) (x  - x  + 1)
   10    9    8    7    6    5    4    3    2
 (x   - x  + x  - x  + x  - x  + x  - x  + x  - x + 1)
   20    19    17    16    14    13    11    10    9    7    6
 (x   + x   - x   - x   + x   + x   - x   - x   - x  + x  + x
    4    3            60    57    51    48    42    39    33
 - x  - x  + x + 1) (x   + x   - x   - x   + x   + x   - x
    30    27    21    18    12    9    3
 - x   - x   + x   + x   - x   - x  + x  + 1)
```

`factor_max_degree` [Option variable]

Default value: `1000`

When `factor_max_degree` is set to a positive integer `n`, it will prevent Maxima from attempting to factor certain polynomials whose degree in any variable exceeds `n`. If `factor_max_degree_print_warning` is true, a warning message will be printed. `factor_max_degree` can be used to prevent excessive memory usage and/or computation time and stack overflows. Note that "obvious" factoring of polynomials such as `x^2000+x^2001` to `x^2000*(x+1)` will still take place. To disable this behavior, set `factor_max_degree` to `0`.

Example:

```
(%i1) factor_max_degree : 100$
(%i2) factor(x^100-1);
                        2        4    3    2
(%o2) (x - 1) (x + 1) (x  + 1) (x  - x  + x  - x + 1)
   4    3    2           8    6    4    2
 (x  + x  + x  + x + 1) (x  - x  + x  - x  + 1)
   20    15    10    5        20    15    10    5
 (x   - x   + x   - x  + 1) (x   + x   + x   + x  + 1)
   40    30    20    10
 (x   - x   + x   - x   + 1)
(%i3) factor(x^101-1);
                                 101
Refusing to factor polynomial x    - 1
              because its degree exceeds factor_max_degree (100)
                               101
(%o3)                         x    - 1
```

See also: factor_max_degree_print_warning

factor_max_degree_print_warning [Option variable]

Default value: true

When factor_max_degree_print_warning is true, then Maxima will print a warning message when the factoring of a polynomial is prevented because its degree exceeds the value of factor_max_degree.

See also: factor_max_degree

factorflag [Option variable]

Default value: false

When factorflag is false, suppresses the factoring of integer factors of rational expressions.

factorout (*expr*, *x_1*, *x_2*, ...) [Function]

Rearranges the sum *expr* into a sum of terms of the form f (*x_1*, *x_2*, ...)*g where g is a product of expressions not containing any *x_i* and f is factored.

Note that the option variable keepfloat is ignored by factorout.

Example:

```
(%i1) expand (a*(x+1)*(x-1)*(u+1)^2);
              2  2          2      2      2
(%o1)     a u  x  + 2 a u x  + a x  - a u  - 2 a u - a
(%i2) factorout(%,x);
            2
(%o2) a u  (x - 1) (x + 1) + 2 a u (x - 1) (x + 1)
                                              + a (x - 1) (x + 1)
```

factorsum (*expr*) [Function]

Tries to group terms in factors of *expr* which are sums into groups of terms such that their sum is factorable. factorsum can recover the result of expand ((x + y)^2 + (z + w)^2) but it can't recover expand ((x + 1)^2 + (x + y)^2) because the terms have variables in common.

Example:

```
(%i1) expand ((x + 1)*((u + v)^2 + a*(w + z)^2));
           2      2                              2      2
(%o1) a x z  + a z  + 2 a w x z + 2 a w z + a w  x + v  x
                                       2        2    2            2
                          + 2 u v x + u  x + a w  + v  + 2 u v + u
(%i2) factorsum (%);
                                     2          2
(%o2)            (x + 1) (a (z + w)  + (v + u) )
```

fasttimes (*p_1*, *p_2*) [Function]

Returns the product of the polynomials *p_1* and *p_2* by using a special algorithm for multiplication of polynomials. p_1 and p_2 should be multivariate, dense, and nearly the same size. Classical multiplication is of order n_1 n_2 where n_1 is the degree of p_1 and n_2 is the degree of p_2. fasttimes is of order max (n_1, n_2)^1.585.

`fullratsimp` (*expr*) [Function]

`fullratsimp` repeatedly applies `ratsimp` followed by non-rational simplification to an expression until no further change occurs, and returns the result.

When non-rational expressions are involved, one call to `ratsimp` followed as is usual by non-rational ("general") simplification may not be sufficient to return a simplified result. Sometimes, more than one such call may be necessary. `fullratsimp` makes this process convenient.

`fullratsimp` (*expr*, *x_1*, ..., *x_n*) takes one or more arguments similar to `ratsimp` and `rat`.

Example:

```
(%i1) expr: (x^(a/2) + 1)^2*(x^(a/2) - 1)^2/(x^a - 1);
                        a/2     2   a/2     2
                      (x    - 1)  (x    + 1)
(%o1)                 ----------------------
                                a
                               x  - 1
(%i2) ratsimp (expr);
                           2 a      a
                          x    - 2 x  + 1
(%o2)                     ---------------
                               a
                              x  - 1
(%i3) fullratsimp (expr);
                               a
(%o3)                         x  - 1
(%i4) rat (expr);
                        a/2 4       a/2 2
                       (x   )  - 2 (x   )  + 1
(%o4)/R/               ----------------------
                                a
                               x  - 1
```

`fullratsubst` (*new*, *old*, *expr*) [Function]

`fullratsubst` (*old* = *new*, *expr*)

`fullratsubst` ([*old_1* = *new_1*, ..., *old_n* = *new_n*], *expr*)

`fullratsubst` applies `lratsubst` repeatedly until *expr* stops changing (or `lrats_max_iter` is reached). This function is useful when the replacement expression and the replaced expression have one or more variables in common.

`fullratsubst` accepts its arguments in the format of `ratsubst` or `lratsubst`.

Examples:

- `subst` can carry out multiple substitutions. `lratsubst` is analogous to `subst`.

```
(%i1) subst ([a = b, c = d], a + c);
(%o1)                         d + b
(%i2) lratsubst ([a^2 = b, c^2 = d], (a + e)*c*(a + c));
(%o2)                (d + a c) e + a d + b c
```

- If only one substitution is desired, then a single equation may be given as first argument.

```
(%i1) lratsubst (a^2 = b, a^3);
(%o1)                          a b
```

- `fullratsubst` is equivalent to `ratsubst` except that it recurses until its result stops changing.

```
(%i1) ratsubst (b*a, a^2, a^3);
                                2
(%o1)                          a  b
(%i2) fullratsubst (b*a, a^2, a^3);
                                  2
(%o2)                          a b
```

- `fullratsubst` also accepts a list of equations or a single equation as first argument.

```
(%i1) fullratsubst ([a^2 = b, b^2 = c, c^2 = a], a^3*b*c);
(%o1)                            b
(%i2) fullratsubst (a^2 = b*a, a^3);
                                  2
(%o2)                          a b
```

- `fullratsubst` catches potential infinite recursions. [lrats_max_iter], page 307.

Warning: fullratsubst2(listofeqns,expr): reached maximum iterations of 15 . Increase 'lrats_max_iter' to increase this limit.

```
(%i1) fullratsubst (b*a^2, a^2, a^3), lrats_max_iter=15;
                                3  15
(%o1)                          a  b
```

See also `lrats_max_iter` and `fullratsubstflag`.

`fullratsubstflag` [Option variable]

Default value: `false`

An option variable that is set to `true` in `fullratsubst`.

`gcd` (*p_1*, *p_2*, *x_1*, ...) [Function]

Returns the greatest common divisor of *p_1* and *p_2*. The flag `gcd` determines which algorithm is employed. Setting `gcd` to `ez`, `subres`, `red`, or `spmod` selects the `ezgcd`, subresultant `prs`, reduced, or modular algorithm, respectively. If `gcd false` then `gcd (p_1, p_2, x)` always returns 1 for all *x*. Many functions (e.g. `ratsimp`, `factor`, etc.) cause gcd's to be taken implicitly. For homogeneous polynomials it is recommended that `gcd` equal to `subres` be used. To take the gcd when an algebraic is present, e.g., `gcd (x^2 - 2*sqrt(2)* x + 2, x - sqrt(2))`, the option variable `algebraic` must be `true` and `gcd` must not be `ez`.

The `gcd` flag, default: `spmod`, if `false` will also prevent the greatest common divisor from being taken when expressions are converted to canonical rational expression (CRE) form. This will sometimes speed the calculation if gcds are not required.

See also `ezgcd`, `gcdex`, `gcdivide`, and `poly_gcd`.

Example:

```
(%i1) p1:6*x^3+19*x^2+19*x+6;
                          3       2
(%o1)                  6 x  + 19 x  + 19 x + 6
(%i2) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
                    5       4       3       2
(%o2)           6 x  + 13 x  + 12 x  + 13 x  + 6 x
(%i3) gcd(p1, p2);
                             2
(%o3)                     6 x  + 13 x + 6
(%i4) p1/gcd(p1, p2), ratsimp;
                               x + 1
(%i5) p2/gcd(p1, p2), ratsimp;
                              3
(%o5)                        x  + x
```

ezgcd returns a list whose first element is the greatest common divisor of the polynomials *p_1* and *p_2*, and whose remaining elements are the polynomials divided by the greatest common divisor.

```
(%i1) p1:6*x^3+19*x^2+19*x+6 $
(%i2) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x $
(%i3) ezgcd(p1, p2);
                     2                       3
(%o3)           [6 x  + 13 x + 6, x + 1, x  + x]
```

gcdex [Function]

gcdex (*f*, *g*)
gcdex (*f*, *g*, *x*)

Returns a list `[a, b, u]` where *u* is the greatest common divisor (gcd) of *f* and *g*, and *u* is equal to `a` *f* `+ b` *g*. The arguments *f* and *g* should be univariate polynomials, or else polynomials in *x* a supplied main variable since we need to be in a principal ideal domain for this to work. The gcd means the gcd regarding *f* and *g* as univariate polynomials with coefficients being rational functions in the other variables.

`gcdex` implements the Euclidean algorithm, where we have a sequence of `L[i]: [a[i], b[i], r[i]]` which are all perpendicular to `[f, g, -1]` and the next one is built as if `q = quotient(r[i]/r[i+1])` then `L[i+2]: L[i] - q L[i+1]`, and it terminates at `L[i+1]` when the remainder `r[i+2]` is zero.

The arguments *f* and *g* can be integers. For this case the function igcdex is called by `gcdex`.

See also ezgcd, gcd, gcdivide, and poly_gcd.

Examples:

```
(%i1) gcdex (x^2 + 1, x^3 + 4);
                          2
                         x  + 4 x - 1  x + 4
(%o1)/R/            [- ------------, -----, 1]
                             17        17
```

```
(%i2) % . [x^2 + 1, x^3 + 4, -1];
(%o2)/R/                          0
```

Note that the gcd in the following is `1` since we work in `k(y)[x]`, not the `y+1` we would expect in `k[y, x]`.

```
(%i1) gcdex (x*(y + 1), y^2 - 1, x);
                                   1
(%o1)/R/                 [0, ------, 1]
                                 2
                                y  - 1
```

`gcfactor` (*n*) [Function]

Factors the Gaussian integer *n* over the Gaussian integers, i.e., numbers of the form `a + b %i` where *a* and *b* are rational integers (i.e., ordinary integers). Factors are normalized by making *a* and *b* non-negative.

`gfactor` (*expr*) [Function]

Factors the polynomial *expr* over the Gaussian integers (that is, the integers with the imaginary unit `%i` adjoined). This is like `factor (expr, a^2+1)` where *a* is `%i`.

Example:

```
(%i1) gfactor (x^4 - 1);
(%o1)           (x - 1) (x + 1) (x - %i) (x + %i)
```

`gfactorsum` (*expr*) [Function]

is similar to `factorsum` but applies `gfactor` instead of `factor`.

`hipow` (*expr*, *x*) [Function]

Returns the highest explicit exponent of *x* in *expr*. *x* may be a variable or a general expression. If *x* does not appear in *expr*, `hipow` returns `0`.

`hipow` does not consider expressions equivalent to `expr`. In particular, `hipow` does not expand `expr`, so `hipow (expr, x)` and `hipow (expand (expr, x))` may yield different results.

Examples:

```
(%i1) hipow (y^3 * x^2 + x * y^4, x);
(%o1)                             2
(%i2) hipow ((x + y)^5, x);
(%o2)                             1
(%i3) hipow (expand ((x + y)^5), x);
(%o3)                             5
(%i4) hipow ((x + y)^5, x + y);
(%o4)                             5
(%i5) hipow (expand ((x + y)^5), x + y);
(%o5)                             0
```

`intfaclim` [Option variable]

Default value: true

If `true`, maxima will give up factorization of integers if no factor is found after trial divisions and Pollard's rho method and factorization will not be complete.

When `intfaclim` is `false` (this is the case when the user calls `factor` explicitly), complete factorization will be attempted. `intfaclim` is set to `false` when factors are computed in `divisors`, `divsum` and `totient`.

Internal calls to `factor` respect the user-specified value of `intfaclim`. Setting `intfaclim` to `true` may reduce the time spent factoring large integers.

`keepfloat` [Option variable]

Default value: `false`

When `keepfloat` is `true`, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.

Note that the function `solve` and those functions calling it (`eigenvalues`, for example) currently ignore this flag, converting floating point numbers anyway.

Examples:

```
(%i1) rat(x/2.0);
rat: replaced 0.5 by 1/2 = 0.5
                                  x
(%o1)/R/                          -
                                  2
(%i2) rat(x/2.0), keepfloat;
(%o2)/R/                       0.5 x
```

`solve` ignores `keepfloat`:

```
(%i1) solve(1.0-x,x), keepfloat;
rat: replaced 1.0 by 1/1 = 1.0
(%o1)                         [x = 1]
```

`lopow` (*expr*, *x*) [Function]

Returns the lowest exponent of *x* which explicitly appears in *expr*. Thus

```
(%i1) lopow ((x+y)^2 + (x+y)^a, x+y);
(%o1)                        min(2, a)
```

`lratsubst` [Function]

`lratsubst` (*new*, *old*, *expr*)
`lratsubst` (*old* = *new*, *expr*)
`lratsubst` ([*old_1* = *new_1*, ..., *old_n* = *new_n*], *expr*)

`lratsubst` is analogous to `subst` except that it uses `ratsubst` to perform substitutions.

When `lratsubst` is given two arguments, the first argument may be an equation, a list of equations, or a list of exactly one element, which is a list of equations.

Substitutions are made in the order given by the list of equations, that is, from left to right.

Multiple substitutions are serial, not parallel. That is, later substitutions are performed on the results of earlier ones.

When `lratsubst` is given three arguments *new*, *old*, and *expr*, it is equivalent to `lratsubst(`*old* = *new*, *expr*`)`.

Examples:

`lratsubst` can carry out multiple substitutions. `lratsubst` is analogous to `subst`.

```
(%i1) lratsubst ([a = b, c = d], a + c);
(%o1)                          d + b
(%i2) lratsubst ([a^2 = b, c^2 = d], (a + e)*c*(a + c));
(%o2)                (d + a c) e + a d + b c
```

If only one substitution is desired, then a single equation may be given as first argument.

```
(%i1) lratsubst (a^2 = b, a^3);
(%o1)                           a b
```

The first argument may be a list of exactly one element, which is a list of equations.

```
(%i1) lratsubst ([[a^2=b*a, b=c]], a^3);
                                 2
(%o1)                           a  c
```

See also `fullratsubst`.

`lrats_max_iter` [Option variable]

Default value: `100000`

The upper limit on the number of iterations that `fullratsubst` and `lratsubst` may perform. It must be set to a positive integer. See the example for `fullratsubst`.

`modulus` [Option variable]

Default value: `false`

When `modulus` is a positive number *p*, operations on canonical rational expressions (CREs, as returned by `rat` and related functions) are carried out modulo *p*, using the so-called "balanced" modulus system in which *n* `modulo` *p* is defined as an integer *k* in `[-(p-1)/2, ..., 0, ..., (p-1)/2]` when *p* is odd, or `[-(p/2 - 1), ..., 0, ..., p/2]` when *p* is even, such that `a p + k` equals *n* for some integer *a*.

If *expr* is already in canonical rational expression (CRE) form when `modulus` is reset, then you may need to re-rat *expr*, e.g., `expr: rat (ratdisrep (expr))`, in order to get correct results.

Typically `modulus` is set to a prime number. If `modulus` is set to a positive non-prime integer, this setting is accepted, but a warning message is displayed. Maxima signals an error, when zero or a negative integer is assigned to `modulus`.

Examples:

```
(%i1) modulus:7;
(%o1)                            7
(%i2) polymod([0,1,2,3,4,5,6,7]);
(%o2)            [0, 1, 2, 3, - 3, - 2, - 1, 0]
(%i3) modulus:false;
(%o3)                          false
(%i4) poly:x^6+x^2+1;
                               6    2
(%o4)                         x  + x  + 1
```

```
(%i5) factor(poly);
                                6    2
(%o5)                          x  + x  + 1
(%i6) modulus:13;
(%o6)                               13
(%i7) factor(poly);
                           2         4      2
(%o7)                 (x  + 6) (x  - 6 x  - 2)
(%i8) polymod(%);
                                6    2
(%o8)                          x  + x  + 1
```

num (*expr*) [Function]

Returns the numerator of *expr* if it is a ratio. If *expr* is not a ratio, *expr* is returned. `num` evaluates its argument.

See also `denom`

```
(%i1) g1:(x+2)*(x+1)/((x+3)^2);
                              (x + 1) (x + 2)
(%o1)                         ---------------
                                        2
                                 (x + 3)
(%i2) num(g1);
                              (x + 1) (x + 2)
(%o2)
(%i3) g2:sin(x)/10*cos(x)/y;
                               cos(x) sin(x)
(%o3)                          -------------
                                   10 y
(%i4) num(g2);
                               cos(x) sin(x)
(%o4)
```

polydecomp (*p*, *x*) [Function]

Decomposes the polynomial p in the variable x into the functional composition of polynomials in x. `polydecomp` returns a list `[p_1, ..., p_n]` such that

```
lambda ([x], p_1) (lambda ([x], p_2) (... (lambda ([x], p_n) (x))
  ...))
```

is equal to p. The degree of `p_i` is greater than 1 for i less than n.

Such a decomposition is not unique.

Examples:

```
(%i1) polydecomp (x^210, x);
                             7   5   3   2
(%o1)                      [x , x , x , x ]
(%i2) p : expand (subst (x^3 - x - 1, x, x^2 - a));
                  6      4      3    2
(%o2)           x  - 2 x  - 2 x  + x  + 2 x - a + 1
(%i3) polydecomp (p, x);
                           2       3
(%o3)                  [x  - a, x  - x - 1]
```

The following function composes `L = [e_1, ..., e_n]` as functions in `x`; it is the inverse of polydecomp:

```
(%i1) compose (L, x) :=
  block ([r : x], for e in L do r : subst (e, x, r), r) $
```

Re-express above example using `compose`:

```
(%i1) compose (L, x) :=
  block ([r : x], for e in L do r : subst (e, x, r), r) $
(%i2) polydecomp (compose ([x^2 - a, x^3 - x - 1], x), x);
                          2       3
(%o2)                   [x  - a, x  - x - 1]
```

Note that though `compose (polydecomp (p, x), x)` always returns p (unexpanded), `polydecomp (compose ([p_1, ..., p_n], x), x)` does *not* necessarily return `[p_1, ..., p_n]`:

```
(%i1) compose (L, x) :=
  block ([r : x], for e in L do r : subst (e, x, r), r) $
(%i2) polydecomp (compose ([x^2 + 2*x + 3, x^2], x), x);
                            2       2
(%o2)                     [x  + 2, x  + 1]
(%i3) polydecomp (compose ([x^2 + x + 1, x^2 + x + 1], x), x);
                         2       2
                        x  + 3  x  + 5
(%o3)                  [------, ------, 2 x + 1]
                          4       2
```

`polymod` [Function]

`polymod (p)`
`polymod (p, m)`

Converts the polynomial p to a modular representation with respect to the current modulus which is the value of the variable `modulus`.

`polymod (p, m)` specifies a modulus m to be used instead of the current value of `modulus`.

See `modulus`.

`polynomialp` [Function]

`polynomialp (p, L, coeffp, exponp)`
`polynomialp (p, L, coeffp)`
`polynomialp (p, L)`

Return `true` if p is a polynomial in the variables in the list L. The predicate *coeffp* must evaluate to `true` for each coefficient, and the predicate *exponp* must evaluate to `true` for all exponents of the variables in L. If you want to use a non-default value for *exponp*, you must supply *coeffp* with a value even if you want to use the default for *coeffp*.

The command `polynomialp (p, L, coeffp)` is equivalent to `polynomialp (p, L, coeffp, 'nonnegintegerp)` and the command `polynomialp (p, L)` is equivalent to `polynomialp (p, L, 'constantp, 'nonnegintegerp)`.

The polynomial needn't be expanded:

```
(%i1) polynomialp ((x + 1)*(x + 2), [x]);
(%o1)                         true
(%i2) polynomialp ((x + 1)*(x + 2)^a, [x]);
(%o2)                         false
```

An example using non-default values for coeffp and exponp:

```
(%i1) polynomialp ((x + 1)*(x + 2)^(3/2), [x], numberp, numberp);
(%o1)                         true
(%i2) polynomialp ((x^(1/2) + 1)*(x + 2)^(3/2), [x], numberp,
                                                          numberp);
(%o2)                         true
```

Polynomials with two variables:

```
(%i1) polynomialp (x^2 + 5*x*y + y^2, [x]);
(%o1)                         false
(%i2) polynomialp (x^2 + 5*x*y + y^2, [x, y]);
(%o2)                         true
```

Polynomial in one variable and accepting any expression free of `x` as a coefficient.

```
(%i1) polynomialp (a*x^2 + b*x + c, [x]);
(%o1)                         false
(%i2) polynomialp (a*x^2 + b*x + c, [x], lambda([ex], freeof(x, ex)));
(%o2)                         true
```

`quotient` [Function]

`quotient` (*p_1*, *p_2*)
`quotient` (*p_1*, *p_2*, *x_1*, ..., *x_n*)

Returns the polynomial *p_1* divided by the polynomial *p_2*. The arguments *x_1*, ..., *x_n* are interpreted as in `ratvars`.

`quotient` returns the first element of the two-element list returned by `divide`.

`rat` [Function]

`rat` (*expr*)
`rat` (*expr*, *x_1*, ..., *x_n*)

Converts *expr* to canonical rational expression (CRE) form by expanding and combining all terms over a common denominator and cancelling out the greatest common divisor of the numerator and denominator, as well as converting floating point numbers to rational numbers within a tolerance of `ratepsilon`. The variables are ordered according to the *x_1*, ..., *x_n*, if specified, as in `ratvars`.

`rat` does not generally simplify functions other than addition `+`, subtraction `-`, multiplication `*`, division `/`, and exponentiation to an integer power, whereas `ratsimp` does handle those cases. Note that atoms (numbers and variables) in CRE form are not the same as they are in the general form. For example, `rat(x)- x` yields `rat(0)` which has a different internal representation than 0.

When `ratfac` is `true`, `rat` yields a partially factored form for CRE. During rational operations the expression is maintained as fully factored as possible without an actual call to the factor package. This should always save space and may save some time

in some computations. The numerator and denominator are still made relatively prime (e.g., `rat((x^2 - 1)^4/(x + 1)^2)` yields `(x - 1)^4 (x + 1)^2` when `ratfac` is `true`), but the factors within each part may not be relatively prime.

`ratprint` if `false` suppresses the printout of the message informing the user of the conversion of floating point numbers to rational numbers.

`keepfloat` if `true` prevents floating point numbers from being converted to rational numbers.

See also `ratexpand` and `ratsimp`.

Examples:

```
(%i1) ((x - 2*y)^4/(x^2 - 4*y^2)^2 + 1)*(y + a)*(2*y + x) /
      (4*y^2 + x^2);
                                          4
                                  (x - 2 y)
                (y + a) (2 y + x) (------------ + 1)
                                    2      2 2
                                   (x  - 4 y )
(%o1)           -----------------------------------
                                 2    2
                               4 y  + x
(%i2) rat (%, y, a, x);
                                2 a + 2 y
(%o2)/R/                        ---------
                                 x + 2 y
```

`ratalgdenom` [Option variable]

Default value: `true`

When `ratalgdenom` is `true`, allows rationalization of denominators with respect to radicals to take effect. `ratalgdenom` has an effect only when canonical rational expressions (CRE) are used in algebraic mode.

`ratcoef` [Function]

`ratcoef` (*expr*, *x*, *n*)
`ratcoef` (*expr*, *x*)

Returns the coefficient of the expression `x^n` in the expression *expr*. If omitted, *n* is assumed to be 1.

The return value is free (except possibly in a non-rational sense) of the variables in *x*. If no coefficient of this type exists, 0 is returned.

`ratcoef` expands and rationally simplifies its first argument and thus it may produce answers different from those of `coeff` which is purely syntactic. Thus `ratcoef ((x + 1)/y + x, x)` returns `(y + 1)/y` whereas `coeff` returns 1.

`ratcoef (expr, x, 0)`, viewing *expr* as a sum, returns a sum of those terms which do not contain *x*. Therefore if *x* occurs to any negative powers, `ratcoef` should not be used.

Since *expr* is rationally simplified before it is examined, coefficients may not appear quite the way they were envisioned.

Example:

```
(%i1) s: a*x + b*x + 5$
(%i2) ratcoef (s, a + b);
(%o2)                           x
```

`ratdenom` (*expr*) [Function]

Returns the denominator of *expr*, after coercing *expr* to a canonical rational expression (CRE). The return value is a CRE.

expr is coerced to a CRE by `rat` if it is not already a CRE. This conversion may change the form of *expr* by putting all terms over a common denominator.

`denom` is similar, but returns an ordinary expression instead of a CRE. Also, `denom` does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by `ratdenom` are not considered ratios by `denom`.

`ratdenomdivide` [Option variable]

Default value: `true`

When `ratdenomdivide` is `true`, `ratexpand` expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, `ratexpand` collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.

Examples:

```
(%i1) expr: (x^2 + x + 1)/(y^2 + 7);
                              2
                             x  + x + 1
(%o1)                        ----------
                                2
                               y  + 7
(%i2) ratdenomdivide: true$
(%i3) ratexpand (expr);
                         2
                        x        x        1
(%o3)                 ------ + ------ + ------
                       2        2        2
                      y  + 7   y  + 7   y  + 7
(%i4) ratdenomdivide: false$
(%i5) ratexpand (expr);
                              2
                             x  + x + 1
(%o5)                        ----------
                                2
                               y  + 7
```

```
(%i6) expr2: a^2/(b^2 + 3) + b/(b^2 + 3);
                                        2
                             b         a
(%o6)                      ------ + ------
                           2         2
                          b  + 3    b  + 3
(%i7) ratexpand (expr2);
                                      2
                                 b + a
(%o7)                            ------
                                  2
                                 b  + 3
```

`ratdiff` (*expr*, *x*) [Function]

Differentiates the rational expression *expr* with respect to *x*. *expr* must be a ratio of polynomials or a polynomial in *x*. The argument *x* may be a variable or a subexpression of *expr*.

The result is equivalent to `diff`, although perhaps in a different form. `ratdiff` may be faster than `diff`, for rational expressions.

`ratdiff` returns a canonical rational expression (CRE) if `expr` is a CRE. Otherwise, `ratdiff` returns a general expression.

`ratdiff` considers only the dependence of *expr* on *x*, and ignores any dependencies established by `depends`.

Example:

```
(%i1) expr: (4*x^3 + 10*x - 11)/(x^5 + 5);
                            3
                       4 x  + 10 x - 11
(%o1)                  ----------------
                              5
                             x  + 5
(%i2) ratdiff (expr, x);
                     7       5       4       2
                  8 x  + 40 x  - 55 x  - 60 x  - 50
(%o2)          - ---------------------------------
                           10       5
                          x   + 10 x  + 25
(%i3) expr: f(x)^3 - f(x)^2 + 7;
                          3       2
(%o3)                    f (x) - f (x) + 7
(%i4) ratdiff (expr, f(x));
                            2
(%o4)                    3 f (x) - 2 f(x)
(%i5) expr: (a + b)^3 + (a + b)^2;
                                3          2
(%o5)                    (b + a)  + (b + a)
```

```
(%i6) ratdiff (expr, a + b);
                      2                    2
(%o6)           3 b  + (6 a + 2) b + 3 a  + 2 a
```

`ratdisrep` (*expr*) [Function]

Returns its argument as a general expression. If *expr* is a general expression, it is returned unchanged.

Typically `ratdisrep` is called to convert a canonical rational expression (CRE) into a general expression. This is sometimes convenient if one wishes to stop the "contagion", or use rational functions in non-rational contexts.

See also `totaldisrep`.

`ratexpand` (*expr*) [Function]

`ratexpand` [Option variable]

Expands *expr* by multiplying out products of sums and exponentiated sums, combining fractions over a common denominator, cancelling the greatest common divisor of the numerator and denominator, then splitting the numerator (if a sum) into its respective terms divided by the denominator.

The return value of `ratexpand` is a general expression, even if *expr* is a canonical rational expression (CRE).

The switch `ratexpand` if `true` will cause CRE expressions to be fully expanded when they are converted back to general form or displayed, while if it is `false` then they will be put into a recursive form. See also `ratsimp`.

When `ratdenomdivide` is `true`, `ratexpand` expands a ratio in which the numerator is a sum into a sum of ratios, all having a common denominator. Otherwise, `ratexpand` collapses a sum of ratios into a single ratio, the numerator of which is the sum of the numerators of each ratio.

When `keepfloat` is `true`, prevents floating point numbers from being rationalized when expressions which contain them are converted to canonical rational expression (CRE) form.

Examples:

```
(%i1) ratexpand ((2*x - 3*y)^3);
                       3         2       2        3
(%o1)           - 27 y  + 54 x y  - 36 x  y + 8 x
(%i2) expr: (x - 1)/(x + 1)^2 + 1/(x - 1);
                          x - 1       1
(%o2)                    -------- + -----
                                 2   x - 1
                          (x + 1)
(%i3) expand (expr);
                        x              1           1
(%o3)           ------------ - ------------ + -----
                 2             2             x - 1
                x  + 2 x + 1   x  + 2 x + 1
```

```
(%i4) ratexpand (expr);
                          2
                       2 x                  2
(%o4)            --------------- + ---------------
                  3    2            3    2
                 x  + x  - x - 1   x  + x  - x - 1
```

`ratfac` [Option variable]

Default value: `false`

When `ratfac` is `true`, canonical rational expressions (CRE) are manipulated in a partially factored form.

During rational operations the expression is maintained as fully factored as possible without calling `factor`. This should always save space and may save time in some computations. The numerator and denominator are made relatively prime, for example `factor ((x^2 - 1)^4/(x + 1)^2)` yields `(x - 1)^4 (x + 1)^2`, but the factors within each part may not be relatively prime.

In the `ctensor` (Component Tensor Manipulation) package, Ricci, Einstein, Riemann, and Weyl tensors and the scalar curvature are factored automatically when `ratfac` is `true`. `ratfac` *should only be set for cases where the tensorial components are known to consist of few terms.*

The `ratfac` and `ratweight` schemes are incompatible and may not both be used at the same time.

`ratnumer` (*expr*) [Function]

Returns the numerator of *expr*, after coercing *expr* to a canonical rational expression (CRE). The return value is a CRE.

expr is coerced to a CRE by `rat` if it is not already a CRE. This conversion may change the form of *expr* by putting all terms over a common denominator.

`num` is similar, but returns an ordinary expression instead of a CRE. Also, `num` does not attempt to place all terms over a common denominator, and thus some expressions which are considered ratios by `ratnumer` are not considered ratios by `num`.

`ratp` (*expr*) [Function]

Returns `true` if *expr* is a canonical rational expression (CRE) or extended CRE, otherwise `false`.

CRE are created by `rat` and related functions. Extended CRE are created by `taylor` and related functions.

`ratprint` [Option variable]

Default value: `true`

When `ratprint` is `true`, a message informing the user of the conversion of floating point numbers to rational numbers is displayed.

`ratsimp` (*expr*) [Function]
`ratsimp` (*expr*, *x_1*, . . . , *x_n*) [Function]

Simplifies the expression *expr* and all of its subexpressions, including the arguments to non-rational functions. The result is returned as the quotient of two polynomials

in a recursive form, that is, the coefficients of the main variable are polynomials in the other variables. Variables may include non-rational functions (e.g., `sin (x^2 + 1)`) and the arguments to any such functions are also rationally simplified.

`ratsimp (`*`expr`*`, `*`x_1`*`, ..., `*`x_n`*`)` enables rational simplification with the specification of variable ordering as in `ratvars`.

When `ratsimpexpons` is `true`, `ratsimp` is applied to the exponents of expressions during simplification.

See also `ratexpand`. Note that `ratsimp` is affected by some of the flags which affect `ratexpand`.

Examples:

```
(%i1) sin (x/(x^2 + x)) = exp ((log(x) + 1)^2 - log(x)^2);
                                                2      2
                     x          (log(x) + 1)  - log (x)
(%o1)        sin(------) = %e
                  2
                 x  + x
(%i2) ratsimp (%);
                               1          2
(%o2)                   sin(-----) = %e x
                            x + 1
(%i3) ((x - 1)^(3/2) - (x + 1)*sqrt(x - 1))/sqrt((x - 1)*(x + 1));
                           3/2
                  (x - 1)    - sqrt(x - 1) (x + 1)
(%o3)             ---------------------------------
                       sqrt((x - 1) (x + 1))
(%i4) ratsimp (%);
                             2 sqrt(x - 1)
(%o4)                      - -------------
                                    2
                             sqrt(x  - 1)
(%i5) x^(a + 1/a), ratsimpexpons: true;
                                 2
                                a  + 1
                                ------
                                  a
(%o5)                          x
```

`ratsimpexpons` [Option variable]

Default value: `false`

When `ratsimpexpons` is `true`, `ratsimp` is applied to the exponents of expressions during simplification.

`radsubstflag` [Option variable]

Default value: `false`

`radsubstflag`, if `true`, permits `ratsubst` to make substitutions such as `u` for `sqrt (x)` in `x`.

`ratsubst` (*a*, *b*, *c*) [Function]

Substitutes *a* for *b* in *c* and returns the resulting expression. *b* may be a sum, product, power, etc.

`ratsubst` knows something of the meaning of expressions whereas `subst` does a purely syntactic substitution. Thus `subst (a, x + y, x + y + z)` returns `x + y + z` whereas `ratsubst` returns `z + a`.

When `radsubstflag` is `true`, `ratsubst` makes substitutions for radicals in expressions which don't explicitly contain them.

`ratsubst` ignores the value `true` of the option variables `keepfloat`, `float`, and `numer`.

Examples:

```
(%i1) ratsubst (a, x*y^2, x^4*y^3 + x^4*y^8);
                                3      4
(%o1)                        a x  y + a
(%i2) cos(x)^4 + cos(x)^3 + cos(x)^2 + cos(x) + 1;
               4         3         2
(%o2)       cos (x) + cos (x) + cos (x) + cos(x) + 1
(%i3) ratsubst (1 - sin(x)^2, cos(x)^2, %);
             4           2                      2
(%o3)    sin (x) - 3 sin (x) + cos(x) (2 - sin (x)) + 3
(%i4) ratsubst (1 - cos(x)^2, sin(x)^2, sin(x)^4);
                          4           2
(%o4)                cos (x) - 2 cos (x) + 1
(%i5) radsubstflag: false$
(%i6) ratsubst (u, sqrt(x), x);
(%o6)                             x
(%i7) radsubstflag: true$
(%i8) ratsubst (u, sqrt(x), x);
                                  2
(%o8)                            u
```

`ratvars` (*x_1*, ..., *x_n*) [Function]

`ratvars` () [Function]

`ratvars` [System variable]

Declares main variables *x_1*, ..., *x_n* for rational expressions. *x_n*, if present in a rational expression, is considered the main variable. Otherwise, *x_[n-1]* is considered the main variable if present, and so on through the preceding variables to *x_1*, which is considered the main variable only if none of the succeeding variables are present.

If a variable in a rational expression is not present in the `ratvars` list, it is given a lower priority than *x_1*.

The arguments to `ratvars` can be either variables or non-rational functions such as `sin(x)`.

The variable `ratvars` is a list of the arguments of the function `ratvars` when it was called most recently. Each call to the function `ratvars` resets the list. `ratvars ()` clears the list.

`ratvarswitch` [Option variable]

Default value: `true`

Maxima keeps an internal list in the Lisp variable `VARLIST` of the main variables for rational expressions. If `ratvarswitch` is `true`, every evaluation starts with a fresh list `VARLIST`. This is the default behavior. Otherwise, the main variables from previous evaluations are not removed from the internal list `VARLIST`.

The main variables, which are declared with the function `ratvars` are not affected by the option variable `ratvarswitch`.

Examples:

If `ratvarswitch` is `true`, every evaluation starts with a fresh list `VARLIST`.

```
(%i1) ratvarswitch:true$
(%i2) rat(2*x+y^2);
                                2
(%o2)/R/                       y  + 2 x
(%i3) :lisp varlist
($X $Y)
(%i3) rat(2*a+b^2);
                                2
(%o3)/R/                       b  + 2 a
(%i4) :lisp varlist
($A $B)
```

If `ratvarswitch` is `false`, the main variables from the last evaluation are still present.

```
(%i1) ratvarswitch:false$
(%i2) rat(2*x+y^2);
                                2
(%o2)/R/                       y  + 2 x
(%i3) :lisp varlist
($X $Y)
(%i3) rat(2*a+b^2);
                                2
(%o3)/R/                       b  + 2 a
(%i4) :lisp varlist
($A $B $X $Y)
```

`ratweight` [Function]

`ratweight` (*x_1*, *w_1*, ..., *x_n*, *w_n*)
`ratweight` ()

Assigns a weight *w_i* to the variable *x_i*. This causes a term to be replaced by 0 if its weight exceeds the value of the variable `ratwtlvl` (default yields no truncation). The weight of a term is the sum of the products of the weight of a variable in the term times its power. For example, the weight of `3 x_1^2 x_2` is `2 w_1 + w_2`. Truncation according to `ratwtlvl` is carried out only when multiplying or exponentiating canonical rational expressions (CRE).

`ratweight ()` returns the cumulative list of weight assignments.

Note: The `ratfac` and `ratweight` schemes are incompatible and may not both be used at the same time.

Examples:

```
(%i1) ratweight (a, 1, b, 1);
(%o1)                       [a, 1, b, 1]
(%i2) expr1: rat(a + b + 1)$
(%i3) expr1^2;
                     2                   2
(%o3)/R/            b  + (2 a + 2) b + a  + 2 a + 1
(%i4) ratwtlvl: 1$
(%i5) expr1^2;
(%o5)/R/                    2 b + 2 a + 1
```

`ratweights` [System variable]

Default value: `[]`

`ratweights` is the list of weights assigned by `ratweight`. The list is cumulative: each call to `ratweight` places additional items in the list.

`kill (ratweights)` and `save (ratweights)` both work as expected.

`ratwtlvl` [Option variable]

Default value: `false`

`ratwtlvl` is used in combination with the `ratweight` function to control the truncation of canonical rational expressions (CRE). For the default value of `false`, no truncation occurs.

`remainder` [Function]

`remainder` (*p_1*, *p_2*)
`remainder` (*p_1*, *p_2*, *x_1*, ..., *x_n*)

Returns the remainder of the polynomial *p_1* divided by the polynomial *p_2*. The arguments *x_1*, ..., *x_n* are interpreted as in `ratvars`.

`remainder` returns the second element of the two-element list returned by `divide`.

`resultant` (*p_1*, *p_2*, *x*) [Function]

The function `resultant` computes the resultant of the two polynomials *p_1* and *p_2*, eliminating the variable *x*. The resultant is a determinant of the coefficients of *x* in *p_1* and *p_2*, which equals zero if and only if *p_1* and *p_2* have a non-constant factor in common.

If *p_1* or *p_2* can be factored, it may be desirable to call `factor` before calling `resultant`.

The option variable `resultant` controls which algorithm will be used to compute the resultant. See the option variable [option_resultant], page 320.

The function `bezout` takes the same arguments as `resultant` and returns a matrix. The determinant of the return value is the desired resultant.

Examples:

```
(%i1) resultant(2*x^2+3*x+1, 2*x^2+x+1, x);
(%o1)                           8
(%i2) resultant(x+1, x+1, x);
(%o2)                           0
```

```
(%i3) resultant((x+1)*x, (x+1), x);
(%o3)                          0
(%i4) resultant(a*x^2+b*x+1, c*x + 2, x);
                          2
(%o4)                    c  - 2 b c + 4 a
(%i5) bezout(a*x^2+b*x+1, c*x+2, x);
                         [ 2 a  2 b - c ]
(%o5)                    [              ]
                         [  c      2    ]
(%i6) determinant(%);
(%o6)                    4 a - (2 b - c) c
```

`resultant` [Option variable]

Default value: `subres`

The option variable `resultant` controls which algorithm will be used to compute the resultant with the function `resultant`. The possible values are:

`subres` for the subresultant polynomial remainder sequence (PRS) algorithm,

`mod` (not enabled) for the modular resultant algorithm, and

`red` for the reduced polynomial remainder sequence (PRS) algorithm.

On most problems the default value `subres` should be best.

`savefactors` [Option variable]

Default value: `false`

When `savefactors` is `true`, causes the factors of an expression which is a product of factors to be saved by certain functions in order to speed up later factorizations of expressions containing some of the same factors.

`showratvars` (*expr*) [Function]

Returns a list of the canonical rational expression (CRE) variables in expression `expr`.

See also `ratvars`.

`sqfr` (*expr*) [Function]

is similar to `factor` except that the polynomial factors are "square-free." That is, they have factors only of degree one. This algorithm, which is also used by the first stage of `factor`, utilizes the fact that a polynomial has in common with its n'th derivative all its factors of degree greater than n. Thus by taking greatest common divisors with the polynomial of the derivatives with respect to each variable in the polynomial, all factors of degree greater than 1 can be found.

Example:

```
(%i1) sqfr (4*x^4 + 4*x^3 - 3*x^2 - 4*x - 1);
                                   2   2
(%o1)                    (2 x + 1)  (x  - 1)
```

`tellrat` [Function]

`tellrat` (*p_1*, ..., *p_n*)
`tellrat` ()

Adds to the ring of algebraic integers known to Maxima the elements which are the solutions of the polynomials *p_1*, ..., *p_n*. Each argument *p_i* is a polynomial with integer coefficients.

`tellrat (x)` effectively means substitute 0 for *x* in rational functions.

`tellrat ()` returns a list of the current substitutions.

`algebraic` must be set to `true` in order for the simplification of algebraic integers to take effect.

Maxima initially knows about the imaginary unit `%i` and all roots of integers.

There is a command `untellrat` which takes kernels and removes `tellrat` properties.

When `tellrat`'ing a multivariate polynomial, e.g., `tellrat (x^2 - y^2)`, there would be an ambiguity as to whether to substitute `y^2` for `x^2` or vice versa. Maxima picks a particular ordering, but if the user wants to specify which, e.g. `tellrat (y^2 = x^2)` provides a syntax which says replace `y^2` by `x^2`.

Examples:

```
(%i1) 10*(%i + 1)/(%i + 3^(1/3));
                             10 (%i + 1)
(%o1)                        -----------
                                    1/3
                              %i + 3
(%i2) ev (ratdisrep (rat(%)), algebraic);
              2/3      1/3              2/3      1/3
(%o2)    (4 3    - 2 3    - 4) %i + 2 3    + 4 3    - 2
(%i3) tellrat (1 + a + a^2);
                              2
(%o3)                        [a  + a + 1]
(%i4) 1/(a*sqrt(2) - 1) + a/(sqrt(3) + sqrt(2));
                        1                 a
(%o4)           ------------- + -----------------
                sqrt(2) a - 1   sqrt(3) + sqrt(2)
(%i5) ev (ratdisrep (rat(%)), algebraic);
                                3/2           3/2
                (7 sqrt(3) - 5 2    + 2) a - 2    - 1
(%o5)           -------------------------------------
                                  7
(%i6) tellrat (y^2 = x^2);
                            2    2   2
(%o6)                  [y  - x , a  + a + 1]
```

`totaldisrep` (*expr*) [Function]

Converts every subexpression of *expr* from canonical rational expressions (CRE) to general form and returns the result. If *expr* is itself in CRE form then `totaldisrep` is identical to `ratdisrep`.

`totaldisrep` may be useful for ratdisrepping expressions such as equations, lists, matrices, etc., which have some subexpressions in CRE form.

`untellrat (x_1, ..., x_n)` [Function]
Removes `tellrat` properties from *x_1*, ..., *x_n*.

16.3 Introduction to algebraic extensions

We assume here that the fields are of characteristic 0 so that irreductible polynomials have simple roots (are separable, thus square free). The base fields K of interest are the field Q of rational numbers, for algebraic numbers, and the fields of rational functions on the real numbers R or the complex numbers C, that is $R(t)$ or $C(t)$, when considering algebraic functions. An extension of degree n is defined by an irreducible degree n polynomial $p(x)$ with coefficients in the base field, and consists of the quotient of the ring $K[x]$ of polynomials by the multiples of $p(x)$. So if $p(x) = x^n + p_0 x^{n-1} + ... + p_n$, each time one encounters x^n one substitutes $-(p_0 x^{n-1} + ... + p_n)$. This is a field because of Bezout's identity, and a vector space of dimension n over K spanned by $1, x, ..., x^{n-1}$. When $K = C(t)$, this field can be identified with the field of algebraic functions on the algebraic curve of equation $p(x, t) = 0$.

In Maxima the process of taking rationals modulo p is obtained by the function `tellrat` when `algebraic` is true. The best way to ensure, in particular when considering the case where p depends on other variables that this simplification property is attached to x is to write (note the polynomial must be monic): `tellrat(x^n = -(p_0*x^(n - 1) + ... + p_n))` where the p_i may depend on other variables. When one wants to remove this tellrat property one then has to write `untellrat(x)`.

In the field $K[x]$ one may do all sorts of algebraic computations, taking quotients, GCD of two elements, etc. by the same algorithms as in the usual case. In particular one can do factorization of polynomials on an extension, using the function `algfac` below. Moreover multiplication by an element f is a linear operation of the vector space $K[x]$ over K and as such has a trace and a determinant. These are called `algtrace` and `algnorm` below. One can see that the trace of an element $f(x)$ in $K[x]$ is the sum of the values $f(a)$ when a runs over roots of p and the norm is the product of the $f(a)$. Both are symmetric in the roots of p and thus belong to K.

The field $K[x]$ is also called the field obtained by adjoining a root a of $p(x)$ to K. One can similarly adjoin a second root b of another polynomial obtaining a new extension $K[a, b]$. In fact there is a "prime element" c in $K[a, b]$ such that $K[a, b] = K[c]$. This is obtained by function `primeelmt` below. Recursively one can thus adjoin any number of elements. In particular adjoining all the roots of $p(x)$ to K one gets the splitting field of p, which is the smallest extension in which p completely splits in linear functions. The function `splitfield` constructs a primitive element of the splitting field, which in general is of very high degree.

The relevant concepts are explained in a concise and self-contained way in the small books edited by Dover: "Algebraic theory of numbers," by Pierre Samuel, "Algebraic curves," by Robert Walker, and the methods presented here are described in the article "Algebraic factoring and rational function integration" by B. Trager, *Proceedings of the 1976 AMS Symposium on Symbolic and Algebraic Computation.*

16.4 Functions and Variables for algebraic extensions

`algfac` (*f*, *p*) [Function]

Returns the factorization of f in the field $K[a]$. Does the same as `factor(f, p)` which in fact calls `algfac`. One can also specify the variable a as in `algfac(f, p, a)`.

Examples:

```
(%i1) algfac(x^4 + 1, a^2 - 2);
                            2             2
(%o1)                     (x  - a x + 1) (x  + a x + 1)
(%i2) algfac(x^4 - t*x^2 + 1, a^2 - t - 2, a);
                            2             2
(%o2)                     (x  - a x + 1) (x  + a x + 1)
```

In the second example note that $a = sqrt(2 + t)$.

`algnorm` (*f*, *p*, `a`) [Function]

Returns the norm of the polynomial $f(a)$ in the extension obtained by a root a of polynomial p. The coefficients of f may depend on other variables.

Examples:

```
(%i1) algnorm(x*a^2 + y*a + z,a^2 - 2, a);
                             2             2      2
(%o1)/R/                    z  + 4 x z - 2 y  + 4 x
```

The norm is also the resultant of polynomials f and p, and the product of the differences of the roots of f and p.

`algtrace` (*f*, *p*, `a`) [Function]

Returns the trace of the polynomial $f(a)$ in the extension obtained by a root a of polynomial p. The coefficients of f may depend on other variables which remain "inert".

Example:

```
(%i1) algtrace(x*a^5 + y*a^3 + z + 1, a^2 + a + 1, a);
(%o1)/R/                        2 z + 2 y - x + 2
```

`bdiscr` (*args*) [Function]

Computes the discriminant of a basis x_i in $K[a]$ as the determinant of the matrix of elements $trace(x_i * x_j)$. The args are the elements of the basis followed by the minimal polynomial.

Example:

```
(%i1) bdiscr(1, x, x^2, x^3 - 2);
(%o1)/R/                                - 108
(%i2) poly_discriminant(x^3 - 2, x);
(%o2)                                   - 108
```

A standard base in an extension of degree n is $1, x, ..., x^{n-1}$. In this case it is known that the discriminant of this base is the discriminant of the minimal polynomial. This is checked in (%o2) above.

`primelmt` (*f_b*, *p_a*, *c*) [Function]

Computes a prime element for the extension of $K[a]$ by a root b of a polynomial $f_b(b)$ whose coefficients may depend on a. One assumes that *f_b* is square free. The function returns an irreducible polynomial, a root of which generates $K[a, b]$, and the expression of this primitive element in terms of a and b.

Examples:

```
(%i1) primelmt(b^2 - a*b - 1, a^2 - 2, c);
                              4      2
(%o1)                        [c  - 12 c  + 9, b + a]
(%i2) solve(b^2 - sqrt(2)*b - 1)[1];
                                   sqrt(6) - sqrt(2)
(%o2)                       b = - -----------------
                                           2
(%i3) primelmt(b^2 - 3, a^2 - 2, c);
                              4      2
(%o3)                        [c  - 10 c  + 1, b + a]
(%i4) factor(c^4 - 12*c^2 + 9, a^4 - 10*a^2 + 1);
                  3    2                       3    2
(%o4) ((4 c - 3 a  - a  + 27 a + 5) (4 c - 3 a  + a  + 27 a - 5)
                            3    2                     3    2
                  (4 c + 3 a  - a  - 27 a + 5) (4 c + 3 a  + a  - 27 a - 5))/256
(%i5) primelmt(b^3 - 3, a^2 - 2, c);
                    6      4      3      2
(%o5)              [c  - 6 c  - 6 c  + 12 c  - 36 c + 1, b + a]
(%i6) factor(b^3 - 3, %[1]);
             5       4        3        2
(%o6) ((48 c  + 27 c  - 320 c  - 468 c  + 124 c + 755 b - 1092)
           5        5         4       4          3        3          2        2
 ((- 48 b c ) - 54 c  - 27 b c  + 64 c  + 320 b c  + 360 c  + 468 b c  + 149 c
                            2
 - 124 b c - 1272 c + 755 b  + 1092 b + 1606))/570025
```

In (%o1), *f_b* depends on `a`. Using `solve`, the solution depends on sqrt(2) and sqrt(3). In (%o3), $K[sqrt(2), sqrt(3)]$ is computed, and we see that the the primitive polynomial in (%o1) factorizes completely here. In (%i5), we compute $K[sqrt(2), 3^{1/3}]$, and we see that `b^3 - 3` gets one factor in this extension. If we assume this extension is real, the two other factors are complex.

`splitfield` (*p*, *x*) [Function]

Computes the splitting field of the polynomial $p(x)$. In the generic case it is of degree $n!$ in terms of the degree n of p, but may be of lower order if the Galois group of p is a strict subgroup of the group of permutations of n elements. The function returns a primitive polynomial for this extension and the expressions of the roots of p as polynomials of a root of this primitive polynomial. The polynomial f may be irreducible or factorizable.

Examples:

```
(%i1) splitfield(x^3 + x + 1, x);
```

```
                                                  4         2
              6         4         2        alg1  + 5 alg1  - 9 alg1 + 4
(%o1)/R/ [alg1  + 6 alg1  + 9 alg1  + 31, ----------------------------,
                                                          18
                                   4         2          4         2
                              alg1  + 5 alg1  + 4  alg1  + 5 alg1  + 9 alg1 + 4
                            - -------------------, ----------------------------]
                                       9                        18
(%i2) splitfield(x^4 + 10*x^2 - 96*x - 71, x)[1];
             8           6           5            4             3
(%o2)/R/ alg2  + 148 alg2  - 576 alg2  + 9814 alg2  - 42624 alg2
                                                     2
                                        + 502260 alg2  + 1109952 alg2 + 18860337
```

In the first case we have the primitive polynomial of degree 6 and the 3 roots of the third degree equations in terms of a variable `alg1` produced by the system. In the second case the primitive polynomial is of degree 8 instead of 24, because the Galois group of the equation is reduced to D8 since there are relations between the roots.

17 Special Functions

17.1 Introduction to Special Functions

Special function notation follows:

```
bessel_j (index, expr)          Bessel function, 1st kind
bessel_y (index, expr)          Bessel function, 2nd kind
bessel_i (index, expr)          Modified Bessel function, 1st kind
bessel_k (index, expr)          Modified Bessel function, 2nd kind

hankel_1 (v,z)                  Hankel function of the 1st kind
hankel_2 (v,z)                  Hankel function of the 2nd kind
struve_h (v,z)                  Struve H function
struve_l (v,z)                  Struve L function

assoc_legendre_p[v,u] (z)       Legendre function of degree v and order u
assoc_legendre_q[v,u] (z)       Legendre function, 2nd kind

%f[p,q] ([], [], expr)          Generalized Hypergeometric function
gamma (z)                       Gamma function
gamma_incomplete_lower (a,z)    Lower incomplete gamma function
gamma_incomplete (a,z)          Tail of incomplete gamma function
hypergeometric (l1, l2, z)      Hypergeometric function
%s[u,v] (z)                     Lommel "small" s function
slommel[u,v] (z)                Lommel "big" S function
%m[u,k] (z)                     Whittaker function, 1st kind
%w[u,k] (z)                     Whittaker function, 2nd kind
erfc (z)                        Complement of the erf function

expintegral_e (v,z)             Exponential integral E
expintegral_e1 (z)              Exponential integral E1
expintegral_ei (z)              Exponential integral Ei
expintegral_li (z)              Logarithmic integral Li
expintegral_si (z)              Exponential integral Si
expintegral_ci (z)              Exponential integral Ci
expintegral_shi (z)             Exponential integral Shi
expintegral_chi (z)             Exponential integral Chi

kelliptic (z)                   Complete elliptic integral of the first
                                kind (K)
parabolic_cylinder_d (v,z)      Parabolic cylinder D function
```

17.2 Bessel Functions

`bessel_j (v, z)` [Function]

The Bessel function of the first kind of order v and argument z. See A&S eqn 9.1.10 and DLMF 10.2.E2.

`bessel_j` is defined as

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{v+2\,k}}{k!\,\Gamma\left(v+k+1\right)}$$

although the infinite series is not used for computations.

When `besselexpand` is `true`, `bessel_j` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

`bessel_y (v, z)` [Function]

The Bessel function of the second kind of order v and argument z. See A&S eqn 9.1.2 and DLMF 10.2.E3.

`bessel_y` is defined as

$$Y_v(z) = \frac{\cos(\pi v)\,J_v(z) - J_{-v}(z)}{\sin \pi v}$$

when v is not an integer. When v is an integer n, the limit as v approaches n is taken.

When `besselexpand` is `true`, `bessel_y` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

`bessel_i (v, z)` [Function]

The modified Bessel function of the first kind of order v and argument z. See A&S eqn 9.6.10 and DLMF 10.25.E2.

`bessel_i` is defined as

$$I_v(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma\left(v+k+1\right)} \left(\frac{z}{2}\right)^{v+2\,k}$$

although the infinite series is not used for computations.

When `besselexpand` is `true`, `bessel_i` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

`bessel_k (v, z)` [Function]

The modified Bessel function of the second kind of order v and argument z. See A&S eqn 9.6.2 and DLMF 10.27.E4.

`bessel_k` is defined as

$$K_v(z) = \frac{1}{2}\pi\,\frac{I_{-v}(z) - I_v(z)}{\sin v\pi}$$

when v is not an integer. If v is an integer n, then the limit as v approaches n is taken.

When `besselexpand` is `true`, `bessel_k` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

`hankel_1` (*v*, *z*) [Function]

The Hankel function of the first kind of order v and argument z. See A&S eqn 9.1.3 and DLMF 10.4.E3.

`hankel_1` is defined as

$$H_v^{(1)}(z) = J_v(z) + iY_v(z)$$

Maxima evaluates `hankel_1` numerically for a complex order v and complex argument z in float precision. The numerical evaluation in bigfloat precision is not supported.

When `besselexpand` is `true`, `hankel_1` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

Maxima knows the derivative of `hankel_1` wrt the argument z.

Examples:

Numerical evaluation:

```
(%i1) hankel_1(1,0.5);
(%o1)      0.24226845767487384 - 1.4714723926702433 %i
(%i2) hankel_1(1,0.5+%i);
(%o2)     - 0.2558287994862166 %i - 0.23957560188301597
```

Expansion of `hankel_1` when `besselexpand` is `true`:

```
(%i1) hankel_1(1/2,z),besselexpand:true;
                 sqrt(2) sin(z) - sqrt(2) %i cos(z)
(%o1)            ----------------------------------
                         sqrt(%pi) sqrt(z)
```

Derivative of `hankel_1` wrt the argument z. The derivative wrt the order v is not supported. Maxima returns a noun form:

```
(%i1) diff(hankel_1(v,z),z);
              hankel_1(v - 1, z) - hankel_1(v + 1, z)
(%o1)         ---------------------------------------
                                 2
(%i2) diff(hankel_1(v,z),v);
                         d
(%o2)                    -- (hankel_1(v, z))
                         dv
```

`hankel_2` (*v*, *z*) [Function]

The Hankel function of the second kind of order v and argument z. See A&S eqn 9.1.4 and DLMF 10.4.E3.

`hankel_2` is defined as

$$H_v^{(2)}(z) = J_v(z) - iY_v(z)$$

Maxima evaluates `hankel_2` numerically for a complex order v and complex argument z in float precision. The numerical evaluation in bigfloat precision is not supported.

When `besselexpand` is `true`, `hankel_2` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

Maxima knows the derivative of `hankel_2` wrt the argument z.

For examples see `hankel_1`.

`besselexpand` [Option variable]

Default value: `false`

Controls expansion of the Bessel, Hankel and Struve functions when the order is half of an odd integer. In this case, the functions can be expanded in terms of other elementary functions. When `besselexpand` is `true`, the Bessel function is expanded.

```
(%i1) besselexpand: false$
(%i2) bessel_j (3/2, z);
                                    3
(%o2)                     bessel_j(-, z)
                                    2
(%i3) besselexpand: true$
(%i4) bessel_j (3/2, z);
                                  sin(z)   cos(z)
                 sqrt(2) sqrt(z) (------ - ------)
                                    2       z
                                   z
(%o4)            --------------------------------
                             sqrt(%pi)
(%i5) bessel_y(3/2,z);
                                  sin(z)   cos(z)
                sqrt(2) sqrt(z) (- ------ - ------)
                                    z         2
                                             z
(%o5)           -----------------------------------
                             sqrt(%pi)
(%i6) bessel_i(3/2,z);
                                 cosh(z)   sinh(z)
                sqrt(2) sqrt(z) (------- - -------)
                                    z         2
                                             z
(%o6)           -----------------------------------
                             sqrt(%pi)
(%i7) bessel_k(3/2,z);
                                   1        - z
                       sqrt(%pi) (- + 1) %e
                                   z
(%o7)                  -----------------------
                           sqrt(2) sqrt(z)
```

`scaled_bessel_i` (*v*, *z*) [Function]

The scaled modified Bessel function of the first kind of order v and argument z. That is,

$$\text{scaled\_bessel\_i}(v, z) = e^{-|z|} I_v(z).$$

This function is particularly useful for calculating $I_v(z)$ for large z, which is large. However, maxima does not otherwise know much about this function. For symbolic work, it is probably preferable to work with the expression `exp(-abs(z))*bessel_i(v, z)`.

`scaled_bessel_i0 (z)` [Function]
Identical to `scaled_bessel_i(0,z)`.

`scaled_bessel_i1 (z)` [Function]
Identical to `scaled_bessel_i(1,z)`.

`%s [u,v] (z)` [Function]
Lommel's little $s_{\mu,\nu}(z)$ function. (DLMF 11.9.E3)(G&R 8.570.1).

This Lommel function is the particular solution of the inhomogeneous Bessel differential equation:

$$\frac{d^2}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w = z^{\mu-1}$$

This can be defined by the series

$$s_{\mu,\nu}(z) = z^{\mu+1} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{a_{k+1}(\mu,\nu)}$$

where

$$a_k(\mu,\nu) = \prod_{m=1}^{k} \left((\mu + 2m - 1)^2 - \nu^2\right) = 4^k \left(\frac{\mu - \nu + 1}{2}\right)_k \left(\frac{\mu + \nu + 1}{2}\right)_k$$

`slommel [u,v] (z)` [Function]
Lommel's big $S_{\mu,\nu}(z)$ function. (DLMF 11.9.E5)(G&R 8.570.2).

Lommels big S function is another particular solution of the inhomogeneous Bessel differential equation (see [%s], page 331) defined for all values of μ and ν, where

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right)\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \times \left(\sin\left(\frac{(\mu-\nu)\pi}{2}\right) J_\nu(z) - \cos\left(\frac{(\mu-\nu)\pi}{2}\right) Y_\nu(z)\right)$$

When $\mu \pm \nu$ is an odd negative integer, the limit must be used.

17.3 Airy Functions

The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Section 10.4 and DLMF 9.

The Airy differential equation is:

$$\frac{d^2y}{dx^2} - xy = 0$$

The numerically satisfactory pair of solutions (DLMF 9.2#T1) on the real line are $y = \mathrm{Ai}(x)$ and $y = \mathrm{Bi}(x)$. These two solutions are oscillatory for $x < 0$. $\mathrm{Ai}(x)$ is the solution subject to the condition that $y \to 0$ as $x \to +\infty$, and $\mathrm{Bi}(x)$ is the second solution with the same amplitude as $\mathrm{Ai}(x)$ as $x \to -\infty$ which differs in phase by $\pi/2$. Also, $\mathrm{Bi}(x)$ is unbounded as $x \to +\infty$.

If the argument x is a real or complex floating point number, the numerical value of the function is returned.

`airy_ai (x)` [Function]

The Airy function $\mathrm{Ai}(x)$. See A&S eqn 10.4.2 and DLMF 9.

See also `airy_bi`, `airy_dai`, and `airy_dbi`.

`airy_dai (x)` [Function]

The derivative of the Airy function $\mathrm{Ai}(x)$:

$$\mathrm{airy\_dai}(x) = \frac{d}{dx}\mathrm{Ai}(x)$$

See `airy_ai`.

`airy_bi (x)` [Function]

The Airy function $\mathrm{Bi}(x)$. See A&S eqn 10.4.3 and DLMF 9.

See `airy_ai`, and `airy_dbi`.

`airy_dbi (x)` [Function]

The derivative of the Airy function $\mathrm{Bi}(x)$:

$$\mathrm{airy\_dbi}(x) = \frac{d}{dx}\mathrm{Bi}(x)$$

See `airy_ai`, and `airy_bi`.

17.4 Gamma and Factorial Functions

The gamma function and the related beta, psi and incomplete gamma functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, Chapter 6.

`bffac (expr, n)` [Function]

Bigfloat version of the factorial (shifted gamma) function. The second argument is how many digits to retain and return, it's a good idea to request a couple of extra.

```
(%i1) bffac(1/2,16);
(%o1)                 8.862269254527584b-1
```

```
(%i2) (1/2)!,numer;
(%o2)                    0.886226925452758
(%i3) bffac(1/2,32);
(%o3)          8.8622692545275801364908374167O7b-1
```

bfpsi (*n*, *z*, *fpprec*) [Function]
bfpsi0 (*z*, *fpprec*) [Function]

`bfpsi` is the polygamma function of real argument *z* and integer order *n*. See [polygamma], page 344, for further information. `bfpsi0` is the digamma function. `bfpsi0(`*z*`, `*fpprec*`)` is equivalent to `bfpsi(0, `*z*`, `*fpprec*`)`.

These functions return bigfloat values. *fpprec* is the bigfloat precision of the return value.

```
(%i1) bfpsi0(1/3, 15);
(%o1)                  - 3.13203378002081b0
(%i2) bfpsi0(1/3, 32);
(%o2)          - 3.13203378002080632299641907428736b0
(%i3) bfpsi(0,1/3,32);
(%o3)          - 3.13203378002080632299641907428736b0
(%i4) psi[0](1/3);
                    3 log(3)      %pi
(%o4)             - -------- - --------- - %gamma
                       2       2 sqrt(3)
(%i5) float(%);
(%o5)                - 3.132033780020806
```

cbffac (*z*, *fpprec*) [Function]

Complex bigfloat factorial.

`load ("bffac")` loads this function.

```
(%i1) cbffac(1+%i,16);
(%o1)    3.430658398165453b-1 %i + 6.529654964201666b-1
(%i2) (1+%i)!,numer;
(%o2)      0.3430658398165453 %i + 0.6529654964201667
```

gamma (*z*) [Function]

The basic definition of the gamma function (DLMF 5.2.E1 and A&S eqn 6.1.1) is

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt$$

Maxima simplifies `gamma` for positive integer and positive and negative rational numbers. For half integral values the result is a rational number times $\sqrt{\pi}$. The simplification for integer values is controlled by `factlim`. For integers greater than `factlim` the numerical result of the factorial function, which is used to calculate `gamma`, will overflow. The simplification for rational numbers is controlled by `gammalim` to avoid internal overflow. See `factlim` and `gammalim`.

For negative integers `gamma` is not defined.

Maxima can evaluate `gamma` numerically for real and complex values in float and bigfloat precision.

`gamma` has mirror symmetry.

When `gamma_expand` is `true`, Maxima expands `gamma` for arguments `z+n` and `z-n` where `n` is an integer.

Maxima knows the derivative of `gamma`.

Examples:

Simplification for integer, half integral, and rational numbers:

```
(%i1) map('gamma,[1,2,3,4,5,6,7,8,9]);
(%o1)        [1, 1, 2, 6, 24, 120, 720, 5040, 40320]
(%i2) map('gamma,[1/2,3/2,5/2,7/2]);
                     sqrt(%pi)  3 sqrt(%pi)  15 sqrt(%pi)
(%o2)   [sqrt(%pi), ---------, -----------, ------------]
                        2           4            8
(%i3) map('gamma,[2/3,5/3,7/3]);
                                    2           1
                             2 gamma(-)  4 gamma(-)
                        2           3           3
(%o3)          [gamma(-), ----------, ----------]
                        3       3           9
```

Numerical evaluation for real and complex values:

```
(%i1) map('gamma,[2.5,2.5b0]);
(%o1)       [1.329340388179137, 1.329340388179137b0]
(%i2) map('gamma,[1.0+%i,1.0b0+%i]);
(%o2) [0.49801566811835596 - 0.15494982830181073 %i,
                  4.980156681183561b-1 - 1.549498283018107b-1 %i]
```

`gamma` has mirror symmetry:

```
(%i1) declare(z,complex)$
(%i2) conjugate(gamma(z));
(%o2)                  gamma(conjugate(z))
```

Maxima expands `gamma(z+n)` and `gamma(z-n)`, when `gamma_expand` is `true`:

```
(%i1) gamma_expand:true$
(%i2) [gamma(z+1),gamma(z-1),gamma(z+2)/gamma(z+1)];
                                 gamma(z)
(%o2)             [z gamma(z), --------, z + 1]
                                  z - 1
```

The derivative of `gamma`:

```
(%i1) diff(gamma(z),z);
(%o1)                   psi (z) gamma(z)
                           0
```

See also `makegamma`.

The Euler-Mascheroni constant is `%gamma`.

`log_gamma (z)` [Function]

The natural logarithm of the gamma function.

```
(%i1) gamma(6);
(%o1)                          120
(%i2) log_gamma(6);
(%o2)                       log(120)
(%i3) log_gamma(0.5);
(%o3)                0.5723649429247004
```

`gamma_incomplete_lower (a, z)` [Function]

The lower incomplete gamma function (DLMF 8.2.E1 and A&S eqn 6.5.2):

$$\gamma\left(a,z\right)=\int_{0}^{z}t^{a-1}\,e^{-t}\,dt$$

See also `gamma_incomplete` (upper incomplete gamma function).

`gamma_incomplete (a, z)` [Function]

The incomplete upper gamma function (DLMF 8.2.E2 and A&S eqn 6.5.3):

$$\Gamma\left(a,z\right)=\int_{z}^{\infty}t^{a-1}\,e^{-t}\,dt$$

See also `gamma_expand` for controlling how `gamma_incomplete` is expressed in terms of elementary functions and `erfc`.

Also see the related functions `gamma_incomplete_regularized` and `gamma_incomplete_generalized`.

`gamma_incomplete_regularized (a, z)` [Function]

The regularized incomplete upper gamma function (DLMF 8.2.E4):

$$Q\left(a,z\right)=\frac{\Gamma\left(a,z\right)}{\Gamma\left(a\right)}$$

See also `gamma_expand` for controlling how `gamma_incomplete` is expressed in terms of elementary functions and `erfc`.

Also see `gamma_incomplete`.

`gamma_incomplete_generalized (a, z1, z1)` [Function]

The generalized incomplete gamma function.

$$\Gamma\left(a,z_1,z_2\right)=\int_{z_1}^{z_2}t^{a-1}\,e^{-t}\,dt$$

Also see `gamma_incomplete` and `gamma_incomplete_regularized`.

gamma_expand [Option variable]

Default value: false

gamma_expand controls expansion of gamma_incomplete. When gamma_expand is true, gamma_incomplete(v,z) is expanded in terms of z, exp(z), and gamma_incomplete or erfc when possible.

```
(%i1) gamma_incomplete(2,z);
(%o1)                gamma_incomplete(2, z)
(%i2) gamma_expand:true;
(%o2)                          true
(%i3) gamma_incomplete(2,z);
                                      - z
(%o3)                      (z + 1) %e
(%i4) gamma_incomplete(3/2,z);
                        - z   sqrt(%pi) erfc(sqrt(z))
(%o4)        sqrt(z) %e    + ----------------------
                                        2
(%i5) gamma_incomplete(4/3,z);
                                                1
                              gamma_incomplete(-, z)
                 1/3   - z                      3
(%o5)           z    %e    + ----------------------
                                          3
(%i6) gamma_incomplete(a+2,z);
       a               - z
(%o6) z  (z + a + 1) %e    + a (a + 1) gamma_incomplete(a, z)
(%i7) gamma_incomplete(a-2, z);
      gamma_incomplete(a, z)    a - 2         z            1
(%o7) ---------------------- - z      (--------------- + -----)
         (1 - a) (2 - a)               (a - 2) (a - 1)   a - 2
                                                             - z
                                                            %e
```

gammalim [Option variable]

Default value: 10000

gammalim controls simplification of the gamma function for integral and rational number arguments. If the absolute value of the argument is not greater than gammalim, then simplification will occur. Note that the factlim switch controls simplification of the result of gamma of an integer argument as well.

makegamma (*expr*) [Function]

Transforms instances of binomial, factorial, and beta functions in *expr* into gamma functions.

See also makefact.

```
(%i1) makegamma(binomial(n,k));
                              gamma(n + 1)
(%o1)              ----------------------------
                   gamma(k + 1) gamma(n - k + 1)
```

```
(%i2) makegamma(x!);
(%o2)                        gamma(x + 1)
(%i3) makegamma(beta(a,b));
                           gamma(a) gamma(b)
(%o3)                      -----------------
                             gamma(b + a)
```

`beta` (*a*, *b*) [Function]

The beta function is defined as

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(DLMF 5.12.E1 and A&S eqn 6.2.1).

Maxima simplifies the beta function for positive integers and rational numbers, which sum to an integer. When `beta_args_sum_to_integer` is `true`, Maxima simplifies also general expressions which sum to an integer.

For *a* or *b* equal to zero the beta function is not defined.

In general the beta function is not defined for negative integers as an argument. The exception is for *a*=-*n*, *n* a positive integer and *b* a positive integer with `b<=n`, it is possible to define an analytic continuation. Maxima gives for this case a result.

When `beta_expand` is `true`, expressions like `beta(a+n,b)` and `beta(a-n,b)` or `beta(a,b+n)` and `beta(a,b-n)` with `n` an integer are simplified.

Maxima can evaluate the beta function for real and complex values in float and bigfloat precision. For numerical evaluation Maxima uses `log_gamma`:

```
                - log_gamma(b + a) + log_gamma(b) + log_gamma(a)
              %e
```

Maxima knows that the beta function is symmetric and has mirror symmetry.

Maxima knows the derivatives of the beta function with respect to *a* or *b*.

To express the beta function as a ratio of gamma functions see `makegamma`.

Examples:

Simplification, when one of the arguments is an integer:

```
(%i1) [beta(2,3),beta(2,1/3),beta(2,a)];
                          1   9      1
(%o1)                    [--, -, ---------]
                          12  4  a (a + 1)
```

Simplification for two rational numbers as arguments which sum to an integer:

```
(%i1) [beta(1/2,5/2),beta(1/3,2/3),beta(1/4,3/4)];
                     3 %pi   2 %pi
(%o1)               [-----, -------, sqrt(2) %pi]
                       8    sqrt(3)
```

When setting `beta_args_sum_to_integer` to `true` more general expression are simplified, when the sum of the arguments is an integer:

```
(%i1) beta_args_sum_to_integer:true$
```

```
(%i2) beta(a+1,-a+2);
                              %pi (a - 1) a
(%o2)                        ------------------
                             2 sin(%pi (2 - a))
```

The possible results, when one of the arguments is a negative integer:

```
(%i1) [beta(-3,1),beta(-3,2),beta(-3,3)];
                                 1  1    1
(%o1)                         [- -, -, - -]
                                 3  6    3
```

`beta(a+n,b)` or `beta(a-n,b)` with `n` an integer simplifies when `beta_expand` is `true`:

```
(%i1) beta_expand:true$
(%i2) [beta(a+1,b),beta(a-1,b),beta(a+1,b)/beta(a,b+1)];
               a beta(a, b)  beta(a, b) (b + a - 1)  a
(%o2)         [------------, ----------------------, -]
                  b + a               a - 1          b
```

Beta is not defined, when one of the arguments is zero:

beta: expected nonzero arguments; found 0, b – an error. To debug this try: debugmode(true);

```
(%i1) beta(0,b);
```

Numerical evaluation for real and complex arguments in float or bigfloat precision:

```
(%i1) beta(2.5,2.3);
(%o1)                   0.08694748611299981
(%i2) beta(2.5,1.4+%i);
(%o2)     0.06401449507966957 - 0.15020780532864159 %i
(%i3) beta(2.5b0,2.3b0);
(%o3)                  8.694748611299965b-2
(%i4) beta(2.5b0,1.4b0+%i);
(%o4)    6.401449507966939b-2 - 1.502078053286414b-1 %i
```

Beta is symmetric and has mirror symmetry:

```
(%i1) beta(a,b)-beta(b,a);
(%o1)                           0
(%i2) declare(a,complex,b,complex)$
(%i3) conjugate(beta(a,b));
(%o3)           beta(conjugate(a), conjugate(b))
```

The derivative of the beta function wrt `a`:

```
(%i1) diff(beta(a,b),a);
(%o1)         - beta(a, b) (psi (b + a) - psi (a))
                                0             0
```

`beta_incomplete` (*a*, *b*, *z*) [Function]

The basic definition of the incomplete beta function (DLMF 8.17.E1 and A&S eqn 6.6.1) is

$$\mathrm{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,dt$$

This definition is possible for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$ and $|z| < 1$. For other values the incomplete beta function can be defined through a generalized hypergeometric function:

```
gamma(a) hypergeometric_generalized([a, 1 - b], [a + 1], z) z
```

(See https://functions.wolfram.com/GammaBetaErf/Beta3/ for a complete definition of the incomplete beta function.)

For negative integers $a = -n$ and positive integers $b = m$ with $m \leq n$ the incomplete beta function is defined through

$$z^{n-1} \sum_{k=0}^{m-1} \frac{(1-m)_k z^k}{k!(n-k)}$$

Maxima uses this definition to simplify `beta_incomplete` for a a negative integer.

For a a positive integer, `beta_incomplete` simplifies for any argument b and z and for b a positive integer for any argument a and z, with the exception of a a negative integer.

For $z = 0$ and $\mathrm{Re}(a) > 0$, `beta_incomplete` has the specific value zero. For $z = 1$ and $\mathrm{Re}(b) > 0$, `beta_incomplete` simplifies to the beta function `beta(a,b)`.

Maxima evaluates `beta_incomplete` numerically for real and complex values in float or bigfloat precision. For the numerical evaluation an expansion of the incomplete beta function in continued fractions is used.

When the option variable `beta_expand` is `true`, Maxima expands expressions like `beta_incomplete(a+n,b,z)` and `beta_incomplete(a-n,b,z)` where n is a positive integer.

Maxima knows the derivatives of `beta_incomplete` with respect to the variables a, b and z and the integral with respect to the variable z.

Examples:

Simplification for a a positive integer:

```
(%i1) beta_incomplete(2,b,z);
                                  b
                      1 - (1 - z)  (b z + 1)
(%o1)                 ----------------------
                            b (b + 1)
```

Simplification for b a positive integer:

```
(%i1) beta_incomplete(a,2,z);
                                        a
                          (a (1 - z) + 1) z
(%o1)                     ------------------
                              a (a + 1)
```

Simplification for a and b a positive integer:

```
(%i1) beta_incomplete(3,2,z);
                                        3
                          (3 (1 - z) + 1) z
(%o1)                     ------------------
                                  12
```

a is a negative integer and $b <= (-a)$, Maxima simplifies:

```
(%i1) beta_incomplete(-3,1,z);
                                 1
(%o1)                        - ----
                                 3
                               3 z
```

For the specific values $z = 0$ and $z = 1$, Maxima simplifies:

```
(%i1) assume(a>0,b>0)$
(%i2) beta_incomplete(a,b,0);
(%o2)                            0
(%i3) beta_incomplete(a,b,1);
(%o3)                       beta(a, b)
```

Numerical evaluation in float or bigfloat precision:

```
(%i1) beta_incomplete(0.25,0.50,0.9);
(%o1)                    4.594959440269333
(%i2) fpprec:25$
(%i3) beta_incomplete(0.25,0.50,0.9b0);
(%o3)             4.5949594402693240869712l6b0
```

For $abs(z) > 1$ `beta_incomplete` returns a complex result:

```
(%i1) beta_incomplete(0.25,0.50,1.7);
(%o1)       5.244115108584249 - 1.4551804778784403 %i
```

Results for more general complex arguments:

```
(%i1) beta_incomplete(0.25+%i,1.0+%i,1.7+%i);
(%o1)      2.7269606756625384 - 0.38311757042691896 %i
(%i2) beta_incomplete(1/2,5/4*%i,2.8+%i);
(%o2)      13.046496351687155 %i - 5.8020679562699975
```

Expansion, when `beta_expand` is `true`:

```
(%i1) beta_incomplete(a+1,b,z),beta_expand:true;
                                                 b  a
            a beta_incomplete(a, b, z)   (1 - z)  z
(%o1)       ------------------------- - -----------
                      b + a                 b + a
(%i2) beta_incomplete(a-1,b,z),beta_expand:true;
                                                      b  a - 1
      beta_incomplete(a, b, z) (- b - a + 1)   (1 - z)  z
(%o2) ------------------------------------- - ---------------
                      1 - a                         1 - a
```

Derivative and integral for `beta_incomplete`:

```
(%i1) diff(beta_incomplete(a, b, z), z);
                              b - 1  a - 1
(%o1)                  (1 - z)      z
(%i2) integrate(beta_incomplete(a, b, z), z);
(%o2) beta_incomplete(a, b, z) z - beta_incomplete(a + 1, b, z)
(%i3) factor(diff(%, z));
(%o3)               beta_incomplete(a, b, z)
```

`beta_incomplete_regularized (a, b, z)` [Function]

The regularized incomplete beta function (DLMF 8.17.E2 and A&S eqn 6.6.2), defined as

$$I_z(a,b) = \frac{\mathrm{B}_z(a,b)}{\mathrm{B}(a,b)}$$

As for `beta_incomplete` this definition is not complete. See `https://functions.wolfram.com/GammaBetaErf/BetaRegularized/` for a complete definition of `beta_incomplete_regularized`.

`beta_incomplete_regularized` simplifies a or b a positive integer.

For $z = 0$ and $\mathrm{Re}(a) > 0$, `beta_incomplete_regularized` has the specific value 0. For $z = 1$ and $\mathrm{Re}(b) > 0$, `beta_incomplete_regularized` simplifies to 1.

Maxima can evaluate `beta_incomplete_regularized` for real and complex arguments in float and bigfloat precision.

When `beta_expand` is `true`, Maxima expands `beta_incomplete_regularized` for arguments $a + n$ or $a - n$, where n is an integer.

Maxima knows the derivatives of `beta_incomplete_regularized` with respect to the variables a, b, and z and the integral with respect to the variable z.

Examples:

Simplification for a or b a positive integer:

```
(%i1) beta_incomplete_regularized(2,b,z);
                                  b
(%o1)                1 - (1 - z)  (b z + 1)
(%i2) beta_incomplete_regularized(a,2,z);
                                        a
(%o2)                  (a (1 - z) + 1) z
(%i3) beta_incomplete_regularized(3,2,z);
                                        3
(%o3)                  (3 (1 - z) + 1) z
```

For the specific values $z = 0$ and $z = 1$, Maxima simplifies:

```
(%i1) assume(a>0,b>0)$
(%i2) beta_incomplete_regularized(a,b,0);
(%o2)                           0
(%i3) beta_incomplete_regularized(a,b,1);
(%o3)                           1
```

Numerical evaluation for real and complex arguments in float and bigfloat precision:

```
(%i1) beta_incomplete_regularized(0.12,0.43,0.9);
(%o1)                 0.9114011367359802
(%i2) fpprec:32$
(%i3) beta_incomplete_regularized(0.12,0.43,0.9b0);
(%o3)         9.1140113673598029169207248506439b-1
(%i4) beta_incomplete_regularized(1+%i,3/3,1.5*%i);
(%o4)      0.2865367499935405 %i - 0.12299596333468409
(%i5) fpprec:20$
```

```
(%i6) beta_incomplete_regularized(1+%i,3/3,1.5b0*%i);
(%o6) 2.8653674999354031589b-1 %i - 1.2299596333468401976b-1
```

Expansion, when beta_expand is true:

```
(%i1) beta_incomplete_regularized(a+1,b,z);
(%o1)       beta_incomplete_regularized(a + 1, b, z)
(%i2) beta_incomplete_regularized(a-1,b,z);
(%o2)       beta_incomplete_regularized(a - 1, b, z)
```

The derivative and the integral wrt z:

```
(%i1) diff(beta_incomplete_regularized(a,b,z),z);
                                b - 1  a - 1
                          (1 - z)      z
(%o1)                     ------------------
                              beta(a, b)
(%i2) integrate(beta_incomplete_regularized(a,b,z),z);
(%o2) beta_incomplete_regularized(a, b, z) z
                           a beta_incomplete_regularized(a + 1, b, z)
                         - ------------------------------------------
                                             b + a
```

beta_incomplete_generalized (a, b, z1, z2) [Function]

The basic definition of the generalized incomplete beta function is

$$\int_{z_1}^{z_2} t^{a-1}(1-t)^{b-1}\,dt$$

Maxima simplifies beta_incomplete_regularized for a and b a positive integer.

For $\mathrm{Re}(a) > 0$ and $z_1 = 0$ or $z_2 = 0$, Maxima simplifies beta_incomplete_generalized to beta_incomplete. For $\mathrm{Re}(b) > 0$ and $z_1 = 1$ or $z_2 = 1$, Maxima simplifies to an expression with beta and beta_incomplete.

Maxima evaluates beta_incomplete_regularized for real and complex values in float and bigfloat precision.

When beta_expand is true, Maxima expands beta_incomplete_generalized for $a + n$ and $a - n$, n a positive integer.

Maxima knows the derivative of beta_incomplete_generalized with respect to the variables *a*, *b*, *z1*, and *z2* and the integrals with respect to the variables *z1* and *z2*.

Examples:

Maxima simplifies beta_incomplete_generalized for a and b a positive integer:

```
(%i1) beta_incomplete_generalized(2,b,z1,z2);
                     b                       b
           (1 - z1)  (b z1 + 1) - (1 - z2)  (b z2 + 1)
(%o1)      ------------------------------------------
                           b (b + 1)
(%i2) beta_incomplete_generalized(a,2,z1,z2);
                              a                          a
           (a (1 - z2) + 1) z2  - (a (1 - z1) + 1) z1
(%o2)      ------------------------------------------
                           a (a + 1)
```

```
(%i3) beta_incomplete_generalized(3,2,z1,z2);
             2      2                        2      2
      (1 - z1)  (3 z1  + 2 z1 + 1) - (1 - z2)  (3 z2  + 2 z2 + 1)
(%o3) ------------------------------------------------------------
                                  12
```

Simplification for specific values $z1 = 0$, $z2 = 0$, $z1 = 1$, or $z2 = 1$:

```
(%i1) assume(a > 0, b > 0)$
(%i2) beta_incomplete_generalized(a,b,z1,0);
(%o2)              - beta_incomplete(a, b, z1)
(%i3) beta_incomplete_generalized(a,b,0,z2);
(%o3)              - beta_incomplete(a, b, z2)
(%i4) beta_incomplete_generalized(a,b,z1,1);
(%o4)        beta(a, b) - beta_incomplete(a, b, z1)
(%i5) beta_incomplete_generalized(a,b,1,z2);
(%o5)        beta_incomplete(a, b, z2) - beta(a, b)
```

Numerical evaluation for real arguments in float or bigfloat precision:

```
(%i1) beta_incomplete_generalized(1/2,3/2,0.25,0.31);
(%o1)                  0.09638178086368676
(%i2) fpprec:32$
(%i3) beta_incomplete_generalized(1/2,3/2,0.25,0.31b0);
(%o3)         9.6381780863686935309170054689964b-2
```

Numerical evaluation for complex arguments in float or bigfloat precision:

```
(%i1) beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31);
(%o1)   - 0.09625463003205387 %i - 0.0033238477353540463
(%i2) fpprec:20$
(%i3) beta_incomplete_generalized(1/2+%i,3/2+%i,0.25,0.31b0);
(%o3) - 9.6254630032054178691b-2 %i - 3.3238477353543591914b-3
```

Expansion for $a + n$ or $a - n$, n a positive integer, when `beta_expand` is `true`:

```
(%i1) beta_expand:true$
(%i2) beta_incomplete_generalized(a+1,b,z1,z2);
             b   a           b   a
      (1 - z1)  z1  - (1 - z2)  z2
(%o2) ------------------------------
                  b + a
                      a beta_incomplete_generalized(a, b, z1, z2)
                    + ------------------------------------------
                                        b + a
(%i3) beta_incomplete_generalized(a-1,b,z1,z2);
      beta_incomplete_generalized(a, b, z1, z2) (- b - a + 1)
(%o3) -------------------------------------------------------
                                1 - a
                                     b   a - 1           b   a - 1
                          (1 - z2)  z2      - (1 - z1)  z1
                        - ------------------------------------
                                          1 - a
```

Derivative wrt the variable *z1* and integrals wrt *z1* and *z2*:

```
(%i1) diff(beta_incomplete_generalized(a,b,z1,z2),z1);
                                  b - 1   a - 1
(%o1)                - (1 - z1)      z1
(%i2) integrate(beta_incomplete_generalized(a,b,z1,z2),z1);
(%o2) beta_incomplete_generalized(a, b, z1, z2) z1
                                     + beta_incomplete(a + 1, b, z1)
(%i3) integrate(beta_incomplete_generalized(a,b,z1,z2),z2);
(%o3) beta_incomplete_generalized(a, b, z1, z2) z2
                                     - beta_incomplete(a + 1, b, z2)
```

`beta_expand` [Option variable]

Default value: false

When `beta_expand` is `true`, `beta(a,b)` and related functions are expanded for arguments like $a + n$ or $a - n$, where n is an integer.

See [beta], page 337, for examples.

`beta_args_sum_to_integer` [Option variable]

Default value: false

When `beta_args_sum_to_integer` is `true`, Maxima simplifies `beta(a,b)`, when the arguments a and b sum to an integer.

See [beta], page 337, for examples.

`psi` [*n*](*x*) [Function]

`psi[n](x)` is the polygamma function (DLMF 5.2E2, DLMF 5.15, A&S eqn 6.3.1 and A&S eqn 6.4.1) defined by

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \log \Gamma(x)$$

Thus, `psi[0](x)` is the first derivative, `psi[1](x)` is the second derivative, etc.

Maxima can compute some exact values for rational args as well for float and bfloat args. Several variables control what range of rational args $\psi^{(n)}(x)$ will return an exact value, if possible. See `maxpsiposint`, `maxpsinegint`, `maxpsifracnum`, and `maxpsifracdenom`. That is, x must lie between `maxpsinegint` and `maxpsiposint`. If the absolute value of the fractional part of x is rational and has a numerator less than `maxpsifracnum` and has a denominator less than `maxpsifracdenom`, $\psi^{(0)}(x)$ will return an exact value.

The function `bfpsi` in the `bffac` package can compute numerical values.

```
(%i1) psi[0](.25);
(%o1)                  - 4.227453533376265
(%i2) psi[0](1/4);
                                    %pi
(%o2)               - 3 log(2) - ------ - %gamma
                                     2
(%i3) float(%);
(%o3)                  - 4.227453533376265
```

```
(%i4) psi[2](0.75);
(%o4)                  - 5.3026332163376395
(%i5) psi[2](3/4);
                                1         3
(%o5)                     psi (-) + 4 %pi
                             2 4
(%i6) float(%);
(%o6)                  - 5.3026332163376395
```

`maxpsiposint` [Option variable]

Default value: 20

`maxpsiposint` is the largest positive integer value for which $\psi^{(n)}(m)$ gives an exact value for rational x.

```
(%i1) psi[0](20);
                        275295799
(%o1)                   --------- - %gamma
                        77597520
(%i2) psi[0](21);
(%o2)                        psi (21)
                                0
(%i3) psi[2](20);
                 1683118856778495358491487
(%o3)         2 (------------------------- - zeta(3))
                 1401731326612193601024000
(%i4) psi[2](21);
(%o4)                        psi (21)
                                2
```

`maxpsinegint` [Option variable]

Default value: -10

`maxpsinegint` is the most negative value for which $\psi^{(0)}(x)$ will try to compute an exact value for rational x. That is if x is less than `maxpsinegint`, $\psi^{(n)}(x)$ will not return simplified answer, even if it could.

```
(%i1) psi[0](-100/9);
                                    100
(%o1)                        psi (- ---)
                                0    9
(%i2) psi[0](-100/11);
                    100 %pi          1     5231385863539
(%o2)      %pi cot(-------) + psi (--) + -------------
                      11           0 11    381905105400
(%i3) psi[2](-100/9);
                                    100
(%o3)                        psi (- ---)
                                2    9
```

```
(%i4) psi[2](-100/11);
           3     100 %pi     2 100 %pi         1
(%o4) 2 %pi  cot(-------) csc (-------) + psi (--)
                   11            11          2 11
                          74191313259470963498957651385614962459
                        + --------------------------------------
                          27850718060013605318710152732000000
```

`maxpsifracnum` [Option variable]

Default value: 6

Let x be a rational number of the form p/q. If p is greater than `maxpsifracnum`, then $\psi^{(0)}(x)$ will not try to return a simplified value.

```
(%i1) psi[0](3/4);
                                  %pi
(%o1)               - 3 log(2) + --- - %gamma
                                   2
(%i2) psi[2](3/4);
                              1         3
(%o2)                    psi (-) + 4 %pi
                              2 4
(%i3) maxpsifracnum:2;
(%o3)                            2
(%i4) psi[0](3/4);
                                  3
(%o4)                        psi (-)
                                  0 4
(%i5) psi[2](3/4);
                              1         3
(%o5)                    psi (-) + 4 %pi
                              2 4
```

`maxpsifracdenom` [Option variable]

Default value: 6

Let x be a rational number of the form p/q. If q is greater than `maxpsifracdenom`, then $\psi^{(0)}(x)$ will not try to return a simplified value.

```
(%i1) psi[0](3/4);
                                  %pi
(%o1)               - 3 log(2) + --- - %gamma
                                   2
(%i2) psi[2](3/4);
                              1         3
(%o2)                    psi (-) + 4 %pi
                              2 4
(%i3) maxpsifracdenom:2;
(%o3)                            2
```

```
(%i4) psi[0](3/4);
                                    3
(%o4)                          psi (-)
                                   0 4
(%i5) psi[2](3/4);
                                   1         3
(%o5)                       psi (-) + 4 %pi
                               2 4
```

makefact (*expr*) [Function]

Transforms instances of binomial, gamma, and beta functions in *expr* into factorials.

See also makegamma.

```
(%i1) makefact(binomial(n,k));
                                    n!
(%o1)                         -----------
                              k! (n - k)!
(%i2) makefact(gamma(x));
(%o2)                          (x - 1)!
(%i3) makefact(beta(a,b));
                            (a - 1)! (b - 1)!
(%o3)                       -----------------
                              (b + a - 1)!
```

numfactor (*expr*) [Function]

Returns the numerical factor multiplying the expression *expr*, which should be a single term.

content returns the greatest common divisor (gcd) of all terms in a sum.

```
(%i1) gamma (7/2);
                               15 sqrt(%pi)
(%o1)                          ------------
                                    8
(%i2) numfactor (%);
                                    15
(%o2)                               --
                                    8
```

17.5 Exponential Integrals

The Exponential Integral and related functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapter 5.

expintegral_e1 (*z*) [Function]

The Exponential Integral E1(z) defined as

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$$

with $|\arg z| < \pi$. (A&S eqn 5.1.1) and (DLMF 6.2E2)

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_ei (x)` [Function]

The Exponential Integral Ei(x) defined as

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^{x} \frac{e^{t}}{t} dt$$

with x real and $x > 0$. (A&S eqn 5.1.2) and (DLMF 6.2E5)

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_li (x)` [Function]

The Exponential Integral li(x) defined as

$$li(x) = \int_{0}^{x} \frac{dt}{\ln t}$$

with x real and $x > 1$. (A&S eqn 5.1.3) and (DLMF 6.2E8)

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_e (n,z)` [Function]

The Exponential Integral En(z) (A&S eqn 5.1.4) defined as

$$E_n(z) = \int_{1}^{\infty} \frac{e^{-zt}}{t^n} dt$$

with $\mathrm{Re}(z) > 1$ and n a non-negative integer.

For half-integral orders, this can be written in terms of `erfc` or `erf`. See [expintexpand], page 353, for examples.

`expintegral_si (z)` [Function]

The Exponential Integral Si(z) (A&S eqn 5.2.1 and DLMF 6.2#E9) defined as

$$\mathrm{Si}(z) = \int_{0}^{z} \frac{\sin t}{t} dt$$

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_ci (z)` [Function]

The Exponential Integral Ci(z) (A&S eqn 5.2.2 and DLMF 6.2#E13) defined as

$$\mathrm{Ci}(z) = \gamma + \log z + \int_{0}^{z} \frac{\cos t - 1}{t} dt$$

with $|\arg z| < \pi$.

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_shi (z)` [Function]

The Exponential Integral Shi(z) (A&S eqn 5.2.3 and DLMF 6.2#E15) defined as

$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt$$

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintegral_chi (z)` [Function]

The Exponential Integral Chi(z) (A&S eqn 5.2.4 and DLMF 6.2#E16) defined as

$$\mathrm{Chi}(z) = \gamma + \log z + \int_0^z \frac{\cosh t - 1}{t} dt$$

with $|\arg z| < \pi$.

This can be written in terms of other functions. See [expintrep], page 349, for examples.

`expintrep` [Option variable]

Default value: false

Change the representation of one of the exponential integrals, `expintegral_e(m, z)`, `expintegral_e1`, or `expintegral_ei` to an equivalent form if possible.

Possible values for `expintrep` are `false`, `gamma_incomplete`, `expintegral_e1`, `expintegral_ei`, `expintegral_li`, `expintegral_trig`, or `expintegral_hyp`.

`false` means that the representation is not changed. Other values indicate the representation is to be changed to use the function specified where `expintegral_trig` means `expintegral_si`, `expintegral_ci`; and `expintegral_hyp` means `expintegral_shi` or `expintegral_chi`.

Here are some examples for `expintrep` set to `gamma_incomplete`:

```
(%i1) expintrep:'gamma_incomplete;
(%o1)                    gamma_incomplete
(%i2) expintegral_e1(z);
(%o2)                 gamma_incomplete(0, z)
(%i3) expintegral_ei(z);
(%o3)     log(z) - log(- z) - gamma_incomplete(0, - z)
(%i4) expintegral_li(z);
(%o4) log(log(z)) - log(- log(z)) - gamma_incomplete(0, - log(z))
(%i5) expintegral_e(n,z);
                                                 n - 1
(%o5)           gamma_incomplete(1 - n, z) z
(%i6) expintegral_si(z);
(%o6) (%i (- log(%i z) + log(- %i z) - gamma_incomplete(0, %i z)
                                  + gamma_incomplete(0, - %i z)))/2
(%i7) expintegral_ci(z);
(%o7) log(z) - (log(%i z) + log(- %i z)
     + gamma_incomplete(0, %i z) + gamma_incomplete(0, - %i z))/2
```

```
(%i8) expintegral_shi(z);
(%o8) (log(z) - log(- z) + gamma_incomplete(0, z)
                                    - gamma_incomplete(0, - z))/2
(%i9) expintegral_chi(z);
(%o9) - (- log(z) + log(- z) + gamma_incomplete(0, z)
                                    + gamma_incomplete(0, - z))/2
```

For `expintrep` set to `expintegral_e1`:

```
(%i1) expintrep:'expintegral_e1;
(%o1)                     expintegral_e1
(%i2) expintegral_ei(z);
(%o2)        log(z) - log(- z) - expintegral_e1(- z)
(%i3) expintegral_li(z);
(%o3) log(log(z)) - log(- log(z)) - expintegral_e1(- log(z))
(%i4) expintegral_e(n,z);
(%o4)                   expintegral_e(n, z)
(%i5) expintegral_si(z);
(%o5) (%i (- log(%i z) - expintegral_e1(%i z) + log(- %i z)
                                     + expintegral_e1(- %i z)))/2
(%i6) expintegral_ci(z);
(%o6) log(z) - (log(- %i z) (expintegral_e1(%i z)
                           + expintegral_e1(- %i z)) log(%i z))/2
(%i7) expintegral_shi(z);
      log(z) + expintegral_e1(z) - log(- z) - expintegral_e1(- z)
(%o7) -------------------------------------------------------
                                  2
(%i8) expintegral_chi(z);
(%o8)
    - log(z) + expintegral_e1(z) + log(- z) + expintegral_e1(- z)
  - -------------------------------------------------------------
                                  2
```

For `expintrep` set to `expintegral_ei`:

```
(%i1) expintrep:'expintegral_ei;
(%o1)                     expintegral_ei
(%i2) expintegral_e1(z);
                                  1
                log(- z) - log(- -)
                                  z
(%o2) - log(z) + ------------------- - expintegral_ei(- z)
                         2
(%i3) expintegral_ei(z);
(%o3)                   expintegral_ei(z)
(%i4) expintegral_li(z);
(%o4)                expintegral_ei(log(z))
(%i5) expintegral_e(n,z);
(%o5)                   expintegral_e(n, z)
```

```
(%i6) expintegral_si(z);
(%o6) (%i (log(%i z) + 2 (expintegral_ei(- %i z)
                                                  %i          %i
  - expintegral_ei(%i z)) - log(- %i z) + log(--) - log(- --)))/4
                                                  z           z
(%i7) expintegral_ci(z);
(%o7) (- log(%i z) + 2 (expintegral_ei(%i z)
                                                   %i          %i
 + expintegral_ei(- %i z)) - log(- %i z) + log(--) + log(- --))/4
                                                   z           z
 + log(z)
(%i8) expintegral_shi(z);
(%o8) (- 2 log(z) + 2 (expintegral_ei(z) - expintegral_ei(- z))
                                                                1
                                            + log(- z) - log(- -))/4
                                                                z
(%i9) expintegral_chi(z);
(%o9) (2 log(z) + 2 (expintegral_ei(z) + expintegral_ei(- z))
                                                                1
                                            - log(- z) + log(- -))/4
                                                                z
```

For `expintrep` set to `expintegral_li`:

```
(%i1) expintrep:'expintegral_li;
(%o1)                     expintegral_li
(%i2) expintegral_e1(z);
                                                              1
                                            log(- z) - log(- -)
                          - z                                 z
(%o2) - expintegral_li(%e   ) - log(z) + -------------------
                                                     2
(%i3) expintegral_ei(z);
                                          z
(%o3)                  expintegral_li(%e )
(%i4) expintegral_li(z);
(%o4)                   expintegral_li(z)
(%i5) expintegral_e(n,z);
(%o5)                  expintegral_e(n, z)
(%i6) expintegral_si(z);
                                %i z                     - %e z
(%o6) - (%i (expintegral_li(%e    ) - expintegral_li(%e      )
                                                  %pi signum(z)
                                                - -------------))/2
                                                         2
```

```
(%i7) expintegral_ci(z);
                      %i z                       - %i z
      expintegral_li(%e    ) + expintegral_li(%e      )
(%o7) ------------------------------------------------
                              2
                                                   - signum(z) + 1
(%i8) expintegral_shi(z);
                            z                      - z
           expintegral_li(%e ) - expintegral_li(%e   )
(%o8)      ----------------------------------------
                              2
(%i9) expintegral_chi(z);
                            z                      - z
           expintegral_li(%e ) + expintegral_li(%e   )
(%o9)      ----------------------------------------
                              2
```

For `expintrep` set to `expintegral_trig`:

```
(%i1) expintrep:'expintegral_trig;
(%o1)                    expintegral_trig
(%i2) expintegral_e1(z);
(%o2) log(%i z) - %i expintegral_si(%i z) - expintegral_ci(%i z)
                                                          - log(z)
(%i3) expintegral_ei(z);
(%o3) - log(%i z) - %i expintegral_si(%i z)
                                + expintegral_ci(%i z) + log(z)
(%i4) expintegral_li(z);
(%o4) - log(%i log(z)) - %i expintegral_si(%i log(z))
                       + expintegral_ci(%i log(z)) + log(log(z))
(%i5) expintegral_e(n,z);
(%o5)                   expintegral_e(n, z)
(%i6) expintegral_si(z);
(%o6)                    expintegral_si(z)
(%i7) expintegral_ci(z);
(%o7)                    expintegral_ci(z)
(%i8) expintegral_shi(z);
(%o8)               - %i expintegral_si(%i z)
(%i9) expintegral_chi(z);
(%o9)      - log(%i z) + expintegral_ci(%i z) + log(z)
```

For `expintrep` set to `expintegral_hyp`:

```
(%i1) expintrep:'expintegral_hyp;
(%o1)                     expintegral_hyp
(%i2) expintegral_e1(z);
(%o2)        expintegral_shi(z) - expintegral_chi(z)
(%i3) expintegral_ei(z);
(%o3)        expintegral_shi(z) + expintegral_chi(z)
```

```
(%i4) expintegral_li(z);
(%o4)   expintegral_shi(log(z)) + expintegral_chi(log(z))
(%i5) expintegral_e(n,z);
(%o5)                  expintegral_e(n, z)
(%i6) expintegral_si(z);
(%o6)              - %i expintegral_shi(%i z)
(%i7) expintegral_ci(z);
(%o7)     - log(%i z) + expintegral_chi(%i z) + log(z)
(%i8) expintegral_shi(z);
(%o8)                  expintegral_shi(z)
(%i9) expintegral_chi(z);
(%o9)                  expintegral_chi(z)
```

`expintexpand` [Option variable]

Default value: false

Expand `expintegral_e(n,z)` for half integral values in terms of `erfc` or `erf` and for positive integers in terms of `expintegral_ei`.

```
(%i1) expintegral_e(1/2,z);
                                     1
(%o1)                  expintegral_e(-, z)
                                     2
(%i2) expintegral_e(1,z);
(%o2)                  expintegral_e(1, z)
(%i3) expintexpand:true;
(%o3)                           true
(%i4) expintegral_e(1/2,z);
                      sqrt(%pi) erfc(sqrt(z))
(%o4)                 -----------------------
                                sqrt(z)
(%i5) expintegral_e(1,z);
                        1
                 log(- -) - log(- z)
                        z
(%o5) - log(z) - -------------------- - expintegral_ei(- z)
                           2
```

17.6 Error Function

The Error function and related functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapter 7 and (DLMF 7)

`erf (z)` [Function]

The Error Function erf(z):

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

(A&S eqn 7.1.1) and (DLMF 7.2.E1).

See also flag `erfflag`. This can also be expressed in terms of a hypergeometric function. See [hypergeometric_representation], page 356.

`erfc (z)` [Function]

The Complementary Error Function erfc(z):

$$\operatorname{erfc} z = 1 - \operatorname{erf} z$$

(A&S eqn 7.1.2) and (DLMF 7.2.E2).

This can also be expressed in terms of a hypergeometric function. See [hypergeometric_representation], page 356.

`erfi (z)` [Function]

The Imaginary Error Function.

$$\operatorname{erfi} z = -i \operatorname{erf}(iz)$$

`erf_generalized (z1,z2)` [Function]

Generalized Error function Erf(z1,z2):

$$\operatorname{erf}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2}\, dt$$

This can also be expressed in terms of a hypergeometric function. See [hypergeometric_representation], page 356.

`fresnel_c (z)` [Function]

The Fresnel Integral

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right)\, dt$$

(A&S eqn 7.3.1) and (DLMF 7.2.E7).

The simplification $C(-x) = -C(x)$ is applied when flag `trigsign` is true.

The simplification $C(ix) = iC(x)$ is applied when flag `%iargs` is true.

See flags `erf_representation` and `hypergeometric_representation`.

`fresnel_s (z)` [Function]

The Fresnel Integral

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right)\, dt$$

(A&S eqn 7.3.2) and (DLMF 7.2.E8).

The simplification $S(-x) = -S(x)$ is applied when flag `trigsign` is true.

The simplification $S(ix) = iS(x)$ is applied when flag `%iargs` is true.

See flags `erf_representation` and `hypergeometric_representation`.

`erf_representation` [Option variable]

Default value: false

`erf_representation` controls how the error functions are represented. It must be set to one of `false`, `erf`, `erfc`, or `erfi`. When set to `false`, the error functions are not modified. When set to `erf`, all error functions (`erfc`, `erfi`, `erf_generalized`, `fresnel_s` and `fresnel_c`) are converted to `erf` functions. Similarly, `erfc` converts error functions to `erfc`. Finally `erfi` converts the functions to `erfi`.

Converting to `erf`:

```
(%i1) erf_representation:erf;
(%o1)                                erf
(%i2) erfc(z);
(%o2)                           1 - erf(z)
(%i3) erfi(z);
(%o3)                         - %i erf(%i z)
(%i4) erf_generalized(z1,z2);
(%o4)                        erf(z2) - erf(z1)
(%i5) fresnel_c(z);
                          sqrt(%pi) (%i + 1) z
(%o5) ((1 - %i) (erf(----------------------)
                                2
                                          sqrt(%pi) (1 - %i) z
                                 + %i erf(----------------------)))/4
                                                   2
(%i6) fresnel_s(z);
                          sqrt(%pi) (%i + 1) z
(%o6) ((%i + 1) (erf(----------------------)
                                2
                                          sqrt(%pi) (1 - %i) z
                                 - %i erf(----------------------)))/4
                                                   2
```

Converting to `erfc`:

```
(%i1) erf_representation:erfc;
(%o1)                               erfc
(%i2) erf(z);
(%o2)                           1 - erfc(z)
(%i3) erfc(z);
(%o3)                             erfc(z)
(%i4) erf_generalized(z1,z2);
(%o4)                        erfc(z1) - erfc(z2)
(%i5) fresnel_s(c);
                           sqrt(%pi) (%i + 1) c
(%o5) ((%i + 1) (- erfc(----------------------)
                                   2
                                     sqrt(%pi) (1 - %i) c
                        - %i (1 - erfc(----------------------)) + 1))/4
                                                2
```

```
(%i6) fresnel_c(c);
                           sqrt(%pi) (%i + 1) c
(%o6) ((1 - %i) (- erfc(--------------------)
                                  2
                                    sqrt(%pi) (1 - %i) c
                     + %i (1 - erfc(--------------------)) + 1))/4
                                           2
```

Converting to `erfc`:

```
(%i1) erf_representation:erfi;
(%o1)                          erfi
(%i2) erf(z);
(%o2)                     - %i erfi(%i z)
(%i3) erfc(z);
(%o3)                    %i erfi(%i z) + 1
(%i4) erfi(z);
(%o4)                         erfi(z)
(%i5) erf_generalized(z1,z2);
(%o5)            %i erfi(%i z1) - %i erfi(%i z2)
(%i6) fresnel_s(z);
                              sqrt(%pi) %i (%i + 1) z
(%o6) ((%i + 1) (- %i erfi(-----------------------)
                                       2
                                      sqrt(%pi) (1 - %i) %i z
                                   - erfi(-----------------------)))/4
                                                 2
(%i7) fresnel_c(z);
                         sqrt(%pi) (1 - %i) %i z
(%o7) ((1 - %i) (erfi(-----------------------)
                                    2
                                        sqrt(%pi) %i (%i + 1) z
                                - %i erfi(-----------------------)))/4
                                                 2
```

`hypergeometric_representation` [Option variable]

Default value: false

Enables transformation to a Hypergeometric representation for `fresnel_s` and `fresnel_c` and other error functions.

```
(%i1) hypergeometric_representation:true;
(%o1)                          true
(%i2) fresnel_s(z);
                                                 2  4
                              3    3  7      %pi  z    3
          %pi hypergeometric([-], [-, -], - -------) z
                              4    2  4        16
(%o2)     ------------------------------------------------
                                   6
```

```
(%i3) fresnel_c(z);
                                                  2  4
                              1    1  5     %pi  z
(%o3)       hypergeometric([-], [-, -], - -------) z
                              4    2  4       16
(%i4) erf(z);
                                                      2
                                      3    2      - z
             2 hypergeometric([1], [-], z ) z %e
                                      2
(%o4)        ---------------------------------------
                             sqrt(%pi)
(%i5) erfi(z);
                                    1    3    2
                2 hypergeometric([-], [-], z ) z
                                    2    2
(%o5)           --------------------------------
                            sqrt(%pi)
(%i6) erfc(z);
                                                        2
                                        3    2      - z
               2 hypergeometric([1], [-], z ) z %e
                                        2
(%o6)      1 - ---------------------------------------
                              sqrt(%pi)
(%i7) erf_generalized(z1,z2);
                                                  2
                              3     2       - z2
      2 hypergeometric([1], [-], z2 ) z2 %e
                              2
(%o7) ------------------------------------------
                      sqrt(%pi)
                                                                2
                                                  3     2       - z1
                       2 hypergeometric([1], [-], z1 ) z1 %e
                                                  2
                     - ------------------------------------------
                                          sqrt(%pi)
```

17.7 Struve Functions

The Struve functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapter 12 and (DLMF 11). The Struve Function $\mathbf{H}_\nu(z)$ is a particular solution of the differential equation:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = \frac{4\left(\frac{1}{2}z\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)}$$

which has the general soution

$$w = aJ_\nu(z) + bY_\nu(z) + \mathbf{H}_\nu(z)$$

`struve_h (v, z)` [Function]

The Struve Function H of order ν and argument z:

$$\mathbf{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

(A&S eqn 12.1.3) and (DLMF 11.2.E1).

When `besselexpand` is `true`, `struve_h` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

`struve_l (v, z)` [Function]

The Modified Struve Function L of order ν and argument z:

$$\mathbf{L}_\nu(z) = -ie^{-\frac{i\nu\pi}{2}}\mathbf{H}_\nu(iz)$$

(A&S eqn 12.2.1) and (DLMF 11.2.E2).

When `besselexpand` is `true`, `struve_l` is expanded in terms of elementary functions when the order v is half of an odd integer. See `besselexpand`.

17.8 Hypergeometric Functions

The Hypergeometric Functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapters 13 and A&S 15.

Maxima has very limited knowledge of these functions. They can be returned from function `hgfred`.

`%m [k,u] (z)` [Function]

Whittaker M function (A&S eqn 13.1.32):

$$M_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} M\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right)$$

where $M(a,b,z)$ is Kummer's solution of the confluent hypergeometric equation.

This can also be expressed by the series (DLMF 13.14.E6):

$$M_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2}+\mu-\kappa\right)_s}{(1+2\mu)_s s!} z^s$$

`%w [k,u] (z)` [Function]

Whittaker W function (A&S eqn 13.1.33):

$$W_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} U\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right)$$

where $U(a,b,z)$ is Kummer's second solution of the confluent hypergeometric equation.

`%f [p,q] ([a],[b],z)` [Function]

The ${}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z)$ hypergeometric function, where a a list of length p and b a list of length q.

`hypergeometric ([a1, ..., ap],[b1, ... ,bq], x)` [Function]

The hypergeometric function. Unlike Maxima's `%f` hypergeometric function, the function `hypergeometric` is a simplifying function; also, `hypergeometric` supports complex double and big floating point evaluation. For the Gauss hypergeometric function, that is $p = 2$ and $q = 1$, floating point evaluation outside the unit circle is supported, but in general, it is not supported.

When the option variable `expand_hypergeometric` is true (default is false) and one of the arguments `a1` through `ap` is a negative integer (a polynomial case), `hypergeometric` returns an expanded polynomial.

Examples:

```
(%i1)  hypergeometric([],[],x);
                                    x
(%o1)                              %e
```

Polynomial cases automatically expand when `expand_hypergeometric` is true:

```
(%i1) hypergeometric([-3],[7],x);
(%o1)              hypergeometric([- 3], [7], x)
(%i2) hypergeometric([-3],[7],x), expand_hypergeometric : true;
                         3       2
                        x     3 x    3 x
(%o2)                 - --- + ---- - --- + 1
                        504    56     7
```

Both double float and big float evaluation is supported:

```
(%i1) hypergeometric([5.1],[7.1 + %i],0.42);
(%o1)      1.3462507863753337 - 0.0559061414208204 %i
(%i2) hypergeometric([5,6],[8], 5.7 - %i);
(%o2)    0.007375824009774945 - 0.0010498136885786736 %i
(%i3) hypergeometric([5,6],[8], 5.7b0 - %i), fpprec : 30;
(%o3) 7.37582400977494674506442010824b-3
                            - 1.04981368857867315858055393376b-3 %i
```

`hypergeometric_simp (e)` [Function]

`hypergeometric_simp` simplifies hypergeometric functions by applying `hgfred` to the arguments of any hypergeometric functions in the expression e.

Only instances of `hypergeometric` are affected; any `%f`, `%w`, and `%m` in the expression e are not affected. Any unsimplified hypergeometric functions are returned unchanged (instead of changing to `%f` as `hgfred` would).

`load("hypergeometric");` loads this function.

See also `hgfred`.

Examples:

```
(%i1) load ("hypergeometric") $
```

```
(%i2) foo : [hypergeometric([1,1], [2], z), hypergeometric([1/2], [1], z)];
(%o2) [hypergeometric([1, 1], [2], z),
                                                        1
                                      hypergeometric([-], [1], z)]
                                                        2
(%i3) hypergeometric_simp (foo);
                  log(1 - z)    z/2             z
(%o3)         [- ----------, %e    bessel_i(0, -)]
                       z                        2
(%i4) bar : hypergeometric([n], [m], z + 1);
(%o4)             hypergeometric([n], [m], z + 1)
(%i5) hypergeometric_simp (bar);
(%o5)             hypergeometric([n], [m], z + 1)
```

`hgfred` (*a*, *b*, *t*) [Function]

Simplify the generalized hypergeometric function in terms of other, simpler, forms. *a* is a list of numerator parameters and *b* is a list of the denominator parameters.

If `hgfred` cannot simplify the hypergeometric function, it returns an expression of the form `%f[p,q]([a], [b], x)` where p is the number of elements in *a*, and q is the number of elements in *b*. This is the usual ${}_pF_q$ generalized hypergeometric function.

```
(%i1) assume(not(equal(z,0)));
(%o1)                     [notequal(z, 0)]
(%i2) hgfred([v+1/2],[2*v+1],2*%i*z);
             v/2                                  %i z
            4    bessel_j(v, z) gamma(v + 1) %e
(%o2)       ---------------------------------------
                                v
                               z
(%i3) hgfred([1,1],[2],z);
                           log(1 - z)
(%o3)                    - ----------
                                z
(%i4) hgfred([a,a+1/2],[3/2],z^2);
                        1 - 2 a          1 - 2 a
                (z + 1)        - (1 - z)
(%o4)           -------------------------------
                         2 (1 - 2 a) z
```

It can be beneficial to load orthopoly too as the following example shows. Note that *L* is the generalized Laguerre polynomial.

```
(%i1) load("orthopoly")$
(%i2) hgfred([-2],[a],z);
                           2
                          z        2 z
(%o2)                  --------- - --- + 1
                       a (a + 1)    a
```

```
(%i3) ev(%);
                                  2
                                 z          2 z
(%o3)                       --------- - --- + 1
                            a (a + 1)     a
```

17.9 Parabolic Cylinder Functions

The Parabolic Cylinder Functions are defined in Abramowitz and Stegun, *Handbook of Mathematical Functions*, A&S Chapter 19.

Maxima has very limited knowledge of these functions. They can be returned from function `specint`.

`parabolic_cylinder_d (v, z)` [Function]

The parabolic cylinder function `parabolic_cylinder_d(v,z)`. (A&S eqn 19.3.1).

The solution of the Weber differential equation

$$y''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right) y(z) = 0$$

has two independent solutions, one of which is $D_\nu(z)$, the parabolic cylinder d function.

Function `specint` can return expressions containing `parabolic_cylinder_d(v,z)` if the option variable `prefer_d` is `true`.

17.10 Functions and Variables for Special Functions

`lambert_w (z)` [Function]

The principal branch of Lambert's W function W(z) (DLMF 4.13), the solution of

$$z = W(z)e^{W(z)}$$

`generalized_lambert_w (k, z)` [Function]

The k-th branch of Lambert's W function W(z) (DLMF 4.13), the solution of $z = W(z)e^{W(z)}$.

The principal branch, denoted $W_p(z)$ in DLMF, is `lambert_w(z) = generalized_lambert_w(0,z)`.

The other branch with real values, denoted $W_m(z)$ in DLMF, is `generalized_lambert_w(-1,z)`.

`kbateman [v] (x)` [Function]

The Bateman k function

$$k_v(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\tan\theta - v\theta)d\theta$$

It is one solution of a differential equation which appears in the theory of turbulence:

$$x\frac{d^2u}{dx^2} = (x-\nu)u$$

It is a special case of the confluent hypergeometric function for $x > 0$:

$$k_v(x) = \frac{e^{-x}}{\Gamma\left(1+\frac{1}{2}\nu\right)} U\left(-\frac{1}{2}\nu, 0, 2x\right)$$

where U is the confluent hypergeometric function. Also, we have

$$k_{2\nu}(z) = \frac{1}{\Gamma(\nu+1)} W_{\nu,1/2}(2z)$$

where W is the [%w], page 358.

Some examples:

```
(%i1) assume(x > 0)$
(%i2) makelist(kbateman[n](0),n,0,5);
                      2          2         2
(%o2)            [0, ---, 0, - -----, 0, -----]
                     %pi       3 %pi     5 %pi
(%i3) kbateman[0](x);
                                - x
(%o3)                          %e
(%i4) kbateman[2](x);
                                - x
(%o4)                       2 %e    x
(%i5) kbateman[4](x);
                               - x
(%o5)                   2 %e    (x - 1) x
(%i6) kbateman[3](x);
(%o6)                     kbateman (x)
                                  3
```

Maxima can calculate the Laplace transform of `kbateman` using `laplace` or `specint`, as shown below:

```
(%i1) assume(s>0)$
(%i2) specint(kbateman[v](z)*exp(-s*z),z);
                                  v        v   s - 1
                2 %f    ([2, 1 - -], [2 - -], -----)
                    2, 1          2        2   s + 1
(%o2)           ------------------------------------
                        2           v        v
                 (s + 1)  gamma(2 - -) gamma(- + 1)
                                  2        2
```

`nzeta` (*z*) [Function]

The Plasma Dispersion Function

$$\mathrm{nzeta}(z) = i\sqrt{\pi}e^{-z^2}(1-\mathrm{erf}(-iz))$$

`nzetar` (*z*) [Function]

Returns `realpart(nzeta(z))`.

`nzetai` (*z*) [Function]

Returns `imagpart(nzeta(z))`.

18 Elliptic Functions

18.1 Introduction to Elliptic Functions and Integrals

Maxima includes support for Jacobian elliptic functions and for complete and incomplete elliptic integrals. This includes symbolic manipulation of these functions and numerical evaluation as well. Definitions of these functions and many of their properties can by found in Abramowitz and Stegun, A&S Chapter 16 and A&S Chapter 17. See also DLMF 22.2. As much as possible, we use the definitions and relationships given in Abramowitz and Stegun.

In particular, all elliptic functions and integrals use the parameter m instead of the modulus k or the modular angle α. The following relationships are true:

$$m = k^2$$
$$k = \sin\alpha$$

Note that Abramowitz and Stegun uses the notation $\mathrm{sn}(u|m)$ where we use $\mathrm{sn}(u, m)$ instead. The DLMF uses modulus k instead of the parameter m.

The elliptic functions and integrals are primarily intended to support symbolic computation. Therefore, most of derivatives of the functions and integrals are known. However, if floating-point values are given, a floating-point result is returned.

Support for most of the other properties of elliptic functions and integrals other than derivatives has not yet been written.

Some examples of elliptic functions:

```
(%i1) jacobi_sn (u, m);
(%o1)                      jacobi_sn(u, m)
(%i2) jacobi_sn (u, 1);
(%o2)                          tanh(u)
(%i3) jacobi_sn (u, 0);
(%o3)                          sin(u)
(%i4) diff (jacobi_sn (u, m), u);
(%o4)             jacobi_cn(u, m) jacobi_dn(u, m)
(%i5) diff (jacobi_sn (u, m), m);
(%o5) (jacobi_cn(u, m) jacobi_dn(u, m)
      elliptic_e(asin(jacobi_sn(u, m)), m)
 (u - ------------------------------------))/(2 m)
                     1 - m
           2
  jacobi_cn (u, m) jacobi_sn(u, m)
+ --------------------------------
              2 (1 - m)
```

Some examples of elliptic integrals:

```
(%i1) elliptic_f (phi, m);
(%o1)                    elliptic_f(phi, m)
(%i2) elliptic_f (phi, 0);
(%o2)                            phi
```

```
(%i3) elliptic_f (phi, 1);
                                 phi   %pi
(%o3)                  log(tan(--- + ---))
                                  2     4
(%i4) elliptic_e (phi, 1);
                    phi                   phi
(%o4)       2 round(---) - sin(%pi round(---) - phi)
                    %pi                   %pi
(%i5) elliptic_e (phi, 0);
(%o5)                            phi
(%i6) elliptic_kc (1/2);
                                  3/2
                               %pi
(%o6)                         -----------
                                     2 3
                              2 gamma (-)
                                      4
(%i7) makegamma (%);
                                  3/2
                               %pi
(%o7)                         -----------
                                     2 3
                              2 gamma (-)
                                      4
(%i8) diff (elliptic_f (phi, m), phi);
                                  1
(%o8)                  ---------------------
                                    2
                       sqrt(1 - m sin (phi))
(%i9) diff (elliptic_f (phi, m), m);
       elliptic_e(phi, m) - (1 - m) elliptic_f(phi, m)
(%o9) (-----------------------------------------------
                             m
                                cos(phi) sin(phi)
                            - --------------------)/(2 (1 - m))
                                          2
                              sqrt(1 - m sin (phi))
```

Support for elliptic functions and integrals was written by Raymond Toy. It is placed under the terms of the General Public License (GPL) that governs the distribution of Maxima.

18.2 Functions and Variables for Elliptic Functions

See A&S Section 6.12 and DLMF 22.2 for more information.

`jacobi_sn` (*u*, *m*) [Function]

The Jacobian elliptic function $\mathrm{sn}(u, m)$.

`jacobi_cn (u, m)` [Function]
The Jacobian elliptic function $\mathrm{cn}(u, m)$.

`jacobi_dn (u, m)` [Function]
The Jacobian elliptic function $\mathrm{dn}(u, m)$.

`jacobi_ns (u, m)` [Function]
The Jacobian elliptic function $\mathrm{ns}(u, m) = 1/\mathrm{sn}(u, m)$.

`jacobi_sc (u, m)` [Function]
The Jacobian elliptic function $\mathrm{sc}(u, m) = \mathrm{sn}(u, m)/\mathrm{cn}(u, m)$.

`jacobi_sd (u, m)` [Function]
The Jacobian elliptic function $\mathrm{sd}(u, m) = \mathrm{sn}(u, m)/\mathrm{dn}(u, m)$.

`jacobi_nc (u, m)` [Function]
The Jacobian elliptic function $\mathrm{nc}(u, m) = 1/\mathrm{cn}(u, m)$.

`jacobi_cs (u, m)` [Function]
The Jacobian elliptic function $\mathrm{cs}(u, m) = \mathrm{cn}(u, m)/\mathrm{sn}(u, m)$.

`jacobi_cd (u, m)` [Function]
The Jacobian elliptic function $\mathrm{cd}(u, m) = \mathrm{cn}(u, m)/\mathrm{dn}(u, m)$.

`jacobi_nd (u, m)` [Function]
The Jacobian elliptic function $\mathrm{nd}(u, m) = 1/\mathrm{dn}(u, m)$.

`jacobi_ds (u, m)` [Function]
The Jacobian elliptic function $\mathrm{ds}(u, m) = \mathrm{dn}(u, m)/\mathrm{sn}(u, m)$.

`jacobi_dc (u, m)` [Function]
The Jacobian elliptic function $\mathrm{dc}(u, m) = \mathrm{dn}(u, m)/\mathrm{cn}(u, m)$.

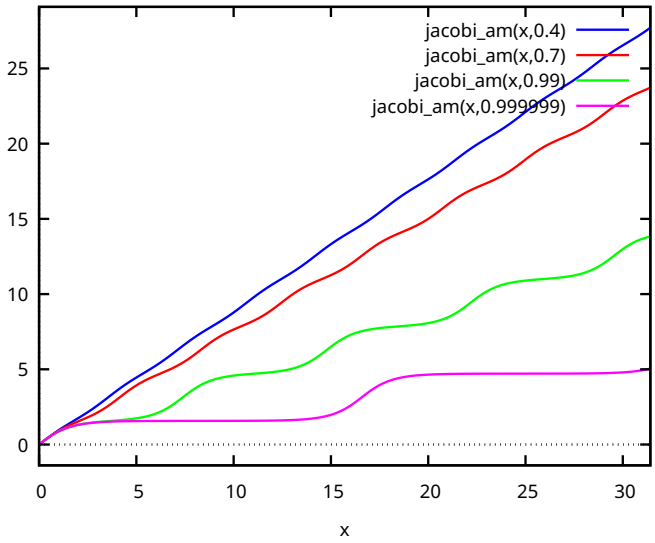
`jacobi_am (u, m)` [Function]
The Jacobi amplitude function, `jacobi_am`, is defined implicitly by (see http://functions.wolfram.com/09.24.02.0001.01) $z = \mathrm{am}(w, m)$ where $w = F(z, m)$ where $F(z, m)$ is the incomplete elliptic integral of the first kind (see [elliptic_f], page 370). It is defined for all real and complex values of z and m. In particular for real z and m with $|m| < 1$, $\mathrm{am}(z, m)$ maps the entire real line to the entire real line. For other values of z and m, the following relationship is used: $\mathrm{am}(z, m) = \sin^{-1}(\mathrm{jacobi\_sn}(z, m))$.

Some examples:

```
(%i1) jacobi_am(z,0);
(%o1)                           z
(%i2) jacobi_am(z,1);
                                  z    %pi
(%o2)                 2 atan(%e ) - ---
                                    2
(%i3) jacobi_am(0,m);
(%o3)                           0
```

```
(%i4) jacobi_am(100, .5);
(%o4)                    84.70311272411382
(%i5) jacobi_am(0.5, 1.5);
(%o5)                   0.4707197897046991
(%i6) jacobi_am(1.5b0, 1.5b0+%i);
(%o6)    9.340542168700782b-1 - 3.723960452146071b-1 %i
(%i1) plot2d([jacobi_am(x,.4),jacobi_am(x,.7),jacobi_am(x,.99),jacobi_am(x,.99999
(%o1)                           false
```

Compare this plot with the plot from DLMF 22.16.iv:



`inverse_jacobi_dn` (*u*, *m*) [Function]

The inverse of the Jacobian elliptic function $\mathrm{dn}(u,m)$. For $\sqrt{1-m} \le u \le 1$, it can also be written (DLMF 22.15.E14):

$$\mathrm{inverse\_jacobi\_dn}(u,m) = \int_u^1 \frac{dt}{\sqrt{(1-t^2)(t^2-(1-m))}}$$

`inverse_jacobi_ns` (*u*, *m*) [Function]

The inverse of the Jacobian elliptic function $\mathrm{ns}(u,m)$. For $1 \le u$, it can also be written (DLMF 22.15.E121):

$$\mathrm{inverse\_jacobi\_ns}(u,m) = \int_u^\infty \frac{dt}{\sqrt{(1-t^2)(t^2-m)}}$$

`inverse_jacobi_sc` (*u*, *m*) [Function]

The inverse of the Jacobian elliptic function $\mathrm{sc}(u,m)$. For all u it can also be written (DLMF 22.15.E20):

$$\mathrm{inverse\_jacobi\_sc}(u,m) = \int_0^u \frac{dt}{\sqrt{(1+t^2)(1+(1-m)t^2)}}$$

`inverse_jacobi_sd (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{sd}(u,m)$. For $-1/\sqrt{1-m} \le u \le 1/\sqrt{1-m}$, it can also be written (DLMF 22.15.E16):

$$\mathrm{inverse\_jacobi\_sd}(u,m) = \int_0^u \frac{dt}{\sqrt{(1-(1-m)t^2)(1+mt^2)}}$$

`inverse_jacobi_nc (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{nc}(u,m)$. For $1 \le u$, it can also be written (DLMF 22.15.E19):

$$\mathrm{inverse\_jacobi\_nc}(u,m) = \int_1^u \frac{dt}{\sqrt{(t^2-1)(m+(1-m)t^2)}}$$

`inverse_jacobi_cs (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{cs}(u,m)$. For all u it can also be written (DLMF 22.15.E23):

$$\mathrm{inverse\_jacobi\_cs}(u,m) = \int_u^\infty \frac{dt}{\sqrt{(1+t^2)(t^2+(1-m))}}$$

`inverse_jacobi_cd (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{cd}(u,m)$. For $-1 \le u \le 1$, it can also be written (DLMF 22.15.E15):

$$\mathrm{inverse\_jacobi\_cd}(u,m) = \int_u^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}$$

`inverse_jacobi_nd (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{nd}(u,m)$. For $1 \le u \le 1/\sqrt{1-m}$, it can also be written (DLMF 22.15.E17):

$$\mathrm{inverse\_jacobi\_nd}(u,m) = \int_1^u \frac{dt}{\sqrt{(t^2-1)(1-(1-m)t^2)}}$$

`inverse_jacobi_ds (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{ds}(u,m)$. For $\sqrt{1-m} \le u$, it can also be written (DLMF 22.15.E22):

$$\mathrm{inverse\_jacobi\_ds}(u,m) = \int_u^\infty \frac{dt}{\sqrt{(t^2+m)(t^2-(1-m))}}$$

`inverse_jacobi_dc (u, m)` [Function]

The inverse of the Jacobian elliptic function $\mathrm{dc}(u,m)$. For $1 \le u$, it can also be written (DLMF 22.15.E18):

$$\mathrm{inverse\_jacobi\_dc}(u,m) = \int_1^u \frac{dt}{\sqrt{(t^2-1)(t^2-m)}}$$

18.3 Functions and Variables for Elliptic Integrals

`elliptic_f` (*phi*, *m*) [Function]

The incomplete elliptic integral of the first kind, defined as

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

See also [elliptic_e], page 370, and [elliptic_kc], page 370.

`elliptic_e` (*phi*, *m*) [Function]

The incomplete elliptic integral of the second kind, defined as

$$\int_0^{\phi} \sqrt{1 - m\sin^2\theta}\, d\theta$$

See also [elliptic_f], page 370, and [elliptic_ec], page 371.

`elliptic_eu` (*u*, *m*) [Function]

The incomplete elliptic integral of the second kind, defined as

$$E(u, m) = \int_0^u \mathrm{dn}(v, m)\, dv = \int_0^{\tau} \sqrt{\frac{1 - mt^2}{1 - t^2}}\, dt$$

where $\tau = \mathrm{sn}(u, m)$.

This is related to `elliptic_e` by

$$E(u, m) = E(\sin^{-1}\mathrm{sn}(u, m), m)$$

See also [elliptic_e], page 370.

`elliptic_pi` (*n*, *phi*, *m*) [Function]

The incomplete elliptic integral of the third kind, defined as

$$\int_0^{\phi} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - m\sin^2\theta}}$$

`elliptic_kc` (*m*) [Function]

The complete elliptic integral of the first kind, defined as

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

For certain values of m, the value of the integral is known in terms of *Gamma* functions. Use `makegamma` to evaluate them.

`elliptic_ec (m)` [Function]

The complete elliptic integral of the second kind, defined as

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \theta}\, d\theta$$

For certain values of m, the value of the integral is known in terms of *Gamma* functions. Use `makegamma` to evaluate them.

`carlson_rc (x, y)` [Function]

Carlson's RC integral is defined by

$$R_C(x, y) = \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{t + x}(t + y)}\, dt$$

See Numerical Computation of Real or Complex Elliptic Integrals for more information.

This integral is related to many elementary functions in the following way:

$$\begin{aligned}
\log x &= (x - 1) R_C\left(\left(\frac{1 + x}{2}\right)^2, x\right), \quad x > 0 \\
\sin^{-1} x &= x R_C(1 - x^2, 1), \quad |x| \le 1 \\
\cos^{-1} x &= \sqrt{1 - x^2} R_C(x^2, 1), \quad 0 \le x \le 1 \\
\tan^{-1} x &= x R_C(1, 1 + x^2) \\
\sinh^{-1} x &= x R_C(1 + x^2, 1) \\
\cosh^{-1} x &= \sqrt{x^2 - 1} R_C(x^2, 1), \quad x \ge 1 \\
\tanh^{-1}(x) &= x R_C(1, 1 - x^2), \quad |x| \le 1
\end{aligned}$$

Also, we have the relationship

$$R_C(x, y) = R_F(x, y, y)$$

Some special values:

$$\begin{aligned}
R_C(0, 1) &= \frac{\pi}{2} \\
R_C(0, 1/4) &= \pi \\
R_C(2, 1) &= \log(\sqrt{2} + 1) \\
R_C(i, i + 1) &= \frac{\pi}{4} + \frac{i}{2} \log(\sqrt{2} - 1) \\
R_C(0, i) &= (1 - i) \frac{\pi}{2\sqrt{2}}
\end{aligned}$$

`carlson_rd (x, y, z)` [Function]

Carlson's RD integral is defined by

$$R_D(x,y,z) = \frac{3}{2}\int_0^\infty \frac{1}{\sqrt{t+x}\sqrt{t+y}\sqrt{t+z}\,(t+z)}\,dt$$

See Numerical Computation of Real or Complex Elliptic Integrals for more information.

We also have the special values

$$R_D(x,x,x) = x^{-\frac{3}{2}}$$
$$R_D(0,y,y) = \frac{3}{4}\pi y^{-\frac{3}{2}}$$
$$R_D(0,2,1) = 3\sqrt{\pi}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}$$

It is also related to the complete elliptic integral of the second kind, E, (`elliptic_ec`) as follows

$$E(m) = R_F(0,1-m,1) - \frac{m}{3}R_D(0,1-m,1)$$

`carlson_rf` (*x*, *y*, *z*) [Function]

Carlson's RF integral is defined by

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t+x}\sqrt{t+y}\sqrt{t+z}}\,dt$$

See Numerical Computation of Real or Complex Elliptic Integrals for more information.

We also have the special values

$$R_F(0,1,2) = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}}$$
$$R_F(i,-i,0) = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{\pi}}$$

It is also related to the complete elliptic integral of the second kind, E, (`elliptic_ec`) as follows

$$E(m) = R_F(0,1-m,1) - \frac{m}{3}R_D(0,1-m,1)$$

`carlson_rj` (*x*, *y*, *z*, *p*) [Function]

Carlson's RJ integral is defined by

$$R_J(x,y,z) = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t+x}\sqrt{t+y}\sqrt{t+z}\,(t+p)}\,dt$$

See Numerical Computation of Real or Complex Elliptic Integrals for more information.

It is related to the elliptic integral of the third kind (`elliptic_pi`) by

$$\int_0^{\phi} \frac{1}{(1+n\sin^2\theta)\sqrt{1-m\sin^2\theta}}\,d\theta = R_F(c-1,c-m,c) - \frac{n}{3}R_j(c-1,c-m,c,c+n)$$

where $c = \csc\phi$. Note that this differs in our definition of `elliptic_pi` by the sign of the parameter n.

19 Limits

19.1 Functions and Variables for Limits

`lhospitallim` [Option variable]

Default value: 4

`lhospitallim` is the maximum number of times L'Hospital's rule is used in `limit`. This prevents infinite looping in cases like `limit (cot(x)/csc(x), x, 0)`.

`limit` [Function]

`limit (`*expr*`, `*x*`, `*val*`, `*dir*`)`
`limit (`*expr*`, `*x*`, `*val*`)`
`limit (`*expr*`)`

Computes the limit of *expr* as the real variable *x* approaches the value *val* from the direction *dir*. *dir* may have the value `plus` for a limit from above, `minus` for a limit from below, or may be omitted (implying a two-sided limit is to be computed).

`limit` uses the following special symbols: `inf` (positive infinity) and `minf` (negative infinity). On output it may also use `und` (undefined), `ind` (indefinite but bounded) and `infinity` (complex infinity).

`infinity` (complex infinity) is returned when the limit of the absolute value of the expression is positive infinity, but the limit of the expression itself is not positive infinity or negative infinity. This includes cases where the limit of the complex argument is a constant, as in `limit(log(x), x, minf)`, cases where the complex argument oscillates, as in `limit((-2)^x, x, inf)`, and cases where the complex argument is different for either side of a two-sided limit, as in `limit(1/x, x, 0)` and `limit(log(x), x, 0)`.

`lhospitallim` is the maximum number of times L'Hospital's rule is used in `limit`. This prevents infinite looping in cases like `limit (cot(x)/csc(x), x, 0)`.

`tlimswitch` when true will allow the `limit` command to use Taylor series expansion when necessary.

`limsubst` prevents `limit` from attempting substitutions on unknown forms. This is to avoid bugs like `limit (f(n)/f(n+1), n, inf)` giving 1. Setting `limsubst` to `true` will allow such substitutions.

`limit` with one argument is often called upon to simplify constant expressions, for example, `limit (inf-1)`.

`example (limit)` displays some examples.

For the method see Wang, P., "Evaluation of Definite Integrals by Symbolic Manipulation", Ph.D. thesis, MAC TR-92, October 1971.

`limsubst` [Option variable]

Default value: `false`

prevents `limit` from attempting substitutions on unknown forms. This is to avoid bugs like `limit (f(n)/f(n+1), n, inf)` giving 1. Setting `limsubst` to `true` will allow such substitutions.

`tlimit` [Function]

`tlimit` (*expr*, *x*, *val*, *dir*)
`tlimit` (*expr*, *x*, *val*)
`tlimit` (*expr*)

Take the limit of the Taylor series expansion of `expr` in `x` at `val` from direction `dir`.

`tlimswitch` [Option variable]

Default value: `true`

When `tlimswitch` is `true`, the `limit` command will use a Taylor series expansion if the limit of the input expression cannot be computed directly. This allows evaluation of limits such as `limit(x/(x-1)-1/log(x),x,1,plus)`. When `tlimswitch` is `false` and the limit of input expression cannot be computed directly, `limit` will return an unevaluated limit expression.

`gruntz` [Function]

`gruntz` (*expr*, *var*, *value*)
`gruntz` (*expr*, *var*, *value*, *direction*)

Compute limit of expression *expr* with respect to variable *var* at *value*. When *value* is not infinite (i.e., not `inf` or `minf`), *direction* must be supplied, either `plus` for a limit from above, or `minus` for a limit from below.

If `gruntz` cannot find the limit, an unevaluated expression `gruntz(...)` is returned.

`gruntz` implements the method described in the dissertation of Dominik Gruntz, "On Computing Limits in a Symbolic Manipulation System" (ETH Zurich, 1996).

The algorithm identifies the most rapidly varying subexpression, replaces it with a new variable, rewrites the expression in terms of the new variable, and then repeats.

The algorithm doesn't handle oscillating functions, so it can't do things like `limit(sin(x)/x, x, inf)`.

To handle limits involving functions such as `gamma(x)` and `erf(x)`, the Gruntz algorithm requires them to be written in terms of asymptotic expansions, which Maxima cannot currently do.

The Gruntz algorithm assumes that variables and expressions are real, so, for example, it can't handle `limit((-2)^x, x, inf)`.

`gruntz` is one of the methods called from `limit`.

20 Differentiation

20.1 Functions and Variables for Differentiation

`antid` (*expr*, *x*, *u*(*x*)) [Function]

Returns a two-element list, such that an antiderivative of *expr* with respect to *x* can be constructed from the list. The expression *expr* may contain an unknown function *u* and its derivatives.

Let *L*, a list of two elements, be the return value of `antid`. Then `L[1] + 'integrate (L[2], x)` is an antiderivative of *expr* with respect to *x*.

When `antid` succeeds entirely, the second element of the return value is zero. Otherwise, the second element is nonzero, and the first element is nonzero or zero. If `antid` cannot make any progress, the first element is zero and the second nonzero.

`load ("antid")` loads this function. The `antid` package also defines the functions `nonzeroandfreeof` and `linear`.

`antid` is related to `antidiff` as follows. Let *L*, a list of two elements, be the return value of `antid`. Then the return value of `antidiff` is equal to `L[1] + 'integrate (L[2], x)` where *x* is the variable of integration.

Examples:

```
(%i1) load ("antid")$
(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);
                        z(x)       d
(%o2)                 %e     y(x) (-- (z(x)))
                                   dx
(%i3) a1: antid (expr, x, z(x));
                   z(x)           z(x)  d
(%o3)          [%e     y(x), - %e     (-- (y(x)))]
                                    dx
(%i4) a2: antidiff (expr, x, z(x));
                               /
                 z(x)          |   z(x)  d
(%o4)         %e     y(x) - | %e     (-- (y(x))) dx
                               |         dx
                               /
(%i5) a2 - (first (a1) + 'integrate (second (a1), x));
(%o5)                              0
(%i6) antid (expr, x, y(x));
                         z(x)       d
(%o6)             [0, %e     y(x) (-- (z(x)))]
                                    dx
```

```
(%i7) antidiff (expr, x, y(x));
                  /
                  |   z(x)       d
(%o7)             | %e     y(x) (-- (z(x))) dx
                  |              dx
                  /
```

antidiff (*expr*, *x*, *u*(*x*)) [Function]

Returns an antiderivative of *expr* with respect to *x*. The expression *expr* may contain an unknown function *u* and its derivatives.

When `antidiff` succeeds entirely, the resulting expression is free of integral signs (that is, free of the `integrate` noun). Otherwise, `antidiff` returns an expression which is partly or entirely within an integral sign. If `antidiff` cannot make any progress, the return value is entirely within an integral sign.

`load ("antid")` loads this function. The `antid` package also defines the functions `nonzeroandfreeof` and `linear`.

`antidiff` is related to `antid` as follows. Let L, a list of two elements, be the return value of `antid`. Then the return value of `antidiff` is equal to `L[1] + 'integrate (L[2], x)` where x is the variable of integration.

Examples:

```
(%i1) load ("antid")$
(%i2) expr: exp (z(x)) * diff (z(x), x) * y(x);
                        z(x)       d
(%o2)                 %e     y(x) (-- (z(x)))
                                    dx
(%i3) a1: antid (expr, x, z(x));
                    z(x)           z(x)  d
(%o3)           [%e     y(x), - %e     (-- (y(x)))]
                                       dx
(%i4) a2: antidiff (expr, x, z(x));
                               /
                 z(x)          |   z(x)  d
(%o4)          %e     y(x) - | %e     (-- (y(x))) dx
                               |         dx
                               /
(%i5) a2 - (first (a1) + 'integrate (second (a1), x));
(%o5)                          0
(%i6) antid (expr, x, y(x));
                         z(x)       d
(%o6)              [0, %e     y(x) (-- (z(x)))]
                                      dx
(%i7) antidiff (expr, x, y(x));
                  /
                  |   z(x)       d
(%o7)             | %e     y(x) (-- (z(x))) dx
                  |              dx
                  /
```

at [Function]

at (*expr*, [*eqn_1*, ..., *eqn_n*])
at (*expr*, *eqn*)

Evaluates the expression *expr* with the variables assuming the values as specified for them in the list of equations [*eqn_1*, ..., *eqn_n*] or the single equation *eqn*.

If a subexpression depends on any of the variables for which a value is specified but there is no atvalue specified and it can't be otherwise evaluated, then a noun form of the at is returned which displays in a two-dimensional form.

at carries out multiple substitutions in parallel.

See also atvalue. For other functions which carry out substitutions, see also subst and ev.

Examples:

```
(%i1) atvalue (f(x,y), [x = 0, y = 1], a^2);
                                    2
(%o1)                              a
(%i2) atvalue ('diff (f(x,y), x), x = 0, 1 + y);
(%o2)                          @2 + 1
(%i3) printprops (all, atvalue);
                                    |
                   d                |
                  --- (f(@1, @2))|       = @2 + 1
                  d@1               |
                                    |@1 = 0

                                         2
                             f(0, 1) = a

(%o3)                           done
(%i4) diff (4*f(x, y)^2 - u(x, y)^2, x);
                    d                                d
(%o4)  8 f(x, y) (-- (f(x, y))) - 2 u(x, y) (-- (u(x, y)))
                    dx                               dx
(%i5) at (%, [x = 0, y = 1]);
                                                    |
                   2              d                 |
(%o5)        16 a  - 2 u(0, 1) (-- (u(x, 1))|     )
                                   dx               |
                                                    |x = 0
```

Note that in the last line y is treated differently to x as y isn't used as a differentiation variable.

The difference between subst, at and ev can be seen in the following example:

```
(%i1) e1:I(t)=C*diff(U(t),t)$
(%i2) e2:U(t)=L*diff(I(t),t)$
```

```
(%i3) at(e1,e2);
                                  |
                        d         |
(%o3)       I(t) = C (-- (U(t))|                    )
                        dt        |          d
                                  |U(t) = L (-- (I(t)))
                                            dt
(%i4) subst(e2,e1);
                              d      d
(%o4)             I(t) = C (-- (L (-- (I(t)))))
                              dt     dt
(%i5) ev(e1,e2,diff);
                                    2
                                   d
(%o5)                I(t) = C L (--- (I(t)))
                                     2
                                   dt
```

atomgrad [Property]

atomgrad is the atomic gradient property of an expression. This property is assigned by gradef.

atvalue [Function]

atvalue (*expr*, [*x_1* = *a_1*, ..., *x_m* = *a_m*], *c*)
atvalue (*expr*, *x_1* = *a_1*, *c*)

Assigns the value *c* to *expr* at the point x = a. Typically boundary values are established by this mechanism.

expr is a function evaluation, f(*x_1*, ..., *x_m*), or a derivative, diff (f(*x_1*, ..., *x_m*), *x_1*, *n_1*, ..., *x_n*, *n_m*) in which the function arguments explicitly appear. *n_i* is the order of differentiation with respect to *x_i*.

The point at which the atvalue is established is given by the list of equations [*x_1* = *a_1*, ..., *x_m* = *a_m*]. If there is a single variable *x_1*, the sole equation may be given without enclosing it in a list.

printprops ([*f_1*, *f_2*, ...], atvalue) displays the atvalues of the functions *f_1*, *f_2*, ... as specified by calls to atvalue. printprops (*f*, atvalue) displays the atvalues of one function *f*. printprops (all, atvalue) displays the atvalues of all functions for which atvalues are defined.

The symbols @1, @2, ... represent the variables *x_1*, *x_2*, ... when atvalues are displayed.

atvalue evaluates its arguments. atvalue returns *c*, the atvalue.

See also at.

Examples:

```
(%i1) atvalue (f(x,y), [x = 0, y = 1], a^2);
                                  2
(%o1)                            a
```

```
(%i2) atvalue ('diff (f(x,y), x), x = 0, 1 + y);
(%o2)                          @2 + 1
(%i3) printprops (all, atvalue);
                                    |
                    d               |
                   --- (f(@1, @2))|       = @2 + 1
                   d@1              |
                                    |@1 = 0

                                        2
                              f(0, 1) = a

(%o3)                           done
(%i4) diff (4*f(x,y)^2 - u(x,y)^2, x);
                    d                               d
(%o4)  8 f(x, y) (-- (f(x, y))) - 2 u(x, y) (-- (u(x, y)))
                    dx                              dx
(%i5) at (%, [x = 0, y = 1]);
                                                   |
                   2                d              |
(%o5)        16 a  - 2 u(0, 1) (-- (u(x, 1)))|     )
                                    dx             |
                                                   |x = 0
```

`del (x)` [Function]

`del (x)` represents the differential of the variable x.

`diff` returns an expression containing `del` if an independent variable is not specified. In this case, the return value is the so-called "total differential".

See also `diff`, `del` and `derivdegree`.

Examples:

```
(%i1) diff (log (x));
                               del(x)
(%o1)                          ------
                                 x
(%i2) diff (exp (x*y));
                     x y                 x y
(%o2)               %e    x del(y) + %e    y del(x)
(%i3) diff (x*y*z);
(%o3)         x y del(z) + x z del(y) + y z del(x)
```

`delta (t)` [Function]

The Dirac Delta function.

Currently only `laplace` knows about the `delta` function.

Example:

```
(%i1) assume(a > 0)$
```

```
(%i2) laplace (delta (t - a) * sin(b*t), t, s);
                  2 %i a b        - a s - %i a b
                (%e         - 1) %e               %i
(%o2)         - ------------------------------------
                                 2
```

`dependencies` [System variable]

`dependencies (f_1, ..., f_n)` [Function]

The variable `dependencies` is the list of atoms which have functional dependencies, assigned by `depends`, the function `dependencies`, or `gradef`. The `dependencies` list is cumulative: each call to `depends`, `dependencies`, or `gradef` appends additional items. The default value of `dependencies` is `[]`.

The function `dependencies(f_1, ..., f_n)` appends *f_1*, ..., *f_n*, to the `dependencies` list, where *f_1*, ..., *f_n* are expressions of the form `f(x_1, ..., x_m)`, and *x_1*, ..., *x_m* are any number of arguments.

`dependencies(f(x_1, ..., x_m))` is equivalent to `depends(f, [x_1, ..., x_m])`.

See also `depends` and `gradef`.

```
(%i1) dependencies;
(%o1)                          []
(%i2) depends (foo, [bar, baz]);
(%o2)                    [foo(bar, baz)]
(%i3) depends ([g, h], [a, b, c]);
(%o3)               [g(a, b, c), h(a, b, c)]
(%i4) dependencies;
(%o4)        [foo(bar, baz), g(a, b, c), h(a, b, c)]
(%i5) dependencies (quux (x, y), mumble (u));
(%o5)                [quux(x, y), mumble(u)]
(%i6) dependencies;
(%o6) [foo(bar, baz), g(a, b, c), h(a, b, c), quux(x, y),
                                                 mumble(u)]
(%i7) remove (quux, dependency);
(%o7)                          done
(%i8) dependencies;
(%o8)  [foo(bar, baz), g(a, b, c), h(a, b, c), mumble(u)]
```

`depends (f_1, x_1, ..., f_n, x_n)` [Function]

Declares functional dependencies among variables for the purpose of computing derivatives. In the absence of declared dependence, `diff (f, x)` yields zero. If `depends (f, x)` is declared, `diff (f, x)` yields a symbolic derivative (that is, a `diff` noun).

Each argument *f_1*, *x_1*, etc., can be the name of a variable or array, or a list of names. Every element of *f_i* (perhaps just a single element) is declared to depend on every element of *x_i* (perhaps just a single element). If some *f_i* is the name of an array or contains the name of an array, all elements of the array depend on *x_i*.

`diff` recognizes indirect dependencies established by `depends` and applies the chain rule in these cases.

`remove (f, dependency)` removes all dependencies declared for *f*.

`depends` returns a list of the dependencies established. The dependencies are appended to the global variable `dependencies`. `depends` evaluates its arguments.

`diff` is the only Maxima command which recognizes dependencies established by `depends`. Other functions (`integrate`, `laplace`, etc.) only recognize dependencies explicitly represented by their arguments. For example, `integrate` does not recognize the dependence of `f` on `x` unless explicitly represented as `integrate (f(x), x)`.

`depends(f, [x_1, ..., x_n])` is equivalent to `dependencies(f(x_1, ..., x_n))`.

See also `diff`, `del`, `derivdegree` and `derivabbrev`.

```
(%i1) depends ([f, g], x);
(%o1)                        [f(x), g(x)]
(%i2) depends ([r, s], [u, v, w]);
(%o2)                [r(u, v, w), s(u, v, w)]
(%i3) depends (u, t);
(%o3)                           [u(t)]
(%i4) dependencies;
(%o4)      [f(x), g(x), r(u, v, w), s(u, v, w), u(t)]
(%i5) diff (r.s, u);
                           dr             ds
(%o5)                     (--) . s + r . --
                           du             du
(%i1) diff (r.s, t);
(%o1)                             0
(%i1) remove (r, dependency);
(%o1)                           done
(%i2) diff (r.s, t);
(%o2)                             0
```

`derivabbrev` [Option variable]

Default value: `false`

When `derivabbrev` is `true`, symbolic derivatives (that is, `diff` nouns) are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation `dy/dx`.

`derivdegree (expr, y, x)` [Function]

Returns the highest degree of the derivative of the dependent variable *y* with respect to the independent variable *x* occurring in *expr*.

Example:

```
(%i1) 'diff (y, x, 2) + 'diff (y, z, 3) + 'diff (y, x) * x^2;
                           3     2
                          d y   d y    2 dy
(%o1)                     --- + --- + x  --
                            3     2     dx
                          dz    dx
(%i2) derivdegree (%, y, x);
(%o2)                             2
```

derivlist (*var_1*, ..., *var_k*) [Function]

Causes only differentiations with respect to the indicated variables, within the `ev` command.

derivsubst [Option variable]

Default value: `false`

When `derivsubst` is `true`, a non-syntactic substitution such as `subst (x, 'diff (y, t), 'diff (y, t, 2))` yields `'diff (x, t)`.

diff [Function]

`diff` (*expr*, *x_1*, *n_1*, ..., *x_m*, *n_m*)
`diff` (*expr*, *x*, *n*)
`diff` (*expr*, *x*)
`diff` (*expr*)

Returns the derivative or differential of *expr* with respect to some or all variables in *expr*.

`diff (`*expr*`, `*x*`, `*n*`)` returns the *n*'th derivative of *expr* with respect to *x*.

`diff (`*expr*`, `*x_1*`, `*n_1*`, ..., `*x_m*`, `*n_m*`)` returns the mixed partial derivative of *expr* with respect to *x_1*, ..., *x_m*. It is equivalent to `diff (... (diff (`*expr*`, `*x_m*`, `*n_m*`) ...), `*x_1*`, `*n_1*`)`.

`diff (`*expr*`, `*x*`)` returns the first derivative of *expr* with respect to the variable *x*.

`diff (`*expr*`)` returns the total differential of *expr*, that is, the sum of the derivatives of *expr* with respect to each its variables times the differential `del` of each variable. No further simplification of `del` is offered.

The noun form of `diff` is required in some contexts, such as stating a differential equation. In these cases, `diff` may be quoted (as `'diff`) to yield the noun form instead of carrying out the differentiation.

When `derivabbrev` is `true`, derivatives are displayed as subscripts. Otherwise, derivatives are displayed in the Leibniz notation, `dy/dx`.

See also `depends`, `del`, `derivdegree`, `derivabbrev`, and `gradef`.

Examples:

```
(%i1) diff (exp (f(x)), x, 2);
                      2
              f(x)  d                f(x)  d         2
(%o1)       %e     (--- (f(x))) + %e     (-- (f(x)))
                       2                    dx
                     dx
(%i2) derivabbrev: true$
(%i3) 'integrate (f(x, y), y, g(x), h(x));
                          h(x)
                         /
                         |
(%o3)                    |     f(x, y) dy
                         |
                         /
                          g(x)
```

```
(%i4) diff (%, x);
       h(x)
      /
      |
(%o4) |     (f(x, y))  dy + f(x, h(x)) (h(x))
      |              x                       x
      /
       g(x)

                                                - f(x, g(x)) (g(x))
                                                                   x
```

For the tensor package, the following modifications have been incorporated:

(1) The derivatives of any indexed objects in *expr* will have the variables *x_i* appended as additional arguments. Then all the derivative indices will be sorted.

(2) The *x_i* may be integers from 1 up to the value of the variable `dimension` [default value: 4]. This will cause the differentiation to be carried out with respect to the *x_i*'th member of the list `coordinates` which should be set to a list of the names of the coordinates, e.g., `[x, y, z, t]`. If `coordinates` is bound to an atomic variable, then that variable subscripted by *x_i* will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like `X[1]`, `X[2]`, ... to be used. If `coordinates` has not been assigned a value, then the variables will be treated as in (1) above.

`diff` [Special symbol]

When `diff` is present as an `evflag` in call to `ev`, all differentiations indicated in `expr` are carried out.

`express` (*expr*) [Function]

Expands differential operator nouns into expressions in terms of partial derivatives. `express` recognizes the operators `grad`, `div`, `curl`, `laplacian`. `express` also expands the cross product `~`.

Symbolic derivatives (that is, `diff` nouns) in the return value of express may be evaluated by including `diff` in the `ev` function call or command line. In this context, `diff` acts as an `evfun`.

`load ("vect")` loads this function.

Examples:

```
(%i1) load ("vect")$
(%i2) grad (x^2 + y^2 + z^2);
                                2    2    2
(%o2)                    grad (z  + y  + x )
(%i3) express (%);
       d    2    2    2   d    2    2    2   d    2    2    2
(%o3) [-- (z  + y  + x ), -- (z  + y  + x ), -- (z  + y  + x )]
       dx                 dy                 dz
(%i4) ev (%, diff);
(%o4)                     [2 x, 2 y, 2 z]
```

```
(%i5) div ([x^2, y^2, z^2]);
                                2   2   2
(%o5)                     div [x , y , z ]
(%i6) express (%);
                    d    2    d    2    d    2
(%o6)               -- (z ) + -- (y ) + -- (x )
                    dz        dy        dx
(%i7) ev (%, diff);
(%o7)                      2 z + 2 y + 2 x
(%i8) curl ([x^2, y^2, z^2]);
                                2   2   2
(%o8)                     curl [x , y , z ]
(%i9) express (%);
      d    2    d    2   d    2    d    2   d    2    d    2
(%o9) [-- (z ) - -- (y ), -- (x ) - -- (z ), -- (y ) - -- (x )]
      dy        dz       dz        dx       dx        dy
(%i10) ev (%, diff);
(%o10)                        [0, 0, 0]
(%i11) laplacian (x^2 * y^2 * z^2);
                                   2  2  2
(%o11)                laplacian (x  y  z )
(%i12) express (%);
        2                2                2
       d     2  2  2    d     2  2  2    d     2  2  2
(%o12)  --- (x  y  z ) + --- (x  y  z ) + --- (x  y  z )
         2                2                2
        dz               dy               dx
(%i13) ev (%, diff);
                       2  2      2  2      2  2
(%o13)              2 y  z  + 2 x  z  + 2 x  y
(%i14) [a, b, c] ~ [x, y, z];
(%o14)                 [a, b, c] ~ [x, y, z]
(%i15) express (%);
(%o15)           [b z - c y, c x - a z, a y - b x]
```

gradef [Function]

`gradef (`*f*`(`*x_1*`, ..., `*x_n*`), `*g_1*`, ..., `*g_m*`)`
`gradef (a, x, expr)`

Defines the partial derivatives (i.e., the components of the gradient) of the function *f* or variable *a*.

`gradef (`*f*`(`*x_1*`, ..., `*x_n*`), `*g_1*`, ..., `*g_m*`)` defines `df/dx_i` as *g_i*, where *g_i* is an expression; *g_i* may be a function call, but not the name of a function. The number of partial derivatives *m* may be less than the number of arguments *n*, in which case derivatives are defined with respect to *x_1* through *x_m* only.

`gradef (a, x, expr)` defines the derivative of variable *a* with respect to *x* as *expr*. This also establishes the dependence of *a* on *x* (via `depends (a, x)`).

The first argument `f(x_1, ..., x_n)` or *a* is quoted, but the remaining arguments *g_1*, ..., *g_m* are evaluated. `gradef` returns the function or variable for which the partial derivatives are defined.

`gradef` can redefine the derivatives of Maxima's built-in functions. For example, `gradef (sin(x), sqrt (1 - sin(x)^2))` redefines the derivative of `sin`.

`gradef` cannot define partial derivatives for a subscripted function.

`printprops ([f_1, ..., f_n], gradef)` displays the partial derivatives of the functions *f_1*, ..., *f_n*, as defined by `gradef`.

`printprops ([a_n, ..., a_n], atomgrad)` displays the partial derivatives of the variables *a_n*, ..., *a_n*, as defined by `gradef`.

`gradefs` is the list of the functions for which partial derivatives have been defined by `gradef`. `gradefs` does not include any variables for which partial derivatives have been defined by `gradef`.

Gradients are needed when, for example, a function is not known explicitly but its first derivatives are and it is desired to obtain higher order derivatives.

`gradefs` [System variable]

Default value: `[]`

`gradefs` is the list of the functions for which partial derivatives have been defined by `gradef`. `gradefs` does not include any variables for which partial derivatives have been defined by `gradef`.

21 Integration

21.1 Introduction to Integration

Maxima has several routines for handling integration. The `integrate` function makes use of most of them. There is also the `antid` package, which handles an unspecified function (and its derivatives, of course). For numerical uses, there is a set of adaptive integrators from QUADPACK, named `quad_qag`, `quad_qags`, etc., which are described under the heading `QUADPACK`. Hypergeometric functions are being worked on, see `specint` for details. Generally speaking, Maxima only handles integrals which are integrable in terms of the "elementary functions" (rational functions, trigonometrics, logs, exponentials, radicals, etc.) and a few extensions (error function, dilogarithm). It does not handle integrals in terms of unknown functions such as `g(x)` and `h(x)`.

21.2 Functions and Variables for Integration

`changevar` (*expr*, *f*(*x*,*y*), *y*, *x*) [Function]

Makes the change of variable given by *f(x,y)* = 0 in all integrals occurring in *expr* with integration with respect to *x*. The new variable is *y*.

The change of variable can also be written *f(x)* = *g(y)*.

```
(%i1) assume(a > 0)$
(%i2) 'integrate (%e**sqrt(a*y), y, 0, 4);
                       4
                      /
                      |    sqrt(a) sqrt(y)
(%o2)                 |  %e                dy
                      |
                      /
                       0
(%i3) changevar (%, y-z^2/a, z, y);
                       0
                      /
                      |              abs(z)
                    2 |            %e       z dz
                      |
                      /
                       - 2 sqrt(a)
(%o3)             - ----------------------------
                                 a
```

An expression containing a noun form, such as the instances of `'integrate` above, may be evaluated by `ev` with the `nouns` flag. For example, the expression returned by `changevar` above may be evaluated by `ev (%o3, nouns)`.

`changevar` may also be used to make changes in the indices of a sum or product. However, it must be realized that when a change is made in a sum or product, this change must be a shift, i.e., `i = j+ ...`, not a higher degree function. E.g.,

```
(%i1) sum (a[i]*x^(i-2), i, 0, inf);
                           inf
                           ____
                           \          i - 2
(%o1)                       >     a  x
                           /       i
                           ----
                           i = 0
(%i2) changevar (%, i-2-n, n, i);
                          inf
                          ____
                          \                n
(%o2)                      >      a      x
                          /        n + 2
                          ----
                          n = - 2
```

`dblint` (*f*, *r*, *s*, a, *b*) [Function]

A double-integral routine which was written in top-level Maxima and then translated and compiled to machine code. Use `load ("dblint")` to access this package. It uses the Simpson's rule method in both the x and y directions to calculate

$$\int_a^b \int_{r(x)}^{s(x)} f\left(x, y\right)\, dy\, dx$$

The function *f* must be a translated or compiled function of two variables, and *r* and *s* must each be a translated or compiled function of one variable, while *a* and *b* must be floating point numbers. The routine has two global variables which determine the number of divisions of the x and y intervals: `dblint_x` and `dblint_y`, both of which are initially 10, and can be changed independently to other integer values (there are `2*dblint_x+1` points computed in the x direction, and `2*dblint_y+1` in the y direction). The routine subdivides the X axis and then for each value of X it first computes *r*`(x)` and *s*`(x)`; then the Y axis between *r*`(x)` and *s*`(x)` is subdivided and the integral along the Y axis is performed using Simpson's rule; then the integral along the X axis is done using Simpson's rule with the function values being the Y-integrals. This procedure may be numerically unstable for a great variety of reasons, but is reasonably fast: avoid using it on highly oscillatory functions and functions with singularities (poles or branch points in the region). The Y integrals depend on how far apart *r*`(x)` and *s*`(x)` are, so if the distance *s*`(x) - `*r*`(x)` varies rapidly with X, there may be substantial errors arising from truncation with different step-sizes in the various Y integrals. One can increase `dblint_x` and `dblint_y` in an effort to improve the coverage of the region, at the expense of computation time. The function values are not saved, so if the function is very time-consuming, you will have to wait for re-computation if you change anything (sorry). It is required that the functions *f*, *r*, and *s* be either translated or compiled prior to calling `dblint`. This will result in orders of magnitude speed improvement over interpreted code in many cases!

`demo ("dblint")` executes a demonstration of `dblint` applied to an example problem.

`defint` (*expr*, *x*, *a*, *b*) [Function]

Attempts to compute a definite integral. `defint` is called by `integrate` when limits of integration are specified, i.e., when `integrate` is called as `integrate (`*expr*`, `*x*`, `*a*`, `*b*`)`. Thus from the user's point of view, it is sufficient to call `integrate`.

`defint` returns a symbolic expression, either the computed integral or the noun form of the integral. See `quad_qag` and related functions for numerical approximation of definite integrals.

`erfflag` [Option variable]

Default value: `true`

When `erfflag` is `false`, prevents `risch` from introducing the `erf` function in the answer if there were none in the integrand to begin with.

`ilt` (*expr*, *s*, *t*) [Function]

Computes the inverse Laplace transform of *expr* with respect to *s* and parameter *t*. *expr* must be a ratio of polynomials whose denominator has only linear and quadratic factors; there is an extension of `ilt`, called `pwilt` (Piece-Wise Inverse Laplace Transform) that handles several other cases where `ilt` fails.

By using the functions `laplace` and `ilt` together with the `solve` or `linsolve` functions the user can solve a single differential or convolution integral equation or a set of them.

```
(%i1) 'integrate (sinh(a*x)*f(t-x), x, 0, t) + b*f(t) = t**2;
               t
              /
              |                                      2
(%o1)         |  f(t - x) sinh(a x) dx + b f(t) = t
              |
              /
               0
(%i2) laplace (%, t, s);
       a laplace(f(t), t, s)                              2
(%o2)  --------------------- + b laplace(f(t), t, s) = --
               2    2                                     3
              s  - a                                     s
(%i3) linsolve ([%], ['laplace(f(t), t, s)]);
                                          2      2
                                       2 s  - 2 a
(%o3)     [laplace(f(t), t, s) = --------------------]
                                     5         2     3
                                  b s  + (a - a  b) s
```

```
(%i4) ilt (rhs (first (%)), s, t);
Is a b (a b - 1) positive, negative or zero?
pos;

                sqrt(a b (a b - 1)) t
        2 cosh(---------------------)       2
                         b               a t
(%o4) - ----------------------------- + -------
              3  2      2               a b - 1
             a  b  - 2 a  b + a
                                                        2
                                              + ------------------
                                                 3  2      2
                                                a  b  - 2 a  b + a
(%i5) pos;
```

`intanalysis` [Option variable]

Default value: `true`

When `true`, definite integration tries to find poles in the integrand in the interval of integration. If there are, then the integral is evaluated appropriately as a principal value integral. If intanalysis is `false`, this check is not performed and integration is done assuming there are no poles.

See also `ldefint`.

Examples:

Maxima can solve the following integrals, when `intanalysis` is set to `false`:

```
(%i1) intanalysis:false;
(%o1)                          false
(%i2) integrate(1/(sqrt(x+1)+1),x,0,1);
                                                  3/2
(%o2)      - 2 log(sqrt(2) + 1) + 2 log(2) + 2    - 2
(%i3) integrate(1/(sqrt(x)+1),x,0,1),intanalysis:false;
(%o3)                      2 - 2 log(2)
(%i4) integrate(cos(a)/sqrt((tan(a))^2+1),a,-%pi/2,%pi/2),intanalysis:false;
(%o4)               %i log(2) - %i log(2 %i)
(%i5) intanalysis:false$
(%i6) integrate(cos(a)/sqrt((tan(a))^2 +1),a,-%pi/2,%pi/2);
(%o6)               %i log(2) - %i log(2 %i)
```

`integrate` [Function]

`integrate` (*expr*, *x*)
`integrate` (*expr*, *x*, *a*, *b*)

Attempts to symbolically compute the integral of *expr* with respect to *x*. `integrate` (*expr*, *x*) is an indefinite integral, while `integrate` (*expr*, *x*, *a*, *b*) is a definite integral, with limits of integration *a* and *b*. The limits should not contain *x*, although `integrate` does not enforce this restriction. *a* need not be less than *b*. If *b* is equal to *a*, `integrate` returns zero.

See quad_qag and related functions for numerical approximation of definite integrals. See residue for computation of residues (complex integration). See antid for an alternative means of computing indefinite integrals.

The integral (an expression free of integrate) is returned if integrate succeeds. Otherwise the return value is the noun form of the integral (the quoted operator 'integrate) or an expression containing one or more noun forms. The noun form of integrate is displayed with an integral sign.

In some circumstances it is useful to construct a noun form by hand, by quoting integrate with a single quote, e.g., 'integrate (*expr*, *x*). For example, the integral may depend on some parameters which are not yet computed. The noun may be applied to its arguments by ev (*i*, nouns) where *i* is the noun form of interest.

integrate handles definite integrals separately from indefinite, and employs a range of heuristics to handle each case. Special cases of definite integrals include limits of integration equal to zero or infinity (inf or minf), trigonometric functions with limits of integration equal to zero and %pi or 2 %pi, rational functions, integrals related to the definitions of the beta and psi functions, and some logarithmic and trigonometric integrals. Processing rational functions may include computation of residues. If an applicable special case is not found, an attempt will be made to compute the indefinite integral and evaluate it at the limits of integration. This may include taking a limit as a limit of integration goes to infinity or negative infinity; see also ldefint.

Special cases of indefinite integrals include trigonometric functions, exponential and logarithmic functions, and rational functions. integrate may also make use of a short table of elementary integrals.

integrate may carry out a change of variable if the integrand has the form f(g(x)) * diff(g(x), x). integrate attempts to find a subexpression g(x) such that the derivative of g(x) divides the integrand. This search may make use of derivatives defined by the gradef function. See also changevar and antid.

If none of the preceding heuristics find the indefinite integral, the Risch algorithm is executed. The flag risch may be set as an evflag, in a call to ev or on the command line, e.g., ev (integrate (*expr*, *x*), risch) or integrate (*expr*, *x*), risch. If risch is present, integrate calls the risch function without attempting heuristics first. See also risch.

integrate works only with functional relations represented explicitly with the f(x) notation. integrate does not respect implicit dependencies established by the depends function.

integrate may need to know some property of a parameter in the integrand. integrate will first consult the assume database, and, if the variable of interest is not there, integrate will ask the user. Depending on the question, suitable responses are yes; or no;, or pos;, zero;, or neg;.

integrate is not, by default, declared to be linear. See declare and linear.

integrate attempts integration by parts only in a few special cases.

Examples:

- Elementary indefinite and definite integrals.

```
(%i1) integrate (sin(x)^3, x);
                            3
                       cos (x)
(%o1)                  ------- - cos(x)
                          3
(%i2) integrate (x/ sqrt (b^2 - x^2), x);
                                  2    2
(%o2)                    - sqrt(b  - x )
(%i3) integrate (cos(x)^2 * exp(x), x, 0, %pi);
                               %pi
                            3 %e      3
(%o3)                       ------- - -
                              5       5
(%i4) integrate (x^2 * exp(-x^2), x, minf, inf);
                             sqrt(%pi)
(%o4)                        ---------
                                 2
```

- Use of `assume` and interactive query.

```
(%i1) assume (a > 1)$
(%i2) integrate (x**a/(x+1)**(5/2), x, 0, inf);
Is a an integer?
no;

Is 2 a - 1 positive, negative or zero?
neg;
                             3
(%o2)                  beta(- - a, a + 1)
                             2
Is 2 a - 1 positive, negative or zero?
no;
                             3
(%o2)                  beta(- - a, a + 1)
                             2
(%i3) neg;
```

- Change of variable. There are two changes of variable in this example: one using a derivative established by `gradef`, and one using the derivation `diff(r(x))` of an unspecified function `r(x)`.

```
(%i1) gradef (q(x), sin(x**2));
(%o1)                          q(x)
(%i2) diff (log (q (r (x))), x);
                       d               2
                      (-- (r(x))) sin(r (x))
                       dx
(%o2)                 ----------------------
                             q(r(x))
```

```
(%i3) integrate (%, x);
(%o3)                       log(q(r(x)))
```

- Return value contains the `'integrate` noun form. In this example, Maxima can extract one factor of the denominator of a rational function, but cannot factor the remainder or otherwise find its integral. `grind` shows the noun form `'integrate` in the result. See also `integrate_use_rootsof` for more on integrals of rational functions.

```
(%i1) expand ((x-4) * (x^3+2*x+1));
                     4      3      2
(%o1)               x  - 4 x  + 2 x  - 7 x - 4
(%i2) integrate (1/%, x);
                                /  2
                                | x  + 4 x + 18
                                | ------------- dx
                                |  3
                  log(x - 4)    / x  + 2 x + 1
(%o2)             ---------- - ------------------
                      73               73
(%i3) grind (%);
log(x-4)/73-('integrate((x^2+4*x+18)/(x^3+2*x+1),x))/73$
(%o3)                          done
```

- Defining a function in terms of an integral. The body of a function is not evaluated when the function is defined. Thus the body of `f_1` in this example contains the noun form of `integrate`. The quote-quote operator `''` causes the integral to be evaluated, and the result becomes the body of `f_2`.

```
(%i1) f_1 (a) := integrate (x^3, x, 1, a);
                                      3
(%o1)           f_1(a) := integrate(x , x, 1, a)
(%i2) ev (f_1 (7), nouns);
(%o2)                          600
(%i3) /* Note parentheses around integrate(...) here */      f_2 (a) := ''(int
                                     4
                                    a    1
(%o3)                   f_2(a) := -- - -
                                    4    4
(%i4) f_2 (7);
(%o4)                          600
```

`integration_constant` [System variable]

Default value: `%c`

When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant` may be assigned any symbol.

Examples:

```
(%i1) integrate (x^2 = 1, x);
                              3
                             x
(%o1)                        -- = x + %c1
                             3
(%i2) integration_constant : 'k;
(%o2)                             k
(%i3) integrate (x^2 = 1, x);
                              3
                             x
(%o3)                        -- = x + k2
                             3
```

`integration_constant_counter` [System variable]

Default value: 0

When a constant of integration is introduced by indefinite integration of an equation, the name of the constant is constructed by concatenating `integration_constant` and `integration_constant_counter`.

`integration_constant_counter` is incremented before constructing the next integration constant.

Examples:

```
(%i1) integrate (x^2 = 1, x);
                              3
                             x
(%o1)                        -- = x + %c1
                             3
(%i2) integrate (x^2 = 1, x);
                              3
                             x
(%o2)                        -- = x + %c2
                             3
(%i3) integrate (x^2 = 1, x);
                              3
                             x
(%o3)                        -- = x + %c3
                             3
(%i4) reset (integration_constant_counter);
(%o4)             [integration_constant_counter]
(%i5) integrate (x^2 = 1, x);
                              3
                             x
(%o5)                        -- = x + %c1
                             3
```

`integrate_use_rootsof` [Option variable]

Default value: `false`

When `integrate_use_rootsof` is `true` and the denominator of a rational function cannot be factored, `integrate` returns the integral in a form which is a sum over the roots (not yet known) of the denominator.

For example, with `integrate_use_rootsof` set to `false`, `integrate` returns an unsolved integral of a rational function in noun form:

```
(%i1) integrate_use_rootsof: false$
(%i2) integrate (1/(1+x+x^5), x);
      /  2
      | x  - 4 x + 5
      | ------------ dx                              2 x + 1
      |  3    2                2             5 atan(-------)
      / x  - x  + 1       log(x  + x + 1)          sqrt(3)
(%o2) ----------------- - --------------- + ---------------
             7                  14              7 sqrt(3)
```

Now we set the flag to be true and the unsolved part of the integral will be expressed as a summation over the roots of the denominator of the rational function:

```
(%i1) integrate_use_rootsof: true$
(%i2) integrate (1/(1+x+x^5), x);
       ----
       \                                        2
(%o2)  ( >                                   ((%r1  - 4 %r1 + 5)
       /
       ----
                          3      2
       %r1 in rootsof(%r1  - %r1  + 1, %r1)
                                            2
                        2                log(x  + x + 1)
 log(x - %r1))/(3 %r1  - 2 %r1))/7 - ---------------
                                           14
          2 x + 1
   5 atan(-------)
          sqrt(3)
 + ---------------
       7 sqrt(3)
```

Alternatively the user may compute the roots of the denominator separately, and then express the integrand in terms of these roots, e.g., `1/((x - a)*(x - b)*(x - c))` or `1/((x^2 - (a+b)*x + a*b)*(x - c))` if the denominator is a cubic polynomial. Sometimes this will help Maxima obtain a more useful result.

`laplace` (*expr*, *t*, *s*) [Function]

Attempts to compute the Laplace transform of *expr* with respect to the variable t and transform parameter s. The Laplace transform of the function `f(t)` is the one-sided transform defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

where $F(s)$ is the transform of $f(t)$, represented by *expr*.

`laplace` recognizes in *expr* the functions `delta`, `exp`, `log`, `sin`, `cos`, `sinh`, `cosh`, and `erf`, as well as `derivative`, `integrate`, `sum`, and `ilt`. If `laplace` fails to find a transform the function `specint` is called. `specint` can find the laplace transform for expressions with special functions like the bessel functions `bessel_j`, `bessel_i`, ... and can handle the `unit_step` function. See also `specint`.

If `specint` cannot find a solution too, a noun `laplace` is returned.

expr may also be a linear, constant coefficient differential equation in which case `atvalue` of the dependent variable is used. The required atvalue may be supplied either before or after the transform is computed. Since the initial conditions must be specified at zero, if one has boundary conditions imposed elsewhere he can impose these on the general solution and eliminate the constants by solving the general solution for them and substituting their values back.

`laplace` recognizes convolution integrals of the form

$$\int_0^t f(x)g(t-x)dx$$

Other kinds of convolutions are not recognized.

Functional relations must be explicitly represented in *expr*; implicit relations, established by `depends`, are not recognized. That is, if f depends on x and y, $f(x,y)$ must appear in *expr*.

See also `ilt`, the inverse Laplace transform.

Examples:

```
(%i1) laplace (exp (2*t + a) * sin(t) * t, t, s);
                              a
                           %e  (2 s - 4)
(%o1)             ------------------------------
                   4      3       2
                  s  - 8 s  + 26 s  - 40 s + 25
(%i2) laplace ('diff (f (x), x), x, s);
(%o2)             s laplace(f(x), x, s) - f(0)
(%i3) diff (diff (delta (t), t), t);
                           2
                          d
(%o3)                     --- (delta(t))
                            2
                          dt
(%i4) laplace (%, t, s);
                               |
                d              |         2
(%o4)         - -- (delta(t))|      + s  - delta(0) s
                dt             |
                               |t = 0
(%i5) assume(a>1)$
(%i6) declare(a, integer)$
```

```
(%i7) laplace(gamma_incomplete(a,t),t,s),gamma_expand:true;
                                              - a - 1
                    gamma(a)   gamma(a) s
(%o7)               -------- - -----------------
                       s                1     a
                                       (- + 1)
                                        s
(%i8) factor(laplace(gamma_incomplete(1/2,t),t,s));
                                           s + 1
                 sqrt(%pi) (sqrt(s) sqrt(-----) - 1)
                                            s
(%o8)            ----------------------------------
                           3/2      s + 1
                          s    sqrt(-----)
                                      s
(%i9) assume(exp(%pi*s)>1, n > 0)$
(%i10) laplace(sum((-1)^n*unit_step(t-n*%pi)*sin(t),n,0,inf),t,s),
  simpsum;
                                 %pi s
                               %e
(%o10)             ------------------------------
                      %pi s       2     %pi s
                   (%e      - 1) s  + %e      - 1
```

`ldefint` (*expr*, *x*, *a*, *b*) [Function]

Attempts to compute the definite integral of *expr* by using `limit` to evaluate the indefinite integral of *expr* with respect to *x* at the upper limit *b* and at the lower limit *a*. If it fails to compute the definite integral, `ldefint` returns an expression containing limits as noun forms.

`ldefint` is not called from `integrate`, so executing `ldefint (`*expr*`, `*x*`, a, `*b*`)` may yield a different result than `integrate (`*expr*`, `*x*`, a, `*b*`)`. `ldefint` always uses the same method to evaluate the definite integral, while `integrate` may employ various heuristics and may recognize some special cases.

`pwilt` (*expr*, *s*, *t*) [Function]

Computes the inverse Laplace transform of *expr* with respect to *s* and parameter *t*. Unlike `ilt`, `pwilt` is able to return piece-wise and periodic functions and can also handle some cases with polynomials of degree greater than 3 in the denominator.

Two examples where `ilt` fails:

```
(%i1) pwilt (exp(-s)*s/(s^3-2*s-s+2), s, t);
                         t - 1                - 2 (t - 1)
                      %e      (t - 1)   2 %e
(%o1)   hstep(t - 1) (--------------- - ---------------)
                             3                 9
```

```
(%i2) pwilt ((s^2+2)/(s^2-1), s, t);
                                      t       - t
                                   3 %e    3 %e
(%o2)                delta(t) + ----- - -------
                                  2        2
```

potential (*givengradient*) [Function]

The calculation makes use of the global variable `potentialzeroloc[0]` which must be `nonlist` or of the form

`[indeterminatej=expressionj, indeterminatek=expressionk, ...]`

the former being equivalent to the nonlist expression for all right-hand sides in the latter. The indicated right-hand sides are used as the lower limit of integration. The success of the integrations may depend upon their values and order. `potentialzeroloc` is initially set to 0.

`prefer_d` [Option variable]

Default value: `false`

When `prefer_d` is `true`, `specint` will prefer to express solutions using `parabolic_cylinder_d` rather than hypergeometric functions.

In the example below, the solution contains `parabolic_cylinder_d` when `prefer_d` is `true`.

```
(%i1) assume(s>0);
(%o1)                          [s > 0]
(%i2) factor(ex:specint(%e^-(t^2/8)*exp(-s*t),t));
                           2
                        2 s
(%o2)    - sqrt(2) %e     sqrt(%pi) (erf(sqrt(2) s) - 1)
(%i3) specint(ex,t),prefer_d=true;
                              2
                           2 s
(%o3) specint(- sqrt(2) %e     sqrt(%pi) erf(sqrt(2) s), t)
                                                  2
                                               2 s
                             + specint(sqrt(2) %e     sqrt(%pi), t)
```

residue (*expr*, *z*, *z_0*) [Function]

Computes the residue in the complex plane of the expression *expr* when the variable *z* assumes the value *z_0*. The residue is the coefficient of `(z - z_0)^(-1)` in the Laurent series for *expr*.

```
(%i1) residue (s/(s**2+a**2), s, a*%i);
                                 1
(%o1)                            -
                                 2
(%i2) residue (sin(a*x)/x**4, x, 0);
                                  3
                                 a
(%o2)                          - --
                                 6
```

risch (*expr*, *x*) [Function]

Integrates *expr* with respect to *x* using the transcendental case of the Risch algorithm. (The algebraic case of the Risch algorithm has not been implemented.) This currently handles the cases of nested exponentials and logarithms which the main part of `integrate` can't do. `integrate` will automatically apply `risch` if given these cases.

`erfflag`, if `false`, prevents `risch` from introducing the `erf` function in the answer if there were none in the integrand to begin with.

```
(%i1) risch (x^2*erf(x), x);
                                2
              3             - x              2
        %pi x  erf(x) + %e     (sqrt(%pi) x  + sqrt(%pi))
(%o1)   ------------------------------------------------
                                3 %pi
(%i2) diff(%, x), ratsimp;
                                2
(%o2)                         x  erf(x)
```

specint (*exp*(- *s***t*) * *expr*, *t*) [Function]

Compute the Laplace transform of *expr* with respect to the variable *t*. The integrand *expr* may contain special functions. The parameter *s* maybe be named something else; it is determined automatically, as the examples below show where *p* is used in some places.

The following special functions are handled by `specint`: incomplete gamma function, error functions (but not the error function `erfi`, it is easy to transform `erfi` e.g. to the error function `erf`), exponential integrals, bessel functions (including products of bessel functions), hankel functions, hermite and the laguerre polynomials.

Furthermore, `specint` can handle the hypergeometric function `%f[p,q]([],[],z)`, the Whittaker function of the first kind `%m[u,k](z)` and of the second kind `%w[u,k](z)`.

The result may be in terms of special functions and can include unsimplified hypergeometric functions. If variable `prefer_d` is `true` then the `parabolic_cylinder_d` function may be used in the result in preference to hypergeometric functions.

When `laplace` fails to find a Laplace transform, `specint` is called. Because `laplace` knows more general rules for Laplace transforms, it is preferable to use `laplace` and not `specint`.

`demo("hypgeo")` displays several examples of Laplace transforms computed by `specint`.

Examples:

```
(%i1) assume (p > 0, a > 0)$
(%i2) specint (t^(1/2) * exp(-a*t/4) * exp(-p*t), t);
                                sqrt(%pi)
(%o2)                         ------------
                                     a 3/2
                              2 (p + -)
                                     4
```

```
(%i3) specint (t^(1/2) * bessel_j(1, 2 * a^(1/2) * t^(1/2))
              * exp(-p*t), t);
                          - a/p
                        %e      sqrt(a)
(%o3)                   ---------------
                              2
                             p
```

Examples for exponential integrals:

```
(%i1) assume(s>0,a>0,s-a>0)$
(%i2) ratsimp(specint(%e^(a*t)*(log(a)+expintegral_e1(a*t))*%e^(-s*t),t));
                             log(s)
(%o2)                        ------
                             s - a
(%i3) logarc:true$
(%i4) gamma_expand:true$
(%i5) radcan(specint((cos(t)*expintegral_si(t) -sin(t)*expintegral_ci(t))*%e^(-s*
                             log(s)
(%o5)                        ------
                              2
                             s  + 1
(%i6) ratsimp(specint((2*t*log(a)+2/a*sin(a*t) -2*t*expintegral_ci(a*t))*%e^(-s*t
                               2    2
                          log(s  + a )
(%o6)                     ------------
                               2
                              s
```

Results when using the expansion of `gamma_incomplete` and when changing the representation to `expintegral_e1`:

```
(%i1) assume(s>0)$
(%i2) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
                                    1
                  gamma_incomplete(-, k s)
                                    2
(%o2)             ------------------------
                     sqrt(%pi) sqrt(s)
(%i3) gamma_expand:true$
(%i4) specint(1/sqrt(%pi*t)*unit_step(t-k)*%e^(-s*t),t);
                    erfc(sqrt(k) sqrt(s))
(%o4)               ---------------------
                             sqrt(s)
(%i5) expintrep:expintegral_e1$
(%i6) ratsimp(specint(1/(t+a)^2*%e^(-s*t),t));
                      a s
                a s %e    expintegral_e1(a s) - 1
(%o6)         - ---------------------------------
                              a
```

`tldefint` (*expr*, *x*, *a*, *b*) [Function]

Equivalent to `ldefint` with `tlimswitch` set to `true`.

21.3 Introduction to QUADPACK

QUADPACK is a collection of functions for the numerical computation of one-dimensional definite integrals. It originated from a joint project of R. Piessens[1], E. de Doncker[2], C. Ueberhuber[3], and D. Kahaner[4].

The QUADPACK library included in Maxima is an automatic translation (via the program `f2cl`) of the Fortran source code of QUADPACK as it appears in the SLATEC Common Mathematical Library, Version 4.1[5]. The SLATEC library is dated July 1993, but the QUADPACK functions were written some years before. There is another version of QUADPACK at Netlib[6]; it is not clear how that version differs from the SLATEC version.

The QUADPACK functions included in Maxima are all automatic, in the sense that these functions attempt to compute a result to a specified accuracy, requiring an unspecified number of function evaluations. Maxima's Lisp translation of QUADPACK also includes some non-automatic functions, but they are not exposed at the Maxima level.

Further information about QUADPACK can be found in the QUADPACK book[7].

21.3.1 Overview

`quad_qag` Integration of a general function over a finite interval. `quad_qag` implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands.

`quad_qags`

Integration of a general function over a finite interval. `quad_qags` implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

`quad_qagi`

Integration of a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in `quad_qags` is applied.

`quad_qawo`

Integration of $\cos(\omega x)f(x)$ or $\sin(\omega x)f(x)$ over a finite interval, where ω is a constant. The rule evaluation component is based on the modified Clenshaw-

[1] Applied Mathematics and Programming Division, K.U. Leuven

[2] Applied Mathematics and Programming Division, K.U. Leuven

[3] Institut für Mathematik, T.U. Wien

[4] National Bureau of Standards, Washington, D.C., U.S.A

[5] https://www.netlib.org/slatec

[6] https://www.netlib.org/quadpack

[7] R. Piessens, E. de Doncker-Kapenga, C.W. Uberhuber, and D.K. Kahaner. *QUADPACK: A Subroutine Package for Automatic Integration.* Berlin: Springer-Verlag, 1983, ISBN 0387125531.

Curtis technique. quad_qawo applies adaptive subdivision with extrapolation, similar to quad_qags.

quad_qawf

Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval. The same approach as in quad_qawo is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

quad_qaws

Integration of $w(x)f(x)$ over a finite interval $[a, b]$, where w is a function of the form $(x-a)^{\alpha}(b-x)^{\beta}v(x)$ and $v(x)$ is 1 or $\log(x-a)$ or $\log(b-x)$ or $\log(x-a)\log(b-x)$, and $\alpha > -1$ and $\beta > -1$.

A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain a or b.

quad_qawc

Computes the Cauchy principal value of $f(x)/(x-c)$ over a finite interval (a, b) and specified c. The strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on the subranges which contain the point $x = c$.

quad_qagp

Basically the same as quad_qags but points of singularity or discontinuity of the integrand must be supplied. This makes it easier for the integrator to produce a good solution.

21.4 Functions and Variables for QUADPACK

quad_qag [Function]

quad_qag (*f*(*x*), *x*, *a*, *b*, *key*, [*epsrel*, *epsabs*, *limit*])
quad_qag (*f*, *x*, *a*, *b*, *key*, [*epsrel*, *epsabs*, *limit*])

Integration of a general function over a finite interval. quad_qag implements a simple globally adaptive integrator using the strategy of Aind (Piessens, 1973). The caller may choose among 6 pairs of Gauss-Kronrod quadrature formulae for the rule evaluation component. The high-degree rules are suitable for strongly oscillating integrands.

quad_qag computes the integral

$$\int_a^b f(x)\,dx$$

The function to be integrated is $f(x)$, with dependent variable x, and the function is to be integrated between the limits a and b. *key* is the integrator to be used and should be an integer between 1 and 6, inclusive. The value of *key* selects the order of the Gauss-Kronrod integration rule. High-order rules are suitable for strongly oscillating integrands.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The numerical integration is done adaptively by subdividing the integration region into sub-intervals until the desired accuracy is achieved.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit* is the maximum number of subintervals to use. Default is 200.

`quad_qag` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` if no problems were encountered;

`1` if too many sub-intervals were done;

`2` if excessive roundoff error is detected;

`3` if extremely bad integrand behavior occurs;

`6` if the input is invalid.

Examples:

```
(%i1) quad_qag (x^(1/2)*log(1/x), x, 0, 1, 3, 'epsrel=5d-8);
(%o1)  [0.44444444445742953, 8.737223570614865e-9, 899, 0]
(%i2) integrate (x^(1/2)*log(1/x), x, 0, 1);
                                4
(%o2)                           -
                                9
```

`quad_qags` [Function]

`quad_qags` (*f*(*x*), *x*, *a*, *b*, [*epsrel*, *epsabs*, *limit*])
`quad_qags` (*f*, *x*, *a*, *b*, [*epsrel*, *epsabs*, *limit*])

Integration of a general function over a finite interval. `quad_qags` implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

`quad_qags` computes the integral

$$\int_a^b f(x)\,dx$$

The function to be integrated is $f(x)$, with dependent variable x, and the function is to be integrated between the limits a and b.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit* is the maximum number of subintervals to use. Default is 200.

`quad_qags` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

`1` too many sub-intervals were done;

`2` excessive roundoff error is detected;

`3` extremely bad integrand behavior occurs;

`4` failed to converge

`5` integral is probably divergent or slowly convergent

`6` if the input is invalid.

Examples:

```
(%i1) quad_qags (x^(1/2)*log(1/x), x, 0, 1, 'epsrel=1d-10);
(%o1) [0.4444444444444475, 1.1102230246251565e-15, 315, 0]
```

Note that `quad_qags` is more accurate and efficient than `quad_qag` for this integrand.

`quad_qagi` [Function]

`quad_qagi (`*f*`(`*x*`), `*x*`, `*a*`, `*b*`, [`*epsrel*`, `*epsabs*`, `*limit*`])`
`quad_qagi (`*f*`, `*x*`, `*a*`, `*b*`, [`*epsrel*`, `*epsabs*`, `*limit*`])`

Integration of a general function over an infinite or semi-infinite interval. The interval is mapped onto a finite interval and then the same strategy as in `quad_qags` is applied.

`quad_qagi` evaluates one of the following integrals

$$\int_a^\infty f(x)\,dx$$

$$\int_\infty^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx$$

using the Quadpack QAGI routine. The function to be integrated is $f(x)$, with dependent variable x, and the function is to be integrated over an infinite range.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

One of the limits of integration must be infinity. If not, then `quad_qagi` will just return the noun form.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit* is the maximum number of subintervals to use. Default is 200.

`quad_qagi` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

`1` too many sub-intervals were done;

`2` excessive roundoff error is detected;

`3` extremely bad integrand behavior occurs;

`4` failed to converge

`5` integral is probably divergent or slowly convergent

`6` if the input is invalid.

Examples:

```
(%i1) quad_qagi (x^2*exp(-4*x), x, 0, inf, 'epsrel=1d-8);
(%o1)       [0.03125, 2.9591610299500215e-11, 105, 0]
(%i2) integrate (x^2*exp(-4*x), x, 0, inf);
                                  1
(%o2)                             --
                                  32
```

`quad_qawc` [Function]

`quad_qawc` (*f*(*x*), *x*, *c*, *a*, *b*, [*epsrel*, *epsabs*, *limit*])
`quad_qawc` (*f*, *x*, *c*, *a*, *b*, [*epsrel*, *epsabs*, *limit*])

Computes the Cauchy principal value of $f(x)/(x-c)$ over a finite interval. The strategy is globally adaptive, and modified Clenshaw-Curtis integration is used on the subranges which contain the point $x = c$.

`quad_qawc` computes the Cauchy principal value of

$$\int_a^b \frac{f\left(x\right)}{x-c}\,dx$$

using the Quadpack QAWC routine. The function to be integrated is $f(x)/(x-c)$, with dependent variable x, and the function is to be integrated over the interval a to b.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit* is the maximum number of subintervals to use. Default is 200.

`quad_qawc` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

`1` too many sub-intervals were done;

`2` excessive roundoff error is detected;

`3` extremely bad integrand behavior occurs;

`6` if the input is invalid.

Examples:

```
(%i1) quad_qawc (2^(-5)*((x-1)^2+4^(-5))^(-1), x, 2, 0, 5, 'epsrel=1d-7);
(%o1) [- 3.130120337415925, 1.3068301402495579e-8, 495, 0]
```

```
(%i2) integrate (2^(-alpha)*(((x-1)^2 + 4^(-alpha))*(x-2))^(-1), x, 0, 5);
Principal Value
          2 alpha - 1      2 alpha + 4
         2            log(2            + 1)
(%o2) (- ----------------------------------
                     2 alpha
                    2        + 1
    3 alpha       alpha + 2     2 alpha - 1      2 alpha
   2        atan(2         )   2            log(2        + 1)
 - ------------------------ + ------------------------------
          2 alpha                          2 alpha
         2        + 1                    2        + 1
    3 alpha + 1       alpha     3 alpha       alpha
   2            atan(2     )   2        atan(2     )
 - ------------------------ + ---------------------
          2 alpha                    2 alpha
         2        + 1               2        + 1
    2 alpha           2 alpha
   2        log(3)   2        log(2)   alpha
 + --------------- - ---------------)/2
     2 alpha           2 alpha
    2        + 1      2        + 1
(%i3) ev (%, alpha=5, numer);
(%o3)                - 3.1301203374159177
```

quad_qawf [Function]

quad_qawf (*f*(*x*), *x*, a, *omega*, *trig*, [*epsabs*, *limit*, *maxp1*, *limlst*])
quad_qawf (*f*, *x*, a, *omega*, *trig*, [*epsabs*, *limit*, *maxp1*, *limlst*])

Calculates a Fourier cosine or Fourier sine transform on a semi-infinite interval using the Quadpack QAWF function. The same approach as in quad_qawo is applied on successive finite intervals, and convergence acceleration by means of the Epsilon algorithm (Wynn, 1956) is applied to the series of the integral contributions.

quad_qawf computes the integral

$$\int_a^\infty f(x)\, w(x)\, dx$$

The weight function w is selected by *trig*:

cos $w(x) = \cos \omega x$

sin $w(x) = \sin \omega x$

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form key=val. The keyword arguments are:

epsabs Desired absolute error of approximation. Default is 1d-10.

`limit` Size of internal work array. (*limit* - *limlst*)/2 is the maximum number of subintervals to use. Default is 200.

`maxp1` Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.

`limlst` Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

`quad_qawf` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

`1` too many sub-intervals were done;

`2` excessive roundoff error is detected;

`3` extremely bad integrand behavior occurs;

`6` if the input is invalid.

Examples:

```
(%i1) quad_qawf (exp(-x^2), x, 0, 1, 'cos, 'epsabs=1d-9);
(%o1)  [0.6901942235215714, 2.848463002545743e-11, 215, 0]
(%i2) integrate (exp(-x^2)*cos(x), x, 0, inf);
                            - 1/4
                          %e      sqrt(%pi)
(%o2)                     -----------------
                                  2
(%i3) ev (%, numer);
(%o3)                   0.6901942235215714
```

`quad_qawo` [Function]

`quad_qawo` (*f*(*x*), *x*, *a*, *b*, *omega*, *trig*, [*epsrel*, *epsabs*, *limit*, *maxp1*, *limlst*])
`quad_qawo` (*f*, *x*, *a*, *b*, *omega*, *trig*, [*epsrel*, *epsabs*, *limit*, *maxp1*, *limlst*])

Integration of $\cos(\omega x)f(x)$ or $\sin(\omega x)$ over a finite interval, where ω is a constant. The rule evaluation component is based on the modified Clenshaw-Curtis technique. `quad_qawo` applies adaptive subdivision with extrapolation, similar to `quad_qags`.

`quad_qawo` computes the integral using the Quadpack QAWO routine:

$$\int_a^b f(x)\, w(x)\, dx$$

The weight function w is selected by *trig*:

`cos` $w(x) = \cos \omega x$

`sin` $w(x) = \sin \omega x$

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit*/2 is the maximum number of subintervals to use. Default is 200.

`maxp1` Maximum number of Chebyshev moments. Must be greater than 0. Default is 100.

`limlst` Upper bound on the number of cycles. Must be greater than or equal to 3. Default is 10.

`quad_qawo` returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

`1` too many sub-intervals were done;

`2` excessive roundoff error is detected;

`3` extremely bad integrand behavior occurs;

`6` if the input is invalid.

Examples:

```
(%i1) quad_qawo (x^(-1/2)*exp(-2^(-2)*x), x, 1d-8, 20*2^2, 1, cos);
(%o1) [1.3760433898776214, 4.7271075942489915e-11, 765, 0]
(%i2) rectform (integrate (x^(-1/2)*exp(-2^(-alpha)*x) * cos(x), x, 0, inf));
         alpha + 1
         --------- - 1
             2                    2 alpha
        2              sqrt(sqrt(2        + 1) %pi + %pi)
(%o2)   ---------------------------------------------------
                              2 alpha
                            sqrt(2        + 1)
(%i3) ev (%, alpha=2, numer);
(%o3)                   1.376043390090716
(%i4) ev (%, alpha=2, numer);
```

quad_qaws [Function]

quad_qaws (*f*(*x*), *x*, a, *b*, *alpha*, *beta*, *wfun*, [*epsrel*, *epsabs*, *limit*])
quad_qaws (*f*, *x*, a, *b*, *alpha*, *beta*, *wfun*, [*epsrel*, *epsabs*, *limit*])

Integration of $w(x)f(x)$ over a finite interval, where $w(x)$ is a certain algebraic or logarithmic function. A globally adaptive subdivision strategy is applied, with modified Clenshaw-Curtis integration on the subintervals which contain the endpoints of the interval of integration.

quad_qaws computes the integral using the Quadpack QAWS routine:

$$\int_a^b f(x)\, w(x)\, dx$$

The weight function w is selected by *wfun*:

1 $w(x) = (x-a)^\alpha (b-x)^\beta$

2 $w(x) = (x-a)^\alpha (b-x)^\beta \log(x-a)$

3 $w(x) = (x-a)^\alpha (b-x)^\beta \log(b-x)$

4 $w(x) = (x-a)^\alpha (b-x)^\beta \log(x-a) \log(b-x)$

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

The keyword arguments are optional and may be specified in any order. They all take the form key=val. The keyword arguments are:

epsrel Desired relative error of approximation. Default is 1d-8.

epsabs Desired absolute error of approximation. Default is 0.

limit Size of internal work array. *limit*is the maximum number of subintervals to use. Default is 200.

quad_qaws returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

0 no problems were encountered;

1 too many sub-intervals were done;

2 excessive roundoff error is detected;

3 extremely bad integrand behavior occurs;

6 if the input is invalid.

Examples:

```
(%i1) quad_qaws (1/(x+1+2^(-4)), x, -1, 1, -0.5, -0.5, 1, 'epsabs=1d-9);
(%o1)  [8.750097361672843, 1.2761903591126173e-8, 130, 0]
(%i2) integrate ((1-x*x)^(-1/2)/(x+1+2^(-alpha)), x, -1, 1);
                                alpha
                               2      %pi
(%o2)                    --------------------
                                 alpha + 1
                         sqrt(2          + 1)
(%i3) ev (%, alpha=4, numer);
(%o3)                     8.75009736167283
```

quad_qagp [Function]

quad_qagp (*f*(*x*), *x*, *a*, *b*, *points*, [*epsrel*, *epsabs*, *limit*])
quad_qagp (*f*, *x*, *a*, *b*, *points*, [*epsrel*, *epsabs*, *limit*])

Integration of a general function over a finite interval. quad_qagp implements globally adaptive interval subdivision with extrapolation (de Doncker, 1978) by the Epsilon algorithm (Wynn, 1956).

quad_qagp computes the integral

$$\int_a^b f(x)\,dx$$

The function to be integrated is $f(x)$, with dependent variable x, and the function is to be integrated between the limits a and b.

The integrand may be specified as the name of a Maxima or Lisp function or operator, a Maxima lambda expression, or a general Maxima expression.

To help the integrator, the user must supply a list of points where the integrand is singular or discontinuous. The list is provided by *points*. It may be an empty list. The elements of the list must be between *a* and *b*, exclusive. An error is thrown if there are elements out of range. The list points may be in any order.

The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`epsrel` Desired relative error of approximation. Default is 1d-8.

`epsabs` Desired absolute error of approximation. Default is 0.

`limit` Size of internal work array. *limit* is the maximum number of subintervals to use. Default is 200.

quad_qagp returns a list of four elements:

- an approximation to the integral,
- the estimated absolute error of the approximation,
- the number integrand evaluations,
- an error code.

The error code (fourth element of the return value) can have the values:

`0` no problems were encountered;

1	too many sub-intervals were done;
2	excessive roundoff error is detected;
3	extremely bad integrand behavior occurs;
4	failed to converge
5	integral is probably divergent or slowly convergent
6	if the input is invalid.

Examples:

```
(%i1) quad_qagp(x^3*log(abs((x^2-1)*(x^2-2))),x,0,3,[1,sqrt(2)]);
(%o1) [52.740748383471434, 2.6247632689546663e-7, 1029, 0]
(%i2) quad_qags(x^3*log(abs((x^2-1)*(x^2-2))), x, 0, 3);
(%o2)  [52.74074847951494, 4.088443219529836e-7, 1869, 0]
```

The integrand has singularities at `1` and `sqrt(2)` so we supply these points to `quad_qagp`. We also note that `quad_qagp` is more accurate and more efficient that `quad_qags`.

`quad_control` (*parameter*, [*value*]) [Function]

Control error handling for quadpack. The parameter should be one of the following symbols:

`current_error`
: The current error number

`control`
: Controls if messages are printed or not. If it is set to zero or less, messages are suppressed.

`max_message`
: The maximum number of times any message is to be printed.

If *value* is not given, then the current value of the *parameter* is returned. If *value* is given, the value of *parameter* is set to the given value.

22 Equations

22.1 Functions and Variables for Equations

`%rnum` [System variable]

Default value: `0`

`%rnum` is the counter for the `%r` variables introduced in solutions by `solve` and `algsys`.. The next `%r` variable is numbered `%rnum+1`.

See also `%rnum_list`.

`%rnum_list` [System variable]

Default value: `[]`

`%rnum_list` is the list of variables introduced in solutions by `solve` and `algsys`. `%r` variables are added to `%rnum_list` in the order they are created. This is convenient for doing substitutions into the solution later on.

See also `%rnum`.

It's recommended to use this list rather than doing `concat ('%r, j)`.

```
(%i1) solve ([x + y = 3], [x,y]);
(%o1)                [[x = 3 - %r1, y = %r1]]
(%i2) %rnum_list;
(%o2)                            [%r1]
(%i3) sol : solve ([x + 2*y + 3*z = 4], [x,y,z]);
(%o3)     [[x = - 2 %r3 - 3 %r2 + 4, y = %r3, z = %r2]]
(%i4) %rnum_list;
(%o4)                         [%r2, %r3]
(%i5) for i : 1 thru length (%rnum_list) do
  sol : subst (t[i], %rnum_list[i], sol)$
(%i6) sol;
(%o6)       [[x = - 2 t  - 3 t  + 4, y = t , z = t ]]
                        2      1           2       1
```

`algepsilon` [Option variable]

Default value: 10^8

`algepsilon` is used by `algsys`.

`algexact` [Option variable]

Default value: `false`

`algexact` affects the behavior of `algsys` as follows:

If `algexact` is `true`, `algsys` always calls `solve` and then uses `realroots` on `solve`'s failures.

If `algexact` is `false`, `solve` is called only if the eliminant was not univariate, or if it was a quadratic or biquadratic.

Thus `algexact: true` does not guarantee only exact solutions, just that `algsys` will first try as hard as it can to give exact solutions, and only yield approximations when all else fails.

algsys [Function]

algsys ([*expr_1*, ..., *expr_m*], [*x_1*, ..., *x_n*])
algsys ([*eqn_1*, ..., *eqn_m*], [*x_1*, ..., *x_n*])

Solves the simultaneous polynomials *expr_1*, ..., *expr_m* or polynomial equations *eqn_1*, ..., *eqn_m* for the variables *x_1*, ..., *x_n*. An expression *expr* is equivalent to an equation *expr* = 0. There may be more equations than variables or vice versa.

`algsys` returns a list of solutions, with each solution given as a list of equations stating values of the variables *x_1*, ..., *x_n* which satisfy the system of equations. If `algsys` cannot find a solution, an empty list `[]` is returned.

The symbols `%r1`, `%r2`, ..., are introduced as needed to represent arbitrary parameters in the solution; these variables are also appended to the list `%rnum_list`.

The method is as follows:

1. First the equations are factored and split into subsystems.
2. For each subsystem *S_i*, an equation *E* and a variable *x* are selected. The variable is chosen to have lowest nonzero degree. Then the resultant of *E* and *E_j* with respect to *x* is computed for each of the remaining equations *E_j* in the subsystem *S_i*. This yields a new subsystem *S_i'* in one fewer variables, as *x* has been eliminated. The process now returns to (1).
3. Eventually, a subsystem consisting of a single equation is obtained. If the equation is multivariate and no approximations in the form of floating point numbers have been introduced, then `solve` is called to find an exact solution.

   In some cases, `solve` is not be able to find a solution, or if it does the solution may be a very large expression.

   If the equation is univariate and is either linear, quadratic, or biquadratic, then again `solve` is called if no approximations have been introduced. If approximations have been introduced or the equation is not univariate and neither linear, quadratic, or biquadratic, then if the switch `realonly` is `true`, the function `realroots` is called to find the real-valued solutions. If `realonly` is `false`, then `allroots` is called which looks for real and complex-valued solutions.

   If `algsys` produces a solution which has fewer significant digits than required, the user can change the value of `algepsilon` to a higher value.

   If `algexact` is set to `true`, `solve` will always be called.
4. Finally, the solutions obtained in step (3) are substituted into previous levels and the solution process returns to (1).

When `algsys` encounters a multivariate equation which contains floating point approximations (usually due to its failing to find exact solutions at an earlier stage), then it does not attempt to apply exact methods to such equations and instead prints the message: "`algsys` cannot solve - system too complicated."

Interactions with `radcan` can produce large or complicated expressions. In that case, it may be possible to isolate parts of the result with `pickapart` or `reveal`.

Occasionally, `radcan` may introduce an imaginary unit `%i` into a solution which is actually real-valued.

Examples:

```
(%i1) e1: 2*x*(1 - a1) - 2*(x - 1)*a2;
(%o1)              2 (1 - a1) x - 2 a2 (x - 1)
(%i2) e2: a2 - a1;
(%o2)                          a2 - a1
(%i3) e3: a1*(-y - x^2 + 1);
                                        2
(%o3)                     a1 (- y - x  + 1)
(%i4) e4: a2*(y - (x - 1)^2);
                                           2
(%o4)                     a2 (y - (x - 1) )
(%i5) algsys ([e1, e2, e3, e4], [x, y, a1, a2]);
(%o5) [[x = 0, y = %r1, a1 = 0, a2 = 0],
                                    [x = 1, y = 0, a1 = 1, a2 = 1]]
(%i6) e1: x^2 - y^2;
                                 2    2
(%o6)                           x  - y
(%i7) e2: -1 - y + 2*y^2 - x + x^2;
                             2        2
(%o7)                   2 y  - y + x  - x - 1
(%i8) algsys ([e1, e2], [x, y]);
                    1            1
(%o8) [[x = - -------, y = -------],
                sqrt(3)      sqrt(3)
          1               1             1         1
[x = -------, y = - -------], [x = - -, y = - -], [x = 1, y = 1]]
     sqrt(3)         sqrt(3)          3         3
```

`allroots` [Function]

`allroots` (*expr*)
`allroots` (*eqn*)

Computes numerical approximations of the real and complex roots of the polynomial *expr* or polynomial equation *eqn* of one variable.

The flag `polyfactor` when `true` causes `allroots` to factor the polynomial over the real numbers if the polynomial is real, or over the complex numbers, if the polynomial is complex.

`allroots` may give inaccurate results in case of multiple roots. If the polynomial is real, `allroots (%i*p)` may yield more accurate approximations than `allroots (p)`, as `allroots` invokes a different algorithm in that case.

`allroots` rejects non-polynomials. It requires that the numerator after `rat`'ing should be a polynomial, and it requires that the denominator be at most a complex number. As a result of this `allroots` will always return an equivalent (but factored) expression, if `polyfactor` is `true`.

For complex polynomials an algorithm by Jenkins and Traub is used (Algorithm 419, *Comm. ACM*, vol. 15, (1972), p. 97). For real polynomials the algorithm used is due to Jenkins (Algorithm 493, *ACM TOMS*, vol. 1, (1975), p.178).

Examples:

```
(%i1) eqn: (1 + 2*x)^3 = 13.5*(1 + x^5);
                               3          5
(%o1)                (2 x + 1)  = 13.5 (x  + 1)
(%i2) soln: allroots (eqn);
(%o2) [x = 0.8296749902129361, x = - 1.0157555438281212,
x = 0.9659625152196369 %i - 0.4069597231924075,
x = - 0.9659625152196369 %i - 0.4069597231924075, x = 1.0]
(%i3) for e in soln
        do (e2: subst (e, eqn), disp (expand (lhs(e2) - rhs(e2))));
                     - 3.552713678800501e-15

                     - 8.43769498715119e-15

          2.6645352591003757e-15 %i - 6.217248937900877e-15

         - 2.6645352591003757e-15 %i - 6.217248937900877e-15

                                0.0

(%o3)                          done
(%i4) polyfactor: true$
(%i5) allroots (eqn);
(%o5) - 13.5 (x - 1.0) (x - 0.8296749902129361)
                               2
 (x + 1.0157555438281212) (x  + 0.813919446384815 x
 + 1.0986997971102883)
```

`bfallroots` [Function]

`bfallroots` (*expr*)
`bfallroots` (*eqn*)

Computes numerical approximations of the real and complex roots of the polynomial *expr* or polynomial equation *eqn* of one variable.

In all respects, `bfallroots` is identical to `allroots` except that `bfallroots` computes the roots using bigfloats. See `allroots` for more information.

`backsubst` [Option variable]

Default value: `true`

When `backsubst` is `false`, prevents back substitution in `linsolve` after the equations have been triangularized. This may be helpful in very big problems where back substitution would cause the generation of extremely large expressions.

```
(%i1) eq1 : x + y + z = 6$
(%i2) eq2 : x - y + z = 2$
(%i3) eq3 : x + y - z = 0$
(%i4) backsubst : false$
(%i5) linsolve ([eq1, eq2, eq3], [x,y,z]);
(%o5)                [x = z - y, y = 2, z = 3]
```

```
(%i6) backsubst : true$
(%i7) linsolve ([eq1, eq2, eq3], [x,y,z]);
(%o7)                 [x = 1, y = 2, z = 3]
```

`breakup` [Option variable]

Default value: `true`

When `breakup` is `true`, `solve` expresses solutions of cubic and quartic equations in terms of common subexpressions, which are assigned to intermediate expression labels (`%t1`, `%t2`, etc.). Otherwise, common subexpressions are not identified.

`breakup: true` has an effect only when `programmode` is `false`.

Examples:

```
(%i1) programmode: false$
(%i2) breakup: true$
(%i3) solve (x^3 + x^2 - 1);
                        sqrt(23)   25 1/3
(%t3)                  (-------- + --)
                             3/2    54
                          2 3
solve: solution:

                                      sqrt(3) %i   - 1
                                      ---------- + ---
            - 1   sqrt(3) %i              2         2    - 1
(%t4)  x = (--- - ----------) %t3 + ----------------- + ---
             2        2                    9 %t3          3

                                      - 1   sqrt(3) %i
                                      --- - ----------
            sqrt(3) %i   - 1           2        2        - 1
(%t5)  x = (---------- + ---) %t3 + ----------------- + ---
               2         2                9 %t3          3

                                   1     - 1
(%t6)                  x = %t3 + ----- + ---
                                 9 %t3    3
(%o6)                     [%t4, %t5, %t6]
(%i7) breakup: false$
```

```
(%i8) solve (x^3 + x^2 - 1);
solve: solution:

                sqrt(3) %i   - 1
                ---------- + ---
                    2         2       sqrt(23)   25 1/3
(%t8) x = -------------------- + (-------- + --)
             sqrt(23)   25 1/3        3/2    54
          9 (-------- + --)        2 3
                3/2    54
               2 3
                                          - 1   sqrt(3) %i    - 1
                                         (--- - ----------) + ---
                                           2        2          3

             sqrt(23)   25 1/3  sqrt(3) %i   - 1
(%t9) x = (-------- + --)    (---------- + ---)
             3/2    54          2         2
            2 3
                                        - 1   sqrt(3) %i
                                        --- - ----------
                                         2        2          - 1
                                    + -------------------- + ---
                                        sqrt(23)   25 1/3    3
                                  9 (-------- + --)
                                        3/2    54
                                       2 3

              sqrt(23)   25 1/3            1             - 1
(%t10) x = (-------- + --)    + -------------------- + ---
              3/2    54          sqrt(23)   25 1/3    3
             2 3               9 (-------- + --)
                                     3/2    54
                                    2 3
(%o10)                  [%t8, %t9, %t10]
```

`dimension` [Function]

`dimension` (*eqn*)
`dimension` (*eqn_1*, ..., *eqn_n*)

`dimen` is a package for dimensional analysis. `load ("dimen")` loads this package. `demo ("dimen")` displays a short demonstration.

`dispflag` [Option variable]

Default value: `true`

If set to `false` within a `block` will inhibit the display of output generated by the solve functions called from within the `block`. Termination of the `block` with a dollar sign, `$`, sets `dispflag` to `false`.

`funcsolve` (*eqn*, *g*(*t*)) [Function]

Returns [*g*(*t*) = ...] or [], depending on whether or not there exists a rational function *g*(*t*) satisfying *eqn*, which must be a first order, linear polynomial in (for this case) *g*(*t*) and *g*(*t*+1)

```
(%i1) eqn: (n + 1)*f(n) - (n + 3)*f(n + 1)/(n + 1) =
     (n - 1)/(n + 2);
                              (n + 3) f(n + 1)   n - 1
(%o1)         (n + 1) f(n) - ---------------- = -----
                                   n + 1         n + 2
(%i2) funcsolve (eqn, f(n));
solve: dependent equations eliminated: (4 3)
                                      n
(%o2)                 f(n) = ---------------
                                (n + 1) (n + 2)
```

Warning: this is a very rudimentary implementation – many safety checks and obvious generalizations are missing.

`globalsolve` [Option variable]

Default value: `false`

When `globalsolve` is `true`, solved-for variables are assigned the solution values found by `linsolve`, and by `solve` when solving two or more linear equations.

When `globalsolve` is `false`, solutions found by `linsolve` and by `solve` when solving two or more linear equations are expressed as equations, and the solved-for variables are not assigned.

When solving anything other than two or more linear equations, `solve` ignores `globalsolve`. Other functions which solve equations (e.g., `algsys`) always ignore `globalsolve`.

Examples:

```
(%i1) globalsolve: true$
(%i2) solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);
                               17        1
(%o2)                   [[x : --, y : - -]]
                               7         7
(%i3) x;
                                 17
(%o3)                            --
                                 7
(%i4) y;
                                   1
(%o4)                            - -
                                   7
(%i5) globalsolve: false$
(%i6) kill (x, y)$
```

```
(%i7) solve ([x + 3*y = 2, 2*x - y = 5], [x, y]);
                                17        1
(%o7)                    [[x = --, y = - -]]
                                7         7
(%i8) x;
(%o8)                           x
(%i9) y;
(%o9)                           y
```

ieqn (*ie*, *unk*, *tech*, *n*, *guess*) [Function]

`inteqn` is a package for solving integral equations. `load ("inteqn")` loads this package.

ie is the integral equation; *unk* is the unknown function; *tech* is the technique to be tried from those given in the lists below; (*tech* = `first` means: try the first technique which finds a solution; *tech* = `all` means: try all applicable techniques); *n* is the maximum number of terms to take for `taylor`, `neumann`, `firstkindseries`, or `fredseries` (it is also the maximum depth of recursion for the differentiation method); *guess* is the initial guess for `neumann` or `firstkindseries`.

Two types of equations are considered. A second-kind equation of the following form,

$$p\left(x\right) = q\left(x, p\left(x\right), \int_{a(x)}^{b(x)} w\left(x, u, p\left(x\right), p\left(u\right)\right)\ du\right)$$

and a first-kind equation with the form

$$f\left(x\right) = \int_{a(x)}^{b(x)} w\left(x, u, p\left(u\right)\right)\ du$$

The different solution techniques used require particular forms of the expressions q and w. The techniques available are the following:

Second-kind equations

- `flfrnk2nd`: For fixed-limit, finite-rank integrands.
- `vlfrnk`: For variable-limit, finite-rank integrands.
- `transform`: Laplace transform for convolution types.
- `fredseries`: Fredholm-Carleman series for linear equations.
- `tailor`: Taylor series for quasi-linear variable-limit equations.
- `neumann`: Neumann series for quasi-second kind equations.
- `collocate`: Collocation using a power series form for p(x) evaluated at equally spaced points.

First-kind equations

- `flfrnk1st`: For fixed-limit, finite-rank integrands.
- `vlfrnk`: For variable-limit, finite-rank integrands.
- `abel`: For singular integrands
- `transform`: See above

- `collocate`: See above
- `firstkindseries`: Iteration technique similar to neumann series.

The default values for the 2nd thru 5th parameters in the calling form are:

unk: `p(x)`, where *p* is the first function encountered in an integrand which is unknown to Maxima and *x* is the variable which occurs as an argument to the first occurrence of *p* found outside of an integral in the case of `secondkind` equations, or is the only other variable besides the variable of integration in `firstkind` equations. If the attempt to search for *x* fails, the user will be asked to supply the independent variable. *tech*: `first`. *n*: 1. *guess*: `none` which will cause `neumann` and `firstkindseries` to use `f(x)` as an initial guess.

Examples:

```
(%i1) load("inteqn")$

(%i2) e: p(x) - 1 -x + cos(x) + 'integrate(cos(x-u)*p(u),u,0,x)$

(%i3) ieqn(e, p(x), 'transform);
default 4th arg, number of iterations or coll. parms.:  1
default 5th arg, initial guess:  none

(%t3)                              [x, transform]
(%o3)                                  [%t3]
(%i4) e: 2*'integrate(p(x*sin(u)), u, 0, %pi/2) - a*x - b$

(%i5) ieqn(e, p(x), 'firstkindseries);
default 4th arg, number of iterations or coll. parms.:  1
default 5th arg, initial guess:  none

(%t5)          [2 a x + %pi b, firstkindseries, 1, approximate]
(%o5)                                  [%t5]
```

`ieqnprint` [Option variable]

Default value: `true`

`ieqnprint` governs the behavior of the result returned by the `ieqn` command. When `ieqnprint` is `false`, the lists returned by the `ieqn` function are of the form

[*solution*, *technique used*, *nterms*, *flag*]

where *flag* is absent if the solution is exact.

Otherwise, it is the word `approximate` or `incomplete` corresponding to an inexact or non-closed form solution, respectively. If a series method was used, *nterms* gives the number of terms taken (which could be less than the n given to `ieqn` if an error prevented generation of further terms).

`lhs` (*expr*) [Function]

Returns the left-hand side (that is, the first argument) of the expression *expr*, when the operator of *expr* is one of the relational operators `< <= = # equal notequal >= >`, one of the assignment operators `:= ::= : ::`, or a user-defined binary infix operator, as declared by `infix`.

When *expr* is an atom or its operator is something other than the ones listed above, `lhs` returns *expr*.

See also `rhs`.

Examples:

```
(%i1) e: aa + bb = cc;
(%o1)                      bb + aa = cc
(%i2) lhs (e);
(%o2)                         bb + aa
(%i3) rhs (e);
(%o3)                           cc
(%i4) [lhs (aa < bb), lhs (aa <= bb), lhs (aa >= bb),
       lhs (aa > bb)];
(%o4)                     [aa, aa, aa, aa]
(%i5) [lhs (aa = bb), lhs (aa # bb), lhs (equal (aa, bb)),
       lhs (notequal (aa, bb))];
(%o5)                     [aa, aa, aa, aa]
(%i6) e1: '(foo(x) := 2*x);
(%o6)                      foo(x) := 2 x
(%i7) e2: '(bar(y) ::= 3*y);
(%o7)                     bar(y) ::= 3 y
(%i8) e3: '(x : y);
(%o8)                          x : y
(%i9) e4: '(x :: y);
(%o9)                         x :: y
(%i10) [lhs (e1), lhs (e2), lhs (e3), lhs (e4)];
(%o10)                [foo(x), bar(y), x, x]
(%i11) infix ("][");
(%o11)                          ][
(%i12) lhs (aa ][ bb);
(%o12)                          aa
```

`linsolve` ([*expr_1*, ..., *expr_m*], [*x_1*, ..., *x_n*]) [Function]

Solves the list of simultaneous linear equations for the list of variables. The expressions must each be polynomials in the variables and may be equations. If the length of the list of variables doesn't match the number of linearly-independent equations to solve the result will be an empty list.

When `globalsolve` is `true`, each solved-for variable is bound to its value in the solution of the equations.

When `backsubst` is `false`, `linsolve` does not carry out back substitution after the equations have been triangularized. This may be necessary in very big problems where back substitution would cause the generation of extremely large expressions.

When `linsolve_params` is `true`, `linsolve` also generates the `%r` symbols used to represent arbitrary parameters described in the manual under `algsys`. Otherwise, `linsolve` solves an under-determined system of equations with some variables expressed in terms of others.

When programmode is false, linsolve displays the solution with intermediate expression (%t) labels, and returns the list of labels.

See also algsys, eliminate. and solve.

Examples:

```
(%i1) e1: x + z = y;
(%o1)                          z + x = y
(%i2) e2: 2*a*x - y = 2*a^2;
                                              2
(%o2)                     2 a x - y = 2 a
(%i3) e3: y - 2*z = 2;
(%o3)                          y - 2 z = 2
(%i4) [globalsolve: false, programmode: true];
(%o4)                         [false, true]
(%i5) linsolve ([e1, e2, e3], [x, y, z]);
(%o5)              [x = a + 1, y = 2 a, z = a - 1]
(%i6) [globalsolve: false, programmode: false];
(%o6)                        [false, false]
(%i7) linsolve ([e1, e2, e3], [x, y, z]);
Solution:

(%t7)                          z = a - 1

(%t8)                           y = 2 a

(%t9)                          x = a + 1
(%o9)                        [%t7, %t8, %t9]
(%i10) ''%;
(%o10)             [z = a - 1, y = 2 a, x = a + 1]
(%i11) [globalsolve: true, programmode: false];
(%o11)                        [true, false]
(%i12) linsolve ([e1, e2, e3], [x, y, z]);
Solution:

(%t12)                         z : a - 1

(%t13)                          y : 2 a

(%t14)                         x : a + 1
(%o14)                     [%t12, %t13, %t14]
(%i15) ''%;
(%o15)             [z : a - 1, y : 2 a, x : a + 1]
(%i16) [x, y, z];
(%o16)                     [a + 1, 2 a, a - 1]
(%i17) [globalsolve: true, programmode: true];
(%o17)                        [true, true]
```

```
(%i18) linsolve ([e1, e2, e3], '[x, y, z]);
(%o18)           [x : a + 1, y : 2 a, z : a - 1]
(%i19) [x, y, z];
(%o19)                  [a + 1, 2 a, a - 1]
```

`linsolvewarn` [Option variable]

Default value: `true`

When `linsolvewarn` is `true`, `linsolve` prints a message "Dependent equations eliminated".

`linsolve_params` [Option variable]

Default value: `true`

When `linsolve_params` is `true`, `linsolve` also generates the `%r` symbols used to represent arbitrary parameters described in the manual under `algsys`. Otherwise, `linsolve` solves an under-determined system of equations with some variables expressed in terms of others.

`multiplicities` [System variable]

Default value: `not_set_yet`

`multiplicities` is set to a list of the multiplicities of the individual solutions returned by `solve` or `realroots`.

`nroots` (*p*, *low*, *high*) [Function]

Returns the number of real roots of the real univariate polynomial *p* in the half-open interval `(`*low*`,` *high*`]`. The endpoints of the interval may be `minf` or `inf`.

`nroots` uses the method of Sturm sequences.

```
(%i1) p: x^10 - 2*x^4 + 1/2$
(%i2) nroots (p, -6, 9.1);
(%o2)                          4
```

`nthroot` (*p*, *n*) [Function]

where *p* is a polynomial with integer coefficients and *n* is a positive integer returns `q`, a polynomial over the integers, such that `q^n = p` or prints an error message indicating that *p* is not a perfect nth power. This routine is much faster than `factor` or even `sqfr`.

`polyfactor` [Option variable]

Default value: `false`

The option variable `polyfactor` when `true` causes `allroots` and `bfallroots` to factor the polynomial over the real numbers if the polynomial is real, or over the complex numbers, if the polynomial is complex.

See `allroots` for an example.

`programmode` [Option variable]

Default value: `true`

When `programmode` is `true`, `solve`, `realroots`, `allroots`, and `linsolve` return solutions as elements in a list. (Except when `backsubst` is set to `false`, in which case `programmode: false` is assumed.)

When `programmode` is `false`, `solve`, etc. create intermediate expression labels `%t1`, `%t2`, etc., and assign the solutions to them.

`realonly` [Option variable]

Default value: `false`

When `realonly` is `true`, `algsys` returns only those solutions which are free of `%i`.

`realroots` [Function]

`realroots` (*expr*, *bound*)
`realroots` (*eqn*, *bound*)
`realroots` (*expr*)
`realroots` (*eqn*)

Computes rational approximations of the real roots of the polynomial *expr* or polynomial equation *eqn* of one variable, to within a tolerance of *bound*. Coefficients of *expr* or *eqn* must be literal numbers; symbol constants such as `%pi` are rejected.

`realroots` assigns the multiplicities of the roots it finds to the global variable `multiplicities`.

`realroots` constructs a Sturm sequence to bracket each root, and then applies bisection to refine the approximations. All coefficients are converted to rational equivalents before searching for roots, and computations are carried out by exact rational arithmetic. Even if some coefficients are floating-point numbers, the results are rational (unless coerced to floats by the `float` or `numer` flags).

When *bound* is less than 1, all integer roots are found exactly. When *bound* is unspecified, it is assumed equal to the global variable `rootsepsilon`.

When the global variable `programmode` is `true`, `realroots` returns a list of the form `[x = x_1, x = x_2, ...]`. When `programmode` is `false`, `realroots` creates intermediate expression labels `%t1`, `%t2`, ..., assigns the results to them, and returns the list of labels.

See also `allroots`, `bfallroots`, `guess_exact_value`, and `lhs`.

Examples:

```
(%i1) realroots (-1 - x + x^5, 5e-6);
                                 612003
(%o1)                      [x = ------]
                                 524288
(%i2) ev (%[1], float);
(%o2)                  x = 1.1673030853271484
(%i3) ev (-1 - x + x^5, %);
(%o3)                  - 7.396496210176906e-6
(%i1) realroots (expand ((1 - x)^5 * (2 - x)^3 * (3 - x)), 1e-20);
(%o1)                  [x = 1, x = 2, x = 3]
(%i2) multiplicities;
(%o2)                        [5, 3, 1]
```

`rhs` (*expr*) [Function]

Returns the right-hand side (that is, the second argument) of the expression *expr*, when the operator of *expr* is one of the relational operators `< <= = # equal notequal`

`>= >`, one of the assignment operators `:= ::= : ::`, or a user-defined binary infix operator, as declared by `infix`.

When *expr* is an atom or its operator is something other than the ones listed above, `rhs` returns 0.

See also `lhs`.

Examples:

```
(%i1) e: aa + bb = cc;
(%o1)                      bb + aa = cc
(%i2) lhs (e);
(%o2)                         bb + aa
(%i3) rhs (e);
(%o3)                           cc
(%i4) [rhs (aa < bb), rhs (aa <= bb), rhs (aa >= bb),
       rhs (aa > bb)];
(%o4)                    [bb, bb, bb, bb]
(%i5) [rhs (aa = bb), rhs (aa # bb), rhs (equal (aa, bb)),
       rhs (notequal (aa, bb))];
(%o5)                    [bb, bb, bb, bb]
(%i6) e1: '(foo(x) := 2*x);
(%o6)                       foo(x) := 2 x
(%i7) e2: '(bar(y) ::= 3*y);
(%o7)                      bar(y) ::= 3 y
(%i8) e3: '(x : y);
(%o8)                           x : y
(%i9) e4: '(x :: y);
(%o9)                          x :: y
(%i10) [rhs (e1), rhs (e2), rhs (e3), rhs (e4)];
(%o10)                   [2 x, 3 y, y, y]
(%i11) infix ("][");
(%o11)                           ][
(%i12) rhs (aa ][ bb);
(%o12)                           bb
```

`rootsconmode` [Option variable]

Default value: `true`

`rootsconmode` governs the behavior of the `rootscontract` command. See `rootscontract` for details.

`rootscontract (`*expr*`)` [Function]

Converts products of roots into roots of products. For example, `rootscontract (sqrt(x)*y^(3/2))` yields `sqrt(x*y^3)`.

When `radexpand` is `true` and `domain` is `real`, `rootscontract` converts `abs` into `sqrt`, e.g., `rootscontract (abs(x)*sqrt(y))` yields `sqrt(x^2*y)`.

There is an option `rootsconmode` affecting `rootscontract` as follows:

Problem	Value of rootsconmode	Result of applying rootscontract

```
x^(1/2)*y^(3/2)      false          (x*y^3)^(1/2)
x^(1/2)*y^(1/4)      false          x^(1/2)*y^(1/4)
x^(1/2)*y^(1/4)      true           (x*y^(1/2))^(1/2)
x^(1/2)*y^(1/3)      true           x^(1/2)*y^(1/3)
x^(1/2)*y^(1/4)      all            (x^2*y)^(1/4)
x^(1/2)*y^(1/3)      all            (x^3*y^2)^(1/6)
```

When `rootsconmode` is `false`, `rootscontract` contracts only with respect to rational number exponents whose denominators are the same. The key to the `rootsconmode: true` examples is simply that 2 divides into 4 but not into 3. `rootsconmode: all` involves taking the least common multiple of the denominators of the exponents.

`rootscontract` uses `ratsimp` in a manner similar to `logcontract`.

Examples:

```
(%i1) rootsconmode: false$
(%i2) rootscontract (x^(1/2)*y^(3/2));
                                    3
(%o2)                        sqrt(x y )
(%i3) rootscontract (x^(1/2)*y^(1/4));
                                    1/4
(%o3)                       sqrt(x) y
(%i4) rootsconmode: true$
(%i5) rootscontract (x^(1/2)*y^(1/4));
(%o5)                      sqrt(x sqrt(y))
(%i6) rootscontract (x^(1/2)*y^(1/3));
                                    1/3
(%o6)                       sqrt(x) y
(%i7) rootsconmode: all$
(%i8) rootscontract (x^(1/2)*y^(1/4));
                                 2   1/4
(%o8)                          (x  y)
(%i9) rootscontract (x^(1/2)*y^(1/3));
                                 3  2 1/6
(%o9)                         (x  y )
(%i10) rootsconmode: false$
(%i11) rootscontract (sqrt(sqrt(x) + sqrt(1 + x))
                     *sqrt(sqrt(1 + x) - sqrt(x)));
(%o11)                             1
(%i12) rootsconmode: true$
(%i13) rootscontract (sqrt(5 + sqrt(5)) - 5^(1/4)*sqrt(1 + sqrt(5)));
(%o13)                             0
```

`rootsepsilon` [Option variable]

Default value: 1.0e-7

`rootsepsilon` is the tolerance which establishes the confidence interval for the roots found by the `realroots` function.

solve [Function]

solve (*expr*, *x*)
solve (*expr*)
solve ([*eqn_1*, ..., *eqn_n*], [*x_1*, ..., *x_n*])

Solves the algebraic equation *expr* for the variable *x* and returns a list of solution equations in *x*. If *expr* is not an equation, the equation *expr* = 0 is assumed in its place. *x* may be a function (e.g. f(x)), or other non-atomic expression except a sum or product. *x* may be omitted if *expr* contains only one variable. *expr* may be a rational expression, and may contain trigonometric functions, exponentials, etc.

The following method is used:

Let E be the expression and X be the variable. If E is linear in X then it is trivially solved for X. Otherwise if E is of the form A*X^N + B then the result is (-B/A)^1/N) times the N'th roots of unity.

If E is not linear in X then the gcd of the exponents of X in E (say N) is divided into the exponents and the multiplicity of the roots is multiplied by N. Then solve is called again on the result. If E factors then solve is called on each of the factors. Finally solve will use the quadratic, cubic, or quartic formulas where necessary.

In the case where E is a polynomial in some function of the variable to be solved for, say F(X), then it is first solved for F(X) (call the result C), then the equation F(X)=C can be solved for X provided the inverse of the function F is known.

breakup if false will cause solve to express the solutions of cubic or quartic equations as single expressions rather than as made up of several common subexpressions which is the default.

multiplicities - will be set to a list of the multiplicities of the individual solutions returned by solve, realroots, or allroots. Try apropos (solve) for the switches which affect solve. describe may then by used on the individual switch names if their purpose is not clear.

solve ([*eqn_1*, ..., *eqn_n*], [*x_1*, ..., *x_n*]) solves a system of simultaneous (linear or non-linear) polynomial equations by calling linsolve or algsys and returns a list of the solution lists in the variables. In the case of linsolve this list would contain a single list of solutions. It takes two lists as arguments. The first list represents the equations to be solved; the second list is a list of the unknowns to be determined. If the total number of variables in the equations is equal to the number of equations, the second argument-list may be omitted.

When programmode is false, solve displays solutions with intermediate expression (%t) labels, and returns the list of labels.

When globalsolve is true and the problem is to solve two or more linear equations, each solved-for variable is bound to its value in the solution of the equations.

Examples:

```
(%i1) solve (asin (cos (3*x))*(f(x) - 1), x);
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
                                %pi
(%o1)                     [x = ---, f(x) = 1]
                                6
```

```
(%i2) ev (solve (5^f(x) = 125, f(x)), solveradcan);
                                log(125)
(%o2)                   [f(x) = --------]
                                 log(5)
(%i3) [4*x^2 - y^2 = 12, x*y - x = 2];
                      2    2
(%o3)             [4 x  - y  = 12, x y - x = 2]
(%i4) solve (%, [x, y]);
(%o4) [[x = 2, y = 2], [x = 0.5202594388652008 %i
 - 0.1331240357358706, y = 0.07678378523787788
 - 3.608003221870287 %i], [x = - 0.5202594388652008 %i
 - 0.1331240357358706, y = 3.608003221870287 %i
 + 0.07678378523787788], [x = - 1.733751846381093,
y = - 0.15356757100196963]]
```

```
(%i5) solve (1 + a*x + x^3, x);
                                                3
            - 1   sqrt(3) %i   sqrt(4 a  + 27)   1 1/3
(%o5) [x = (--- - ----------) (---------------- - -)
             2        2               3/2        2
                                    2 3
       sqrt(3) %i   - 1
      (---------- + ---) a
           2         2
 - -------------------------, x =
              3
      sqrt(4 a  + 27)   1 1/3
   3 (--------------- - -)
             3/2        2
          2 3
                            3
 sqrt(3) %i   - 1   sqrt(4 a  + 27)   1 1/3
(---------- + ---) (---------------- - -)
     2         2           3/2        2
                         2 3
       - 1   sqrt(3) %i
      (--- - ----------) a
        2        2
 - -------------------------, x =
              3
      sqrt(4 a  + 27)   1 1/3
   3 (--------------- - -)
             3/2        2
          2 3
         3
 sqrt(4 a  + 27)   1 1/3               a
(---------------- - -)    - --------------------------]
        3/2        2                  3
     2 3                     sqrt(4 a  + 27)   1 1/3
                           3 (---------------- - -)
                                     3/2        2
                                  2 3
(%i6) solve (x^3 - 1);
             sqrt(3) %i - 1        sqrt(3) %i + 1
(%o6)   [x = --------------, x = - --------------, x = 1]
                   2                     2
```

```
(%i7) solve (x^6 - 1);
           sqrt(3) %i + 1      sqrt(3) %i - 1
(%o7) [x = --------------, x = --------------, x = - 1,
                 2                   2
                     sqrt(3) %i + 1        sqrt(3) %i - 1
               x = - --------------, x = - --------------, x = 1]
                           2                     2
(%i8) ev (x^6 - 1, %[1]);
                                     6
                      (sqrt(3) %i + 1)
(%o8)                 ---------------- - 1
                             64
(%i9) expand (%);
(%o9)                           0
(%i10) x^2 - 1;
                                2
(%o10)                         x  - 1
(%i11) solve (%, x);
(%o11)                  [x = - 1, x = 1]
(%i12) ev (%th(2), %[1]);
(%o12)                          0
```

The symbols `%r` are used to denote arbitrary constants in a solution.

```
(%i1) solve([x+y=1,2*x+2*y=2],[x,y]);
solve: dependent equations eliminated: (2)
(%o1)               [[x = 1 - %r1, y = %r1]]
```

See `algsys` and `%rnum_list` for more information.

`solvedecomposes` [Option variable]

Default value: `true`

When `solvedecomposes` is `true`, `solve` calls `polydecomp` if asked to solve polynomials.

`solveexplicit` [Option variable]

Default value: `false`

When `solveexplicit` is `true`, inhibits `solve` from returning implicit solutions, that is, solutions of the form `F(x) = 0` where `F` is some function.

`solvefactors` [Option variable]

Default value: `true`

When `solvefactors` is `false`, `solve` does not try to factor the expression. The `false` setting may be desired in some cases where factoring is not necessary.

`solvenullwarn` [Option variable]

Default value: `true`

When `solvenullwarn` is `true`, `solve` prints a warning message if called with either a null equation list or a null variable list. For example, `solve ([], [])` would print two warning messages and return `[]`.

`solveradcan` [Option variable]

Default value: `false`

When `solveradcan` is `true`, `solve` calls `radcan` which makes `solve` slower but will allow certain problems containing exponentials and logarithms to be solved.

`solvetrigwarn` [Option variable]

Default value: `true`

When `solvetrigwarn` is `true`, `solve` may print a message saying that it is using inverse trigonometric functions to solve the equation, and thereby losing solutions.

22.2 Examples for algsys

algsys often fails to solve problems that have solutions. The following examples provide some solution strategies.

22.2.1 Strategies

Many users try to solve the most general problem imaginable, with multiple free parameters. Unfortunately the size and complexity of individual solutions increases rapidly with the number of parameters, as can be seen in the examples below. This 'expression swell' makes it difficult for a computer algebra system to confirm symbolically that solutions satisfy the input equations so solutions can be incorrectly rejected by a solver. Every effort should be made to reduce the complexity of, and number of parameters in, the equations to be solved.

22.2.2 Example 1: Intersection of two circles

We wish to find the intersection points of two circles: one centered at (a,b) with radius r and the second centered at (c,d) with radius s.

algsys does not find a solution to the original problem.

```
(%i1) eq1: (x-a)^2+(y-b)^2-r^2;
                             2          2    2
(%o1)                (y - b)  + (x - a)  - r
(%i2) eq2: (x-c)^2+(y-d)^2-s^2;
                             2          2    2
(%o2)                (y - d)  + (x - c)  - s
(%i3) algsys([eq1,eq2],[x,y]);
(%o3)                           []
```

Simplify the problem

This problem can be transformed to eliminate parameters by: translating the center of the first circle to the origin; scaling so that the first circle has radius 1; and, rotating about the new origin so that the center of the second circle is located on the positive x-axis.

```
(%i1) eq1a:x^2+y^2-1;
                              2    2
(%o1)                        y  + x  - 1
(%i2) eq2a:(x-C)^2+y^2-S^2;
                            2          2    2
(%o2)                      y  + (x - C)  - S
```

```
(%i3) algsys([eq1a,eq2a],[x,y]);
                2    2
               S  - C  - 1
(%o3) [[x = - -----------, y =
                    2 C
          4       2       2    4      2
  sqrt(- S  + (2 C  + 2) S  - C  + 2 C  - 1)
- ----------------------------------------],
                      2 C
        2    2
       S  - C  - 1
[x = - -----------, y =
            2 C
         4       2       2    4      2
sqrt(- S  + (2 C  + 2) S  - C  + 2 C  - 1)
----------------------------------------]]
                    2 C
```

It is not necessary to rescale these equations to obtain a solution.

```
(%i1) eq1b:x^2+y^2-r^2;
                            2    2    2
(%o1)                      y  + x  - r
(%i2) eq2b:(x-C)^2+y^2-s^2;
                          2          2    2
(%o2)                    y  + (x - C)  - s
(%i3) algsys([eq1b,eq2b],[x,y]);
                2    2    2
               s  - r  - C
(%o3) [[x = - ------------, y =
                    2 C
          4       2      2   2    4      2  2    4
  sqrt(- s  + (2 r  + 2 C ) s  - r  + 2 C  r  - C )
- ------------------------------------------------],
                       2 C
        2    2    2
       s  - r  - C
[x = - ------------, y =
             2 C
         4       2      2   2    4      2  2    4
sqrt(- s  + (2 r  + 2 C ) s  - r  + 2 C  r  - C )
------------------------------------------------]]
                       2 C
```

Simplify the equations

Another strategy is to simplify the equations. The expression *eq1-eq2* is linear in unknowns x and y. *algsys* can solve the simpler system of equations *[eq1,eq1-eq2]*.

Note the complexity of the solution at (%o4).

```
(%i1) eq1: (x-a)^2+(y-b)^2-r^2;
                           2          2    2
(%o1)                (y - b)  + (x - a)  - r
(%i2) eq2: (x-c)^2+(y-d)^2-s^2;
                           2          2    2
(%o2)                (y - d)  + (x - c)  - s
(%i3) eq3: expand(eq1-eq2);
                                      2    2    2    2    2    2
(%o3) 2 d y - 2 b y + 2 c x - 2 a x + s  - r  - d  - c  + b  + a
```

```
(%i4) soln:algsys([eq1,eq3],[x,y]);
                                4
(%o4) [[x = - ((d - b) sqrt(- s
      2     2             2             2     2   2    4
 + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) s  - r
      2            2            2     2   2    4       3
 + (2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) r  - d  + 4 b d
        2             2     2   2
 + (- 2 c  + 4 a c - 6 b  - 2 a ) d
        2              3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
        2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
          2           2            2
 + (c - a) s  + (a - c) r  + (- c - a) d  + (2 b c + 2 a b) d
   3     2    2   2        2   3
 - c  + a c  + (a  - b ) c - a b  - a )
    2            2            2     2
/(2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ),
                   4      2     2             2
y = ((c - a) sqrt(- s  + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c
     2     2   2    4      2            2            2
 + 2 b  + 2 a ) s  - r  + (2 d  - 4 b d + 2 c  - 4 a c + 2 b
     2   2    4       3        2             2     2   2
 + 2 a ) r  - d  + 4 b d  + (- 2 c  + 4 a c - 6 b  - 2 a ) d
        2              3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
        2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
          2           2   3     2    2          2   2
 + (b - d) s  + (d - b) r  + d  - b d  + (c  - 2 a c - b  + a ) d
     2            3   2       2            2
 + b c  - 2 a b c + b  + a  b)/(2 d  - 4 b d + 2 c  - 4 a c
     2     2                       4
 + 2 b  + 2 a )], [x = ((d - b) sqrt(- s
      2     2             2             2     2   2    4
 + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) s  - r
      2            2            2     2   2    4       3
 + (2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) r  - d  + 4 b d
        2             2     2   2
 + (- 2 c  + 4 a c - 6 b  - 2 a ) d
        2              3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
        2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
          2           2            2
 + (a - c) s  + (c - a) r  + (c + a) d  + (- 2 b c - 2 a b) d
   3     2    2   2        2   3
 + c  - a c  + (b  - a ) c + a b  + a )
    2            2            2     2
/(2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ),
                     4      2     2             2
y = - ((c - a) sqrt(- s  + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c
```

```
(%i5) ratsimp(subst(soln[1],[eq1,eq2]));
(%o5)                          [0, 0]
(%i6) ratsimp(subst(soln[2],[eq1,eq2]));
(%o6)                          [0, 0]
```

Simplify the equations using poly_reduced_grobner

Sometimes the system of equations can be solved after preprocessing with the function `poly_reduced_grobner` from the *grobner* package.

The reduced Gröbner basis returned at (%o4) consists of two polynomials in unknowns x and y: the first is bilinear in the unknowns; the second is quadratic in y only. This confirms there are two (perhaps multiple) solutions, as we know geometrically.

```
(%i1) eq1: (x-a)^2+(y-b)^2-r^2;
                             2          2    2
(%o1)                 (y - b)  + (x - a)  - r
(%i2) eq2: (x-c)^2+(y-d)^2-s^2;
                             2          2    2
(%o2)                 (y - d)  + (x - c)  - s
(%i3) /* load grobner package */
 load("grobner")$
(%i4) poly_reduced_grobner([eq1,eq2],[x,y]);
                                        2    2    2    2    2
(%o4) [(2 d - 2 b) y + (2 c - 2 a) x + s  - r  - d  - c  + b
    2      2              2              2      2   2
 + a , (4 d  - 8 b d + 4 c  - 8 a c + 4 b  + 4 a ) y
         2      2      2          2      2       3        2
 + (d (4 s  - 4 r  + 4 b ) + b (4 r  - 4 s ) - 4 d  + 4 b d
                    2                  2                     3
 + a c (8 d + 8 b) + c  (- 4 d - 4 b) + a  (- 4 d - 4 b) - 4 b )
     4           2      2      2      2       3
 y + s  + c (a (4 s  + 4 r  - 4 d  - 4 b ) - 4 a )
    2     2      2       2  2    2       2      2      2
 + b  (2 s  - 2 r ) - 2 r  s  + d  (- 2 s  + 2 r  - 2 b )
    2       2      2      2      2      2
 + c  (- 2 s  - 2 r  + 2 d  + 2 b  + 6 a )
    2       2      2      2      2     4    4    4        3    4
 + a  (- 2 s  - 2 r  + 2 d  + 2 b ) + r  + d  + c  - 4 a c  + b
    4
 + a ]
```

```
(%i5) soln:algsys(%,[x,y]);
                                4
(%o5) [[x = - ((d - b) sqrt(- s
       2     2             2             2     2   2    4
 + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) s  - r
       2             2             2     2   2    4       3
 + (2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) r  - d  + 4 b d
         2             2     2   2
 + (- 2 c  + 4 a c - 6 b  - 2 a ) d
         2               3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
         2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
           2           2             2
 + (c - a) s  + (a - c) r  + (- c - a) d  + (2 b c + 2 a b) d
    3     2    2   2        2   3
 - c  + a c  + (a  - b ) c - a b  - a )
     2             2             2     2
/(2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ),
                    4      2     2             2
y = ((c - a) sqrt(- s  + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c
      2     2   2    4      2             2             2
 + 2 b  + 2 a ) s  - r  + (2 d  - 4 b d + 2 c  - 4 a c + 2 b
     2   2    4       3        2             2     2   2
 + 2 a ) r  - d  + 4 b d  + (- 2 c  + 4 a c - 6 b  - 2 a ) d
         2               3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
         2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
           2           2   3     2    2          2   2
 + (b - d) s  + (d - b) r  + d  - b d  + (c  - 2 a c - b  + a ) d
     2             3   2       2             2
 + b c  - 2 a b c + b  + a  b)/(2 d  - 4 b d + 2 c  - 4 a c
      2     2                        4
 + 2 b  + 2 a )], [x = ((d - b) sqrt(- s
       2     2             2             2     2   2    4
 + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) s  - r
       2             2             2     2   2    4       3
 + (2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ) r  - d  + 4 b d
         2             2     2   2
 + (- 2 c  + 4 a c - 6 b  - 2 a ) d
         2               3     2        4       3
 + (4 b c  - 8 a b c + 4 b  + 4 a  b) d - c  + 4 a c
         2     2   2        2     3      4     2  2    4
 + (- 2 b  - 6 a ) c  + (4 a b  + 4 a ) c - b  - 2 a  b  - a )
           2           2           2
 + (a - c) s  + (c - a) r  + (c + a) d  + (- 2 b c - 2 a b) d
    3     2    2   2        2   3
 + c  - a c  + (b  - a ) c + a b  + a )
     2             2             2     2
/(2 d  - 4 b d + 2 c  - 4 a c + 2 b  + 2 a ),
                     4      2     2             2
y = - ((c - a) sqrt(- s  + (2 r  + 2 d  - 4 b d + 2 c  - 4 a c
```

```
(%i6) ratsimp(subst(soln[1],[eq1,eq2]));
(%o6)                          [0, 0]
(%i7) ratsimp(subst(soln[2],[eq1,eq2]));
(%o7)                          [0, 0]
```

22.2.3 Example 2: Equations with irrational coefficients

The presence of irrational coefficients can prevent *algsys* from finding solutions. Replacing the constant with a symbol allows a solution to be found.

```
(%i1) eq1: %pi*y + x - 1;
(%o1)                       %pi y + x - 1
(%i2) eq2: 1 - x^2*y;
                                    2
(%o2)                          1 - x  y
(%i3) algsys([eq1,eq2],[x,y]);
(%o3)                              []
(%i4) /* Substitute p for %p and magically algsys finds solutions */
 eq1a:subst(p,%pi,eq1);
(%o4)                        p y + x - 1
```

```
(%i5) s:algsys([eq1a,eq2],[x,y]);
                  5/3                5/3
(%o5) [[x = - ((2    sqrt(3) %i + 2   )
   3/2          2                    1/3
 (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
      1/3       1/3  3/2           2
 ((9 2    %i - 2    3   ) sqrt(27 p  - 4 p)
     1/3  7/2          1/3       4/3               4/3
 + (2    3    %i - 27 2   ) p - 2    sqrt(3) %i + 2   ) - 8)/24,
       5/3               5/3
y = ((2    sqrt(3) %i + 2   )
   3/2          2                    1/3
 (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
      1/3       1/3  3/2           2
 ((9 2    %i - 2    3   ) sqrt(27 p  - 4 p)
     1/3  7/2          1/3       4/3               4/3
 + (2    3    %i - 27 2   ) p - 2    sqrt(3) %i + 2   ) + 16)
                   5/3               5/3
/(24 p)], [x = ((2    sqrt(3) %i - 2   )
   3/2          2                    1/3
 (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
      1/3       1/3  3/2           2
 ((9 2    %i + 2    3   ) sqrt(27 p  - 4 p)
     1/3  7/2          1/3       4/3               4/3
 + (2    3    %i + 27 2   ) p - 2    sqrt(3) %i - 2   ) + 8)/24,
         5/3               5/3
y = - ((2    sqrt(3) %i - 2   )
   3/2          2                    1/3
 (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
      1/3       1/3  3/2           2
 ((9 2    %i + 2    3   ) sqrt(27 p  - 4 p)
     1/3  7/2          1/3       4/3               4/3
 + (2    3    %i + 27 2   ) p - 2    sqrt(3) %i - 2   ) - 16)
                     5/3   3/2          2                    1/3
/(24 p)], [x = - (- 2    (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
   1/3  3/2          2              1/3      4/3
 (2    3    sqrt(27 p  - 4 p) + 27 2    p - 2   ) - 4)/12,
        5/3   3/2          2                    1/3
y = (- 2    (3    sqrt(27 p  - 4 p) - 27 p + 2)
     3/2          2                    2/3
 + (3    sqrt(27 p  - 4 p) - 27 p + 2)
   1/3  3/2          2              1/3      4/3
 (2    3    sqrt(27 p  - 4 p) + 27 2    p - 2   ) + 8)/(12 p)]]
```

```
(%i6) /* substitute %pi for p */
 s:subst(%pi,p,s)$
(%i7) /* Check the solution */
 ratsimp(subst(s[1],[eq1,eq2]));
(%o7)                         [0, 0]
(%i8) ratsimp(subst(s[2],[eq1,eq2]));
(%o8)                         [0, 0]
(%i9) ratsimp(subst(s[3],[eq1,eq2]));
(%o9)                         [0, 0]
(%i10) /* numerical values of solution */
 rectform(float(s));
(%o10) [[x = 1.0980298583288306 - 1.1920356507617584 %i,
y = 0.37943673232099623 %i - 0.031203873047263193],
[x = 1.1920356507617584 %i + 1.0980298583288306,
y = - 0.37943673232099623 %i - 0.031203873047263193],
[x = - 1.1960597166576612, y = 0.6990275184621078]]
```

22.2.4 Example 3: Assumptions on parameters

algsys may require assumptions on parameters to obtain a solution. No general guidance can be provided. Users may be able to use their knowledge of the problem to restrict the range of parameters appropriately.

```
(%i1) eqs:[y-x=0, g*x*y-h=0, z+(x+1)/y-x-1=0];
                                          x + 1
(%o1)   [y - x = 0, g x y - h = 0, z + ----- - x - 1 = 0]
                                           y
(%i2) /* without assumptions no solution is found */
 algsys(eqs,[x,y,z]);
(%o2)                          []
(%i3) assume(h>0);
(%o3)                        [h > 0]
(%i4) algsys(eqs,[x,y,z]);
            sqrt(h)      sqrt(h)           h - g
(%o4) [[x = -------, y = -------, z = ---------------],
            sqrt(g)      sqrt(g)      sqrt(g) sqrt(h)
                  sqrt(h)        sqrt(h)             h - g
           [x = - -------, y = - -------, z = - ---------------]]
                  sqrt(g)        sqrt(g)        sqrt(g) sqrt(h)
(%i5) /* With a different assumption the solution has a different form */
 (forget(h>0),assume(h<0));
(%o5)                        [h < 0]
```

```
(%i6) algsys(eqs,[x,y,z]);
                %i sqrt(- h)          %i sqrt(- h)
(%o6) [[x = - ------------, y = - ------------,
                  sqrt(g)               sqrt(g)
        sqrt(- h) (%i h - %i g)
z = - ----------------------],
              sqrt(g) h
       %i sqrt(- h)        %i sqrt(- h)
[x = ------------, y = ------------,
         sqrt(g)             sqrt(g)
      sqrt(- h) (%i h - %i g)
z = ----------------------]]
            sqrt(g) h
```

22.2.5 Example 4: Preproccess equations with poly_reduced_grobner

In this example the direct solution of the system of equations using *algsys* finds four real solutions. After preprocessing with `poly_reduced_grobner`, ten solutions are found.

Inspection of the reduced Gröbner basis returned at (%o5) shows that it comprises a degree-10 polynomial in y and an expression linear in x and of degree-8 in y. There can be at most 10 solutions for y, and exactly one x solution for each y solution, for a maximum of ten solutions overall.

```
(%i1) p1:-x*y^3+y^2+x^4-9*x/8;
                           3    2    4   9 x
(%o1)                - x y  + y  + x  - ---
                                        8
(%i2) p2:y^4-x^3*y-9*y/8+x^2;
                         4    3     9 y    2
(%o2)                   y  - x  y - --- + x
                                    8
(%i3) algsys([p1,p2],[x,y]);
            1               9      9               1
(%o3) [[x = -, y = 1], [x = -, y = -], [x = 1, y = -],
            2               8      8               2
                                                [x = 0, y = 0]]
(%i4) /* load grobner package */
 load("grobner")$
(%i5) eqs:poly_reduced_grobner([p1,p2],[x,y]);
                  8           5           2
(%o5) [- 200704 y  + 437832 y  - 263633 y  + 53010 x,
                                  10          7         4
                            4096 y   - 10440 y  + 7073 y  - 729 y]
```

```
(%i6) algsys(eqs,[x,y]);
                                      1        9      9
(%o6) [[x = 0, y = 0], [x = 1, y = -], [x = -, y = -],
                                      2        8      8
     1               sqrt(3) %i - 1        sqrt(3) %i + 1
[x = -, y = 1], [x = --------------, y = - --------------],
     2                     4                      2
       sqrt(3) %i + 1      sqrt(3) %i - 1
[x = - --------------, y = --------------],
             4                   2
      5/2                5/2
     3    %i - 9        3    %i + 9
[x = -----------, y = - -----------],
        16                 16
       5/2                5/2
       3    %i + 9        3    %i - 9
[x = - -----------, y = -----------],
          16                 16
     sqrt(3) %i - 1        sqrt(3) %i + 1
[x = --------------, y = - --------------],
          2                     4
       sqrt(3) %i + 1      sqrt(3) %i - 1
[x = - --------------, y = --------------]]
             2                   4
```

22.2.6 Example 5: Intersection of a sphere with two paraboloids

This problem is the intersection of a sphere with two paraboloids ([Allgower et al 1992], page 445, Example 2, p841).

There are two real solutions y = z = (sqrt(5)-1)/2, x = +/- sqrt(z-y^2) = +/- sqrt(sqrt(5)-2).

```
(%i1) algsys([x^2+y^2+z^2-1,z-x^2-y^2,y-x^2-z^2],[x,y,z]);
                              sqrt(5) - 1        sqrt(5) + 1
(%o1) [[x = sqrt(2) %i, y = -----------, z = - -----------],
                                   2                  2
                         sqrt(5) + 1      sqrt(5) - 1
[x = sqrt(2) %i, y = - -----------, z = -----------],
                              2                2
                         sqrt(5) - 1        sqrt(5) + 1
[x = - sqrt(2) %i, y = -----------, z = - -----------],
                              2                  2
                           sqrt(5) + 1      sqrt(5) - 1
[x = - sqrt(2) %i, y = - -----------, z = -----------],
                                2                2
                                  sqrt(5) + 1        sqrt(5) + 1
[x = sqrt(- sqrt(5) - 2), y = - -----------, z = - -----------],
                                       2                  2
                                    sqrt(5) + 1
[x = - sqrt(- sqrt(5) - 2), y = - -----------,
                                         2
      sqrt(5) + 1                                      sqrt(5) - 1
z = - -----------], [x = sqrt(sqrt(5) - 2), y = -----------,
           2                                             2
    sqrt(5) - 1                                      sqrt(5) - 1
z = -----------], [x = - sqrt(sqrt(5) - 2), y = -----------,
         2                                             2
    sqrt(5) - 1
z = -----------]]
         2
```

22.3 References for equations

- (Beyer 1984) Beyer, William A., Solution of Simultaneous Polynomial Equations by Elimination in MACSYMA, Proceedings of the 1984 MACSYMA Users' Conference, pp 110-120
- (Allgower et al 1992) Eugene L. Allgower, Kurt Georg and Rick Miranda, The Method of Resultants for Computing Real Solutions of Polynomial Systems, SIAM Journal on Numerical Analysis, Vol 29 No 3, Jun 1992, pp 831-844 doi:10.1137/0729051

23 Differential Equations

23.1 Introduction to Differential Equations

This section describes the functions available in Maxima to obtain analytic solutions for some specific types of first and second-order equations. To obtain a numerical solution for a system of differential equations, see the additional package `dynamics`. For graphical representations in phase space, see the additional package `plotdf`.

23.2 Functions and Variables for Differential Equations

`bc2` (*solution*, *xval1*, *yval1*, *xval2*, *yval2*) [Function]

Solves a boundary value problem for a second order differential equation. Here: *solution* is a general solution to the equation, as found by `ode2`; *xval1* specifies the value of the independent variable in a first point, in the form `x = x1`, and *yval1* gives the value of the dependent variable in that point, in the form `y = y1`. The expressions *xval2* and *yval2* give the values for these variables at a second point, using the same form.

See `ode2` for an example of its usage.

`desolve` [Function]

`desolve` (*eqn*, *y*)
`desolve` ([*eqn_1*, ..., *eqn_n*], [*y_1*, ..., *y_n*])

The function `desolve` solves systems of linear ordinary differential equations using Laplace transform. Here the *eqn*'s are differential equations in the dependent variables *y_1*, ..., *y_n*. The functional dependence of *y_1*, ..., *y_n* on an independent variable, for instance *x*, must be explicitly indicated in the variables and its derivatives. For example, the *correct* way to define the differential equations would be:

```
eqn_1: 'diff(f(x),x,2) = sin(x) + 'diff(g(x),x);
eqn_2: 'diff(f(x),x) + x^2 - f(x) = 2*'diff(g(x),x,2);
```

The call to the function `desolve` would then be:

```
desolve([eqn_1, eqn_2], [f(x),g(x)]);
```

If initial conditions at `x=0` are known, they can be supplied before calling `desolve` by using `atvalue`.

```
(%i1) 'diff(f(x),x)='diff(g(x),x)+sin(x);
                    d           d
(%o1)               -- (f(x)) = -- (g(x)) + sin(x)
                    dx          dx
(%i2) 'diff(g(x),x,2)='diff(f(x),x)-cos(x);
                     2
                    d            d
(%o2)               --- (g(x)) = -- (f(x)) - cos(x)
                      2          dx
                    dx
(%i3) atvalue('diff(g(x),x),x=0,a);
(%o3)                            a
```

```
(%i4) atvalue(f(x),x=0,1);
(%o4)                                1
(%i5) desolve([%o1,%o2],[f(x),g(x)]);
                     x
(%o5) [f(x) = %e  a - a + 1, g(x) =
                                                        x
                                          cos(x) + %e  a - a + g(0) - 1]
(%i6) [%o1,%o2],%o5,diff;
             x       x      x                 x
(%o6)   [%e  a = %e  a, %e  a - cos(x) = %e  a - cos(x)]
```

If `desolve` cannot obtain a solution, it returns `false`.

See also `ode2`, `drawdf` and `rk`.

`ic1` (*solution*, *xval*, *yval*) [Function]

Solves initial value problems for first order differential equations. Here *solution* is a general solution to the equation, as found by `ode2`, *xval* gives an initial value for the independent variable in the form *x = x0*, and *yval* gives the initial value for the dependent variable in the form *y = y0*.

See `ode2` for an example of its usage.

`ic2` (*solution*, *xval*, *yval*, *dval*) [Function]

Solves initial value problems for second-order differential equations. Here *solution* is a general solution to the equation, as found by `ode2`, *xval* gives the initial value for the independent variable in the form *x = x0*, *yval* gives the initial value of the dependent variable in the form *y = y0*, and *dval* gives the initial value for the first derivative of the dependent variable with respect to independent variable, in the form `'diff(y,x) = dy0`.

See `ode2` for an example of its usage.

`ode2` (*eqn*, *dvar*, *ivar*) [Function]

The function `ode2` solves an ordinary differential equation (ODE) of first or second order. It takes three arguments: an ODE given by *eqn*, the dependent variable *dvar*, and the independent variable *ivar*. When successful, it returns either an explicit or implicit solution for the dependent variable. `%c` is used to represent the integration constant in the case of first-order equations, and `%k1` and `%k2` the constants for second-order equations. The dependence of the dependent variable on the independent variable does not have to be written explicitly, as in the case of `desolve`, but the independent variable must always be given as the third argument.

If `ode2` cannot obtain a solution for whatever reason, it returns `false`, after perhaps printing out an error message. The methods implemented for first order equations in the order in which they are tested are: linear, separable, exact - perhaps requiring an integrating factor, homogeneous, Bernoulli's equation, and a generalized homogeneous method. The types of second-order equations which can be solved are: constant coefficients, exact, linear homogeneous with non-constant coefficients which can be transformed to constant coefficients, the Euler or equi-dimensional equation, equations solvable by the method of variation of parameters, and equations which are free

of either the independent or of the dependent variable so that they can be reduced to two first order linear equations to be solved sequentially.

In the course of solving ODE's, several variables are set purely for informational purposes: `method` denotes the method of solution used (e.g., `linear`), `intfactor` denotes any integrating factor used, `odeindex` denotes the index for Bernoulli's method or for the generalized homogeneous method, and `yp` denotes the particular solution for the variation of parameters technique.

In order to solve initial value problems (IVP) functions `ic1` and `ic2` are available for first and second order equations, and to solve second-order boundary value problems (BVP) the function `bc2` can be used.

See also `desolve`, `drawdf` and `rk`.

Example:

```
(%i1) x^2*'diff(y,x) + 3*y*x = sin(x)/x;
                       2 dy           sin(x)
(%o1)                 x  -- + 3 x y = ------
                         dx             x
(%i2) soln1: ode2(%,y,x);
                              %c - cos(x)
(%o2)                      y = -----------
                                    3
                                   x
(%i3) ic1 (soln1,x=%pi,y=0);
                                cos(x) + 1
(%o3)                     y = - ----------
                                     3
                                    x
(%i4) 'diff(y,x,2) + y*'diff(y,x)^3 = 0;
                           2
                          d y      dy 3
(%o4)                     --- + y (--)  = 0
                            2      dx
                          dx
(%i5) soln2: ode2(%,y,x);
                        3
                       y  + 6 %k1 y
(%o5)                  ------------ = x + %k2
                            6
(%i6) ratsimp (ic2(soln2,x=0,y=0,'diff(y,x)=2));
                            3
                           y  + 3 y
(%o6)                      -------- = x
                              6
```

```
(%i7) bc2 (soln2,x=0,y=1,x=1,y=3);
                              3
                             y  - 10 y       3
(%o7)                        --------- = x - -
                                 6           2
```

`method` [System variable]

The variable `method` is set by `ode2` to the successful solution method.

`%c` [Variable]

`%c` is the integration constant in the solutions of first order ODEs returned from `ode2`.

`%k1` [Variable]

`%k1` is the first integration constant in the solutions of second order ODEs returned from `ode2`.

`%k2` [Variable]

`%k2` is the second integration constant in the solutions of second order ODEs returned from `ode2`.

`yp` [Variable]

`yp` is the particular solution of an ODE found by `ode2` when using the variation of parameters technique.

24 Numerical

24.1 Introduction to fast Fourier transform

The `fft` package comprises functions for the numerical (not symbolic) computation of the fast Fourier transform. This is limited to sequences whit length that is a power of two. For more general lengths, consider the `fftpack5` package that supports sequences of any length, but is most efficient if the length is a product of small primes.

24.2 Functions and Variables for fft

`polartorect (r, t)` [Function]

Translates complex values of the form re^{it} to the form $a+bi$, where r is the magnitude and t is the phase. r and t are 1-dimensional arrays of the same size. The array size need not be a power of 2.

The original values of the input arrays are replaced by the real and imaginary parts, a and b, on return. The outputs are calculated as

$$a = r\cos t$$
$$b = r\sin t$$

`polartorect` is the inverse function of `recttopolar`.

`load("fft")` loads this function. See also `fft`.

`recttopolar (a, b)` [Function]

Translates complex values of the form $a+bi$ to the form re^{it}, where a is the real part and b is the imaginary part. a and b are 1-dimensional arrays of the same size. The array size need not be a power of 2.

The original values of the input arrays are replaced by the magnitude and angle, r and t, on return. The outputs are calculated as

$$r = \sqrt{a^2+b^2}$$
$$t = \operatorname{atan2}(b, a)$$

The computed angle is in the range $-\pi$ to π.

`recttopolar` is the inverse function of `polartorect`.

`load("fft")` loads this function. See also `fft`.

`inverse_fft (y)` [Function]

Computes the inverse complex fast Fourier transform. y is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions $a+bi$ where a and b are literal numbers or symbolic constants.

`inverse_fft` returns a new object of the same type as y, which is not modified. Results are always computed as floats or expressions `a + b*%i` where `a` and `b` are floats.

If bigfloat precision is needed the function bf_inverse_fft can be used instead as a drop-in replacement of inverse_fft that is slower, but supports bfloats.

The inverse discrete Fourier transform is defined as follows. Let x be the output of the inverse transform. Then for j from 0 through $n-1$,

$$x[j] = \sum_{k=0}^{n-1} y[k] e^{-2i\pi jk/n}$$

As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

load("fft") loads this function.

See also fft (forward transform), recttopolar, and polartorect.

Examples:

Real data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $
(%i4) L1 : inverse_fft (L);
(%o4) [0.0, - 14.49 %i - 0.8284, 0.0, 4.828 - 2.485 %i, 0.0,
                          2.485 %i + 4.828, 0.0, 14.49 %i - 0.8284]
(%i5) L2 : fft (L1);
(%o5) [1.0, 2.0 - 4.441e-16 %i, 3.0, 4.441e-16 %i + 4.0, - 1.0,
                  4.441e-16 %i - 2.0, - 3.0, - 4.441e-16 %i - 4.0]
(%i6) lmax (abs (L2 - L));
(%o6)                         4.441e-16
```

Complex data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1] $
(%i4) L1 : inverse_fft (L);
(%o4) [4.0, 2.828 %i + 2.828, - 2.0 %i - 2.0, 4.0, 0.0,
                             - 2.828 %i - 2.828, 2.0 %i - 2.0, 4.0]
(%i5) L2 : fft (L1);
(%o5) [1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, - 1.0, - 1.0,
                                  1.0 - 1.0 %i, 1.0 %i + 1.0, 1.0]
(%i6) lmax (abs (L2 - L));
(%o6)                            0.0
```

fft (*x*) [Function]

Computes the complex fast Fourier transform. *x* is a list or array (named or unnamed) which contains the data to transform. The number of elements must be a power of 2. The elements must be literal numbers (integers, rationals, floats, or bigfloats) or symbolic constants, or expressions a + b*%i where a and b are literal numbers or symbolic constants.

fft returns a new object of the same type as *x*, which is not modified. Results are always computed as floats or expressions a + b*%i where a and b are floats. If bigfloat

precision is needed the function bf_fft can be used instead as a drop-in replacement of fft that is slower, but supports bfloats. In addition if it is known that the input consists of only real values (no imaginary parts), real_fft can be used which is potentially faster.

The discrete Fourier transform is defined as follows. Let y be the output of the transform. Then for k from 0 through $n-1$,

$$y[k] = \frac{1}{n}\sum_{j=0}^{n-1} x[j]e^{+2i\pi jk/n}$$

As there are various sign and normalization conventions possible, this definition of the transform may differ from that used by other mathematical software.

When the data x are real, real coefficients a and b can be computed such that

$$x[j] = \sum_{k=0}^{n/2}\left(a[k]\cos\frac{2\pi jk}{n} + b[k]\sin\frac{2\pi jk}{n}\right)$$

with

$$\begin{aligned} a[0] &= \text{realpart}(y[0]) \\ b[0] &= 0 \end{aligned}$$

and, for k from 1 through $n/2-1$,

$$\begin{aligned} a[k] &= \text{realpart}(y[k]+y[n-k]) \\ b[k] &= \text{imagpart}(y[n-k]-y[k]) \end{aligned}$$

and

$$\begin{aligned} a\left[\frac{n}{2}\right] &= \text{realpart}\left(y\left[\frac{n}{2}\right]\right) \\ b\left[\frac{n}{2}\right] &= 0 \end{aligned}$$

load("fft") loads this function.

See also inverse_fft (inverse transform), recttopolar, and polartorect.. See real_fft for FFTs of a real-valued input, and bf_fft and bf_real_fft for operations on bigfloat values. Finally, for transforms of any size (but limited to float values), see fftpack5_fft and fftpack5_real_fft.

Examples:

Real data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, -1, -2, -3, -4] $
```

```
(%i4) L1 : fft (L);
(%o4) [0.0, 1.811 %i - 0.1036, 0.0, 0.3107 %i + 0.6036, 0.0,
                   0.6036 - 0.3107 %i, 0.0, - 1.811 %i - 0.1036]
(%i5) L2 : inverse_fft (L1);
(%o5) [1.0, 4.441e-16 %i + 2.0, 3.0, 4.0 - 4.441e-16 %i, - 1.0,
                - 4.441e-16 %i - 2.0, - 3.0, 4.441e-16 %i - 4.0]
(%i6) lmax (abs (L2 - L));
(%o6)                        4.441e-16
```

Complex data.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1] $
(%i4) L1 : fft (L);
(%o4) [0.5, 0.5, 0.25 %i - 0.25, - 0.3536 %i - 0.3536, 0.0, 0.5,
                            - 0.25 %i - 0.25, 0.3536 %i + 0.3536]
(%i5) L2 : inverse_fft (L1);
(%o5) [1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, - 1.0, - 1.0,
                                1.0 - 1.0 %i, 1.0 %i + 1.0, 1.0]
(%i6) lmax (abs (L2 - L));
(%o6)                           0.0
```

Computation of sine and cosine coefficients.

```
(%i1) load ("fft") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, 5, 6, 7, 8] $
(%i4) n : length (L) $
(%i5) x : make_array (any, n) $
(%i6) fillarray (x, L) $
(%i7) y : fft (x) $
(%i8) a : make_array (any, n/2 + 1) $
(%i9) b : make_array (any, n/2 + 1) $
(%i10) a[0] : realpart (y[0]) $
(%i11) b[0] : 0 $
(%i12) for k : 1 thru n/2 - 1 do
   (a[k] : realpart (y[k] + y[n - k]),
    b[k] : imagpart (y[n - k] - y[k]));
(%o12)                         done
(%i13) a[n/2] : y[n/2] $
(%i14) b[n/2] : 0 $
(%i15) listarray (a);
(%o15)          [4.5, - 1.0, - 1.0, - 1.0, - 0.5]
(%i16) listarray (b);
(%o16)             [0, 2.414, 1.0, 0.4142, 0]
(%i17) f(j) := sum (a[k] * cos (2*%pi*j*k / n) + b[k] * sin (2*%pi*j*k / n), k, 0
(%i18) makelist (float (f (j)), j, 0, n - 1);
(%o18)      [1.0, 8.0, 7.0, 6.0, 5.0, 4.0, 3.0, 2.0]
```

real_fft (*x*) [Function]

Computes the fast Fourier transform of a real-valued sequence *x*. This is equivalent to performing `fft(x)`, except that only the first `N/2+1` results are returned, where `N` is the length of *x*. `N` must be power of two.

No check is made that *x* contains only real values.

The symmetry properties of the Fourier transform of real sequences to reduce he complexity. In particular the first and last output values of `real_fft` are purely real. For larger sequences, `real_fft` may be computed more quickly than `fft`.

Since the output length is short, the normal `inverse_fft` cannot be directly used. Use `inverse_real_fft` to compute the inverse.

inverse_real_fft (*y*) [Function]

Computes the inverse Fourier transform of *y*, which must have a length of `N/2+1` where `N` is a power of two. That is, the input *x* is expected to be the output of `real_fft`.

No check is made to ensure that the input has the correct format. (The first and last elements must be purely real.)

bf_inverse_fft (*y*) [Function]

Computes the inverse complex fast Fourier transform. This is the bigfloat version of `inverse_fft` that converts the input to bigfloats and returns a bigfloat result.

bf_fft (*y*) [Function]

Computes the forward complex fast Fourier transform. This is the bigfloat version of `fft` that converts the input to bigfloats and returns a bigfloat result.

bf_real_fft (*x*) [Function]

Computes the forward fast Fourier transform of a real-valued input returning a bigfloat result. This is the bigfloat version of `real_fft`.

bf_inverse_real_fft (*y*) [Function]

Computes the inverse fast Fourier transform with a real-valued bigfloat output. This is the bigfloat version of `inverse_real_fft`.

24.3 Functions and Variables for FFTPACK5

`FFTPACK5` provides several routines to compute Fourier transforms for both real and complex sequences and their inverses. The forward transform is defined the same as for `fft`. The major difference is the length of the sequence is not constrained to be a power of two. In fact, any length is supported, but it is most efficient when the length has the form $2^r * 3^s * 5^t$.

`load("fftpack5")` loads this function.

fftpack5_fft (*x*) [Function]

Like `fft` (`fft`), this computes the fast Fourier transform of a complex sequence. However, the length of *x* is not limited to a power of 2.

`load("fftpack5")` loads this function.

Examples:

Real data.

```
(%i1) load("fftpack5") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 2, 3, 4, -1, -2 ,-3, -4] $
(%i4) L1 : fftpack5_fft(L);
(%o4) [0.0, 1.811 %i - 0.1036, 0.0, 0.3107 %i + 0.6036, 0.0,
                                  0.6036 - 0.3107 %i, 0.0, (- 1.811 %i) - 0.1036]
(%i5) L2 : fftpack5_inverse_fft(L1);
(%o5) [1.0, 4.441e-16 %i + 2.0, 1.837e-16 %i + 3.0, 4.0 - 4.441e-16 %i,
     - 1.0, (- 4.441e-16 %i) - 2.0, (- 1.837e-16 %i) - 3.0, 4.441e-16
       %i - 4.0]
(%i6) lmax (abs (L2-L));
(%o6)                         4.441e-16
(%i7) L : [1, 2, 3, 4, 5, 6]$
(%i8) L1 : fftpack5_fft(L);
(%o8) [3.5, (- 0.866 %i) - 0.5, (- 0.2887 %i) - 0.5, (- 1.48e-16 %i) - 0.5,
                                                  0.2887 %i - 0.5, 0.866
                                                         %%i - 0.5]
(%i9) L2 : fftpack5_inverse_fft (L1);
(%o9) [1.0 - 1.48e-16 %i, 3.701e-17 %i + 2.0, 3.0 - 1.48e-16 %i,
                      4.0 - 1.811e-16 %i, 5.0 - 1.48e-16 %i, 5.881e-16
                            %i + 6.0]
(%i10) lmax (abs (L2-L));
(%o10)                              9.064e-16
```

Complex data.

```
(%i1) load("fftpack5") $
(%i2) fpprintprec : 4 $
(%i3) L : [1, 1 + %i, 1 - %i, -1, -1, 1 - %i, 1 + %i, 1] $
(%i4) L1 : fftpack5_inverse_fft (L);
(%o4) [4.0, 2.828 %i + 2.828, (- 2.0 %i) - 2.0, 4.0, 0.0,
                                      (- 2.828 %i) - 2.828, 2.0 %i - 2.0, 4.0]
(%i5) L2 : fftpack5_fft(L1);
(%o5) [1.0, 1.0 %i + 1.0, 1.0 - 1.0 %i, (- 2.776e-17 %i) - 1.0, - 1.0,
                                  1.0 - 1.0 %i, 1.0 %i + 1.0, 1.0 -
                                             %2.776e-17 %i]
(%i6) lmax(abs(L2-L));
(%o6)                               1.11e-16
```

`fftpack5_inverse_fft` (*y*) [Function]

Computes the inverse complex Fourier transform, like `inverse_fft`, but is not constrained to be a power of two.

`fftpack5_real_fft` (*x*) [Function]

Computes the fast Fourier transform of a real-valued sequence *x*, just like `real_fft`, except the length is not constrained to be a power of two.

Examples:

```
(%i1) fpprintprec : 4 $
```

```
(%i2) L : [1, 2, 3, 4, 5, 6] $
(%i3) L1 : fftpack5_real_fft(L);
(%o3)       [3.5, (- 0.866 %i) - 0.5, (- 0.2887 %i) - 0.5, - 0.5]
(%i4) L2 : fftpack5_inverse_real_fft(L1, 6);
(%o4)                  [1.0, 2.0, 3.0, 4.0, 5.0, 6.0]
(%i5) lmax(abs(L2-L));
(%o5)                              1.332e-15
(%i6) fftpack5_inverse_real_fft(L1, 7);
(%o6)            [0.5, 2.083, 2.562, 3.7, 4.3, 5.438, 5.917]
```

The last example shows how important it to set the length correctly for `fftpack5_inverse_real_fft`.

`fftpack5_inverse_real_fft` (*y*, *n*) [Function]

Computes the inverse Fourier transform of *y*, which must have a length of `floor(n/2) + 1`. The length of sequence produced by the inverse transform must be specified by *n*. This is required because the length of *y* does not uniquely determine *n*. The last element of *y* is always real if *n* is even, but it can be complex when *n* is odd.

24.4 Functions for numerical solution of equations

`horner` [Function]

`horner` (*expr*, *x*)
`horner` (*expr*)

Returns a rearranged representation of *expr* as in Horner's rule, using *x* as the main variable if it is specified. `x` may be omitted in which case the main variable of the canonical rational expression form of *expr* is used.

`horner` sometimes improves stability if `expr` is to be numerically evaluated. It is also useful if Maxima is used to generate programs to be run in Fortran. See also `stringout`.

```
(%i1) expr: 1e-155*x^2 - 5.5*x + 5.2e155;
                                2
(%o1)             1.0e-155 x  - 5.5 x + 5.2e155
(%i2) expr2: horner (%, x), keepfloat: true;
(%o2)         1.0 ((1.0e-155 x - 5.5) x + 5.2e155)
(%i3) ev (expr, x=1e155);
Maxima encountered a Lisp error:

 Arithmetic error FLOATING-POINT-OVERFLOW signalled.
Operation was *, operands (1.0e155 NIL).

Automatically continuing.
To enable the Lisp debugger set *debugger-hook* to nil.
(%i4) ev (expr2, x=1e155);
(%o4)                 7.000000000000006e154
```

`find_root` (*expr*, *x*, *a*, *b*, [*abserr*, *relerr*]) [Function]
`find_root` (*f*, *a*, *b*, [*abserr*, *relerr*]) [Function]

bf_find_root (*expr*, *x*, a, *b*, [*abserr*, *relerr*]) [Function]
bf_find_root (*f*, a, *b*, [*abserr*, *relerr*]) [Function]
find_root_error [Option variable]
find_root_abs [Option variable]
find_root_rel [Option variable]

Finds a root of the expression *expr* or the function *f* over the closed interval $[a, b]$. The expression *expr* may be an equation, in which case find_root seeks a root of lhs(*expr*) - rhs(*expr*).

Given that Maxima can evaluate *expr* or *f* over $[a, b]$ and that *expr* or *f* is continuous, find_root is guaranteed to find the root, or one of the roots if there is more than one.

find_root initially applies binary search. If the function in question appears to be smooth enough, find_root applies linear interpolation instead.

bf_find_root is a bigfloat version of find_root. The function is computed using bigfloat arithmetic and a bigfloat result is returned. Otherwise, bf_find_root is identical to find_root, and the following description is equally applicable to bf_find_root.

The accuracy of find_root is governed by abserr and relerr, which are optional keyword arguments to find_root. These keyword arguments take the form key=val. The keyword arguments are

abserr Desired absolute error of function value at root. Default is find_root_abs.

relerr Desired relative error of root. Default is find_root_rel.

find_root stops when the function in question evaluates to something less than or equal to abserr, or if successive approximants *x_0*, *x_1* differ by no more than relerr * max(abs(x_0), abs(x_1)). The default values of find_root_abs and find_root_rel are both zero.

find_root expects the function in question to have a different sign at the endpoints of the search interval. When the function evaluates to a number at both endpoints and these numbers have the same sign, the behavior of find_root is governed by find_root_error. When find_root_error is true, find_root prints an error message. Otherwise find_root returns the value of find_root_error. The default value of find_root_error is true.

If *f* evaluates to something other than a number at any step in the search algorithm, find_root returns a partially-evaluated find_root expression.

The order of *a* and *b* is ignored; the region in which a root is sought is $[min(a, b), max(a, b)]$.

Examples:

```
(%i1) f(x) := sin(x) - x/2;
                                          x
(%o1)                   f(x) := sin(x) - -
                                          2
(%i2) find_root (sin(x) - x/2, x, 0.1, %pi);
(%o2)                    1.895494267033981
```

```
(%i3) find_root (sin(x) = x/2, x, 0.1, %pi);
(%o3)                    1.895494267033981
(%i4) find_root (f(x), x, 0.1, %pi);
(%o4)                    1.895494267033981
(%i5) find_root (f, 0.1, %pi);
(%o5)                    1.895494267033981
(%i6) find_root (exp(x) = y, x, 0, 100);
                              x
(%o6)           find_root(%e  = y, x, 0.0, 100.0)
(%i7) find_root (exp(x) = y, x, 0, 100), y = 10;
(%o7)                    2.302585092994046
(%i8) log (10.0);
(%o8)                    2.302585092994046
(%i9) fpprec:32;
(%o9)                           32
(%i10) 32;
(%o10)                          32
(%i11) bf_find_root (exp(x) = y, x, 0, 100), y = 10;
(%o11)         2.3025850929940456840179914546844b0
(%i12) log(10b0);
(%o12)         2.3025850929940456840179914546844b0
```

newton (*expr*, *x*, *x_0*, *eps*) [Function]

Returns an approximate solution of *expr* = 0 by Newton's method, considering *expr* to be a function of one variable, *x*. The search begins with *x* = *x_0* and proceeds until abs(*expr*) < *eps* (with *expr* evaluated at the current value of *x*).

newton allows undefined variables to appear in *expr*, so long as the termination test abs(*expr*) < *eps* evaluates to true or false. Thus it is not necessary that *expr* evaluate to a number.

load("newton1") loads this function.

See also realroots, allroots, find_root and mnewton.

Examples:

```
(%i1) load ("newton1");
(%o1) /maxima/share/numeric/newton1.mac
(%i2) newton (cos (u), u, 1, 1/100);
(%o2)                   1.5706752771612507
(%i3) ev (cos (u), u = %);
(%o3)                  1.2104963335033528e-4
(%i4) assume (a > 0);
(%o4)                         [a > 0]
(%i5) newton (x^2 - a^2, x, a/2, a^2/100);
(%o5)                  1.0003048780487804 a
(%i6) ev (x^2 - a^2, x = %);
                                              2
(%o6)               6.098490481853958e-4 a
```

24.5 Introduction to numerical solution of differential equations

The Ordinary Differential Equations (ODE) solved by the functions in this section should have the form,

$$\frac{dy}{dx} = F(x, y)$$

which is a first-order ODE. Higher order differential equations of order n must be written as a system of n first-order equations of that kind. For instance, a second-order ODE should be written as a system of two equations

$$\frac{dx}{dt} = G(x, y, t) \qquad \frac{dy}{dt} = F(x, y, t)$$

The first argument in the functions will be a list with the expressions on the right-side of the ODE's. The variables whose derivatives are represented by those expressions should be given in a second list. In the case above those variables are x and y. The independent variable, t in the examples above, might be given in a separated option. If the expressions given do not depend on that independent variable, the system is called autonomous.

24.6 Functions for numerical solution of differential equations

`plotdf` [Function]

`plotdf` (*dydx*, *options...*)
`plotdf` (*dvdu*, [*u*,*v*], *options...*)
`plotdf` ([*dxdt*,*cdydt*], *options...*)
`plotdf` ([*dudt*,*cdvdt*], [*u*,*cv*], *options...*)

The function `plotdf` creates a two-dimensional plot of the direction field (also called slope field) for a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE's.

Plotdf requires Xmaxima, even if its run from a Maxima session in a console, since the plot will be created by the Tk scripts in Xmaxima. If Xmaxima is not installed plotdf will not work.

dydx, *dxdt* and *dydt* are expressions that depend on x and y. *dvdu*, *dudt* and *dvdt* are expressions that depend on u and v. In addition to those two variables, the expressions can also depend on a set of parameters, with numerical values given with the `parameters` option (the option syntax is given below), or with a range of allowed values specified by a *sliders* option.

Several other options can be given within the command, or selected in the menu. Integral curves can be obtained by clicking on the plot, or with the option `trajectory_at`. The direction of the integration can be controlled with the `direction` option, which can have values of *forward*, *backward* or *both*. The number of integration steps is given by `nsteps`; at each integration step the time increment will be adjusted automatically to produce displacements much smaller than the size of the plot window. The numerical method used is 4th order Runge-Kutta with variable time steps.

Plot window menu:

The menu bar of the plot window has the following five buttons:

Close: can be used to close the plot window.

Config: opens the configuration menu with several fields that show the ODE(s) in use and various other settings. If a pair of coordinates are entered in the field *Trajectory at* and the `enter` key is pressed, a new integral curve will be shown, in addition to the ones already shown.

Save: used to save a copy of the plot, in Postscript format, in the file specified in a field of the window that appears when that button is clicked.

Replot: replots the direction field with the new settings defined in the configuration menu and replots only the last integral curve that was previously plotted. If you just resized the plot window, the size and width of the arrows and curves will be adapted to the new size if you click on Replot.

Time plot: creates two new window showing the plots of the two variables in terms of time, for the last integral curve that was plotted.

Plot options:

Options can also be given within the `plotdf` itself, each one being a list of two or more elements. The first element in each option is the name of the option, and the remainder is the value or values assigned to the option.

The options which are recognized by `plotdf` are the following:

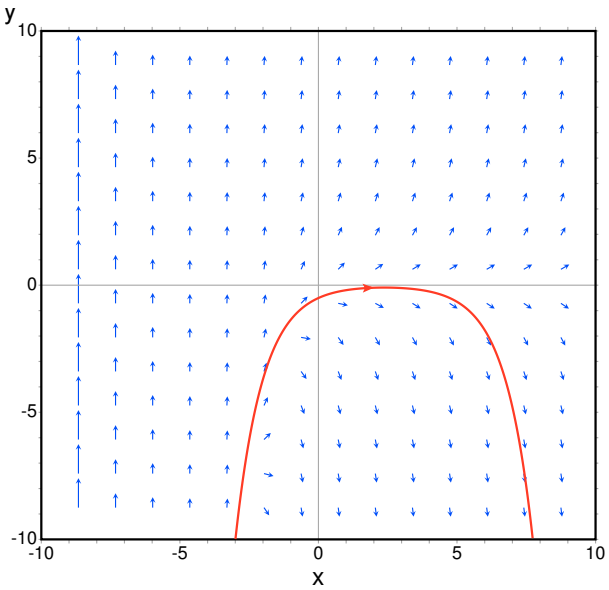
- *nsteps* defines the number of steps that will be used for the independent variable, to compute an integral curve. The default value is 100.
- *direction* defines the direction of the independent variable that will be followed to compute an integral curve. Possible values are `forward`, to make the independent variable increase `nsteps` times, with increments `tstep`, `backward`, to make the independent variable decrease, or `both` that will lead to an integral curve that extends `nsteps` forward, and `nsteps` backward. The keywords `right` and `left` can be used as synonyms for `forward` and `backward`. The default value is `both`.
- *tinitial* defines the initial value of variable t used to compute integral curves. Since the differential equations are autonomous, that setting will only appear in the plot of the curves as functions of t. The default value is 0.
- *versus_t* is used to create a second plot window, with a plot of an integral curve, as two functions x, y, of the independent variable t. If `versus_t` is given any value different from 0, the second plot window will be displayed. The second plot window includes another menu, similar to the menu of the main plot window. The default value is 0.
- *trajectory_at* defines the coordinates *xinitial* and *yinitial* for the starting point of an integral curve. The option is empty by default.
- *parameters* defines a list of parameters, and their numerical values, used in the definition of the differential equations. The name and values of the parameters must be given in a string with a comma-separated sequence of pairs `name=value`.
- *sliders* defines a list of parameters that will be changed interactively using slider buttons, and the range of variation of those parameters. The names and ranges of the parameters must be given in a string with a comma-separated sequence of elements `name=min:max`

- *tstep* sets the value of the time intervals used in the integration algorithm. It must be a floating point number; you might have to adjust its value to get good results for the integral curves. If not given, a default value will be chosen according to the region to be plotted.
- *xfun* defines a string with semi-colon-separated sequence of functions of x to be displayed, on top of the direction field. Those functions will be parsed by Tcl and not by Maxima.
- x should be followed by two numbers, which will set up the minimum and maximum values shown on the horizontal axis. If the variable on the horizontal axis is not x, then this option should have the name of the variable on the horizontal axis. The default horizontal range is from -10 to 10.
- y should be followed by two numbers, which will set up the minimum and maximum values shown on the vertical axis. If the variable on the vertical axis is not y, then this option should have the name of the variable on the vertical axis. The default vertical range is from -10 to 10.
- *xaxislabel* will be used to identify the horizontal axis. Its default value is the name of the first state variable.
- *yaxislabel* will be used to identify the vertical axis. Its default value is the name of the second state variable.
- *number_of_arrows* should be set to a square number and defines the approximate density of the arrows being drawn. The default value is 225.

Examples:

- To show the direction field of the differential equation $y' = e^{-x} + y$ and the solution that goes through $(2, -0.1)$:

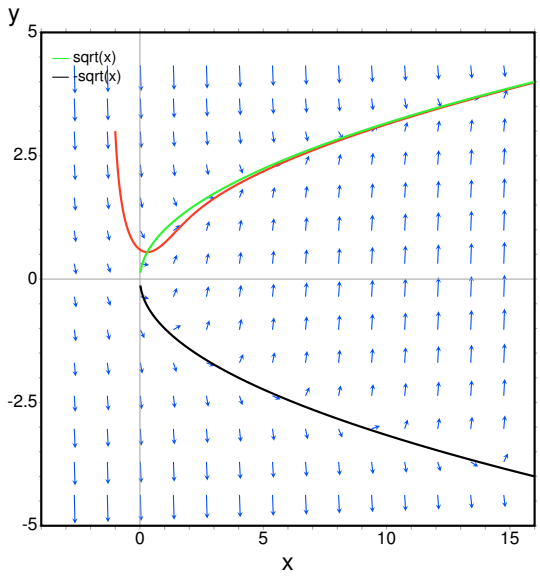
```
(%i1) plotdf(exp(-x)+y,[trajectory_at,2,-0.1])$
```



- To obtain the direction field for the equation $dy/dx = x - y^2$ and the solution with initial condition $y(-1) = 3$, we can use the command:

```
(%i1) plotdf(x-y^2,[xfun,"sqrt(x);-sqrt(x)"],
         [trajectory_at,-1,3], [direction,forward],
         [y,-5,5], [x,-4,16])$
```

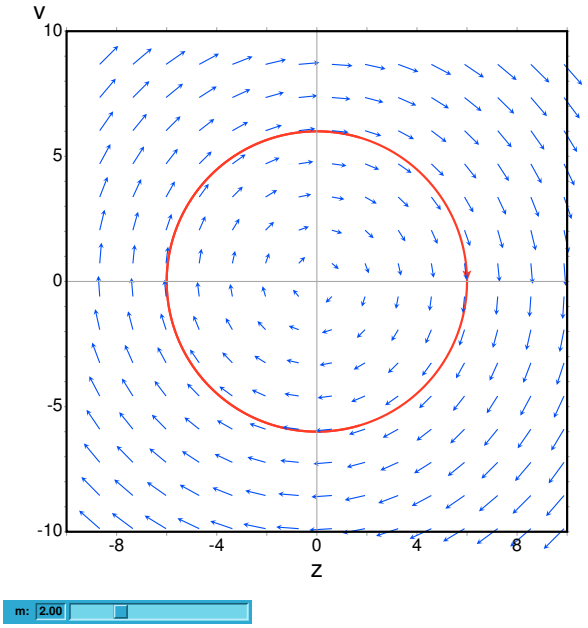
The graph also shows the function $y = \sqrt{x}$.



- The following example shows the direction field of a harmonic oscillator, defined by the two equations $dz/dt = v$ and $dv/dt = -kz/m$,

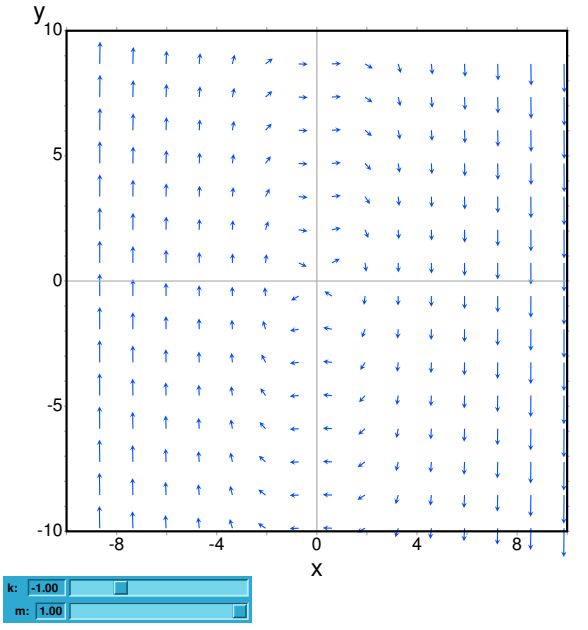
  and the integral curve through $(z, v) = (6, 0)$, with a slider that will allow you to change the value of m interactively (k is fixed at 2):

```
(%i1) plotdf([v,-k*z/m], [z,v], [parameters,"m=2,k=2"],
          [sliders,"m=1:5"], [trajectory_at,6,0])$
```



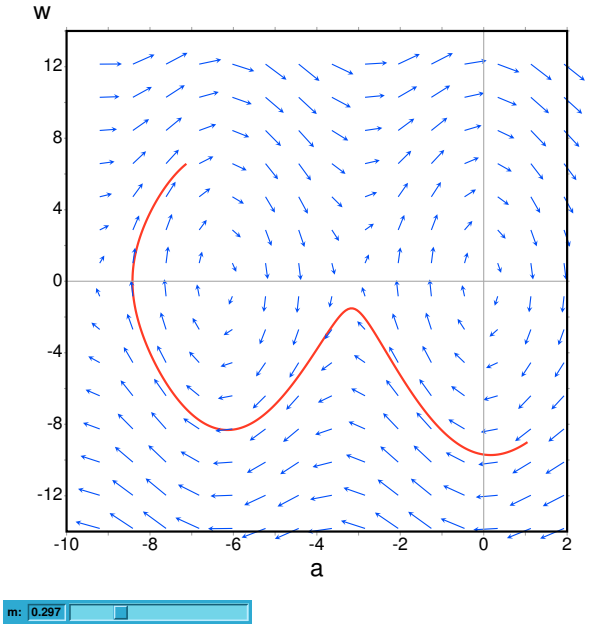
- To plot the direction field of the Duffing equation, $mx'' + cx' + kx + bx^3 = 0$, we introduce the variable $y = x'$ and use:

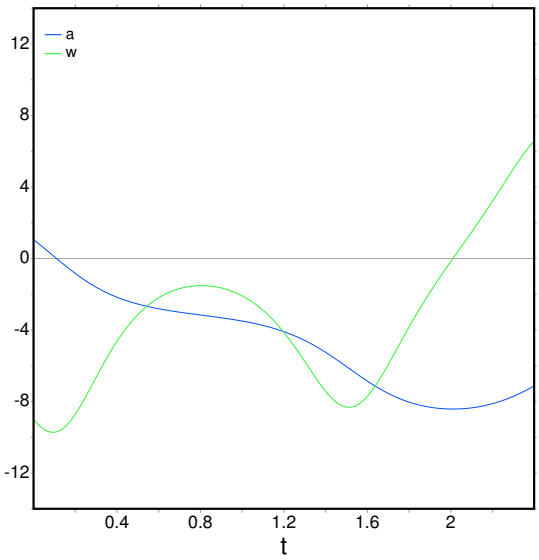
```
(%i1) plotdf([y,-(k*x + c*y + b*x^3)/m],
             [parameters,"k=-1,m=1.0,c=0,b=1"],
             [sliders,"k=-2:2,m=-1:1"],[tstep,0.1])$
```



- The direction field for a damped pendulum, including the solution for the given initial conditions, with a slider that can be used to change the value of the mass m, and with a plot of the two state variables as a function of time:

```
(%i1) plotdf([w,-g*sin(a)/l - b*w/m/l], [a,w],
        [parameters,"g=9.8,l=0.5,m=0.3,b=0.05"],
        [trajectory_at,1.05,-9],[tstep,0.01],
        [a,-10,2], [w,-14,14], [direction,forward],
        [nsteps,300], [sliders,"m=0.1:1"], [versus_t,1])$
```





`ploteq` (*exp*, *...options...*) [Function]

Plots equipotential curves for *exp*, which should be an expression depending on two variables. The curves are obtained by integrating the differential equation that define the orthogonal trajectories to the solutions of the autonomous system obtained from

the gradient of the expression given. The plot can also show the integral curves for that gradient system (option fieldlines).

This program also requires Xmaxima, even if its run from a Maxima session in a console, since the plot will be created by the Tk scripts in Xmaxima. By default, the plot region will be empty until the user clicks in a point (or gives its coordinate with in the set-up menu or via the trajectory_at option).

Most options accepted by plotdf can also be used for ploteq and the plot interface is the same that was described in plotdf.

Example:

```
(%i1) V: 900/((x+1)^2+y^2)^(1/2)-900/((x-1)^2+y^2)^(1/2)$
(%i2) ploteq(V,[x,-2,2],[y,-2,2],[fieldlines,"blue"])$
```

Clicking on a point will plot the equipotential curve that passes by that point (in red) and the orthogonal trajectory (in blue).

`rk` [Function]

`rk` (*ODE*, *var*, *initial*, *domain*)
`rk` ([*ODE1*, ..., *ODEm*], [*v1*, ..., *vm*], [*init1*, ..., *initm*], *domain*)

The first form solves numerically one first-order ordinary differential equation, and the second form solves a system of m of those equations, using the 4th order Runge-Kutta method. *var* represents the dependent variable. *ODE* must be an expression that depends only on the independent and dependent variables and defines the derivative of the dependent variable with respect to the independent variable.

The independent variable is specified with `domain`, which must be a list of four elements as, for instance:

```
[t, 0, 10, 0.1]
```

the first element of the list identifies the independent variable, the second and third elements are the initial and final values for that variable, and the last element sets the increments that should be used within that interval.

If *m* equations are going to be solved, there should be *m* dependent variables *v1*, *v2*, ..., *vm*. The initial values for those variables will be *init1*, *init2*, ..., *initm*. There will still be just one independent variable defined by `domain`, as in the previous case. *ODE1*, ..., *ODEm* are the expressions that define the derivatives of each dependent variable in terms of the independent variable. The only variables that may appear in those expressions are the independent variable and any of the dependent variables. It is important to give the derivatives *ODE1*, ..., *ODEm* in the list in exactly the same order used for the dependent variables; for instance, the third element in the list will be interpreted as the derivative of the third dependent variable.

The program will try to integrate the equations from the initial value of the independent variable until its last value, using constant increments. If at some step one of the dependent variables takes an absolute value too large, the integration will be interrupted at that point. The result will be a list with as many elements as the number of iterations made. Each element in the results list is itself another list with *m*+1 elements: the value of the independent variable, followed by the values of the dependent variables corresponding to that point.

See also `drawdf`, `desolve` and `ode2`.

Examples:

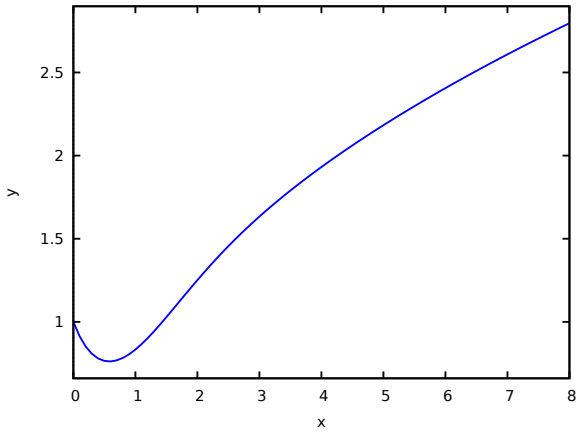
To solve numerically the differential equation

$$\frac{dx}{dt} = t - x^2$$

With initial value $x(t = 0) = 1$, in the interval of t from 0 to 8 and with increments of 0.1 for t, use:

```
(%i1) results: rk(t-x^2,x,1,[t,0,8,0.1])$
(%i2) plot2d ([discrete, results])$
```

The results will be saved in the list `results` and the plot will show the solution obtained, with t on the horizontal axis and x on the vertical axis.



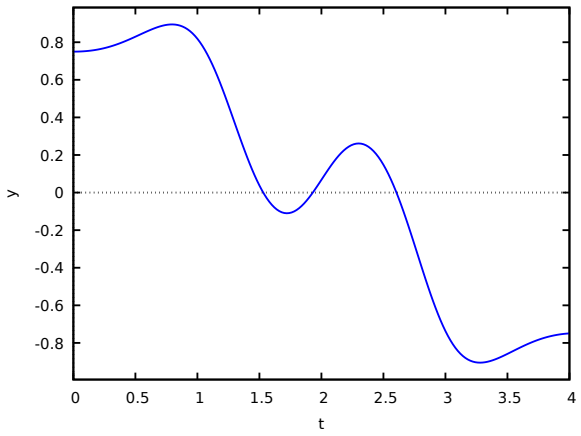
To solve numerically the system:

$$\frac{dx}{dy} = 4 - x^2 - 4y^2$$
$$\frac{dy}{dt} = y^2 - x^2 + 1$$

for t between 0 and 4, and with values of -1.25 and 0.75 for x and y at $t = 0$:

```
(%i1) sol: rk([4-x^2-4*y^2, y^2-x^2+1], [x, y], [-1.25, 0.75],
              [t,0,4,0.02])$
(%i2) plot2d([discrete, makelist([p[1], p[3]], p, sol)], [xlabel, "t"],
             [ylabel, "y"])$
```

The plot will show the solution for variable y as a function of t.


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
y
0
0.5
1
1.5
2
2.5
3
3.5
4
t

25 Matrices and Linear Algebra

25.1 Introduction to Matrices and Linear Algebra

25.1.1 Dot

The operator `.` represents noncommutative multiplication and scalar product. When the operands are 1-column or 1-row matrices `a` and `b`, the expression `a.b` is equivalent to `sum (a[i]*b[i], i, 1, length(a))`. If `a` and `b` are not complex, this is the scalar product, also called the inner product or dot product, of `a` and `b`. The scalar product is defined as `conjugate(a).b` when `a` and `b` are complex; `innerproduct` in the `eigen` package provides the complex scalar product.

When the operands are more general matrices, the product is the matrix product `a` and `b`. The number of rows of `b` must equal the number of columns of `a`, and the result has number of rows equal to the number of rows of `a` and number of columns equal to the number of columns of `b`.

To distinguish `.` as an arithmetic operator from the decimal point in a floating point number, it may be necessary to leave spaces on either side. For example, `5.e3` is `5000.0` but `5 . e3` is `5` times `e3`.

There are several flags which govern the simplification of expressions involving `.`, namely `dot0nscsimp`, `dot0simp`, `dot1simp`, `dotassoc`, `dotconstrules`, `dotdistrib`, `dotexptsimp`, `dotident`, and `dotscrules`.

25.1.2 Matrices

Matrices are handled with speed and memory-efficiency in mind. This means that assigning a matrix to a variable will create a reference to, not a copy of the matrix. If the matrix is modified all references to the matrix point to the modified object (See `copymatrix` for a way of avoiding this):

```
(%i1) M1: matrix([0,0],[0,0]);
                              [ 0  0 ]
(%o1)                         [      ]
                              [ 0  0 ]
(%i2) M2: M1;
                              [ 0  0 ]
(%o2)                         [      ]
                              [ 0  0 ]
(%i3) M1[1][1]: 2;
(%o3)                            2
(%i4) M2;
                              [ 2  0 ]
(%o4)                         [      ]
                              [ 0  0 ]
```

Converting a matrix to nested lists and vice versa works the following way:

```
(%i1) l: [[1,2],[3,4]];
(%o1)                    [[1, 2], [3, 4]]
```

```
(%i2) M1: apply('matrix,l);
                              [ 1  2 ]
(%o2)                         [      ]
                              [ 3  4 ]
(%i3) M2: transpose(M1);
                              [ 1  3 ]
(%o3)                         [      ]
                              [ 2  4 ]
(%i4) args(M2);
(%o4)                    [[1, 3], [2, 4]]
```

25.1.3 Vectors

`vect` is a package of functions for vector analysis. `load ("vect")` loads this package, and `demo ("vect")` displays a demonstration.

The vector analysis package can combine and simplify symbolic expressions including dot products and cross products, together with the gradient, divergence, curl, and Laplacian operators. The distribution of these operators over sums or products is governed by several flags, as are various other expansions, including expansion into components in any specific orthogonal coordinate systems. There are also functions for deriving the scalar or vector potential of a field.

The `vect` package contains these functions: `vectorsimp`, `scalefactors`, `express`, `potential`, and `vectorpotential`.

By default the `vect` package does not declare the dot operator to be a commutative operator. To get a commutative dot operator `.`, the command `declare(".", commutative)` must be executed.

25.1.4 eigen

The package `eigen` contains several functions devoted to the symbolic computation of eigenvalues and eigenvectors. Maxima loads the package automatically if one of the functions `eigenvalues` or `eigenvectors` is invoked. The package may be loaded explicitly as `load ("eigen")`.

`demo ("eigen")` displays a demonstration of the capabilities of this package. `batch ("eigen")` executes the same demonstration, but without the user prompt between successive computations.

The functions in the `eigen` package are:
`innerproduct`, `unitvector`, `columnvector`, `gramschmidt`, `eigenvalues`, `eigenvectors`, `uniteigenvectors`, and `similaritytransform`.

25.2 Functions and Variables for Matrices and Linear Algebra

`addcol` (*M*, *list_1*, ..., *list_n*) [Function]

Appends the column(s) given by the one or more lists (or matrices) onto the matrix *M*.

See also `addrow` and `append`.

addrow (*M*, *list_1*, ..., *list_n*) [Function]

Appends the row(s) given by the one or more lists (or matrices) onto the matrix *M*.

See also addcol and append.

adjoint (*M*) [Function]

Returns the adjoint of the matrix *M*. The adjoint matrix is the transpose of the matrix of cofactors of *M*.

augcoefmatrix ([*eqn_1*, ..., *eqn_m*], [*x_1*, ..., *x_n*]) [Function]

Returns the augmented coefficient matrix for the variables *x_1*, ..., *x_n* of the system of linear equations *eqn_1*, ..., *eqn_m*. This is the coefficient matrix with a column adjoined for the constant terms in each equation (i.e., those terms not dependent upon *x_1*, ..., *x_n*).

```
(%i1) m: [2*x - (a - 1)*y = 5*b, c + b*y + a*x = 0]$
(%i2) augcoefmatrix (m, [x, y]);
                          [ 2  1 - a  - 5 b ]
(%o2)                     [                 ]
                          [ a    b      c   ]
```

cauchy_matrix [Function]

cauchy_matrix ([*x_1*, *x_2*, ..., *x_m*], [*y_1*, *y_2*, ..., *y_n*])

cauchy_matrix ([*x_1*, *x_2*, ..., *x_n*])

Returns a n by *m* Cauchy matrix with the elements *a[i,j]* = 1/(*x_i*+*y_i*). The second argument of cauchy_matrix is optional. For this case the elements of the Cauchy matrix are *a[i,j]* = 1/(*x_i*+*x_j*).

Remark: In the literature the Cauchy matrix can be found defined in two forms. A second definition is *a[i,j]* = 1/(*x_i*-*y_i*).

Examples:

```
(%i1) cauchy_matrix([x1, x2], [y1, y2]);
                          [    1         1    ]
                          [ -------   ------- ]
                          [ y1 + x1   y2 + x1 ]
(%o1)                     [                   ]
                          [    1         1    ]
                          [ -------   ------- ]
                          [ y1 + x2   y2 + x2 ]
(%i2) cauchy_matrix([x1, x2]);
                          [   1         1     ]
                          [  ----    -------  ]
                          [  2 x1    x2 + x1  ]
(%o2)                     [                   ]
                          [    1        1     ]
                          [ -------    ----   ]
                          [ x2 + x1    2 x2   ]
```

charpoly (*M*, *x*) [Function]

Returns the characteristic polynomial for the matrix *M* with respect to variable *x*. That is, determinant (*M* - diagmatrix (length (*M*), *x*)).

```
(%i1) a: matrix ([3, 1], [2, 4]);
                              [ 3  1 ]
(%o1)                         [      ]
                              [ 2  4 ]
(%i2) expand (charpoly (a, lambda));
                             2
(%o2)                lambda  - 7 lambda + 10
(%i3) (programmode: true, solve (%));
(%o3)               [lambda = 5, lambda = 2]
(%i4) matrix ([x1], [x2]);
                               [ x1 ]
(%o4)                          [    ]
                               [ x2 ]
(%i5) ev (a . % - lambda*%, %th(2)[1]);
                             [ x2 - 2 x1 ]
(%o5)                        [           ]
                             [ 2 x1 - x2 ]
(%i6) %[1, 1] = 0;
(%o6)                        x2 - 2 x1 = 0
(%i7) x2^2 + x1^2 = 1;
                               2     2
(%o7)                        x2  + x1  = 1
(%i8) solve ([%th(2), %], [x1, x2]);
                    1                2
(%o8) [[x1 = - -------, x2 = - -------],
                sqrt(5)          sqrt(5)
                                                 1              2
                                       [x1 = -------, x2 = -------]]
                                             sqrt(5)        sqrt(5)
```

`coefmatrix` ([*eqn_1*, ..., *eqn_m*], [*x_1*, ..., *x_n*]) [Function]

Returns the coefficient matrix for the variables *x_1*, ..., *x_n* of the system of linear equations *eqn_1*, ..., *eqn_m*.

```
(%i1) coefmatrix([2*x-(a-1)*y+5*b = 0, b*y+a*x = 3], [x,y]);
                             [ 2  1 - a ]
(%o1)                        [          ]
                             [ a    b   ]
```

`col` (*M*, *i*) [Function]

Returns the *i*'th column of the matrix *M*. The return value is a matrix.

The matrix returned by `col` does not share memory with the argument *M*; a modification to the return value does not modify *M*.

Examples:

`col` returns the *i*'th column of the matrix *M*.

```
(%i1) abc: matrix ([12, 14, -4], [2, x, b], [3*y, -7, 9]);
                        [ 12   14   - 4 ]
                        [               ]
(%o1)                   [  2    x    b  ]
                        [               ]
                        [ 3 y  - 7   9  ]
(%i2) col (abc, 1);
                             [ 12  ]
                             [     ]
(%o2)                        [  2  ]
                             [     ]
                             [ 3 y ]
(%i3) col (abc, 2);
                             [ 14  ]
                             [     ]
(%o3)                        [  x  ]
                             [     ]
                             [ - 7 ]
(%i4) col (abc, 3);
                             [ - 4 ]
                             [     ]
(%o4)                        [  b  ]
                             [     ]
                             [  9  ]
```

The matrix returned by `col` does not share memory with the argument. In this example, assigning a new value to `aa2` does not modify `aa`.

```
(%i1) aa: matrix ([1, 2, x], [7, y, 3]);
                           [ 1  2  x ]
(%o1)                      [         ]
                           [ 7  y  3 ]
(%i2) aa2: col (aa, 2);
                              [ 2 ]
(%o2)                         [   ]
                              [ y ]
(%i3) aa2[2, 1]: 123;
(%o3)                          123
(%i4) aa2;
                              [  2  ]
(%o4)                         [     ]
                              [ 123 ]
(%i5) aa;
                           [ 1  2  x ]
(%o5)                      [         ]
                           [ 7  y  3 ]
```

`columnvector (L)` [Function]
`covect (L)` [Function]

Returns a matrix of one column and `length (L)` rows, containing the elements of the list L.

`covect` is a synonym for `columnvector`.

`load ("eigen")` loads this function.

This is useful if you want to use parts of the outputs of the functions in this package in matrix calculations.

Example:

```
(%i1) load ("eigen")$
(%i2) columnvector ([aa, bb, cc, dd]);
                                [ aa ]
                                [    ]
                                [ bb ]
(%o2)                           [    ]
                                [ cc ]
                                [    ]
                                [ dd ]
```

`copymatrix (M)` [Function]

Returns a copy of the matrix M. This is the only way to make a copy aside from copying M element by element.

Note that an assignment of one matrix to another, as in `m2: m1`, does not copy `m1`. An assignment `m2 [i,j]: x` or `setelmx(x, i, j, m2)` also modifies `m1 [i,j]`. Creating a copy with `copymatrix` and then using assignment creates a separate, modified copy.

`determinant (M)` [Function]

Computes the determinant of M by a method similar to Gaussian elimination.

The form of the result depends upon the setting of the switch `ratmx`.

There is a special routine for computing sparse determinants which is called when the switches `ratmx` and `sparse` are both `true`.

`display_determinant_bars` governs the display of determinants.

`display_determinant_bars` [Option variable]

Default value: `true`

When `display_determinant_bars` is `true`, a determinant noun expression which has a literal matrix as its sole argument is displayed with a vertical bar on either side.

Otherwise, `display_determinant_bars` is `false`, or the determinant is not a noun expression, or its argument is not a literal matrix; in these cases, the expression is displayed as an ordinary function call.

`detout` [Option variable]

Default value: `false`

When `detout` is `true`, the determinant of a matrix whose inverse is computed is factored out of the inverse.

For this switch to have an effect doallmxops and doscmxops should be false (see their descriptions). Alternatively this switch can be given to ev which causes the other two to be set correctly.

Example:

```
(%i1) m: matrix ([a, b], [c, d]);
                              [ a  b ]
(%o1)                         [      ]
                              [ c  d ]
(%i2) detout: true$
(%i3) doallmxops: false$
(%i4) doscmxops: false$
(%i5) invert (m);
                             [  d   - b ]
                             [          ]
                             [ - c   a  ]
(%o5)                        ------------
                              a d - b c
```

diagmatrix (*n*, *x*) [Function]

Returns a diagonal matrix of size *n* by *n* with the diagonal elements all equal to *x*. diagmatrix (*n*, 1) returns an identity matrix (same as ident (*n*)).

n must evaluate to an integer, otherwise diagmatrix complains with an error message.

x can be any kind of expression, including another matrix. If *x* is a matrix, it is not copied; all diagonal elements refer to the same instance, *x*.

doallmxops [Option variable]

Default value: true

When doallmxops is true, all operations relating to matrices are carried out. When it is false then the setting of the individual dot switches govern which operations are performed.

domxexpt [Option variable]

Default value: true

When domxexpt is true, a matrix exponential, exp (*M*) where *M* is a matrix, is interpreted as a matrix with element [i,j] equal to exp (m[i,j]). Otherwise exp (*M*) evaluates to exp (*ev(M)*).

domxexpt affects all expressions of the form *base*^*power* where *base* is an expression assumed scalar or constant, and *power* is a list or matrix.

Example:

```
(%i1) m: matrix ([1, %i], [a+b, %pi]);
                            [   1    %i  ]
(%o1)                       [            ]
                            [ b + a  %pi ]
(%i2) domxexpt: false$
```

```
(%i3) (1 - c)^m;
                                [   1    %i  ]
                                [            ]
                                [ b + a  %pi ]
(%o3)                    (1 - c)
(%i4) domxexpt: true$
(%i5) (1 - c)^m;
                     [                       %i  ]
                     [    1 - c      (1 - c)    ]
(%o5)                [                          ]
                     [        b + a         %pi ]
                     [ (1 - c)       (1 - c)    ]
```

`domxmxops` [Option variable]

Default value: `true`

When `domxmxops` is `true`, all matrix-matrix or matrix-list operations are carried out (but not scalar-matrix operations); if this switch is `false` such operations are not carried out.

`domxnctimes` [Option variable]

Default value: `false`

When `domxnctimes` is `true`, non-commutative products of matrices are carried out.

`dontfactor` [Option variable]

Default value: `[]`

`dontfactor` may be set to a list of variables with respect to which factoring is not to occur. (The list is initially empty.) Factoring also will not take place with respect to any variables which are less important, according the variable ordering assumed for canonical rational expression (CRE) form, than those on the `dontfactor` list.

`doscmxops` [Option variable]

Default value: `false`

When `doscmxops` is `true`, scalar-matrix operations are carried out.

`doscmxplus` [Option variable]

Default value: `false`

When `doscmxplus` is `true`, scalar-matrix operations yield a matrix result. This switch is not subsumed under `doallmxops`.

`dot0nscsimp` [Option variable]

Default value: `true`

When `dot0nscsimp` is `true`, a non-commutative product of zero and a nonscalar term is simplified to a commutative product.

`dot0simp` [Option variable]

Default value: `true`

When `dot0simp` is `true`, a non-commutative product of zero and a scalar term is simplified to a commutative product.

`dot1simp` [Option variable]

Default value: `true`

When `dot1simp` is `true`, a non-commutative product of one and another term is simplified to a commutative product.

`dotassoc` [Option variable]

Default value: `true`

When `dotassoc` is `true`, an expression `(A.B).C` simplifies to `A.(B.C)`.

`dotconstrules` [Option variable]

Default value: `true`

When `dotconstrules` is `true`, a non-commutative product of a constant and another term is simplified to a commutative product. Turning on this flag effectively turns on `dot0simp`, `dot0nscsimp`, and `dot1simp` as well.

`dotdistrib` [Option variable]

Default value: `false`

When `dotdistrib` is `true`, an expression `A.(B + C)` simplifies to `A.B + A.C`.

`dotexptsimp` [Option variable]

Default value: `true`

When `dotexptsimp` is `true`, an expression `A.A` simplifies to `A^^2`.

`dotident` [Option variable]

Default value: 1

`dotident` is the value returned by `X^^0`.

`dotscrules` [Option variable]

Default value: `false`

When `dotscrules` is `true`, an expression `A.SC` or `SC.A` simplifies to `SC*A` and `A.(SC*B)` simplifies to `SC*(A.B)`.

`echelon` (*M*) [Function]

Returns the echelon form of the matrix *M*, as produced by Gaussian elimination. The echelon form is computed from *M* by elementary row operations such that the first non-zero element in each row in the resulting matrix is one and the column elements under the first one in each row are all zero.

`triangularize` also carries out Gaussian elimination, but it does not normalize the leading non-zero element in each row.

`lu_factor` and `cholesky` are other functions which yield triangularized matrices.

```
(%i1) M: matrix ([3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]);
                          [  3   7  aa  bb ]
                          [                ]
(%o1)                     [ - 1  8  5   2  ]
                          [                ]
                          [  9   2  11  4  ]
```

```
(%i2) echelon (M);
                    [ 1   - 8   - 5      - 2     ]
                    [                            ]
                    [          28        11      ]
                    [ 0    1   --        --      ]
(%o2)               [          37        37      ]
                    [                            ]
                    [                37 bb - 119 ]
                    [ 0    0    1   ----------- ]
                    [                37 aa - 313 ]
```

`eigenvalues` (*M*) [Function]
`eivals` (*M*) [Function]

Returns a list of two lists containing the eigenvalues of the matrix *M*. The first sublist of the return value is the list of eigenvalues of the matrix, and the second sublist is the list of the multiplicities of the eigenvalues in the corresponding order.

`eivals` is a synonym for `eigenvalues`.

`eigenvalues` calls the function `solve` to find the roots of the characteristic polynomial of the matrix. Sometimes `solve` may not be able to find the roots of the polynomial; in that case some other functions in this package (except `innerproduct`, `unitvector`, `columnvector` and `gramschmidt`) will not work. Sometimes `solve` may find only a subset of the roots of the polynomial. This may happen when the factoring of the polynomial contains polynomials of degree 5 or more. In such cases a warning message is displayed and the only the roots found and their corresponding multiplicities are returned.

In some cases the eigenvalues found by `solve` may be complicated expressions. (This may happen when `solve` returns a not-so-obviously real expression for an eigenvalue which is known to be real.) It may be possible to simplify the eigenvalues using some other functions.

The package `eigen.mac` is loaded automatically when `eigenvalues` or `eigenvectors` is referenced. If `eigen.mac` is not already loaded, `load ("eigen")` loads it. After loading, all functions and variables in the package are available.

For matrices consisting of only floating-point values, see also `dgeev`.

`eigenvectors` (*M*) [Function]
`eivects` (*M*) [Function]

Computes eigenvectors of the matrix *M*. The return value is a list of two elements. The first is a list of the eigenvalues of *M* and a list of the multiplicities of the eigenvalues. The second is a list of lists of eigenvectors. There is one list of eigenvectors for each eigenvalue. There may be one or more eigenvectors in each list.

`eivects` is a synonym for `eigenvectors`.

The package `eigen.mac` is loaded automatically when `eigenvalues` or `eigenvectors` is referenced. If `eigen.mac` is not already loaded, `load ("eigen")` loads it. After loading, all functions and variables in the package are available.

Note that `eigenvectors` internally calls `eigenvalues` to obtain eigenvalues. So, when `eigenvalues` returns a subset of all the eigenvalues, the `eigenvectors` returns

the corresponding subset of the all the eigenvectors, with the same warning displayed as `eigenvalues`.

The flags that affect this function are:

`nondiagonalizable` is set to `true` or `false` depending on whether the matrix is nondiagonalizable or diagonalizable after `eigenvectors` returns.

`hermitianmatrix` when `true`, causes the degenerate eigenvectors of the Hermitian matrix to be orthogonalized using the Gram-Schmidt algorithm.

`knowneigvals` when `true` causes the `eigen` package to assume the eigenvalues of the matrix are known to the user and stored under the global name `listeigvals`. `listeigvals` should be set to a list similar to the output `eigenvalues`.

The function `algsys` is used here to solve for the eigenvectors. Sometimes if the eigenvalues are messy, `algsys` may not be able to find a solution. In some cases, it may be possible to simplify the eigenvalues by first finding them using `eigenvalues` command and then using other functions to reduce them to something simpler. Following simplification, `eigenvectors` can be called again with the `knowneigvals` flag set to `true`.

See also `eigenvalues`.

For matrices consisting of only floating-point values, see also `dgeev`.

Examples:

A matrix which has just one eigenvector per eigenvalue.

```
(%i1) M1: matrix ([11, -1], [1, 7]);
                              [ 11  - 1 ]
(%o1)                         [         ]
                              [ 1    7  ]
(%i2) [vals, vecs] : eigenvectors (M1);
(%o2) [[[9 - sqrt(3), sqrt(3) + 9], [1, 1]],
                          [[[1, sqrt(3) + 2]], [[1, 2 - sqrt(3)]]]]
```

```
(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i],
  mult[i] = vals[2][i], vec[i] = vecs[i]);
                        val  = 9 - sqrt(3)
                           1

                              mult  = 1
                                  1

                       vec  = [[1, sqrt(3) + 2]]
                          1

                          val  = sqrt(3) + 9
                             2

                              mult  = 1
                                  2

                       vec  = [[1, 2 - sqrt(3)]]
                          2

(%o3)                            done
```

A matrix which has two eigenvectors for one eigenvalue (namely 2).

```
(%i1) M1 : matrix ([0, 1, 0, 0], [0, 0, 0, 0], [0, 0, 2, 0], [0, 0, 0, 2]);
                             [ 0  1  0  0 ]
                             [            ]
                             [ 0  0  0  0 ]
(%o1)                        [            ]
                             [ 0  0  2  0 ]
                             [            ]
                             [ 0  0  0  2 ]
(%i2) [vals, vecs] : eigenvectors (M1);
(%o2) [[[0, 2], [2, 2]], [[1, 0, 0, 0]],
                                    [[0, 0, 1, 0], [0, 0, 0, 1]]]]
```

```
(%i3) for i thru length (vals[1]) do disp (val[i] = vals[1][i],
  mult[i] = vals[2][i], vec[i] = vecs[i]);
                              val  = 0
                                 1

                              mult  = 2
                                  1

                           vec  = [[1, 0, 0, 0]]
                              1

                              val  = 2
                                 2

                              mult  = 2
                                  2

                     vec  = [[0, 0, 1, 0], [0, 0, 0, 1]]
                        2

(%o3)                           done
```

`ematrix` (*m*, *n*, *x*, *i*, *j*) [Function]

Returns an *m* by *n* matrix, all elements of which are zero except for the `[i, j]` element which is *x*.

`entermatrix` (*m*, *n*) [Function]

Returns an *m* by *n* matrix, reading the elements interactively.

If *n* is equal to *m*, Maxima prompts for the type of the matrix (diagonal, symmetric, antisymmetric, or general) and for each element. Each response is terminated by a semicolon `;` or dollar sign `$`.

If *n* is not equal to *m*, Maxima prompts for each element.

The elements may be any expressions, which are evaluated. `entermatrix` evaluates its arguments.

```
(%i1) n: 3$
(%i2) m: entermatrix (n, n)$

Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric
4. General
Answer 1, 2, 3 or 4 :
1$
Row 1 Column 1:
(a+b)^n$
Row 2 Column 2:
(a+b)^(n+1)$
Row 3 Column 3:
(a+b)^(n+2)$
```

```
Matrix entered.
(%i3) m;
                  [        3                    ]
                  [ (b + a)      0         0     ]
                  [                              ]
(%o3)             [                  4           ]
                  [    0      (b + a)      0     ]
                  [                              ]
                  [                           5  ]
                  [    0         0      (b + a)  ]
```

genmatrix [Function]

`genmatrix` (`a`, *i_2*, *j_2*, *i_1*, *j_1*)
`genmatrix` (`a`, *i_2*, *j_2*, *i_1*)
`genmatrix` (`a`, *i_2*, *j_2*)

Returns a matrix generated from *a*, taking element `a[`*i_1*`, `*j_1*`]` as the upper-left element and `a[`*i_2*`, `*j_2*`]` as the lower-right element of the matrix. Here *a* is a declared array (created by `array` but not by `make_array`) or a `hashed array`, or a `memoizing function`, or a lambda expression of two arguments. (A `memoizing function` is created like other functions with `:=` or `define`, but arguments are enclosed in square brackets instead of parentheses.)

If *j_1* is omitted, it is assumed equal to *i_1*. If both *j_1* and *i_1* are omitted, both are assumed equal to 1.

If a selected element `i,j` of the array is undefined, the matrix will contain a symbolic element `a[i,j]`.

Examples:

```
(%i1) h [i, j] := 1 / (i + j - 1);
                                      1
(%o1)                  h     := ---------
                        i, j    i + j - 1
(%i2) genmatrix (h, 3, 3);
                              [    1  1 ]
                              [ 1  -  - ]
                              [    2  3 ]
                              [         ]
                              [ 1  1  1 ]
(%o2)                         [ -  -  - ]
                              [ 2  3  4 ]
                              [         ]
                              [ 1  1  1 ]
                              [ -  -  - ]
                              [ 3  4  5 ]
(%i3) array (a, fixnum, 2, 2);
(%o3)                          a
(%i4) a [1, 1] : %e;
(%o4)                         %e
```

```
(%i5) a [2, 2] : %pi;
(%o5)                           %pi
(%i6) genmatrix (a, 2, 2);
                              [ %e   0  ]
(%o6)                         [         ]
                              [ 0   %pi ]
(%i7) genmatrix (lambda ([i, j], j - i), 3, 3);
                             [  0    1   2 ]
                             [             ]
(%o7)                        [ - 1   0   1 ]
                             [             ]
                             [ - 2  - 1  0 ]
(%i8) genmatrix (B, 2, 2);
                            [ B      B     ]
                            [  1, 1   1, 2 ]
(%o8)                       [              ]
                            [ B      B     ]
                            [  2, 1   2, 2 ]
```

`gramschmidt` [Function]

`gramschmidt` (*x*)
`gramschmidt` (*x*, *F*)

Carries out the Gram-Schmidt orthogonalization algorithm on *x*, which is either a matrix or a list of lists. *x* is not modified by `gramschmidt`. The inner product employed by `gramschmidt` is *F*, if present, otherwise the inner product is the function `innerproduct`.

If *x* is a matrix, the algorithm is applied to the rows of *x*. If *x* is a list of lists, the algorithm is applied to the sublists, which must have equal numbers of elements. In either case, the return value is a list of lists, the sublists of which are orthogonal and span the same space as *x*. If the dimension of the span of *x* is less than the number of rows or sublists, some sublists of the return value are zero.

`factor` is called at each stage of the algorithm to simplify intermediate results. As a consequence, the return value may contain factored integers.

`load("eigen")` loads this function.

Example:

Gram-Schmidt algorithm using default inner product function.

```
(%i1) load ("eigen")$
(%i2) x: matrix ([1, 2, 3], [9, 18, 30], [12, 48, 60]);
                            [ 1   2   3  ]
                            [            ]
(%o2)                       [ 9   18  30 ]
                            [            ]
                            [ 12  48  60 ]
```

```
(%i3) y: gramschmidt (x);
                        2      2            4     3
                       3      3   3 5      2  3  2  3
(%o3)  [[1, 2, 3], [- ---, - --, ---], [- ----, ----, 0]]
                       2 7    7   2 7       5     5
(%i4) map (innerproduct, [y[1], y[2], y[3]], [y[2], y[3], y[1]]);
(%o4)                         [0, 0, 0]
```

Gram-Schmidt algorithm using a specified inner product function.

```
(%i1) load ("eigen")$
(%i2) ip (f, g) := integrate (f * g, u, a, b);
(%o2)          ip(f, g) := integrate(f g, u, a, b)
(%i3) y: gramschmidt ([1, sin(u), cos(u)], ip), a=-%pi/2, b=%pi/2;
                                  %pi cos(u) - 2
(%o3)              [1, sin(u), --------------]
                                       %pi
(%i4) map (ip, [y[1], y[2], y[3]], [y[2], y[3], y[1]]), a=-%pi/2,
        b=%pi/2;
(%o4)                         [0, 0, 0]
```

`ident` (*n*) [Function]

Returns an *n* by *n* identity matrix.

`innerproduct` (*x*, *y*) [Function]

`inprod` (*x*, *y*) [Function]

Returns the inner product (also called the scalar product or dot product) of *x* and *y*, which are lists of equal length, or both 1-column or 1-row matrices of equal length. The return value is `conjugate (x) . y`, where `.` is the noncommutative multiplication operator.

`load ("eigen")` loads this function.

`inprod` is a synonym for `innerproduct`.

`invert_by_adjoint` (*M*) [Function]

Returns the inverse of the matrix *M*. The inverse is computed by the adjoint method.

`invert_by_adjoint` honors the `ratmx` and `detout` flags, the same as `invert`.

`invert` (*M*) [Function]

Returns the inverse of the matrix *M*. The inverse is computed via the LU decomposition.

When `ratmx` is `true`, elements of *M* are converted to canonical rational expressions (CRE), and the elements of the return value are also CRE.

When `ratmx` is `false`, elements of *M* are not converted to a common representation. In particular, float and bigfloat elements are not converted to rationals.

When `detout` is `true`, the determinant is factored out of the inverse. The global flags `doallmxops` and `doscmxops` must be `false` to prevent the determinant from being absorbed into the inverse. `xthru` can multiply the determinant into the inverse.

`invert` does not apply any simplifications to the elements of the inverse apart from the default arithmetic simplifications. `ratsimp` and `expand` can apply additional

simplifications. In particular, when M has polynomial elements, `expand(invert(M))` might be preferable.

`invert(M)` is equivalent to `M^^-1`.

`list_matrix_entries (M)` [Function]

Returns a list containing the elements of the matrix M.

Example:

```
(%i1) list_matrix_entries(matrix([a,b],[c,d]));
(%o1)                     [a, b, c, d]
```

`lmxchar` [Option variable]

Default value: `[`

`lmxchar` is the character displayed as the left delimiter of a matrix. See also `rmxchar`.

`lmxchar` is only used when `display2d_unicode` is `false`.

Example:

```
(%i1) display2d_unicode: false $
(%i2) lmxchar: "|"$
(%i3) matrix ([a, b, c], [d, e, f], [g, h, i]);
                             | a  b  c |
                             |         |
(%o3)                        | d  e  f |
                             |         |
                             | g  h  i |
```

`display_matrix_brackets` [Option variable]

Default value: `true`

When `display_matrix_brackets` is `true`, matrices are displayed with brackets (square braces) to the left and right.

When `display_matrix_brackets` is `false`, matrices are not displayed with brackets; only the matrix elements are displayed.

`display_matrix_padding_horizontal` [Option variable]

Default value: `true`

When `display_matrix_padding_horizontal` is `true`, matrices are displayed with spaces between successive columns, and a space before the first column and a space after the last column.

When `display_matrix_padding_horizontal` is `false`, matrices are not displayed with spaces between successive columns, and no space before the first column and no space after the last column. Successive columns are immediately adjacent to each other, and the first column is immediately adjacent to the left bracket, and the last column is immediately adjacent to the right bracket, if the brackets are present (see `display_matrix_brackets`).

See also `display_matrix_padding_vertical`.

Examples:

```
(%i1) display_matrix_padding_horizontal;
(%o1)                          true
(%i2) foo: matrix ([a, b, c], [d, e, f], [g, h, i]);

                               a  b  c

(%o2)                          d  e  f

                               g  h  i

(%i3) display_matrix_padding_horizontal: false;
(%o3)                          false
(%i4) foo;

                               abc

(%o4)                          def

                               ghi
```

`display_matrix_padding_vertical` [Option variable]

Default value: `true`

When `display_matrix_padding_vertical` is `true`, matrices are displayed with an empty line between successive rows.

When `display_matrix_padding_vertical` is `false`, matrices are not displayed with an empty line between successive rows; successive rows are immediately adjacent to each other.

See also `display_matrix_padding_horizontal`.

Examples:

```
(%i1) display_matrix_padding_vertical;
(%o1)                          true
(%i2) foo: matrix ([a, b, c], [d, e, f], [g, h, i]);

                               a  b  c

(%o2)                          d  e  f

                               g  h  i
```

```
(%i3) display_matrix_padding_vertical: false;
(%o3)                          false
(%i4) foo;

                               a  b  c
(%o4)                          d  e  f
                               g  h  i
```

`matrix` (*row_1*, ..., *row_n*) [Function]

Returns a rectangular matrix which has the rows *row_1*, ..., *row_n*. Each row is a list of expressions. All rows must be the same length.

The operations `+` (addition), `-` (subtraction), `*` (multiplication), and `/` (division), are carried out element by element when the operands are two matrices, a scalar and a matrix, or a matrix and a scalar. The operation `^` (exponentiation, equivalently `**`) is carried out element by element if the operands are a scalar and a matrix or a matrix and a scalar, but not if the operands are two matrices. All operations are normally carried out in full, including `.` (noncommutative multiplication).

Matrix multiplication is represented by the noncommutative multiplication operator `.`. The corresponding noncommutative exponentiation operator is `^^`. For a matrix *A*, `A.A` = `A^^2` and `A^^-1` is the inverse of *A*, if it exists. `A^^-1` is equivalent to `invert(A)`.

There are switches for controlling simplification of expressions involving dot and matrix-list operations. These are `doallmxops`, `domxexpt`, `domxmxops`, `doscmxops`, and `doscmxplus`.

There are additional options which are related to matrices. These are: `lmxchar`, `rmxchar`, `ratmx`, `listarith`, `detout`, `scalarmatrix` and `sparse`.

There are a number of functions which take matrices as arguments or yield matrices as return values. See `eigenvalues`, `eigenvectors`, `determinant`, `charpoly`, `genmatrix`, `addcol`, `addrow`, `copymatrix`, `transpose`, `echelon`, and `rank`.

Option variables `display_matrix_brackets` and `display_matrix_padding_vertical` govern the display of matrices.

Examples:

- Construction of matrices from lists.

```
(%i1) x: matrix ([17, 3], [-8, 11]);
                               [ 17   3  ]
(%o1)                          [         ]
                               [ - 8  11 ]
(%i2) y: matrix ([%pi, %e], [a, b]);
                               [ %pi  %e ]
(%o2)                          [         ]
                               [  a   b  ]
```

- Addition, element by element.

```
(%i3) x + y;
                          [ %pi + 17  %e + 3 ]
```

```
(%o3)                  [                  ]
                       [  a - 8    b + 11 ]
```

- Subtraction, element by element.

```
(%i4) x - y;
                       [ 17 - %pi  3 - %e ]
(%o4)                  [                  ]
                       [ - a - 8   11 - b ]
```

- Multiplication, element by element.

```
(%i5) x * y;
                         [ 17 %pi  3 %e ]
(%o5)                    [              ]
                         [ - 8 a   11 b ]
```

- Division, element by element.

```
(%i6) x / y;
                         [ 17       - 1 ]
                         [ ---  3 %e    ]
                         [ %pi          ]
(%o6)                    [              ]
                         [   8    11    ]
                         [ - -    --    ]
                         [   a    b     ]
```

- Matrix to a scalar exponent, element by element.

```
(%i7) x ^ 3;
                          [ 4913    27  ]
(%o7)                     [             ]
                          [ - 512  1331 ]
```

- Scalar base to a matrix exponent, element by element.

```
(%i8) exp(y);
                          [   %pi    %e ]
                          [ %e     %e   ]
(%o8)                     [             ]
                          [    a     b  ]
                          [  %e    %e   ]
```

- Matrix base to a matrix exponent. This is not carried out element by element. See also `matrixexp`.

```
(%i9) x ^ y;
                                    [ %pi  %e ]
                                    [         ]
                                    [  a   b  ]
                         [ 17   3  ]
(%o9)                    [         ]
                         [ - 8  11 ]
```

- Noncommutative matrix multiplication.

```
(%i10) x . y;
                    [ 3 a + 17 %pi  3 b + 17 %e ]
(%o10)              [                           ]
                    [ 11 a - 8 %pi  11 b - 8 %e ]
(%i11) y . x;
                  [ 17 %pi - 8 %e  3 %pi + 11 %e ]
(%o11)            [                              ]
                  [  17 a - 8 b     11 b + 3 a   ]
```

- Noncommutative matrix exponentiation. A scalar base b to a matrix power M is carried out element by element and so `b^^m` is the same as `b^m`.

```
(%i12) x ^^ 3;
                           [  3833   1719 ]
(%o12)                     [              ]
                           [ - 4584  395  ]
(%i13) %e ^^ y;
                            [   %pi    %e ]
                            [ %e     %e   ]
(%o13)                      [              ]
                            [    a     b  ]
                            [  %e    %e   ]
```

- A matrix raised to a -1 exponent with noncommutative exponentiation is the matrix inverse, if it exists.

```
(%i14) x ^^ -1;
                            [ 11      3  ]
                            [ ---  - --- ]
                            [ 211    211 ]
(%o14)                      [             ]
                            [  8    17   ]
                            [ ---   ---  ]
                            [ 211   211  ]
(%i15) x . (x ^^ -1);
                               [ 1  0 ]
(%o15)                         [      ]
                               [ 0  1 ]
```

`matrixexp` [Function]

`matrixexp` (*M*)
`matrixexp` (*M*, *n*)
`matrixexp` (*M*, *V*)

Calculates the matrix exponential e^{MV} . Instead of the vector V a number n can be specified as the second argument. If this argument is omitted `matrixexp` replaces it by `1`.

The matrix exponential of a matrix M can be expressed as a power series:

$$e^M = \sum_{k=0}^{\infty} \left(\frac{M^k}{k!} \right)$$

`matrixmap` (*f*, *M*) [Function]

Returns a matrix with element `i,j` equal to *f*(*M*`[i,j]`).

See also `map`, `fullmap`, `fullmapl`, and `apply`.

`matrixp` (*expr*) [Function]

Returns `true` if *expr* is a matrix, otherwise `false`.

`matrix_element_add` [Option variable]

Default value: `+`

`matrix_element_add` is the operation invoked in place of addition in a matrix multiplication. `matrix_element_add` can be assigned any n-ary operator (that is, a function which handles any number of arguments). The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

See also `matrix_element_mult` and `matrix_element_transpose`.

Example:

```
(%i1) matrix_element_add: "*"$
(%i2) matrix_element_mult: "^"$
(%i3) aa: matrix ([a, b, c], [d, e, f]);
                             [ a  b  c ]
(%o3)                        [         ]
                             [ d  e  f ]
(%i4) bb: matrix ([u, v, w], [x, y, z]);
                             [ u  v  w ]
(%o4)                        [         ]
                             [ x  y  z ]
(%i5) aa . transpose (bb);
                       [  u  v  w   x  y  z ]
                       [ a  b  c   a  b  c  ]
(%o5)                  [                    ]
                       [  u  v  w   x  y  z ]
                       [ d  e  f   d  e  f  ]
```

`matrix_element_mult` [Option variable]

Default value: `*`

`matrix_element_mult` is the operation invoked in place of multiplication in a matrix multiplication. `matrix_element_mult` can be assigned any binary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

The dot operator `.` is a useful choice in some contexts.

See also `matrix_element_add` and `matrix_element_transpose`.

Example:

```
(%i1) matrix_element_add: lambda ([[x]], sqrt (apply ("+", x)))$
(%i2) matrix_element_mult: lambda ([x, y], (x - y)^2)$
```

```
(%i3) [a, b, c] . [x, y, z];
                            2          2          2
(%o3)          sqrt((c - z)  + (b - y)  + (a - x) )
(%i4) aa: matrix ([a, b, c], [d, e, f]);
                             [ a  b  c ]
(%o4)                        [         ]
                             [ d  e  f ]
(%i5) bb: matrix ([u, v, w], [x, y, z]);
                             [ u  v  w ]
(%o5)                        [         ]
                             [ x  y  z ]
(%i6) aa . transpose (bb);
                [             2          2          2  ]
                [ sqrt((c - w)  + (b - v)  + (a - u) ) ]
(%o6)  Col 1 = [                                       ]
                [             2          2          2  ]
                [ sqrt((f - w)  + (e - v)  + (d - u) ) ]
                          [             2          2          2  ]
                          [ sqrt((c - z)  + (b - y)  + (a - x) ) ]
                  Col 2 = [                                      ]
                          [             2          2          2  ]
                          [ sqrt((f - z)  + (e - y)  + (d - x) ) ]
```

`matrix_element_transpose` [Option variable]

Default value: `false`

`matrix_element_transpose` is the operation applied to each element of a matrix when it is transposed. `matrix_element_mult` can be assigned any unary operator. The assigned value may be the name of an operator enclosed in quote marks, the name of a function, or a lambda expression.

When `matrix_element_transpose` equals `transpose`, the `transpose` function is applied to every element. When `matrix_element_transpose` equals `nonscalars`, the `transpose` function is applied to every nonscalar element. If some element is an atom, the `nonscalars` option applies `transpose` only if the atom is declared nonscalar, while the `transpose` option always applies `transpose`.

The default value, `false`, means no operation is applied.

See also `matrix_element_add` and `matrix_element_mult`.

Examples:

```
(%i1) declare (a, nonscalar)$
(%i2) transpose ([a, b]);
                              [ a ]
(%o2)                         [   ]
                              [ b ]
(%i3) matrix_element_transpose: nonscalars$
```

```
(%i4) transpose ([a, b]);
                           [ transpose(a) ]
(%o4)                      [              ]
                           [      b       ]
(%i5) matrix_element_transpose: transpose$
(%i6) transpose ([a, b]);
                           [ transpose(a) ]
(%o6)                      [              ]
                           [ transpose(b) ]
(%i7) matrix_element_transpose: lambda ([x], realpart(x) - %i*imagpart(x))$
(%i8) m: matrix ([1 + 5*%i, 3 - 2*%i], [7*%i, 11]);
                         [ 5 %i + 1  3 - 2 %i ]
(%o8)                    [                    ]
                         [   7 %i       11    ]
(%i9) transpose (m);
                          [ 1 - 5 %i  - 7 %i ]
(%o9)                     [                  ]
                          [ 2 %i + 3    11   ]
```

`mattrace` (*M*) [Function]

Returns the trace (that is, the sum of the elements on the main diagonal) of the square matrix *M*.

`mattrace` is called by `ncharpoly`, an alternative to Maxima's `charpoly`.

`load ("nchrpl")` loads this function.

`minor` (*M*, *i*, *j*) [Function]

Returns the *i*, *j* minor of the matrix *M*. That is, *M* with row *i* and column *j* removed.

`ncharpoly` (*M*, *x*) [Function]

Returns the characteristic polynomial of the matrix *M* with respect to *x*. This is an alternative to Maxima's `charpoly`.

`ncharpoly` works by computing traces of powers of the given matrix, which are known to be equal to sums of powers of the roots of the characteristic polynomial. From these quantities the symmetric functions of the roots can be calculated, which are nothing more than the coefficients of the characteristic polynomial. `charpoly` works by forming the determinant of *x* `* ident [n] - a`. Thus `ncharpoly` wins, for example, in the case of large dense matrices filled with integers, since it avoids polynomial arithmetic altogether.

`load ("nchrpl")` loads this file.

`newdet` (*M*) [Function]

Computes the determinant of the matrix *M* by the Johnson-Gentleman tree minor algorithm. `newdet` returns the result in CRE form.

`permanent` (*M*) [Function]

Computes the permanent of the matrix *M* by the Johnson-Gentleman tree minor algorithm. A permanent is like a determinant but with no sign changes. `permanent` returns the result in CRE form.

See also `newdet`.

`rank` (*M*) [Function]

Computes the rank of the matrix *M*. That is, the order of the largest non-singular subdeterminant of *M*.

rank may return the wrong answer if it cannot determine that a matrix element that is equivalent to zero is indeed so.

`ratmx` [Option variable]

Default value: `false`

When `ratmx` is `false`, determinant and matrix addition, subtraction, and multiplication are performed in the representation of the matrix elements and cause the result of matrix inversion to be left in general representation.

When `ratmx` is `true`, the 4 operations mentioned above are performed in CRE form and the result of matrix inverse is in CRE form. Note that this may cause the elements to be expanded (depending on the setting of `ratfac`) which might not always be desired.

`row` (*M*, *i*) [Function]

Returns the *i*'th row of the matrix *M*. The return value is a matrix.

The matrix returned by `row` shares memory with the argument *M*; a modification to the return value modifies *M*.

Examples:

`row` returns the *i*'th row of the matrix *M*.

```
(%i1) abc: matrix ([12, 14, -4], [2, x, b], [3*y, -7, 9]);
                          [ 12   14   - 4 ]
                          [               ]
(%o1)                     [  2    x    b  ]
                          [               ]
                          [ 3 y  - 7   9  ]
(%i2) row (abc, 1);
(%o2)                      [ 12  14  - 4 ]
(%i3) row (abc, 2);
(%o3)                        [ 2  x  b ]
(%i4) row (abc, 3);
(%o4)                      [ 3 y  - 7  9 ]
```

The matrix returned by `row` shares memory with the argument. In this example, assigning a new value to `aa2` also modifies `aa`.

```
(%i1) aa: matrix ([1, 2, x], [7, y, 3]);
                            [ 1  2  x ]
(%o1)                       [         ]
                            [ 7  y  3 ]
(%i2) aa2: row (aa, 2);
(%o2)                       [ 7  y  3 ]
(%i3) aa2[1, 3]: 123;
(%o3)                          123
```

```
(%i4) aa2;
(%o4)                        [ 7  y  123 ]
(%i5) aa;
                             [ 1  2   x  ]
(%o5)                        [           ]
                             [ 7  y  123 ]
```

`rmxchar` [Option variable]

Default value: `]`

`rmxchar` is the character drawn on the right-hand side of a matrix.

`rmxchar` is only used when `display2d_unicode` is `false`.

See also `lmxchar`.

`scalarmatrixp` [Option variable]

Default value: `true`

When `scalarmatrixp` is `true`, then whenever a 1 x 1 matrix is produced as a result of computing the dot product of matrices it is simplified to a scalar, namely the sole element of the matrix.

When `scalarmatrixp` is `all`, then all 1 x 1 matrices are simplified to scalars.

When `scalarmatrixp` is `false`, 1 x 1 matrices are not simplified to scalars.

`scalefactors` (*coordinatetransform*) [Function]

Here the argument *coordinatetransform* evaluates to the form `[[expression1, expression2, ...], indeterminate1, indeterminat2, ...]`, where the variables *indeterminate1*, *indeterminate2*, etc. are the curvilinear coordinate variables and where a set of rectangular Cartesian components is given in terms of the curvilinear coordinates by `[expression1, expression2, ...]`. `coordinates` is set to the vector `[indeterminate1, indeterminate2,...]`, and `dimension` is set to the length of this vector. SF[1], SF[2], ..., SF[DIMENSION] are set to the coordinate scale factors, and `sfprod` is set to the product of these scale factors. Initially, `coordinates` is `[X, Y, Z]`, `dimension` is 3, and SF[1]=SF[2]=SF[3]=SFPROD=1, corresponding to 3-dimensional rectangular Cartesian coordinates. To expand an expression into physical components in the current coordinate system, there is a function with usage of the form

`setelmx` (*x*, *i*, *j*, *M*) [Function]

Assigns *x* to the (*i*, *j*)'th element of the matrix *M*, and returns the altered matrix.

`M [i, j]: x` has the same effect, but returns *x* instead of *M*.

`similaritytransform` (*M*) [Function]
`simtran` (*M*) [Function]

`similaritytransform` computes a similarity transform of the matrix `M`. It returns a list which is the output of the `uniteigenvectors` command. In addition if the flag `nondiagonalizable` is `false` two global matrices `leftmatrix` and `rightmatrix` are computed. These matrices have the property that `leftmatrix . M . rightmatrix` is a diagonal matrix with the eigenvalues of *M* on the diagonal. If `nondiagonalizable` is `true` the left and right matrices are not computed.

If the flag `hermitianmatrix` is `true` then `leftmatrix` is the complex conjugate of the transpose of `rightmatrix`. Otherwise `leftmatrix` is the inverse of `rightmatrix`.

`rightmatrix` is the matrix the columns of which are the unit eigenvectors of *M*. The other flags (see `eigenvalues` and `eigenvectors`) have the same effects since `similaritytransform` calls the other functions in the package in order to be able to form `rightmatrix`.

`load ("eigen")` loads this function.

`simtran` is a synonym for `similaritytransform`.

`sparse` [Option variable]

Default value: `false`

When `sparse` is `true`, and if `ratmx` is `true`, then `determinant` will use special routines for computing sparse determinants.

`submatrix` [Function]

`submatrix` (*i_1*, ..., *i_m*, *M*, *j_1*, ..., *j_n*)
`submatrix` (*i_1*, ..., *i_m*, *M*)
`submatrix` (*M*, *j_1*, ..., *j_n*)

Returns a new matrix composed of the matrix *M* with rows *i_1*, ..., *i_m* deleted, and columns *j_1*, ..., *j_n* deleted.

`transpose` (*M*) [Function]

Returns the transpose of *M*.

If *M* is a matrix, the return value is another matrix *N* such that `N[i,j] = M[j,i]`.

If *M* is a list, the return value is a matrix *N* of `length (m)` rows and 1 column, such that `N[i,1] = M[i]`.

Otherwise *M* is a symbol, and the return value is a noun expression `'transpose (M)`.

`triangularize` (*M*) [Function]

Returns the upper triangular form of the matrix `M`, as produced by Gaussian elimination. The return value is the same as `echelon`, except that the leading nonzero coefficient in each row is not normalized to 1.

`lu_factor` and `cholesky` are other functions which yield triangularized matrices.

```
(%i1) M: matrix ([3, 7, aa, bb], [-1, 8, 5, 2], [9, 2, 11, 4]);
                        [  3   7  aa  bb ]
                        [                ]
(%o1)                   [ - 1  8  5   2  ]
                        [                ]
                        [  9   2  11  4  ]
(%i2) triangularize (M);
                [ - 1   8         5            2       ]
                [                                      ]
(%o2)           [  0   - 74     - 56         - 22      ]
                [                                      ]
                [  0    0    626 - 74 aa  238 - 74 bb ]
```

`uniteigenvectors` (*M*) [Function]
`ueivects` (*M*) [Function]

Computes unit eigenvectors of the matrix *M*. The return value is a list of lists, the first sublist of which is the output of the `eigenvalues` command, and the other sublists of which are the unit eigenvectors of the matrix corresponding to those eigenvalues respectively.

The flags mentioned in the description of the `eigenvectors` command have the same effects in this one as well.

When `knowneigvects` is `true`, the `eigen` package assumes that the eigenvectors of the matrix are known to the user and are stored under the global name `listeigvects`. `listeigvects` should be set to a list similar to the output of the `eigenvectors` command.

If `knowneigvects` is set to `true` and the list of eigenvectors is given the setting of the flag `nondiagonalizable` may not be correct. If that is the case please set it to the correct value. The author assumes that the user knows what he is doing and will not try to diagonalize a matrix the eigenvectors of which do not span the vector space of the appropriate dimension.

`load ("eigen")` loads this function.

`ueivects` is a synonym for `uniteigenvectors`.

`unitvector` (*x*) [Function]
`uvect` (*x*) [Function]

Returns $x/norm(x)$; this is a unit vector in the same direction as x.

`load ("eigen")` loads this function.

`uvect` is a synonym for `unitvector`.

`vectorpotential` (*givencurl*) [Function]

Returns the vector potential of a given curl vector, in the current coordinate system. `potentialzeroloc` has a similar role as for `potential`, but the order of the left-hand sides of the equations must be a cyclic permutation of the coordinate variables.

`vectorsimp` (*expr*) [Function]

Applies simplifications and expansions according to the following global flags:

`expandall`, `expanddot`, `expanddotplus`, `expandcross`, `expandcrossplus`, `expandcrosscross`, `expandgrad`, `expandgradplus`, `expandgradprod`, `expanddiv`, `expanddivplus`, `expanddivprod`, `expandcurl`, `expandcurlplus`, `expandcurlcurl`, `expandlaplacian`, `expandlaplacianplus`, and `expandlaplacianprod`.

All these flags have default value `false`. The `plus` suffix refers to employing additivity or distributivity. The `prod` suffix refers to the expansion for an operand that is any kind of product.

`expandcrosscross`
: Simplifies $p \sim (q \sim r)$ to $(p.r)q - (p.q)r$.

`expandcurlcurl`
: Simplifies curl curl p to grad div $p +$ div grad p.

`expandlaplaciantodivgrad`
: Simplifies laplacian p to div grad p.

`expandcross`
: Enables `expandcrossplus` and `expandcrosscross`.

`expandplus`
: Enables `expanddotplus`, `expandcrossplus`, `expandgradplus`, `expanddivplus`, `expandcurlplus`, and `expandlaplacianplus`.

`expandprod`
: Enables `expandgradprod`, `expanddivprod`, and `expandlaplacianprod`.

These flags have all been declared `evflag`.

`vect_cross` [Option variable]

Default value: `false`

When `vect_cross` is `true`, it allows DIFF(X~Y,T) to work where `~` is defined in SHARE;VECT (where VECT_CROSS is set to `true`, anyway.)

`zeromatrix` (*m*, *n*) [Function]

Returns an m by n matrix, all elements of which are zero.

26 Sums, Products, and Series

26.1 Functions and Variables for Sums and Products

`bashindices` (*expr*) [Function]

Transforms the expression *expr* by giving each summation and product a unique index. This gives `changevar` greater precision when it is working with summations or products. The form of the unique index is `j`*number*. The quantity *number* is determined by referring to `gensumnum`, which can be changed by the user. For example, `gensumnum:0$` resets it.

`lsum` (*expr*, *x*, *L*) [Function]

Represents the sum of *expr* for each element *x* in *L*. A noun form `'lsum` is returned if the argument *L* does not evaluate to a list.

Examples:

```
(%i1) lsum (x^i, i, [1, 2, 7]);
                              7    2
(%o1)                        x  + x  + x
(%i2) lsum (i^2, i, rootsof (x^3 - 1, x));
                     ----
                     \                      2
(%o2)                 >                    i
                     /
                     ----
                                  3
                     i in rootsof(x  - 1, x)
```

`intosum` (*expr*) [Function]

Moves multiplicative factors outside a summation to inside. If the index is used in the outside expression, then the function tries to find a reasonable index, the same as it does for `sumcontract`. This is essentially the reverse idea of the `outative` property of summations, but note that it does not remove this property, it only bypasses it.

In some cases, a `scanmap (multthru, `*expr*`)` may be necessary before the `intosum`.

`simpproduct` [Option variable]

Default value: `false`

When `simpproduct` is `true`, the result of a `product` is simplified. This simplification may sometimes be able to produce a closed form. If `simpproduct` is `false` or if the quoted form `'product` is used, the value is a product noun form which is a representation of the pi notation used in mathematics.

`product` (*expr*, *i*, *i_0*, *i_1*) [Function]

Represents a product of the values of *expr* as the index *i* varies from *i_0* to *i_1*. The noun form `'product` is displayed as an uppercase letter pi.

`product` evaluates *expr* and lower and upper limits *i_0* and *i_1*, `product` quotes (does not evaluate) the index *i*.

If the upper and lower limits differ by an integer, *expr* is evaluated for each value of the index *i*, and the result is an explicit product.

Otherwise, the range of the index is indefinite. Some rules are applied to simplify the product. When the global variable `simpproduct` is `true`, additional rules are applied. In some cases, simplification yields a result which is not a product; otherwise, the result is a noun form `'product`.

See also `nouns` and `evflag`.

Examples:

```
(%i1) product (x + i*(i+1)/2, i, 1, 4);
(%o1)           (x + 1) (x + 3) (x + 6) (x + 10)
(%i2) product (i^2, i, 1, 7);
(%o2)                        25401600
(%i3) product (a[i], i, 1, 7);
(%o3)                  a  a  a  a  a  a  a
                        1  2  3  4  5  6  7
(%i4) product (a(i), i, 1, 7);
(%o4)          a(1) a(2) a(3) a(4) a(5) a(6) a(7)
(%i5) product (a(i), i, 1, n);
                              n
                             -----
                             |   |
(%o5)                        |   | a(i)
                             |   |
                             i = 1
(%i6) product (k, k, 1, n);
                                n
                               -----
                               |   |
(%o6)                          |   | k
                               |   |
                               k = 1
(%i7) product (k, k, 1, n), simpproduct;
(%o7)                            n!
(%i8) product (integrate (x^k, x, 0, 1), k, 1, n);
                              n
                             -----
                             |   |   1
(%o8)                        |   | -----
                             |   | k + 1
                             k = 1
(%i9) product (if k <= 5 then a^k else b^k, k, 1, 10);
                                15  40
(%o9)                          a   b
```

`simpsum` [Option variable]

Default value: `false`

When `simpsum` is `true`, the result of a `sum` is simplified. This simplification may sometimes be able to produce a closed form. If `simpsum` is `false` or if the quoted form `'sum` is used, the value is a sum noun form which is a representation of the sigma notation used in mathematics.

`sum` (*expr*, *i*, *i_0*, *i_1*) [Function]

Represents a summation of the values of *expr* as the index *i* varies from *i_0* to *i_1*. The noun form `'sum` is displayed as an uppercase letter sigma.

`sum` evaluates its summand *expr* and lower and upper limits *i_0* and *i_1*, `sum` quotes (does not evaluate) the index *i*.

If the upper and lower limits differ by an integer, the summand *expr* is evaluated for each value of the summation index *i*, and the result is an explicit sum.

Otherwise, the range of the index is indefinite. Some rules are applied to simplify the summation. When the global variable `simpsum` is `true`, additional rules are applied. In some cases, simplification yields a result which is not a summation; otherwise, the result is a noun form `'sum`.

When the `evflag` (evaluation flag) `cauchysum` is `true`, a product of summations is expressed as a Cauchy product, in which the index of the inner summation is a function of the index of the outer one, rather than varying independently.

The global variable `genindex` is the alphabetic prefix used to generate the next index of summation, when an automatically generated index is needed.

`gensumnum` is the numeric suffix used to generate the next index of summation, when an automatically generated index is needed. When `gensumnum` is `false`, an automatically-generated index is only `genindex` with no numeric suffix.

Note that `sum` is slow for symbolic sums that result in many terms, such as `sum(x^i, i, 1, 10000)`. For such cases, using `tree_reduce` and `makelist` is significantly faster, e.g. `tree_reduce("+", makelist(x^i, i, 1, 10000))`.

See also `lsum`, `sumcontract`, `intosum`, `bashindices`, `niceindices`, `nouns`, `evflag`, and Chapter 102 [Package zeilberger], page 1391,

Examples:

```
(%i1) sum (i^2, i, 1, 7);
(%o1)                          140
(%i2) sum (a[i], i, 1, 7);
(%o2)           a  + a  + a  + a  + a  + a  + a
                 7    6    5    4    3    2    1
(%i3) sum (a(i), i, 1, 7);
(%o3)    a(7) + a(6) + a(5) + a(4) + a(3) + a(2) + a(1)
(%i4) sum (a(i), i, 1, n);
                             n
                           ____
                           \
(%o4)                       >    a(i)
                           /
                           ----
                           i = 1
```

```
(%i5) sum (2^i + i^2, i, 0, n);
                          n
                         ____
                         \       2    i
(%o5)                     >    (i  + 2 )
                         /
                         ----
                         i = 0
(%i6) sum (2^i + i^2, i, 0, n), simpsum;
                     3      2
                  2 n  + 3 n  + n    n + 1
(%o6)             --------------- + 2      - 1
                         6
(%i7) sum (1/3^i, i, 1, inf);
                            inf
                            ____
                            \     1
(%o7)                        >    --
                            /      i
                            ----  3
                            i = 1
(%i8) sum (1/3^i, i, 1, inf), simpsum;
                               1
(%o8)                          -
                               2
(%i9) sum (i^2, i, 1, 4) * sum (1/i^2, i, 1, inf);
                               inf
                               ____
                               \     1
(%o9)                       30  >    --
                               /      2
                               ----  i
                               i = 1
(%i10) sum (i^2, i, 1, 4) * sum (1/i^2, i, 1, inf), simpsum;
                                    2
(%o10)                           5 %pi
(%i11) sum (integrate (x^k, x, 0, 1), k, 1, n);
                           n
                          ____
                          \       1
(%o11)                     >    -----
                          /     k + 1
                          ----
                          k = 1
(%i12) sum (if k <= 5 then a^k else b^k, k, 1, 10);
         10    9    8    7    6    5    4    3    2
(%o12)   b   + b  + b  + b  + b  + a  + a  + a  + a  + a
```

`sumcontract (expr)` [Function]

Combines all sums of an addition that have upper and lower bounds that differ by constants. The result is an expression containing one summation for each set of such summations added to all appropriate extra terms that had to be extracted to form this sum. `sumcontract` combines all compatible sums and uses one of the indices from one of the sums if it can, and then try to form a reasonable index if it cannot use any supplied.

It may be necessary to do an `intosum (expr)` before the `sumcontract`.

`sumexpand` [Option variable]

Default value: `false`

When `sumexpand` is `true`, products of sums and exponentiated sums simplify to nested sums.

See also `cauchysum`.

Examples:

```
(%i1) sumexpand: true$
(%i2) sum (f (i), i, 0, m) * sum (g (j), j, 0, n);
                       m      n
                      ----   ----
                      \      \
(%o2)                  >      >     f(i1) g(i2)
                      /      /
                      ----   ----
                      i1 = 0 i2 = 0
(%i3) sum (f (i), i, 0, m)^2;
                       m      m
                      ----   ----
                      \      \
(%o3)                  >      >     f(i3) f(i4)
                      /      /
                      ----   ----
                      i3 = 0 i4 = 0
```

26.2 Introduction to Series

Maxima contains functions `taylor` and `powerseries` for finding the series of differentiable functions. It also has tools such as `nusum` capable of finding the closed form of some series. Operations such as addition and multiplication work as usual on series. This section presents the global variables which control the expansion.

26.3 Functions and Variables for Series

`cauchysum` [Option variable]

Default value: `false`

When multiplying together sums with `inf` as their upper limit, if `sumexpand` is `true` and `cauchysum` is `true` then the Cauchy product will be used rather than the usual

product. In the Cauchy product the index of the inner summation is a function of the index of the outer one rather than varying independently.

Example:

```
(%i1) sumexpand: false$
(%i2) cauchysum: false$
(%i3) s: sum (f(i), i, 0, inf) * sum (g(j), j, 0, inf);
                       inf          inf
                       ____         ____
                       \            \
(%o3)                   >    f(i))   >    g(j)
                       /            /
                       ____         ____
                       i = 0        j = 0
(%i4) sumexpand: true$
(%i5) cauchysum: true$
(%i6) expand(s,0,0);
                  inf     i1
                  ____   ____
                  \      \
(%o6)              >      >     g(i1 - i2) f(i2)
                  /      /
                  ____   ____
                  i1 = 0 i2 = 0
```

`deftaylor` (*f_1*(*x_1*), *expr_1*, ..., *f_n*(*x_n*), *expr_n*) [Function]

For each function *f_i* of one variable *x_i*, `deftaylor` defines *expr_i* as the Taylor series about zero. *expr_i* is typically a polynomial in *x_i* or a summation; more general expressions are accepted by `deftaylor` without complaint.

`powerseries (f_i(x_i), x_i, 0)` returns the series defined by `deftaylor`.

`deftaylor` returns a list of the functions *f_1*, ..., *f_n*. `deftaylor` evaluates its arguments.

Example:

```
(%i1) deftaylor (f(x), x^2 + sum(x^i/(2^i*i!^2), i, 4, inf));
(%o1)                          [f]
(%i2) powerseries (f(x), x, 0);
                       inf
                       ____      i1
                       \        x         2
(%o2)                   >     -------- + x
                       /       i1    2
                       ____   2   i1!
                       i1 = 4
```

```
(%i3) taylor (exp (sqrt (f(x))), x, 0, 4);
                       2         3          4
                      x    3073 x    12817 x
(%o3)/T/     1 + x + -- + ------- + -------- + . . .
                      2     18432     307200
```

`maxtayorder` [Option variable]

Default value: `true`

When `maxtayorder` is `true`, then during algebraic manipulation of (truncated) Taylor series, `taylor` tries to retain as many terms as are known to be correct.

`niceindices` (*expr*) [Function]

Renames the indices of sums and products in *expr*. `niceindices` attempts to rename each index to the value of `niceindicespref[1]`, unless that name appears in the summand or multiplicand, in which case `niceindices` tries the succeeding elements of `niceindicespref` in turn, until an unused variable is found. If the entire list is exhausted, additional indices are constructed by appending integers to the value of `niceindicespref[1]`, e.g., `i0`, `i1`, `i2`, ...

`niceindices` returns an expression. `niceindices` evaluates its argument.

Example:

```
(%i1) niceindicespref;
(%o1)                   [i, j, k, l, m, n]
(%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);
                 inf    inf
                -----   ----
                |   |   \
(%o2)           |   |    >      f(bar i j + foo)
                |   |   /
                |   |   /
                bar = 1 ----
                        foo = 1
(%i3) niceindices (%);
                     inf  inf
                    ----- ----
                    |   | \
(%o3)               |   |  >    f(i j l + k)
                    |   | /
                    |   | /
                    l = 1 ----
                          k = 1
```

`niceindicespref` [Option variable]

Default value: `[i, j, k, l, m, n]`

`niceindicespref` is the list from which `niceindices` takes the names of indices for sums and products.

The elements of `niceindicespref` are must be names of simple variables.

Example:

```
(%i1) niceindicespref: [p, q, r, s, t, u]$
```

```
(%i2) product (sum (f (foo + i*j*bar), foo, 1, inf), bar, 1, inf);
                 inf    inf
                -----   ----
                |   |   \
(%o2)           |   |    >      f(bar i j + foo)
                |   |   /
                bar = 1 ----
                        foo = 1
(%i3) niceindices (%);
                     inf  inf
                    ----- ----
                    |   | \
(%o3)               |   |  >    f(i j q + p)
                    |   | /
                    q = 1 ----
                          p = 1
```

nusum (*expr*, *x*, *i_0*, *i_1*) [Function]

Carries out indefinite hypergeometric summation of *expr* with respect to *x* using a decision procedure due to R.W. Gosper. *expr* and the result must be expressible as products of integer powers, factorials, binomials, and rational functions.

The terms "definite" and "indefinite summation" are used analogously to "definite" and "indefinite integration". To sum indefinitely means to give a symbolic result for the sum over intervals of variable length, not just e.g. 0 to inf. Thus, since there is no formula for the general partial sum of the binomial series, nusum can't do it.

nusum and unsum know a little about sums and differences of finite products. See also unsum.

Examples:

```
(%i1) nusum (n*n!, n, 0, n);
solve: dependent equations eliminated: (1)
(%o1)                      (n + 1)! - 1
(%i2) nusum (n^4*4^n/binomial(2*n,n), n, 0, n);
         n              4        3       2
      2 4  (n + 1) (63 n  + 112 n  + 18 n  - 22 n + 3)    2
(%o2) ------------------------------------------------ - ---
                    693 binomial(2 n, n)                 231
(%i3) unsum (%, n);
                             n  4
                            4  n
(%o3)                     ---------------
                          binomial(2 n, n)
```

```
(%i4) unsum (prod (i^2, i, 1, n), n);
                     n - 1
                     -----
                     |   |  2
(%o4)               (|   | i ) (n - 1) (n + 1)
                     |   |
                     i = 1
(%i5) nusum (%, n, 1, n);
solve: dependent equations eliminated: (2 3)
                              n
                            -----
                            |   |  2
(%o5)                       |   | i  - 1
                            |   |
                            i = 1
```

pade (*taylor_series*, *numer_deg_bound*, *denom_deg_bound*) [Function]

Returns a list of all rational functions which have the given Taylor series expansion where the sum of the degrees of the numerator and the denominator is less than or equal to the truncation level of the power series, i.e. are "best" approximants, and which additionally satisfy the specified degree bounds.

taylor_series is an univariate Taylor series. *numer_deg_bound* and *denom_deg_bound* are positive integers specifying degree bounds on the numerator and denominator.

taylor_series can also be a Laurent series, and the degree bounds can be `inf` which causes all rational functions whose total degree is less than or equal to the length of the power series to be returned. Total degree is defined as *numer_deg_bound* + *denom_deg_bound*. Length of a power series is defined as `"truncation level" + 1 - min(0, "order of series")`.

```
(%i1) taylor (1 + x + x^2 + x^3, x, 0, 3);
                                2    3
(%o1)/T/             1 + x + x  + x  + . . .
(%i2) pade (%, 1, 1);
                                 1
(%o2)                       [- -----]
                               x - 1
```

```
(%i3) t: taylor(-(83787*x^10 - 45552*x^9 - 187296*x^8
             + 387072*x^7 + 86016*x^6 - 1507328*x^5
             + 1966080*x^4 + 4194304*x^3 - 25165824*x^2
             + 67108864*x - 134217728)
 /134217728, x, 0, 10);
                     2    3       4       5       6        7
             x   3 x    x    15 x    23 x    21 x    189 x
(%o3)/T/ 1 - - + ---- - -- - ----- + ----- - ----- - ------
             2    16    32   1024    2048    32768   65536
                                8         9          10
                            5853 x    2847 x    83787 x
                          + ------- + ------- - --------- + . . .
                            4194304   8388608   134217728
(%i4) pade (t, 4, 4);
(%o4)                          []
```

There is no rational function of degree 4 numerator/denominator, with this power series expansion. You must in general have degree of the numerator and degree of the denominator adding up to at least the degree of the power series, in order to have enough unknown coefficients to solve.

```
(%i5) pade (t, 5, 5);
                      5                4                 3
(%o5) [- (520256329 x  - 96719020632 x  - 489651410240 x

                   2
 - 1619100813312 x  - 2176885157888 x - 2386516803584)

                5                 4                   3
/(47041365435 x  + 381702613848 x  + 1360678489152 x

                   2
 + 2856700692480 x  + 3370143559680 x + 2386516803584)]
```

`powerseries` (*expr*, *x*, *a*) [Function]

Returns the general form of the power series expansion for *expr* in the variable *x* about the point *a* (which may be `inf` for infinity):

```
inf
====
\              n
 >    b  (x - a)
/      n
====
n = 0
```

If `powerseries` is unable to expand *expr*, `taylor` may give the first several terms of the series.

When `verbose` is `true`, `powerseries` prints progress messages.

```
(%i1) verbose: true$
```

```
(%i2) powerseries (log(sin(x)/x), x, 0);
trigreduce: failed to expand.

                              sin(x)
                          log(------)
                                x

trigreduce: try again after applying rule:
                                 d   sin(x)
                                / -- (------)
                    sin(x)      | dx    x
                log(------) = | ------------ dx
                      x       |   sin(x)
                              /   ------
                                    x


powerseries: first simplification returned
    x
   /
   |  csc(g3955) sin(g3955) - g3955 cos(g3955) csc(g3955)
 - |  ---------------------------------------------------- dg3955
   |                        g3955
   /
    0


powerseries: first simplification returned
                      g3955 cot(g3955) - 1
                    - --------------------
                              g3955

powerseries: attempt rational function expansion of
                                1
                               -----
                               g3955
            inf
            ____         i2  2 i2 - 1             2 i2
            \      (- 1)   2         bern(2 i2) x
(%o2)        >     ---------------------------------
            /                  i2 (2 i2)!
            ----
            i2 = 1
```

psexpand [Option variable]

Default value: `false`

When `psexpand` is `true`, an extended rational function expression is displayed fully expanded. The switch `ratexpand` has the same effect.

When `psexpand` is `false`, a multivariate expression is displayed just as in the rational function package.

When `psexpand` is `multi`, then terms with the same total degree in the variables are grouped together.

`revert` (*expr*, *x*) [Function]
`revert2` (*expr*, *x*, *n*) [Function]

These functions return the reversion of *expr*, a Taylor series about zero in the variable *x*. `revert` returns a polynomial of degree equal to the highest power in *expr*. `revert2` returns a polynomial of degree *n*, which may be greater than, equal to, or less than the degree of *expr*.

`load ("revert")` loads these functions.

Examples:

```
(%i1) load ("revert")$
(%i2) t: taylor (exp(x) - 1, x, 0, 6);
                    2    3    4    5     6
                   x    x    x    x     x
(%o2)/T/      x + -- + -- + -- + --- + --- + . . .
                   2    6    24   120   720
(%i3) revert (t, x);
                6       5       4       3       2
           10 x  - 12 x  + 15 x  - 20 x  + 30 x  - 60 x
(%o3)/R/ - ----------------------------------------
                               60
(%i4) ratexpand (%);
                      6    5    4    3    2
                     x    x    x    x    x
(%o4)              - -- + -- - -- + -- - -- + x
                     6    5    4    3    2
(%i5) taylor (log(x+1), x, 0, 6);
                     2    3    4    5    6
                    x    x    x    x    x
(%o5)/T/       x - -- + -- - -- + -- - -- + . . .
                    2    3    4    5    6
(%i6) ratsimp (revert (t, x) - taylor (log(x+1), x, 0, 6));
(%o6)                           0
(%i7) revert2 (t, x, 4);
                          4    3    2
                         x    x    x
(%o7)                  - -- + -- - -- + x
                         4    3    2
```

`taylor` [Function]

`taylor (expr, x, a, n)`
`taylor (expr, [x_1, x_2, ...], a, n)`
`taylor (expr, [x, a, n, 'asymp])`
`taylor (expr, [x_1, x_2, ...], [a_1, a_2, ...], [n_1, n_2, ...])`
`taylor (expr, [x_1, a_1, n_1], [x_2, a_2, n_2], ...)`

`taylor (expr, x, a, n)` expands the expression *expr* in a truncated Taylor or Laurent series in the variable *x* around the point *a*, containing terms through `(x - a)^n`.

If *expr* is of the form `f(x)/g(x)` and `g(x)` has no terms up to degree *n* then `taylor` attempts to expand `g(x)` up to degree `2 n`. If there are still no nonzero terms, `taylor` doubles the degree of the expansion of `g(x)` so long as the degree of the expansion is less than or equal to `n 2^taylordepth`.

`taylor (expr, [x_1, x_2, ...], a, n)` returns a truncated power series of degree *n* in all variables *x_1*, *x_2*, ... about the point `(a, a, ...)`.

`taylor (expr, [x_1, a_1, n_1], [x_2, a_2, n_2], ...)` returns a truncated power series in the variables *x_1*, *x_2*, ... about the point `(a_1, a_2, ...)`, truncated at *n_1*, *n_2*, ...

`taylor (expr, [x_1, x_2, ...], [a_1, a_2, ...], [n_1, n_2, ...])` returns a truncated power series in the variables *x_1*, *x_2*, ... about the point `(a_1, a_2, ...)`, truncated at *n_1*, *n_2*, ...

`taylor (expr, [x, a, n, 'asymp])` returns an expansion of *expr* in negative powers of `x - a`. The highest order term is `(x - a)^-n`.

When `maxtayorder` is `true`, then during algebraic manipulation of (truncated) Taylor series, `taylor` tries to retain as many terms as are known to be correct.

When `psexpand` is `true`, an extended rational function expression is displayed fully expanded. The switch `ratexpand` has the same effect. When `psexpand` is `false`, a multivariate expression is displayed just as in the rational function package. When `psexpand` is `multi`, then terms with the same total degree in the variables are grouped together.

See also the `taylor_logexpand` switch for controlling expansion.

Examples:

```
(%i1) taylor (sqrt (sin(x) + a*x + 1), x, 0, 3);
                              2             2
                 (a + 1) x   (a  + 2 a + 1) x
(%o1)/T/ 1 + --------- - -----------------
                    2               8
                                      3      2             3
                                  (3 a  + 9 a  + 9 a - 1) x
                                + ------------------------- + . . .
                                             48
(%i2) %^2;
                                       3
                                      x
(%o2)/T/           1 + (a + 1) x - -- + . . .
                                      6
```

```
(%i3) taylor (sqrt (x + 1), x, 0, 5);
                       2    3      4      5
                   x   x    x    5 x    7 x
(%o3)/T/      1 + - - -- + -- - ---- + ---- + . . .
                   2   8    16   128    256
(%i4) %^2;
(%o4)/T/                   1 + x + . . .
(%i5) product ((1 + x^i)^2.5, i, 1, inf)/(1 + x^2);
                        inf
                        _____
                        |   |   i     2.5
                        |   | (x  + 1)
                        |   |
                        i = 1
(%o5)                   -----------------
                              2
                             x  + 1
(%i6) ev (taylor(%, x,  0, 3), keepfloat);
                                2           3
(%o6)/T/    1 + 2.5 x + 3.375 x  + 6.5625 x  + . . .
(%i7) taylor (1/log (x + 1), x, 0, 3);
                                2       3
                1   1   x    x    19 x
(%o7)/T/        - + - - -- + -- - ----- + . . .
                x   2   12   24    720
(%i8) taylor (cos(x) - sec(x), x, 0, 5);
                              4
                         2   x
(%o8)/T/                - x  - -- + . . .
                              6
(%i9) taylor ((cos(x) - sec(x))^3, x, 0, 5);
(%o9)/T/                    0 + . . .
(%i10) taylor (1/(cos(x) - sec(x))^3, x, 0, 5);
                                                 2          4
           1     1       11      347    6767 x    15377 x
(%o10)/T/ - -- + ---- + ------ - ----- - ------- - --------
           6      4        2   15120   604800    7983360
          x    2 x    120 x
                                                          + . . .
```

```
(%i11) taylor (sqrt (1 - k^2*sin(x)^2), x, 0, 6);
                2  2       4      2   4
               k  x    (3 k  - 4 k ) x
(%o11)/T/ 1 - ----- - ----------------
                2            24
                                    6       4       2   6
                              (45 k  - 60 k  + 16 k ) x
                            - ------------------------- + . . .
                                        720
(%i12) taylor ((x + 1)^n, x, 0, 4);
                      2       2     3      2         3
                    (n  - n) x    (n  - 3 n  + 2 n) x
(%o12)/T/ 1 + n x + ----------- + --------------------
                         2                6
                                4      3       2         4
                              (n  - 6 n  + 11 n  - 6 n) x
                            + --------------------------- + . . .
                                          24
(%i13) taylor (sin (y + x), x, 0, 3, y, 0, 3);
               3                       2
              y                       y
(%o13)/T/ (- -- + y + . . .) + (1 - -- + . . .) x
              6                       2
                     3                       2
                y   y            2      1   y            3
           + (- - + -- + . . .) x  + (- - + -- + . . .) x  + . . .
                2   12                  6   12
(%i14) taylor (sin (y + x), [x, y], 0, 3);
                       3       2     2     3
                      x  + 3 y x  + 3 y  x + y
(%o14)/T/  (y + x) - ------------------------ + . . .
                                6
(%i15) taylor (1/sin (y + x), x, 0, 3, y, 0, 3);
            y   1               1    1
(%o15)/T/ (- + - + . . .) + (- -- + - + . . .) x
            6   y               2   6
                               y
                      1            2      1            3
                   + (-- + . . .) x  + (- -- + . . .) x  + . . .
                        3                   4
                       y                   y
(%i16) taylor (1/sin (y + x), [x, y], 0, 3);
                               3         2       2        3
             1     x + y   7 x  + 21 y x  + 21 y  x + 7 y
(%o16)/T/ ----- + ----- + ------------------------------ + . . .
          x + y     6                  360
```

`taylordepth` [Option variable]

Default value: 3

If there are still no nonzero terms, `taylor` doubles the degree of the expansion of *g*(*x*) so long as the degree of the expansion is less than or equal to *n* `2^taylordepth`.

`taylorinfo` (*expr*) [Function]

Returns information about the Taylor series *expr*. The return value is a list of lists. Each list comprises the name of a variable, the point of expansion, and the degree of the expansion.

`taylorinfo` returns `false` if *expr* is not a Taylor series.

Example:

```
(%i1) taylor ((1 - y^2)/(1 - x), x, 0, 3, [y, a, inf]);
                    2                           2
(%o1)/T/ ((1 - a ) - 2 a (y - a) - (y - a) )
              2                         2
 + (- (y - a)  - 2 a (y - a) + (1 - a )) x
              2                         2    2
 + (- (y - a)  - 2 a (y - a) + (1 - a )) x
              2                         2    3
 + (- (y - a)  - 2 a (y - a) + (1 - a )) x  + . . .
(%i2) taylorinfo(%);
(%o2)               [[x, 0, 3], [y, a, inf]]
```

`taylorp` (*expr*) [Function]

Returns `true` if *expr* is a Taylor series, and `false` otherwise.

`taylor_logexpand` [Option variable]

Default value: `true`

`taylor_logexpand` controls expansions of logarithms in `taylor` series.

When `taylor_logexpand` is `true`, all logarithms are expanded fully so that zero-recognition problems involving logarithmic identities do not disturb the expansion process. However, this scheme is not always mathematically correct since it ignores branch information.

When `taylor_logexpand` is set to `false`, then the only expansion of logarithms that occur is that necessary to obtain a formal power series.

`taylor_order_coefficients` [Option variable]

Default value: `true`

`taylor_order_coefficients` controls the ordering of coefficients in a Taylor series.

When `taylor_order_coefficients` is `true`, coefficients of taylor series are ordered canonically.

`taylor_simplifier` (*expr*) [Function]

Simplifies coefficients of the power series *expr*. `taylor` calls this function.

`taylor_truncate_polynomials` [Option variable]

Default value: `true`

When `taylor_truncate_polynomials` is `true`, polynomials are truncated based upon the input truncation levels.

Otherwise, polynomials input to `taylor` are considered to have infinite precision.

`taytorat` (*expr*) [Function]

Converts *expr* from `taylor` form to canonical rational expression (CRE) form. The effect is the same as `rat (ratdisrep (`*expr*`))`, but faster.

`trunc` (*expr*) [Function]

Annotates the internal representation of the general expression *expr* so that it is displayed as if its sums were truncated Taylor series. *expr* is not otherwise modified.

Example:

```
(%i1) expr: x^2 + x + 1;
                              2
(%o1)                        x  + x + 1
(%i2) trunc (expr);
                                  2
(%o2)                    1 + x + x  + . . .
(%i3) is (expr = trunc (expr));
(%o3)                           true
```

`unsum` (*f*, *n*) [Function]

Returns the first backward difference `f(n) - f(n - 1)`. Thus `unsum` in a sense is the inverse of `sum`.

See also `nusum`.

Examples:

```
(%i1) g(p) := p*4^n/binomial(2*n,n);
                                       n
                                    p 4
(%o1)                g(p) := ----------------
                                binomial(2 n, n)
(%i2) g(n^4);
                                  n  4
                                 4  n
(%o2)                       ----------------
                            binomial(2 n, n)
(%i3) nusum (%, n, 0, n);
          n               4        3       2
       2 4  (n + 1) (63 n  + 112 n  + 18 n  - 22 n + 3)    2
(%o3) ------------------------------------------------ - ---
                     693 binomial(2 n, n)                 231
```

```
(%i4) unsum (%, n);
                                    n  4
                                   4  n
(%o4)                         ----------------
                              binomial(2 n, n)
```

verbose [Option variable]

Default value: false

When verbose is true, powerseries prints progress messages.

26.4 Introduction to Fourier series

The fourie package comprises functions for the symbolic computation of Fourier series. There are functions in the fourie package to calculate Fourier integral coefficients and some functions for manipulation of expressions.

26.5 Functions and Variables for Fourier series

equalp (*x*, *y*) [Function]

Returns true if equal (*x*, *y*) otherwise false (doesn't give an error message like equal (x, y) would do in this case).

remfun [Function]

remfun (*f*, *expr*)
remfun (*f*, *expr*, *x*)

remfun (*f*, *expr*) replaces all occurrences of *f* (*arg*) by *arg* in *expr*.

remfun (*f*, *expr*, *x*) replaces all occurrences of *f* (*arg*) by *arg* in *expr* only if *arg* contains the variable *x*.

funp [Function]

funp (*f*, *expr*)
funp (*f*, *expr*, *x*)

funp (*f*, *expr*) returns true if *expr* contains the function *f*.

funp (*f*, *expr*, *x*) returns true if *expr* contains the function *f* and the variable *x* is somewhere in the argument of one of the instances of *f*.

absint [Function]

absint (*f*, *x*, *halfplane*)
absint (*f*, *x*)
absint (*f*, *x*, *a*, *b*)

absint (*f*, *x*, *halfplane*) returns the indefinite integral of *f* with respect to *x* in the given halfplane (pos, neg, or both). *f* may contain expressions of the form abs (x), abs (sin (x)), abs (a) * exp (-abs (b) * abs (x)).

absint (*f*, *x*) is equivalent to absint (*f*, *x*, pos).

absint (*f*, *x*, *a*, *b*) returns the definite integral of *f* with respect to *x* from *a* to *b*. *f* may include absolute values.

`fourier (f, x, p)` [Function]
Returns a list of the Fourier coefficients of `f(x)` defined on the interval `[-p, p]`.

`foursimp (l)` [Function]
Simplifies `sin (n %pi)` to 0 if `sinnpiflag` is `true` and `cos (n %pi)` to `(-1)^n` if `cosnpiflag` is `true`.

`sinnpiflag` [Option variable]
Default value: `true`

See `foursimp`.

`cosnpiflag` [Option variable]
Default value: `true`

See `foursimp`.

`fourexpand (l, x, p, limit)` [Function]
Constructs and returns the Fourier series from the list of Fourier coefficients *l* up through *limit* terms (*limit* may be `inf`). *x* and *p* have same meaning as in `fourier`.

`fourcos (f, x, p)` [Function]
Returns the Fourier cosine coefficients for `f(x)` defined on `[0, p]`.

`foursin (f, x, p)` [Function]
Returns the Fourier sine coefficients for `f(x)` defined on `[0, p]`.

`totalfourier (f, x, p)` [Function]
Returns `fourexpand (foursimp (fourier (f, x, p)), x, p, 'inf)`.

`fourint (f, x)` [Function]
Constructs and returns a list of the Fourier integral coefficients of `f(x)` defined on `[minf, inf]`.

`fourintcos (f, x)` [Function]
Returns the Fourier cosine integral coefficients for `f(x)` on `[0, inf]`.

`fourintsin (f, x)` [Function]
Returns the Fourier sine integral coefficients for `f(x)` on `[0, inf]`.

26.6 Functions and Variables for Poisson series

`intopois (a)` [Function]
Converts *a* into a Poisson encoding.

`outofpois (a)` [Function]
Converts *a* from Poisson encoding to general representation. If *a* is not in Poisson form, `outofpois` carries out the conversion, i.e., the return value is `outofpois (intopois (a))`. This function is thus a canonical simplifier for sums of powers of sine and cosine terms of a particular type.

`poisdiff (a, b)` [Function]
Differentiates *a* with respect to *b*. *b* must occur only in the trig arguments or only in the coefficients.

`poisexpt (a, b)` [Function]
Functionally identical to `intopois (a^b)`. *b* must be a positive integer.

`poisint (a, b)` [Function]
Integrates in a similarly restricted sense (to `poisdiff`). Non-periodic terms in *b* are dropped if *b* is in the trig arguments.

`poislim` [Option variable]
Default value: 5

`poislim` determines the domain of the coefficients in the arguments of the trig functions. The initial value of 5 corresponds to the interval [-2^(5-1)+1,2^(5-1)], or [-15,16], but it can be set to [-2^(n-1)+1, 2^(n-1)].

`poismap (series, sinfn, cosfn)` [Function]
will map the functions *sinfn* on the sine terms and *cosfn* on the cosine terms of the Poisson series given. *sinfn* and *cosfn* are functions of two arguments which are a coefficient and a trigonometric part of a term in series respectively.

`poisplus (a, b)` [Function]
Is functionally identical to `intopois (a + b)`.

`poissimp (a)` [Function]
Converts *a* into a Poisson series for *a* in general representation.

`poisson` [Special symbol]
The symbol `/P/` follows the line label of Poisson series expressions.

`poissubst (a, b, c)` [Function]
Substitutes *a* for *b* in *c*. *c* is a Poisson series.

(1) Where *B* is a variable *u*, *v*, *w*, *x*, *y*, or *z*, then *a* must be an expression linear in those variables (e.g., `6*u + 4*v`).

(2) Where *b* is other than those variables, then *a* must also be free of those variables, and furthermore, free of sines or cosines.

`poissubst (a, b, c, d, n)` is a special type of substitution which operates on *a* and *b* as in type (1) above, but where *d* is a Poisson series, expands `cos(d)` and `sin(d)` to order *n* so as to provide the result of substituting `a + d` for *b* in *c*. The idea is that *d* is an expansion in terms of a small parameter. For example, `poissubst (u, v, cos(v), %e, 3)` yields `cos(u)*(1 - %e^2/2) - sin(u)*(%e - %e^3/6)`.

`poistimes (a, b)` [Function]
Is functionally identical to `intopois (a*b)`.

`poistrim ()` [Function]
is a reserved function name which (if the user has defined it) gets applied during Poisson multiplication. It is a predicate function of 6 arguments which are the coefficients of the *u*, *v*, ..., *z* in a term. Terms for which `poistrim` is `true` (for the coefficients of that term) are eliminated during multiplication.

`printpois (a)` [Function]

Prints a Poisson series in a readable format. In common with `outofpois`, it will convert *a* into a Poisson encoding first, if necessary.

27 Number Theory

27.1 Functions and Variables for Number Theory

`bern` (*n*) [Function]

Returns the *n*'th Bernoulli number for integer *n*. Bernoulli numbers equal to zero are suppressed if `zerobern` is `false`.

See also `burn`.

```
(%i1) zerobern: true$
(%i2) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);
                    1  1       1      1        1
(%o2)         [1, - -, -, 0, - --, 0, --, 0, - --]
                    2  6       30     42       30
(%i3) zerobern: false$
(%i4) map (bern, [0, 1, 2, 3, 4, 5, 6, 7, 8]);
                  1  1    1   1     1   5     691   7
(%o4)      [1, - -, -, - --, --, - --, --, - ----, -]
                  2  6    30  42    30  66    2730  6
```

`bernpoly` (*x*, *n*) [Function]

Returns the *n*'th Bernoulli polynomial in the variable *x*.

`bfzeta` (*s*, *n*) [Function]

Returns the Riemann zeta function for the argument *s*. The return value is a big float (bfloat); *n* is the number of digits in the return value.

`bfhzeta` (*s*, *h*, *n*) [Function]

Returns the Hurwitz zeta function for the arguments *s* and *h*. The return value is a big float (bfloat); *n* is the number of digits in the return value.

The Hurwitz zeta function is defined as

$$\zeta(s,h) = \sum_{k=0}^{\infty} \frac{1}{(k+h)^s}$$

`load ("bffac")` loads this function.

`burn` (*n*) [Function]

Returns a rational number, which is an approximation of the *n*'th Bernoulli number for integer *n*. `burn` exploits the observation that (rational) Bernoulli numbers can be approximated by (transcendental) zetas with tolerable efficiency:

```
                    n - 1  1 - 2 n
             (- 1)      2        zeta(2 n) (2 n)!
B(2 n) = ------------------------------------
                                2 n
                             %pi
```

`burn` may be more efficient than `bern` for large, isolated *n* as `bern` computes all the Bernoulli numbers up to index *n* before returning. `burn` invokes the approximation for even integers `n > 255`. For odd integers and `n <= 255` the function `bern` is called.

`load ("bffac")` loads this function. See also `bern`.

`solve_congruences` ([*r_1*, ..., *r_n*], [*m_1*, ..., *m_n*]) [Function]

Solves the system of congruences `x = r_1 mod m_1`, ..., `x = r_n mod m_n`. The remainders *r_n* may be arbitrary integers while the moduli *m_n* have to be positive and pairwise coprime integers.

```
(%i1) mods : [1000, 1001, 1003, 1007];
(%o1)                [1000, 1001, 1003, 1007]
(%i2) lreduce('gcd, mods);
(%o2)                                1
(%i3) x : random(apply("*", mods));
(%o3)                        685124877004
(%i4) rems : map(lambda([z], mod(x, z)), mods);
(%o4)                      [4, 568, 54, 624]
(%i5) solve_congruences(rems, mods);
(%o5)                        685124877004
(%i6) solve_congruences([1, 2], [3, n]);
(%o6)            solve_congruences([1, 2], [3, n])
(%i7) %, n = 4;
(%o7)                               10
```

`cf` (*expr*) [Function]

Computes a continued fraction approximation. *expr* is an expression comprising continued fractions, square roots of integers, and literal real numbers (integers, rational numbers, ordinary floats, and bigfloats). `cf` computes exact expansions for rational numbers, but expansions are truncated at `ratepsilon` for ordinary floats and `10^(-fpprec)` for bigfloats.

Operands in the expression may be combined with arithmetic operators. Maxima does not know about operations on continued fractions outside of `cf`.

`cf` evaluates its arguments after binding `listarith` to `false`. `cf` returns a continued fraction, represented as a list.

A continued fraction `a + 1/(b + 1/(c + ...))` is represented by the list `[a, b, c, ...]`. The list elements `a`, `b`, `c`, ... must evaluate to integers. *expr* may also contain `sqrt (n)` where `n` is an integer. In this case `cf` will give as many terms of the continued fraction as the value of the variable `cflength` times the period.

A continued fraction can be evaluated to a number by evaluating the arithmetic representation returned by `cfdisrep`. See also `cfexpand` for another way to evaluate a continued fraction.

See also `cfdisrep`, `cfexpand`, and `cflength`.

Examples:

- *expr* is an expression comprising continued fractions and square roots of integers.

```
(%i1) cf ([5, 3, 1]*[11, 9, 7] + [3, 7]/[4, 3, 2]);
```

```
(%o1)               [59, 17, 2, 1, 1, 1, 27]
(%i2) cf ((3/17)*[1, -2, 5]/sqrt(11) + (8/13));
(%o2)        [0, 1, 1, 1, 3, 2, 1, 4, 1, 9, 1, 9, 2]
```

- `cflength` controls how many periods of the continued fraction are computed for algebraic, irrational numbers.

```
(%i1) cflength: 1$
(%i2) cf ((1 + sqrt(5))/2);
(%o2)                     [1, 1, 1, 1, 2]
(%i3) cflength: 2$
(%i4) cf ((1 + sqrt(5))/2);
(%o4)               [1, 1, 1, 1, 1, 1, 1, 2]
(%i5) cflength: 3$
(%i6) cf ((1 + sqrt(5))/2);
(%o6)           [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]
```

- A continued fraction can be evaluated by evaluating the arithmetic representation returned by `cfdisrep`.

```
(%i1) cflength: 3$
(%i2) cfdisrep (cf (sqrt (3)))$
(%i3) ev (%, numer);
(%o3)                  1.7317073170731707
```

- Maxima does not know about operations on continued fractions outside of `cf`.

```
(%i1) cf ([1,1,1,1,1,2] * 3);
(%o1)                      [4, 1, 5, 2]
(%i2) cf ([1,1,1,1,1,2]) * 3;
(%o2)                   [3, 3, 3, 3, 3, 6]
```

`cfdisrep` (*list*) [Function]

Constructs and returns an ordinary arithmetic expression of the form `a + 1/(b + 1/(c + ...))` from the list representation of a continued fraction `[a, b, c, ...]`.

```
(%i1) cf ([1, 2, -3] + [1, -2, 1]);
(%o1)                     [1, 1, 1, 2]
(%i2) cfdisrep (%);
                                  1
(%o2)                     1 + ---------
                                    1
                              1 + -----
                                    1
                                1 + -
                                    2
```

`cfexpand` (*x*) [Function]

Returns a matrix of the numerators and denominators of the last (column 1) and next-to-last (column 2) convergents of the continued fraction *x*.

```
(%i1) cf (rat (ev (%pi, numer)));
rat: replaced 3.141592653589793 by 80143857/25510582 = 3.1415926535897927
(%o1)      [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]
```

```
(%i2) cfexpand (%);
                          [ 80143857  5419351 ]
(%o2)                     [                   ]
                          [ 25510582  1725033 ]
(%i3) %[1,1]/%[2,1], numer;
(%o3)                   3.1415926535897927
```

`cflength` [Option variable]

Default value: 1

`cflength` controls the number of terms of the continued fraction the function `cf` will give, as the value `cflength` times the period. Thus the default is to give one period.

```
(%i1) cflength: 1$
(%i2) cf ((1 + sqrt(5))/2);
(%o2)                      [1, 1, 1, 1, 2]
(%i3) cflength: 2$
(%i4) cf ((1 + sqrt(5))/2);
(%o4)                [1, 1, 1, 1, 1, 1, 1, 2]
(%i5) cflength: 3$
(%i6) cf ((1 + sqrt(5))/2);
(%o6)           [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]
```

`divsum` [Function]

`divsum` (*n*, *k*)
`divsum` (*n*)

`divsum (n, k)` returns the sum of the divisors of *n* raised to the *k*'th power.

`divsum (n)` returns the sum of the divisors of *n*.

```
(%i1) divsum (12);
(%o1)                           28
(%i2) 1 + 2 + 3 + 4 + 6 + 12;
(%o2)                           28
(%i3) divsum (12, 2);
(%o3)                           210
(%i4) 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2;
(%o4)                           210
```

`euler (n)` [Function]

Returns the *n*'th Euler number for nonnegative integer *n*. Euler numbers equal to zero are suppressed if `zerobern` is `false`.

For the Euler-Mascheroni constant, see `%gamma`.

```
(%i1) zerobern: true$
(%i2) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o2)              [1, 0, - 1, 0, 5, 0, - 61]
(%i3) zerobern: false$
(%i4) map (euler, [0, 1, 2, 3, 4, 5, 6]);
(%o4)       [1, - 1, 5, - 61, 1385, - 50521, 2702765]
```

`factors_only` [Option variable]

Default value: `false`

Controls the value returned by `ifactors`. The default `false` causes `ifactors` to provide information about multiplicities of the computed prime factors. If `factors_only` is set to `true`, `ifactors` returns nothing more than a list of prime factors.

Example: See `ifactors`.

`fib` (*n*) [Function]

Returns the *n*'th Fibonacci number. `fib(0)` is equal to 0 and `fib(1)` equal to 1, and `fib (-n)` equal to `(-1)^(n + 1) * fib(n)`.

```
(%i1) map (fib, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o1)     [- 3, 2, - 1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21]
```

`fibtophi` (*expr*) [Function]

Expresses Fibonacci numbers in *expr* in terms of the constant `%phi`, which is `(1 + sqrt(5))/2`, approximately 1.61803399.

Examples:

```
(%i1) fibtophi (fib (n));
                            n             n
                         %phi  - (1 - %phi)
(%o1)                    ------------------
                             2 %phi - 1
(%i2) fib (n-1) + fib (n) - fib (n+1);
(%o2)          - fib(n + 1) + fib(n) + fib(n - 1)
(%i3) fibtophi (%);
            n + 1             n + 1       n             n
        %phi      - (1 - %phi)        %phi  - (1 - %phi)
(%o3) - ------------------------- + ------------------
                2 %phi - 1               2 %phi - 1
                                         n - 1             n - 1
                                   %phi      - (1 - %phi)
                                 + --------------------------
                                            2 %phi - 1
(%i4) ratsimp (%);
(%o4)                            0
```

`ifactors` (*n*) [Function]

For a positive integer *n* returns the factorization of *n*. If `n=p1^e1..pk^nk` is the decomposition of *n* into prime factors, ifactors returns `[[p1, e1], ... , [pk, ek]]`.

Factorization methods used are trial divisions by primes up to 9973, Pollard's rho and p-1 method and elliptic curves.

If the variable `ifactor_verbose` is set to `true` ifactor produces detailed output about what it is doing including immediate feedback as soon as a factor has been found.

The value returned by `ifactors` is controlled by the option variable `factors_only`. The default `false` causes `ifactors` to provide information about the multiplicities

of the computed prime factors. If `factors_only` is set to `true`, `ifactors` simply returns the list of prime factors.

```
(%i1) ifactors(51575319651600);
(%o1)   [[2, 4], [3, 2], [5, 2], [1583, 1], [9050207, 1]]
(%i2) apply("*", map(lambda([u], u[1]^u[2]), %));
(%o2)                     51575319651600
(%i3) ifactors(51575319651600), factors_only : true;
(%o3)               [2, 3, 5, 1583, 9050207]
```

`igcdex` (*n*, *k*) [Function]

Returns a list `[a, b, u]` where *u* is the greatest common divisor of *n* and *k*, and *u* is equal to `a n + b k`. The arguments *n* and *k* must be integers.

`igcdex` implements the Euclidean algorithm. See also `gcdex`.

The command `load("gcdex")` loads the function.

Examples:

```
(%i1) load("gcdex")$
(%i2) igcdex(30,18);
(%o2)                        [- 1, 2, 6]
(%i3) igcdex(1526757668, 7835626735736);
(%o3)            [845922341123, - 164826435, 4]
(%i4) igcdex(fib(20), fib(21));
(%o4)                    [4181, - 2584, 1]
```

`inrt` (*x*, *n*) [Function]

Returns the integer *n*'th root of the absolute value of *x*.

```
(%i1) l: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
(%i2) map (lambda ([a], inrt (10^a, 3)), l);
(%o2) [2, 4, 10, 21, 46, 100, 215, 464, 1000, 2154, 4641, 10000]
```

`inv_mod` (*n*, *m*) [Function]

Computes the inverse of *n* modulo *m*. `inv_mod (n,m)` returns `false`, if *n* is a zero divisor modulo *m*.

```
(%i1) inv_mod(3, 41);
(%o1)                           14
(%i2) ratsimp(3^-1), modulus = 41;
(%o2)                           14
(%i3) inv_mod(3, 42);
(%o3)                         false
```

`isqrt` (*x*) [Function]

Returns the "integer square root" of the absolute value of *x*, which is an integer.

`jacobi` (*p*, *q*) [Function]

Returns the Jacobi symbol of *p* and *q*.

```
(%i1) l: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$
(%i2) map (lambda ([a], jacobi (a, 9)), l);
(%o2)         [1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0]
```

lcm (*expr_1*, ..., *expr_n*) [Function]

Returns the least common multiple of its arguments. The arguments may be general expressions as well as integers.

`load ("functs")` loads this function.

lucas (*n*) [Function]

Returns the *n*'th Lucas number. `lucas(0)` is equal to 2 and `lucas(1)` equal to 1, and in general, `lucas(n) = lucas(n-1) + lucas(n-2)`. Also `lucas(-n)` is equal to `(-1)^(-n) * lucas(n)`.

```
(%i1) map (lucas, [-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8]);
(%o1)    [7, - 4, 3, - 1, 2, 1, 3, 4, 7, 11, 18, 29, 47]
```

mod (*x*, *y*) [Function]

If *x* and *y* are real numbers and *y* is nonzero, return `x - y * floor(x / y)`. Further for all real *x*, we have `mod (x, 0) = x`. For a discussion of the definition `mod (x, 0) = x`, see Section 3.4, of "Concrete Mathematics," by Graham, Knuth, and Patashnik. The function `mod (x, 1)` is a sawtooth function with period 1 with `mod (1, 1) = 0` and `mod (0, 1) = 0`.

To find the principal argument (a number in the interval `(-%pi, %pi]`) of a complex number, use the function `x |-> %pi - mod (%pi - x, 2*%pi)`, where *x* is an argument.

When *x* and *y* are constant expressions (`10 * %pi`, for example), `mod` uses the same big float evaluation scheme that `floor` and `ceiling` uses. Again, it's possible, although unlikely, that `mod` could return an erroneous value in such cases.

For nonnumerical arguments *x* or *y*, `mod` knows several simplification rules:

```
(%i1) mod (x, 0);
(%o1)                           x
(%i2) mod (a*x, a*y);
(%o2)                      a mod(x, y)
(%i3) mod (0, x);
(%o3)                           0
```

next_prime (*n*) [Function]

Returns the smallest prime bigger than *n*.

```
(%i1) next_prime(27);
(%o1)                          29
```

partfrac (*expr*, *var*) [Function]

Expands the expression *expr* in partial fractions with respect to the main variable *var*. `partfrac` does a complete partial fraction decomposition. The algorithm employed is based on the fact that the denominators of the partial fraction expansion (the factors of the original denominator) are relatively prime. The numerators can be written as linear combinations of denominators, and the expansion falls out.

`partfrac` ignores the value `true` of the option variable `keepfloat`.

```
(%i1) 1/(1+x)^2 - 2/(1+x) + 2/(2+x);
                        2       2        1
(%o1)                 ----- - ----- + --------
                      x + 2   x + 1          2
                                     (x + 1)
```

```
(%i2) ratsimp (%);
                                    x
(%o2)                  - ------------------
                            3      2
                           x  + 4 x  + 5 x + 2
(%i3) partfrac (%, x);
                         2       2        1
(%o3)                  ----- - ----- + --------
                       x + 2   x + 1          2
                                       (x + 1)
```

power_mod (a, n, m) [Function]

Uses a modular algorithm to compute `a^n mod m` where *a* and *n* are integers and *m* is a positive integer. If *n* is negative, `inv_mod` is used to find the modular inverse.

```
(%i1) power_mod(3, 15, 5);
(%o1)                           2
(%i2) mod(3^15,5);
(%o2)                           2
(%i3) power_mod(2, -1, 5);
(%o3)                           3
(%i4) inv_mod(2,5);
(%o4)                           3
```

primep (n) [Function]

Primality test. If `primep (n)` returns `false`, *n* is a composite number and if it returns `true`, *n* is a prime number with very high probability.

For *n* less than 331704406467988738596 1981 a deterministic version of Miller-Rabin's test is used. If `primep (n)` returns `true`, then *n* is a prime number.

For *n* bigger than 331704406467988738596 1981 `primep` uses `primep_number_of_tests` Miller-Rabin's pseudo-primality tests and one Lucas pseudo-primality test. The probability that a non-prime *n* will pass one Miller-Rabin test is less than 1/4. Using the default value 25 for `primep_number_of_tests`, the probability of *n* being composite is much smaller that 10^-15.

primep_number_of_tests [Option variable]

Default value: 25

Number of Miller-Rabin's tests used in `primep`.

primes (start, end) [Function]

Returns the list of all primes from *start* to *end*.

```
(%i1) primes(3, 7);
(%o1)                        [3, 5, 7]
```

prev_prime (n) [Function]

Returns the greatest prime smaller than *n*.

```
(%i1) prev_prime(27);
(%o1)                           23
```

`qunit (n)` [Function]

Returns the principal unit of the real quadratic number field `sqrt (n)` where *n* is an integer, i.e., the element whose norm is unity. This amounts to solving Pell's equation `a^2 - n b^2 = 1`.

```
(%i1) qunit (17);
(%o1)                      sqrt(17) + 4
(%i2) expand (% * (sqrt(17) - 4));
(%o2)                            1
```

`totient (n)` [Function]

Returns the number of integers less than or equal to *n* which are relatively prime to *n*.

`zerobern` [Option variable]

Default value: `true`

When `zerobern` is `false`, `bern` excludes the Bernoulli numbers and `euler` excludes the Euler numbers which are equal to zero. See `bern` and `euler`.

`zeta (n)` [Function]

Returns the Riemann zeta function. If *n* is a negative integer, 0, or a positive even integer, the Riemann zeta function simplifies to an exact value. For a positive even integer the option variable `zeta%pi` has to be `true` in addition (See `zeta%pi`). For a floating point or bigfloat number the Riemann zeta function is evaluated numerically. Maxima returns a noun form `zeta (n)` for all other arguments, including rational noninteger, and complex arguments, or for even integers, if `zeta%pi` has the value `false`.

`zeta(1)` is undefined, but Maxima knows the limit `limit(zeta(x), x, 1)` from above and below.

The Riemann zeta function distributes over lists, matrices, and equations.

See also `bfzeta` and `zeta%pi`.

Examples:

```
(%i1) zeta([-2, -1, 0, 0.5, 2, 3,1+%i]);
                                                  2
            1     1                            %pi
(%o1) [0, - --, - -, - 1.4603545088095862, ----, zeta(3),
            12    2                             6
                                                        zeta(%i + 1)]
(%i2) limit(zeta(x),x,1,plus);
(%o2)                          inf
(%i3) limit(zeta(x),x,1,minus);
(%o3)                         minf
```

`zeta%pi` [Option variable]

Default value: `true`

When `zeta%pi` is `true`, `zeta` returns an expression proportional to `%pi^n` for even integer `n`. Otherwise, `zeta` returns a noun form `zeta (n)` for even integer `n`.

Examples:

```
(%i1) zeta%pi: true$
(%i2) zeta (4);
                                  4
                               %pi
(%o2)                          ----
                                90
(%i3) zeta%pi: false$
(%i4) zeta (4);
(%o4)                       zeta(4)
```

`zn_add_table` (*n*) [Function]

Shows an addition table of all elements in (Z/*n*Z).

See also `zn_mult_table`, `zn_power_table`.

`zn_characteristic_factors` (*n*) [Function]

Returns a list containing the characteristic factors of the totient of *n*.

Using the characteristic factors a multiplication group modulo *n* can be expressed as a group direct product of cyclic subgroups.

In case the group itself is cyclic the list only contains the totient and using `zn_primroot` a generator can be computed. If the totient splits into more than one characteristic factors `zn_factor_generators` finds generators of the corresponding subgroups.

Each of the `r` factors in the list divides the right following factors. For the last factor `f_r` therefore holds `a^f_r = 1 (mod n)` for all `a` coprime to *n*. This factor is also known as Carmichael function or Carmichael lambda.

If `n > 2`, then `totient(n)/2^r` is the number of quadratic residues, and each of these has `2^r` square roots.

See also `totient`, `zn_primroot`, `zn_factor_generators`.

Examples:

The multiplication group modulo `14` is cyclic and its `6` elements can be generated by a primitive root.

```
(%i1) [zn_characteristic_factors(14), phi: totient(14)];
(%o1)                        [[6], 6]
(%i2) [zn_factor_generators(14), g: zn_primroot(14)];
(%o2)                        [[3], 3]
(%i3) M14: makelist(power_mod(g,i,14), i,0,phi-1);
(%o3)                 [1, 3, 9, 13, 11, 5]
```

The multiplication group modulo `15` is not cyclic and its `8` elements can be generated by two factor generators.

```
(%i1) [[f1,f2]: zn_characteristic_factors(15), totient(15)];
(%o1)                       [[2, 4], 8]
(%i2) [[g1,g2]: zn_factor_generators(15), zn_primroot(15)];
(%o2)                    [[11, 7], false]
```

```
(%i3) UG1: makelist(power_mod(g1,i,15), i,0,f1-1);
(%o3)                         [1, 11]
(%i4) UG2: makelist(power_mod(g2,i,15), i,0,f2-1);
(%o4)                       [1, 7, 4, 13]
(%i5) M15: create_list(mod(i*j,15), i,UG1, j,UG2);
(%o5)              [1, 7, 4, 13, 11, 2, 14, 8]
```

For the last characteristic factor `4` it holds that `a^4 = 1 (mod 15)` for all `a` in `M15`.

`M15` has two characteristic factors and therefore `8/2^2` quadratic residues, and each of these has `2^2` square roots.

```
(%i1) zn_power_table(15);
                          [ 1   1  1   1 ]
                          [              ]
                          [ 2   4  8   1 ]
                          [              ]
                          [ 4   1  4   1 ]
                          [              ]
                          [ 7   4  13  1 ]
(%o1)                     [              ]
                          [ 8   4  2   1 ]
                          [              ]
                          [ 11  1  11  1 ]
                          [              ]
                          [ 13  4  7   1 ]
                          [              ]
                          [ 14  1  14  1 ]
(%i2) map(lambda([i], zn_nth_root(i,2,15)), [1,4]);
(%o2)             [[1, 4, 11, 14], [2, 7, 8, 13]]
```

`zn_carmichael_lambda` (*n*) [Function]

Returns `1` if *n* is `1` and otherwise the greatest characteristic factor of the totient of *n*.

For remarks and examples see `zn_characteristic_factors`.

`zn_determinant` (*matrix*, *p*) [Function]

Uses the technique of LU-decomposition to compute the determinant of *matrix* over (Z/*p*Z). *p* must be a prime.

However if the determinant is equal to zero the LU-decomposition might fail. In that case `zn_determinant` computes the determinant non-modular and reduces thereafter.

See also `zn_invert_by_lu`.

Examples:

```
(%i1) m : matrix([1,3],[2,4]);
                              [ 1  3 ]
(%o1)                         [      ]
                              [ 2  4 ]
(%i2) zn_determinant(m, 5);
(%o2)                            3
```

```
(%i3) m : matrix([2,4,1],[3,1,4],[4,3,2]);
                              [ 2  4  1 ]
                              [         ]
(%o3)                         [ 3  1  4 ]
                              [         ]
                              [ 4  3  2 ]
(%i4) zn_determinant(m, 5);
(%o4)                              0
```

`zn_factor_generators` (*n*) [Function]

Returns a list containing factor generators corresponding to the characteristic factors of the totient of *n*.

For remarks and examples see `zn_characteristic_factors`.

`zn_invert_by_lu` (*matrix*, *p*) [Function]

Uses the technique of LU-decomposition to compute the modular inverse of *matrix* over (Z/*p*Z). *p* must be a prime and *matrix* invertible. `zn_invert_by_lu` returns `false` if *matrix* is not invertible.

See also `zn_determinant`.

Example:

```
(%i1) m : matrix([1,3],[2,4]);
                              [ 1  3 ]
(%o1)                         [      ]
                              [ 2  4 ]
(%i2) zn_determinant(m, 5);
(%o2)                              3
(%i3) mi : zn_invert_by_lu(m, 5);
                              [ 3  4 ]
(%o3)                         [      ]
                              [ 1  2 ]
(%i4) matrixmap(lambda([a], mod(a, 5)), m . mi);
                              [ 1  0 ]
(%o4)                         [      ]
                              [ 0  1 ]
```

`zn_log` [Function]

`zn_log` (`a`, *g*, *n*)
`zn_log` (`a`, *g*, *n*, [[*p1*, *e1*], . . . , [*pk*, *ek*]])

Computes the discrete logarithm. Let (Z/*n*Z)* be a cyclic group, *g* a primitive root modulo *n* or a generator of a subgroup of (Z/*n*Z)* and let *a* be a member of this group. `zn_log (a, g, n)` then solves the congruence `g^x = a mod n`. Please note that if *a* is not a power of *g* modulo *n*, `zn_log` will not terminate.

The applied algorithm needs a prime factorization of `zn_order(g)` resp. `totient(n)` in case *g* is a primitive root modulo *n*. A precomputed list of factors of `zn_order(g)` might be used as the optional fourth argument. This list must be of the same form as the list returned by `ifactors(zn_order(g))` using the default option `factors_`

`only : false`. However, compared to the running time of the logarithm algorithm providing the list of factors has only a quite small effect.

The algorithm uses a Pohlig-Hellman-reduction and Pollard's Rho-method for discrete logarithms. The running time of `zn_log` primarily depends on the bitlength of the greatest prime factor of `zn_order(g)`.

See also `zn_primroot`, `zn_order`, `ifactors`, `totient`.

Examples:

`zn_log (a, g, n)` solves the congruence `g^x = a mod n`.

```
(%i1) n : 22$
(%i2) g : zn_primroot(n);
(%o2)                          7
(%i3) ord_7 : zn_order(7, n);
(%o3)                         10
(%i4) powers_7 : makelist(power_mod(g, x, n), x, 0, ord_7 - 1);
(%o4)          [1, 7, 5, 13, 3, 21, 15, 17, 9, 19]
(%i5) zn_log(9, g, n);
(%o5)                          8
(%i6) map(lambda([x], zn_log(x, g, n)), powers_7);
(%o6)            [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
(%i7) ord_5 : zn_order(5, n);
(%o7)                          5
(%i8) powers_5 : makelist(power_mod(5,x,n), x, 0, ord_5 - 1);
(%o8)                    [1, 5, 3, 15, 9]
(%i9) zn_log(9, 5, n);
(%o9)                          4
```

The optional fourth argument must be of the same form as the list returned by `ifactors(zn_order(g))`. The running time primarily depends on the bitlength of the totient's greatest prime factor.

```
(%i1) (p : 2^127-1, primep(p));
(%o1)                         true
(%i2) ifs : ifactors(p - 1)$
(%i3) g : zn_primroot(p, ifs);
(%o3)                          43
(%i4) a : power_mod(g, 4711, p)$
(%i5) zn_log(a, g, p, ifs);
(%o5)                         4711
(%i6) f_max : last(ifs);
(%o6)                    [77158673929, 1]
(%i7) ord_5 : zn_order(5,p,ifs)$
(%i8) (p - 1)/ord_5;
(%o8)                          73
(%i9) ifs_5 : ifactors(ord_5)$
(%i10) a : power_mod(5, 4711, p)$
(%i11) zn_log(a, 5, p, ifs_5);
(%o11)                        4711
```

`zn_mult_table` [Function]

`zn_mult_table` (*n*)

`zn_mult_table` (*n*, *gcd*)

Without the optional argument *gcd* `zn_mult_table(n)` shows a multiplication table of all elements in (Z/*n*Z)* which are all elements coprime to *n*.

The optional second argument *gcd* allows to select a specific subset of (Z/*n*Z). If *gcd* is an integer, a multiplication table of all residues `x` with `gcd(x,n) =` *gcd* are returned. Additionally row and column headings are added for better readability. If necessary, these can be easily removed by `submatrix(1, table, 1)`.

If *gcd* is set to `all`, the table is printed for all non-zero elements in (Z/*n*Z).

The second example shows an alternative way to create a multiplication table for subgroups.

See also `zn_add_table`, `zn_power_table`.

Examples:

The default table shows all elements in (Z/*n*Z)* and allows to demonstrate and study basic properties of modular multiplication groups. E.g. the principal diagonal contains all quadratic residues, each row and column contains every element, the tables are symmetric, etc..

If *gcd* is set to `all`, the table is printed for all non-zero elements in (Z/*n*Z).

```
(%i1) zn_mult_table(8);
                          [ 1  3  5  7 ]
                          [            ]
                          [ 3  1  7  5 ]
(%o1)                     [            ]
                          [ 5  7  1  3 ]
                          [            ]
                          [ 7  5  3  1 ]
(%i2) zn_mult_table(8, all);
                      [ 1  2  3  4  5  6  7 ]
                      [                     ]
                      [ 2  4  6  0  2  4  6 ]
                      [                     ]
                      [ 3  6  1  4  7  2  5 ]
                      [                     ]
(%o2)                 [ 4  0  4  0  4  0  4 ]
                      [                     ]
                      [ 5  2  7  4  1  6  3 ]
                      [                     ]
                      [ 6  4  2  0  6  4  2 ]
                      [                     ]
                      [ 7  6  5  4  3  2  1 ]
```

If *gcd* is an integer, row and column headings are added for better readability.

If the subset chosen by *gcd* is a group there is another way to create a multiplication table. An isomorphic mapping from a group with `1` as identity builds a table which is easy to read. The mapping is accomplished via CRT.

In the second version of `T36_4` the identity, here `28`, is placed in the top left corner, just like in table `T9`.

```
(%i1) T36_4: zn_mult_table(36,4);
                  [ *   4   8   16  20  28  32 ]
                  [                            ]
                  [ 4   16  32  28  8   4   20 ]
                  [                            ]
                  [ 8   32  28  20  16  8   4  ]
                  [                            ]
(%o1)             [ 16  28  20  4   32  16  8  ]
                  [                            ]
                  [ 20  8   16  32  4   20  28 ]
                  [                            ]
                  [ 28  4   8   16  20  28  32 ]
                  [                            ]
                  [ 32  20  4   8   28  32  16 ]
(%i2) T9: zn_mult_table(36/4);
                       [ 1  2  4  5  7  8 ]
                       [                  ]
                       [ 2  4  8  1  5  7 ]
                       [                  ]
                       [ 4  8  7  2  1  5 ]
(%o2)                  [                  ]
                       [ 5  1  2  7  8  4 ]
                       [                  ]
                       [ 7  5  1  8  4  2 ]
                       [                  ]
                       [ 8  7  5  4  2  1 ]
(%i3) T36_4: matrixmap(lambda([x], solve_congruences([0,x],[4,9])), T9);
                     [ 28  20  4   32  16  8  ]
                     [                        ]
                     [ 20  4   8   28  32  16 ]
                     [                        ]
                     [ 4   8   16  20  28  32 ]
(%o3)                [                        ]
                     [ 32  28  20  16  8   4  ]
                     [                        ]
                     [ 16  32  28  8   4   20 ]
                     [                        ]
                     [ 8   16  32  4   20  28 ]
```

`zn_nth_root` [Function]

`zn_nth_root` (*x*, *n*, *m*)
`zn_nth_root` (*x*, *n*, *m*, [[*p1*, *e1*], ..., [*pk*, *ek*]])

Returns a list with all *n*-th roots of *x* from the multiplication subgroup of $(\mathrm{Z}/m\mathrm{Z})$ which contains *x*, or `false`, if *x* is no *n*-th power modulo *m* or not contained in any multiplication subgroup of $(\mathrm{Z}/m\mathrm{Z})$.

x is an element of a multiplication subgroup modulo m, if the greatest common divisor `g = gcd(x,m)` is coprime to `m/g`.

`zn_nth_root` is based on an algorithm by Adleman, Manders and Miller and on theorems about modulo multiplication groups by Daniel Shanks.

The algorithm needs a prime factorization of the modulus m. So in case the factorization of m is known, the list of factors can be passed as the fourth argument. This optional argument must be of the same form as the list returned by `ifactors(m)` using the default option `factors_only: false`.

Examples:

A power table of the multiplication group modulo `14` followed by a list of lists containing all n-th roots of `1` with n from `1` to `6`.

```
(%i1) zn_power_table(14);
                     [ 1   1   1   1   1   1 ]
                     [                       ]
                     [ 3   9   13  11  5   1 ]
                     [                       ]
                     [ 5   11  13  9   3   1 ]
(%o1)                [                       ]
                     [ 9   11  1   9   11  1 ]
                     [                       ]
                     [ 11  9   1   11  9   1 ]
                     [                       ]
                     [ 13  1   13  1   13  1 ]
(%i2) makelist(zn_nth_root(1,n,14), n,1,6);
(%o2) [[1], [1, 13], [1, 9, 11], [1, 13], [1],
                                          [1, 3, 5, 9, 11, 13]]
```

In the following example x is not coprime to m, but is a member of a multiplication subgroup of $(\mathrm{Z}/m\mathrm{Z})$ and any n-th root is a member of the same subgroup.

The residue class `3` is no member of any multiplication subgroup of (Z/63Z) and is therefore not returned as a third root of `27`.

Here `zn_power_table` shows all residues `x` in (Z/63Z) with `gcd(x,63) = 9`. This subgroup is isomorphic to $(\mathrm{Z}/7\mathrm{Z})^*$ and its identity `36` is computed via CRT.

```
(%i1) m: 7*9$
(%i2) zn_power_table(m,9);
                    [ 9   18  36  9   18  36 ]
                    [                        ]
                    [ 18  9   36  18  9   36 ]
                    [                        ]
                    [ 27  36  27  36  27  36 ]
(%o2)               [                        ]
                    [ 36  36  36  36  36  36 ]
                    [                        ]
                    [ 45  9   27  18  54  36 ]
                    [                        ]
                    [ 54  18  27  9   45  36 ]
```

```
(%i3) zn_nth_root(27,3,m);
(%o3)                      [27, 45, 54]
(%i4) id7:1$  id63_9: solve_congruences([id7,0],[7,9]);
(%o5)                           36
```

In the following RSA-like example, where the modulus `N` is squarefree, i.e. it splits into exclusively first power factors, every `x` from `0` to `N-1` is contained in a multiplication subgroup.

The process of decryption needs the `e`-th root. `e` is coprime to `totient(N)` and therefore the `e`-th root is unique. In this case `zn_nth_root` effectively performs CRT-RSA. (Please note that `flatten` removes braces but no solutions.)

```
(%i1) [p,q,e]: [5,7,17]$  N: p*q$
(%i3) xs: makelist(x,x,0,N-1)$
(%i4) ys: map(lambda([x],power_mod(x,e,N)),xs)$
(%i5) zs: flatten(map(lambda([y], zn_nth_root(y,e,N)), ys))$
(%i6) is(zs = xs);
(%o6)                           true
```

In the following example the factorization of the modulus is known and passed as the fourth argument.

```
(%i1) p: 2^107-1$  q: 2^127-1$  N: p*q$
(%i4) ibase: obase: 16$
(%i5) msg: 11223344556677889900aabbccddeeff$
(%i6) enc: power_mod(msg, 10001, N);
(%o6) 1A8DB7892AE588BDC2BE25DD5107A425001FE9C82161ABC673241C8B383
(%i7) zn_nth_root(enc, 10001, N, [[p,1],[q,1]]);
(%o7)          [11223344556677889900AABBCCDDEEFF]
```

`zn_order` [Function]

`zn_order` (*x*, *n*)
`zn_order` (*x*, *n*, [[*p1*, *e1*], . . . , [*pk*, *ek*]])

Returns the order of *x* if it is a unit of the finite group $(\mathrm{Z}/n\mathrm{Z})^*$ or returns `false`. *x* is a unit modulo *n* if it is coprime to *n*.

The applied algorithm needs a prime factorization of `totient(n)`. This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to `zn_log` as the third argument. The list must be of the same form as the list returned by `ifactors(totient(n))` using the default option `factors_only : false`.

See also `zn_primroot`, `ifactors`, `totient`.

Examples:

`zn_order` computes the order of the unit *x* in $(\mathrm{Z}/n\mathrm{Z})^*$.

```
(%i1) n: 22$
(%i2) g: zn_primroot(n);
(%o2)                            7
(%i3) units_22: sublist(makelist(i,i,1,21), lambda([x], gcd(x,n)=1));
(%o3)          [1, 3, 5, 7, 9, 13, 15, 17, 19, 21]
```

```
(%i4) (ord_7: zn_order(7, n)) = totient(n);
(%o4)                          10 = 10
(%i5) powers_7: makelist(power_mod(g,i,n), i,0,ord_7 - 1);
(%o5)          [1, 7, 5, 13, 3, 21, 15, 17, 9, 19]
(%i6) map(lambda([x], zn_order(x, n)), powers_7);
(%o6)          [1, 10, 5, 10, 5, 2, 5, 10, 5, 10]
(%i7) map(lambda([x], ord_7/gcd(x,ord_7)), makelist(i,i,0,ord_7-1));
(%o7)          [1, 10, 5, 10, 5, 2, 5, 10, 5, 10]
(%i8) totient(totient(n));
(%o8)                            4
```

The optional third argument must be of the same form as the list returned by `ifactors(totient(n))`.

```
(%i1) (p : 2^142 + 217, primep(p));
(%o1)                          true
(%i2) ifs: ifactors( totient(p) )$
(%i3) g: zn_primroot(p, ifs);
(%o3)                           3
(%i4) is( (ord_3 : zn_order(g, p, ifs)) = totient(p) );
(%o4)                          true
(%i5) map(lambda([x], ord_3/zn_order(x,p,ifs)), makelist(i,i,2,15));
(%o5)    [22, 1, 44, 10, 5, 2, 22, 2, 8, 2, 1, 1, 20, 1]
```

`zn_power_table` [Function]

`zn_power_table` (*n*)
`zn_power_table` (*n*, *gcd*)
`zn_power_table` (*n*, *gcd*, *max_exp*)

Without any optional argument `zn_power_table(n)` shows a power table of all elements in (Z/*n*Z)* which are all residue classes coprime to *n*. The exponent loops from `1` to the greatest characteristic factor of `totient(n)` (also known as Carmichael function or Carmichael lambda) and the table ends with a column of ones on the right side.

The optional second argument *gcd* allows to select powers of a specific subset of (Z/*n*Z). If *gcd* is an integer, powers of all residue classes `x` with `gcd(x,n) =` *gcd* are returned, i.e. the default value for *gcd* is `1`. If *gcd* is set to `all`, the table contains powers of all elements in (Z/*n*Z).

If the optional third argument *max_exp* is given, the exponent loops from `1` to *max_exp*.

See also `zn_add_table`, `zn_mult_table`.

Examples:

The default which is *gcd* `= 1` allows to demonstrate and study basic theorems of e.g. Fermat and Euler.

The argument *gcd* allows to select subsets of (Z/*n*Z) and to study multiplication subgroups and isomorphisms. E.g. the groups `G10` and `G10_2` are under multiplication both isomorphic to `G5`. `1` is the identity in `G5`. So are `1` resp. `6` the identities in `G10` resp. `G10_2`. There are corresponding mappings for primitive roots, n-th roots, etc..

```
(%i1) zn_power_table(10);
                          [ 1  1  1  1 ]
                          [             ]
                          [ 3  9  7  1 ]
(%o1)                     [             ]
                          [ 7  9  3  1 ]
                          [             ]
                          [ 9  1  9  1 ]
(%i2) zn_power_table(10,2);
                          [ 2  4  8  6 ]
                          [             ]
                          [ 4  6  4  6 ]
(%o2)                     [             ]
                          [ 6  6  6  6 ]
                          [             ]
                          [ 8  4  2  6 ]
(%i3) zn_power_table(10,5);
(%o3)                     [ 5  5  5  5 ]
(%i4) zn_power_table(10,10);
(%o4)                     [ 0  0  0  0 ]
(%i5) G5: [1,2,3,4];
(%o5)                      [1, 2, 3, 4]
(%i6) G10_2: map(lambda([x], solve_congruences([0,x],[2,5])), G5);
(%o6)                      [6, 2, 8, 4]
(%i7) G10: map(lambda([x], power_mod(3, zn_log(x,2,5), 10)), G5);
(%o7)                      [1, 3, 7, 9]
```

If *gcd* is set to `all`, the table contains powers of all elements in $(\mathrm{Z}/n\mathrm{Z})$.

The third argument *max_exp* allows to set the highest exponent. The following table shows a very small example of RSA.

```
(%i1) linel:100$ N:2*5$ phi:totient(N)$ e:7$ d:inv_mod(e,phi)$
```

```
(%i6) zn_power_table(N, all, e*d);
                  [ 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
                  [
                  [ 1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
                  [
                  [ 2  4  8  6  2  4  8  6  2  4  8  6  2  4  8  6  2  4  8  6  2  4
                  [
                  [ 3  9  7  1  3  9  7  1  3  9  7  1  3  9  7  1  3  9  7  1  3  9
                  [
                  [ 4  6  4  6  4  6  4  6  4  6  4  6  4  6  4  6  4  6  4  6  4  6
(%o6)             [
                  [ 5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5  5
                  [
                  [ 6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6  6
                  [
                  [ 7  9  3  1  7  9  3  1  7  9  3  1  7  9  3  1  7  9  3  1  7  9
                  [
                  [ 8  4  2  6  8  4  2  6  8  4  2  6  8  4  2  6  8  4  2  6  8  4
                  [
                  [ 9  1  9  1  9  1  9  1  9  1  9  1  9  1  9  1  9  1  9  1  9  1
```

zn_primroot [Function]

zn_primroot (*n*)

zn_primroot (*n*, [[*p1*, *e1*], ..., [*pk*, *ek*]])

If the multiplicative group (Z/*n*Z)* is cyclic, `zn_primroot` computes the smallest primitive root modulo *n*. (Z/*n*Z)* is cyclic if *n* is equal to `2`, `4`, `p^k` or `2*p^k`, where `p` is prime and greater than `2` and `k` is a natural number. `zn_primroot` performs an according pretest if the option variable `zn_primroot_pretest` (default: `false`) is set to `true`. In any case the computation is limited by the upper bound `zn_primroot_limit`.

If (Z/*n*Z)* is not cyclic or if there is no primitive root up to `zn_primroot_limit`, `zn_primroot` returns `false`.

The applied algorithm needs a prime factorization of `totient(n)`. This factorization might be time consuming in some cases and it can be useful to factor first and then to pass the list of factors to `zn_log` as an additional argument. The list must be of the same form as the list returned by `ifactors(totient(n))` using the default option `factors_only : false`.

See also `zn_primroot_p`, `zn_order`, `ifactors`, `totient`.

Examples:

`zn_primroot` computes the smallest primitive root modulo *n* or returns `false`.

```
(%i1) n : 14$
(%i2) g : zn_primroot(n);
(%o2)                          3
(%i3) zn_order(g, n) = totient(n);
(%o3)                        6 = 6
(%i4) n : 15$
```

```
(%i5) zn_primroot(n);
(%o5)                          false
```

The optional second argument must be of the same form as the list returned by `ifactors(totient(n))`.

```
(%i1) (p : 2^142 + 217, primep(p));
(%o1)                          true
(%i2) ifs : ifactors( totient(p) )$
(%i3) g : zn_primroot(p, ifs);
(%o3)                           3
(%i4) [time(%o2), time(%o3)];
(%o4)                     [[3.2], [0.0]]
(%i5) is(zn_order(g, p, ifs) = p - 1);
(%o5)                          true
(%i6) n : 2^142 + 216$
(%i7) ifs : ifactors(totient(n))$
(%i8) zn_primroot(n, ifs), zn_primroot_limit : 200, zn_primroot_verbose : true;
`zn_primroot' stopped at zn_primroot_limit = 200
(%o8)                          false
```

`zn_primroot_limit` [Option variable]

Default value: `1000`

If `zn_primroot` cannot find a primitive root, it stops at this upper bound. If the option variable `zn_primroot_verbose` (default: `false`) is set to `true`, a message will be printed when `zn_primroot_limit` is reached.

`zn_primroot_p` [Function]

`zn_primroot_p` (*x*, *n*)
`zn_primroot_p` (*x*, *n*, [[*p1*, *e1*], ..., [*pk*, *ek*]])

Checks whether x is a primitive root in the multiplicative group $(\mathrm{Z}/n\mathrm{Z})^*$.

The applied algorithm needs a prime factorization of `totient(n)`. This factorization might be time consuming and in case `zn_primroot_p` will be consecutively applied to a list of candidates it can be useful to factor first and then to pass the list of factors to `zn_log` as a third argument. The list must be of the same form as the list returned by `ifactors(totient(n))` using the default option `factors_only : false`.

See also `zn_primroot`, `zn_order`, `ifactors`, `totient`.

Examples:

`zn_primroot_p` as a predicate function.

```
(%i1) n : 14$
(%i2) units_14 : sublist(makelist(i,i,1,13), lambda([i], gcd(i, n) = 1));
(%o2)                  [1, 3, 5, 9, 11, 13]
(%i3) zn_primroot_p(13, n);
(%o3)                          false
(%i4) sublist(units_14, lambda([x], zn_primroot_p(x, n)));
(%o4)                         [3, 5]
(%i5) map(lambda([x], zn_order(x, n)), units_14);
(%o5)                   [1, 6, 6, 3, 3, 2]
```

The optional third argument must be of the same form as the list returned by `ifactors(totient(n))`.

```
(%i1) (p: 2^142 + 217, primep(p));
(%o1)                          true
(%i2) ifs: ifactors( totient(p) )$
(%i3) sublist(makelist(i,i,1,50), lambda([x], zn_primroot_p(x,p,ifs)));
(%o3)  [3, 12, 13, 15, 21, 24, 26, 27, 29, 33, 38, 42, 48]
(%i4) [time(%o2), time(%o3)];
(%o4)                    [[3.01], [0.03]]
```

`zn_primroot_pretest` [Option variable]

Default value: `false`

The multiplicative group $(\mathrm{Z}/n\mathrm{Z})^*$ is cyclic if n is equal to `2`, `4`, `p^k` or `2*p^k`, where `p` is prime and greater than `2` and `k` is a natural number.

`zn_primroot_pretest` controls whether `zn_primroot` will check if one of these cases occur before it computes the smallest primitive root. Only if `zn_primroot_pretest` is set to `true` this pretest will be performed.

`zn_primroot_verbose` [Option variable]

Default value: `false`

Controls whether `zn_primroot` prints a message when reaching `zn_primroot_limit`.

28 Groups

28.1 Functions and Variables for Groups

todd_coxeter [Function]

todd_coxeter (*relations*, *subgroup*)
todd_coxeter (*relations*)

Find the order of G/H where G is the Free Group modulo *relations*, and H is the subgroup of G generated by *subgroup*. *subgroup* is an optional argument, defaulting to []. In doing this it produces a multiplication table for the right action of G on G/H, where the cosets are enumerated [H,Hg2,Hg3,...]. This can be seen internally in the variable todd_coxeter_state.

Example:

```
(%i1) symet(n):=create_list(
        if (j - i) = 1 then (p(i,j))^^3 else
            if (not i = j) then (p(i,j))^^2 else
                p(i,i) , j, 1, n-1, i, 1, j);
                                                              <3>
(%o1) symet(n) := create_list(if j - i = 1 then p(i, j)
                                    <2>
 else (if not i = j then p(i, j)    else p(i, i)), j, 1, n - 1,
i, 1, j)
(%i2) p(i,j) := concat(x,i).concat(x,j);
(%o2)        p(i, j) := concat(x, i) . concat(x, j)
(%i3) symet(5);
         <2>           <3>    <2>           <2>           <3>
(%o3) [x1   , (x1 . x2)   , x2   , (x1 . x3)   , (x2 . x3)   ,
            <2>           <2>           <2>           <3>    <2>
          x3   , (x1 . x4)   , (x2 . x4)   , (x3 . x4)   , x4   ]
(%i4) todd_coxeter(%o3);
Rows tried 426
(%o4)                          120
(%i5) todd_coxeter(%o3,[x1]);
Rows tried 213
(%o5)                          60
(%i6) todd_coxeter(%o3,[x1,x2]);
Rows tried 71
(%o6)                          20
```

29 Miscellaneous Options

29.1 Introduction to Miscellaneous Options

In this section various options are discussed which have a global effect on the operation of Maxima. Also various lists such as the list of all user defined functions, are discussed.

29.2 Share

The Maxima "share" directory contains programs and other files of interest to Maxima users, but not part of the core implementation of Maxima. These programs are typically loaded via `load` or `setup_autoload`.

`:lisp *maxima-sharedir*` displays the location of the share directory within the user's file system.

29.3 Functions and Variables for Miscellaneous Options

`askexp` [System variable]

When `asksign` is called, `askexp` is the expression `asksign` is testing.

At one time, it was possible for a user to inspect `askexp` by entering a Maxima break with control-A.

`genindex` [Option variable]

Default value: `i`

`genindex` is the alphabetic prefix used to generate the next variable of summation when necessary.

`gensumnum` [Option variable]

Default value: 0

`gensumnum` is the numeric suffix used to generate the next variable of summation. If it is set to `false` then the index will consist only of `genindex` with no numeric suffix.

`gensym` [Function]

`gensym ()`
`gensym (`*x*`)`

`gensym()` creates and returns a fresh symbol.

The name of the new symbol is the concatenation of a prefix, which defaults to "g", and a suffix, which is an integer that defaults to the value of an internal counter.

If *x* is supplied, and is a string, then that string is used as a prefix instead of "g" for this call to gensym only.

If *x* is supplied, and is a nonnegative integer, then that integer, instead of the value of the internal counter, is used as the suffix for this call to gensym only.

If and only if no explicit suffix is supplied, the internal counter is incremented after it is used.

Examples:

```
(%i1) gensym();
```

```
(%o1)                          g887
(%i2) gensym("new");
(%o2)                         new888
(%i3) gensym(123);
(%o3)                          g123
```

`packagefile` [Option variable]

Default value: `false`

Package designers who use `save` or `translate` to create packages (files) for others to use may want to set `packagefile: true` to prevent information from being added to Maxima's information-lists (e.g. `values`, `functions`) except where necessary when the file is loaded in. In this way, the contents of the package will not get in the user's way when he adds his own data. Note that this will not solve the problem of possible name conflicts. Also note that the flag simply affects what is output to the package file. Setting the flag to `true` is also useful for creating Maxima init files.

`remvalue` [Function]

`remvalue` (*name_1*, ..., *name_n*)
`remvalue` (*all*)

Removes the values of user variables *name_1*, ..., *name_n* (which can be subscripted) from the system.

`remvalue (all)` removes the values of all variables in `values`, the list of all variables given names by the user (as opposed to those which are automatically assigned by Maxima).

See also `values`.

`rncombine` (*expr*) [Function]

Transforms *expr* by combining all terms of *expr* that have identical denominators or denominators that differ from each other by numerical factors only. This is slightly different from the behavior of `combine`, which collects terms that have identical denominators.

Setting `pfeformat: true` and using `combine` yields results similar to those that can be obtained with `rncombine`, but `rncombine` takes the additional step of cross-multiplying numerical denominator factors. This results in neater forms, and the possibility of recognizing some cancellations.

`load("rncomb")` loads this function.

`setup_autoload` (*filename*, *function_1*, ..., *function_n*) [Function]

Specifies that if any of *function_1*, ..., *function_n* are referenced and not yet defined, *filename* is loaded via `load`. *filename* usually contains definitions for the functions specified, although that is not enforced.

`setup_autoload` does not work for `memoizing functions`.

`setup_autoload` quotes its arguments.

Example:

```
(%i1) legendre_p (1, %pi);
(%o1)                  legendre_p(1, %pi)
```

```
(%i2) setup_autoload ("specfun.mac", legendre_p, ultraspherical);
(%o2)                          done
(%i3) ultraspherical (2, 1/2, %pi);
Warning - you are redefining the Macsyma function ultraspherical
Warning - you are redefining the Macsyma function legendre_p
                                2
                  3 (%pi - 1)
(%o3)             ------------ + 3 (%pi - 1) + 1
                       2
(%i4) legendre_p (1, %pi);
(%o4)                           %pi
(%i5) legendre_q (1, %pi);
                                %pi + 1
                          %pi log(-------)
                                  1 - %pi
(%o5)                     ---------------- - 1
                                 2
```

`tcl_output` [Function]

`tcl_output (`*list*`, `*i0*`, `*skip*`)`
`tcl_output (`*list*`, `*i0*`)`
`tcl_output ([`*list_1*`, ..., `*list_n*`], `*i*`)`

Prints elements of a list enclosed by curly braces `{ }`, suitable as part of a program in the Tcl/Tk language.

`tcl_output (`*list*`, `*i0*`, `*skip*`)` prints *list*, beginning with element *i0* and printing elements *i0* + *skip*, *i0* + 2 *skip*, etc.

`tcl_output (`*list*`, `*i0*`)` is equivalent to `tcl_output (`*list*`, `*i0*`, 2)`.

`tcl_output ([`*list_1*`, ..., `*list_n*`], `*i*`)` prints the *i*'th elements of *list_1*, ..., *list_n*.

Examples:

```
(%i1) tcl_output ([1, 2, 3, 4, 5, 6], 1, 3)$

 {1.000000000     4.000000000
 }
(%i2) tcl_output ([1, 2, 3, 4, 5, 6], 2, 3)$

 {2.000000000     5.000000000
 }
(%i3) tcl_output ([3/7, 5/9, 11/13, 13/17], 1)$

 {((RAT SIMP) 3 7) ((RAT SIMP) 11 13)
 }
(%i4) tcl_output ([x1, y1, x2, y2, x3, y3], 2)$

 {$Y1 $Y2 $Y3
 }
```

```
(%i5) tcl_output ([[1, 2, 3], [11, 22, 33]], 1)$

 {SIMP 1.000000000     11.00000000
 }
```

30 Rules and Patterns

30.1 Introduction to Rules and Patterns

This section describes user-defined pattern matching and simplification rules. There are two groups of functions which implement somewhat different pattern matching schemes. In one group are `tellsimp`, `tellsimpafter`, `defmatch`, `defrule`, `apply1`, `applyb1`, and `apply2`. In the other group are `let` and `letsimp`. Both schemes define patterns in terms of pattern variables declared by `matchdeclare`.

Pattern-matching rules defined by `tellsimp` and `tellsimpafter` are applied automatically by the Maxima simplifier. Rules defined by `defmatch`, `defrule`, and `let` are applied by an explicit function call.

There are additional mechanisms for rules applied to polynomials by `tellrat`, and for commutative and noncommutative algebra in `affine` package.

30.2 Functions and Variables for Rules and Patterns

`apply1` (*expr*, *rule_1*, ..., *rule_n*) [Function]

Repeatedly applies *rule_1* to *expr* until it fails, then repeatedly applies the same rule to all subexpressions of *expr*, left to right, until *rule_1* has failed on all subexpressions. Call the result of transforming *expr* in this manner *expr_2*. Then *rule_2* is applied in the same fashion starting at the top of *expr_2*. When *rule_n* fails on the final subexpression, the result is returned.

`maxapplydepth` is the depth of the deepest subexpressions processed by `apply1` and `apply2`.

See also `applyb1`, `apply2` and `let`.

`apply2` (*expr*, *rule_1*, ..., *rule_n*) [Function]

If *rule_1* fails on a given subexpression, then *rule_2* is repeatedly applied, etc. Only if all rules fail on a given subexpression is the whole set of rules repeatedly applied to the next subexpression. If one of the rules succeeds, then the same subexpression is reprocessed, starting with the first rule.

`maxapplydepth` is the depth of the deepest subexpressions processed by `apply1` and `apply2`.

See also `apply1` and `let`.

`applyb1` (*expr*, *rule_1*, ..., *rule_n*) [Function]

Repeatedly applies *rule_1* to the deepest subexpression of *expr* until it fails, then repeatedly applies the same rule one level higher (i.e., larger subexpressions), until *rule_1* has failed on the top-level expression. Then *rule_2* is applied in the same fashion to the result of *rule_1*. After *rule_n* has been applied to the top-level expression, the result is returned.

`applyb1` is similar to `apply1` but works from the bottom up instead of from the top down.

`maxapplyheight` is the maximum height which `applyb1` reaches before giving up.

See also `apply1`, `apply2` and `let`.

current_let_rule_package [Option variable]

Default value: default_let_rule_package

current_let_rule_package is the name of the rule package that is used by functions in the let package (letsimp, etc.) if no other rule package is specified. This variable may be assigned the name of any rule package defined via the let command.

If a call such as letsimp (expr, rule_pkg_name) is made, the rule package rule_pkg_name is used for that function call only, and the value of current_let_rule_package is not changed.

default_let_rule_package [Option variable]

Default value: default_let_rule_package

default_let_rule_package is the name of the rule package used when one is not explicitly set by the user with let or by changing the value of current_let_rule_package.

defmatch [Function]

defmatch (*progname*, *pattern*, *x_1*, ..., *x_n*)
defmatch (*progname*, *pattern*)

Defines a function *progname*(*expr*, *x_1*, ..., *x_n*) which tests *expr* to see if it matches *pattern*.

pattern is an expression containing the pattern arguments *x_1*, ..., *x_n* (if any) and some pattern variables (if any). The pattern arguments are given explicitly as arguments to defmatch while the pattern variables are declared by the matchdeclare function. Any variable not declared as a pattern variable in matchdeclare or as a pattern argument in defmatch matches only itself.

The first argument to the created function *progname* is an expression to be matched against the pattern and the other arguments are the actual arguments which correspond to the dummy variables *x_1*, ..., *x_n* in the pattern.

If the match is successful, *progname* returns a list of equations whose left sides are the pattern arguments and pattern variables, and whose right sides are the subexpressions which the pattern arguments and variables matched. The pattern variables, but not the pattern arguments, are assigned the subexpressions they match. If the match fails, *progname* returns false.

A literal pattern (that is, a pattern which contains neither pattern arguments nor pattern variables) returns true if the match succeeds.

See also matchdeclare, defrule, tellsimp and tellsimpafter.

Examples:

Define a function linearp(expr, x) which tests expr to see if it is of the form a*x + b such that a and b do not contain x and a is nonzero. This match function matches expressions which are linear in any variable, because the pattern argument x is given to defmatch.

```
(%i1) matchdeclare (a, lambda ([e], e#0 and freeof(x, e)), b,
                    freeof(x));
(%o1)                          done
(%i2) defmatch (linearp, a*x + b, x);
(%o2)                         linearp
```

```
(%i3) linearp (3*z + (y + 1)*z + y^2, z);
                          2
(%o3)              [b = y , a = y + 4, x = z]
(%i4) a;
(%o4)                          y + 4
(%i5) b;
                                 2
(%o5)                           y
(%i6) x;
(%o6)                            x
```

Define a function `linearp(expr)` which tests `expr` to see if it is of the form `a*x + b` such that `a` and `b` do not contain `x` and `a` is nonzero. This match function only matches expressions linear in `x`, not any other variable, because no pattern argument is given to `defmatch`.

```
(%i1) matchdeclare (a, lambda ([e], e#0 and freeof(x, e)), b,
                    freeof(x));
(%o1)                          done
(%i2) defmatch (linearp, a*x + b);
(%o2)                         linearp
(%i3) linearp (3*z + (y + 1)*z + y^2);
(%o3)                          false
(%i4) linearp (3*x + (y + 1)*x + y^2);
                               2
(%o4)                  [b = y , a = y + 4]
```

Define a function `checklimits(expr)` which tests `expr` to see if it is a definite integral.

```
(%i1) matchdeclare ([a, f], true);
(%o1)                          done
(%i2) constinterval (l, h) := constantp (h - l);
(%o2)        constinterval(l, h) := constantp(h - l)
(%i3) matchdeclare (b, constinterval (a));
(%o3)                          done
(%i4) matchdeclare (x, atom);
(%o4)                          done
(%i5) simp : false;
(%o5)                          false
(%i6) defmatch (checklimits, 'integrate (f, x, a, b));
(%o6)                       checklimits
(%i7) simp : true;
(%o7)                          true
```

```
(%i8) 'integrate (sin(t), t, %pi + x, 2*%pi + x);
                       x + 2 %pi
                      /
                      |
(%o8)                 |          sin(t) dt
                      |
                      /
                       x + %pi
(%i9) checklimits (%);
(%o9)    [b = x + 2 %pi, a = x + %pi, x = t, f = sin(t)]
```

defrule (*rulename*, *pattern*, *replacement*) [Function]

Defines and names a replacement rule for the given pattern. If the rule named *rulename* is applied to an expression (by `apply1`, `applyb1`, or `apply2`), every subexpression matching the pattern will be replaced by the replacement. All variables in the replacement which have been assigned values by the pattern match are assigned those values in the replacement which is then simplified.

The rules themselves can be treated as functions which transform an expression by one operation of the pattern match and replacement. If the match fails, the rule function returns `false`.

disprule [Function]

`disprule` (*rulename_1*, ..., *rulename_2*)
`disprule` (*all*)

Display rules with the names *rulename_1*, ..., *rulename_n*, as returned by `defrule`, `tellsimp`, or `tellsimpafter`, or a pattern defined by `defmatch`. Each rule is displayed with an intermediate expression label (`%t`).

`disprule (all)` displays all rules.

`disprule` quotes its arguments. `disprule` returns the list of intermediate expression labels corresponding to the displayed rules.

See also `letrules`, which displays rules defined by `let`.

Examples:

```
(%i1) tellsimpafter (foo (x, y), bar (x) + baz (y));
(%o1)                    [foorule1, false]
(%i2) tellsimpafter (x + y, special_add (x, y));
(%o2)                    [+rule1, simplus]
(%i3) defmatch (quux, mumble (x));
(%o3)                           quux
(%i4) disprule (foorule1, ?\+rule1, quux);
(%t4)        foorule1 : foo(x, y) -> baz(y) + bar(x)

(%t5)          +rule1 : y + x -> special_add(x, y)

(%t6)                quux : mumble(x) -> []

(%o6)                    [%t4, %t5, %t6]
```

```
(%i7) ev(%);
(%o7) [foorule1 : foo(x, y) -> baz(y) + bar(x),
     +rule1 : y + x -> special_add(x, y), quux : mumble(x) -> []]
```

`let` [Function]

`let` (*prod*, *repl*, *predname*, *arg_1*, ..., *arg_n*)
`let` ([*prod*, *repl*, *predname*, *arg_1*, ..., *arg_n*], *package_name*)

Defines a substitution rule for `letsimp` such that *prod* is replaced by *repl*. *prod* is a product of positive or negative powers of the following terms:

- Atoms which `letsimp` will search for literally unless previous to calling `letsimp` the `matchdeclare` function is used to associate a predicate with the atom. In this case `letsimp` will match the atom to any term of a product satisfying the predicate.
- Kernels such as `sin(x)`, `n!`, `f(x,y)`, etc. As with atoms above `letsimp` will look for a literal match unless `matchdeclare` is used to associate a predicate with the argument of the kernel.

A term to a positive power will only match a term having at least that power. A term to a negative power on the other hand will only match a term with a power at least as negative. In the case of negative powers in *prod* the switch `letrat` must be set to `true`. See also `letrat`.

If a predicate is included in the `let` function followed by a list of arguments, a tentative match (i.e. one that would be accepted if the predicate were omitted) is accepted only if `predname (arg_1', ..., arg_n')` evaluates to `true` where *arg_i'* is the value matched to *arg_i*. The *arg_i* may be the name of any atom or the argument of any kernel appearing in *prod*. *repl* may be any rational expression. If any of the atoms or arguments from *prod* appear in *repl* the appropriate substitutions are made.

The global flag `letrat` controls the simplification of quotients by `letsimp`. When `letrat` is `false`, `letsimp` simplifies the numerator and denominator of *expr* separately, and does not simplify the quotient. Substitutions such as `n!/n` goes to `(n-1)!` then fail. When `letrat` is `true`, then the numerator, denominator, and the quotient are simplified in that order.

These substitution functions allow you to work with several rule packages at once. Each rule package can contain any number of `let` rules and is referenced by a user-defined name. The command `let ([prod, repl, predname, arg_1, ..., arg_n], package_name)` adds the rule *predname* to the rule package *package_name*. The command `letsimp (expr, package_name)` applies the rules in *package_name*. `letsimp (expr, package_name1, package_name2, ...)` is equivalent to `letsimp (expr, package_name1)` followed by `letsimp (%, package_name2)`, ...

`current_let_rule_package` is the name of the rule package that is presently being used. This variable may be assigned the name of any rule package defined via the `let` command. Whenever any of the functions comprising the `let` package are called with no package name, the package named by `current_let_rule_package` is used. If a call such as `letsimp (expr, rule_pkg_name)` is made, the rule package *rule_pkg_name* is used for that `letsimp` command only, and `current_let_rule_package` is not changed. If not otherwise specified, `current_let_rule_package` defaults to `default_let_rule_package`.

```
(%i1) matchdeclare ([a, a1, a2], true)$
(%i2) oneless (x, y) := is (x = y-1)$
(%i3) let (a1*a2!, a1!, oneless, a2, a1);
(%o3)         a1 a2! --> a1! where oneless(a2, a1)
(%i4) letrat: true$
(%i5) let (a1!/a1, (a1-1)!);
                          a1!
(%o5)                     --- --> (a1 - 1)!
                          a1
(%i6) letsimp (n*m!*(n-1)!/m);
                             (m - 1)! n!
(%o6)                        ------------
                                  m
(%i7) let (sin(a)^2, 1 - cos(a)^2);
                          2               2
(%o7)                 sin (a) --> 1 - cos (a)
(%i8) letsimp (sin(x)^4);
                          4           2
(%o8)                 cos (x) - 2 cos (x) + 1
```

`letrat` [Option variable]

Default value: `false`

When `letrat` is `false`, `letsimp` simplifies the numerator and denominator of a ratio separately, and does not simplify the quotient.

When `letrat` is `true`, the numerator, denominator, and their quotient are simplified in that order.

```
(%i1) matchdeclare (n, true)$
(%i2) let (n!/n, (n-1)!);
                           n!
(%o2)                      -- --> (n - 1)!
                           n
(%i3) letrat: false$
(%i4) letsimp (a!/a);
                                 a!
(%o4)                            --
                                 a
(%i5) letrat: true$
(%i6) letsimp (a!/a);
(%o6)                         (a - 1)!
```

`letrules` [Function]

`letrules ()`
`letrules (`*package_name*`)`

Displays the rules in a rule package. `letrules ()` displays the rules in the current rule package. `letrules (`*package_name*`)` displays the rules in *package_name*.

The current rule package is named by `current_let_rule_package`. If not otherwise specified, `current_let_rule_package` defaults to `default_let_rule_package`.

See also `disprule`, which displays rules defined by `tellsimp` and `tellsimpafter`.

letsimp [Function]

letsimp (*expr*)
letsimp (*expr*, *package_name*)
letsimp (*expr*, *package_name_1*, ..., *package_name_n*)

Repeatedly applies the substitution rules defined by `let` until no further change is made to *expr*.

`letsimp (expr)` uses the rules from `current_let_rule_package`.

`letsimp (expr, package_name)` uses the rules from *package_name* without changing `current_let_rule_package`.

`letsimp (expr, package_name_1, ..., package_name_n)` is equivalent to `letsimp (expr, package_name_1)`, followed by `letsimp (%, package_name_2)`, and so on.

See also `let`. For other ways to do substitutions see also `subst`, `psubst`, `at` and `ratsubst`.

```
(%i1) e0: e(k) = -(9*y(k))/(5*z)-u(k-1)/(5*z)+(4*y(k))/(5*z^2)
                 +(3*u(k-1))/(5*z^2)+y(k)-(2*u(k-1))/5;
                9 y(k)   u(k - 1)   4 y(k)   3 u(k - 1)
(%o1) e(k) = - ------ - -------- + ------ + ---------- + y(k)
                5 z       5 z         2          2
                                     5 z        5 z

                                                    2 u(k - 1)
                                                  - ----------
                                                        5
(%i2) matchdeclare(h,any)$
(%i3) let(u(h)/z,u(h-1));
                        u(h)
(%o3)                   ---- --> u(h - 1)
                         z
(%i4) let(y(h)/z,y(h-1));
                        y(h)
(%o4)                   ---- --> y(h - 1)
                         z
(%i5) e1:letsimp(e0);
                     9 y(k - 1)   2 u(k - 1)   4 y(k - 2)
(%o5) e(k) = y(k) - ---------- - ---------- + ----------
                        5            5            5

                                                u(k - 2)   3 u(k - 3)
                                              - -------- + ----------
                                                   5           5
```

let_rule_packages [Option variable]

Default value: `[default_let_rule_package]`

`let_rule_packages` is a list of all user-defined let rule packages plus the default package `default_let_rule_package`.

matchdeclare (*a_1*, *pred_1*, ..., *a_n*, *pred_n*) [Function]

Associates a predicate *pred_k* with a variable or list of variables *a_k* so that *a_k* matches expressions for which the predicate returns anything other than `false`.

A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or `true` or `all`. Any expression matches `true` or `all`. If the predicate is specified as a function call or lambda call, the expression to be tested is appended to the list of arguments; the arguments are evaluated at the time the match is evaluated. Otherwise, the predicate is specified as a function name or lambda expression, and the expression to be tested is the sole argument. A predicate function need not be defined when `matchdeclare` is called; the predicate is not evaluated until a match is attempted.

A predicate may return a Boolean expression as well as `true` or `false`. Boolean expressions are evaluated by `is` within the constructed rule function, so it is not necessary to call `is` within the predicate.

If an expression satisfies a match predicate, the match variable is assigned the expression, except for match variables which are operands of addition `+` or multiplication `*`. Only addition and multiplication are handled specially; other n-ary operators (both built-in and user-defined) are treated like ordinary functions.

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions. Such multiple-term matching is greedy: predicates are evaluated in the order in which their associated variables appear in the match pattern, and a term which satisfies more than one predicate is taken by the first predicate which it satisfies. Each predicate is tested against all operands of the sum or product before the next predicate is evaluated. In addition, if 0 or 1 (respectively) satisfies a match predicate, and there are no other terms which satisfy the predicate, 0 or 1 is assigned to the match variable associated with the predicate.

The algorithm for processing addition and multiplication patterns makes some match results (for example, a pattern in which a "match anything" variable appears) dependent on the ordering of terms in the match pattern and in the expression to be matched. However, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

Calling `matchdeclare` with a variable *a* as an argument changes the `matchdeclare` property for *a*, if one was already declared; only the most recent `matchdeclare` is in effect when a rule is defined. Later changes to the `matchdeclare` property (via `matchdeclare` or `remove`) do not affect existing rules.

`propvars (matchdeclare)` returns the list of all variables for which there is a `matchdeclare` property. `printprops (a, matchdeclare)` returns the predicate for variable `a`. `printprops (all, matchdeclare)` returns the list of predicates for all `matchdeclare` variables. `remove (a, matchdeclare)` removes the `matchdeclare` property from *a*.

The functions `defmatch`, `defrule`, `tellsimp`, `tellsimpafter`, and `let` construct rules which test expressions against patterns.

`matchdeclare` quotes its arguments. `matchdeclare` always returns `done`.

Examples:

A predicate is the name of a function, or a lambda expression, or a function call or lambda call missing the last argument, or `true` or `all`.

```
(%i1) matchdeclare (aa, integerp);
(%o1)                         done
(%i2) matchdeclare (bb, lambda ([x], x > 0));
(%o2)                         done
(%i3) matchdeclare (cc, freeof (%e, %pi, %i));
(%o3)                         done
(%i4) matchdeclare (dd, lambda ([x, y], gcd (x, y) = 1) (1728));
(%o4)                         done
(%i5) matchdeclare (ee, true);
(%o5)                         done
(%i6) matchdeclare (ff, all);
(%o6)                         done
```

If an expression satisfies a match predicate, the match variable is assigned the expression.

```
(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1)                         done
(%i2) defrule (r1, bb^aa, ["integer" = aa, "atom" = bb]);
                       aa
(%o2)         r1 : bb   -> [integer = aa, atom = bb]
(%i3) r1 (%pi^8);
(%o3)                [integer = 8, atom = %pi]
```

In the case of addition and multiplication, the match variable may be assigned a single expression which satisfies the match predicate, or a sum or product (respectively) of such expressions.

```
(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));
(%o1)                         done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" =
                bb]);
(%o2)  r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
(%i3) r1 (8 + a*b + sin(x));
(%o3)     [all atoms = 8, all nonatoms = sin(x) + a b]
(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" =
                bb]);
(%o4)   r2 : aa bb -> [all atoms = aa, all nonatoms = bb]
(%i5) r2 (8 * (a + b) * sin(x));
(%o5)    [all atoms = 8, all nonatoms = (b + a) sin(x)]
```

When matching arguments of `+` and `*`, if all match predicates are mutually exclusive, the match result is insensitive to ordering, as one match predicate cannot accept terms matched by another.

```
(%i1) matchdeclare (aa, atom, bb, lambda ([x], not atom(x)));
(%o1)                         done
(%i2) defrule (r1, aa + bb, ["all atoms" = aa, "all nonatoms" =
                bb]);
(%o2)  r1 : bb + aa -> [all atoms = aa, all nonatoms = bb]
```

```
(%i3) r1 (8 + a*b + %pi + sin(x) - c + 2^n);
                                                                    n
(%o3) [all atoms = %pi + 8, all nonatoms = sin(x) - c + a b + 2 ]
(%i4) defrule (r2, aa * bb, ["all atoms" = aa, "all nonatoms" =
                bb]);
(%o4)   r2 : aa bb -> [all atoms = aa, all nonatoms = bb]
(%i5) r2 (8 * (a + b) * %pi * sin(x) / c * 2^n);
                                          n + 3
                                        2      (b + a) sin(x)
(%o5) [all atoms = %pi, all nonatoms = ----------------------]
                                                 c
```

The functions `propvars` and `printprops` return information about match variables.

```
(%i1) matchdeclare ([aa, bb, cc], atom, [dd, ee], integerp);
(%o1)                         done
(%i2) matchdeclare (ff, floatnump, gg, lambda ([x], x > 100));
(%o2)                         done
(%i3) propvars (matchdeclare);
(%o3)             [aa, bb, cc, dd, ee, ff, gg]
(%i4) printprops (ee, matchdeclare);
(%o4)                     [integerp(ee)]
(%i5) printprops (gg, matchdeclare);
(%o5)               [lambda([x], x > 100)(gg)]
(%i6) printprops (all, matchdeclare);
(%o6) [atom(aa), atom(bb), atom(cc), integerp(dd), integerp(ee),
                          floatnump(ff), lambda([x], x > 100)(gg)]
```

`maxapplydepth` [Option variable]

Default value: 10000

`maxapplydepth` is the maximum depth to which `apply1` and `apply2` will delve.

`maxapplyheight` [Option variable]

Default value: 10000

`maxapplyheight` is the maximum height to which `applyb1` will reach before giving up.

`remlet` [Function]

`remlet` (*prod*, *name*)
`remlet` ()
`remlet` (*all*)
`remlet` (*all*, *name*)

Deletes the substitution rule, `prod --> repl`, most recently defined by the `let` function. If name is supplied the rule is deleted from the rule package name.

`remlet()` and `remlet(all)` delete all substitution rules from the current rule package. If the name of a rule package is supplied, e.g. `remlet (all, name)`, the rule package *name* is also deleted.

If a substitution is to be changed using the same product, `remlet` need not be called, just redefine the substitution using the same product (literally) with the `let` function

and the new replacement and/or predicate name. Should `remlet (prod)` now be called the original substitution rule is revived.

See also `remrule`, which removes a rule defined by `tellsimp` or `tellsimpafter`.

`remrule` [Function]

`remrule` (*op*, *rulename*)
`remrule` (*op*, *all*)

Removes rules defined by `tellsimp` or `tellsimpafter`.

`remrule (op, rulename)` removes the rule with the name *rulename* from the operator *op*. When *op* is a built-in or user-defined operator (as defined by `infix`, `prefix`, etc.), *op* and *rulename* must be enclosed in double quote marks.

`remrule (op, all)` removes all rules for the operator *op*.

See also `remlet`, which removes a rule defined by `let`.

Examples:

```
(%i1) tellsimp (foo (aa, bb), bb - aa);
(%o1)                    [foorule1, false]
(%i2) tellsimpafter (aa + bb, special_add (aa, bb));
(%o2)                    [+rule1, simplus]
(%i3) infix ("@@");
(%o3)                             @@
(%i4) tellsimp (aa @@ bb, bb/aa);
(%o4)                    [@@rule1, false]
(%i5) tellsimpafter (quux (%pi, %e), %pi - %e);
(%o5)                   [quuxrule1, false]
(%i6) tellsimpafter (quux (%e, %pi), %pi + %e);
(%o6)              [quuxrule2, quuxrule1, false]
(%i7) [foo (aa, bb), aa + bb, aa @@ bb, quux (%pi, %e),
       quux (%e, %pi)];
                                          bb
(%o7) [bb - aa, special_add(aa, bb), --, %pi - %e, %pi + %e]
                                          aa
(%i8) remrule (foo, foorule1);
(%o8)                            foo
(%i9) remrule ("+", ?\+rule1);
(%o9)                             +
(%i10) remrule ("@@", ?\@\@rule1);
(%o10)                           @@
(%i11) remrule (quux, all);
(%o11)                          quux
(%i12) [foo (aa, bb), aa + bb, aa @@ bb, quux (%pi, %e),
        quux (%e, %pi)];
(%o12) [foo(aa, bb), bb + aa, aa @@ bb, quux(%pi, %e),
                                                quux(%e, %pi)]
```

`tellsimp` (*pattern*, *replacement*) [Function]

is similar to `tellsimpafter` but places new information before old so that it is applied before the built-in simplification rules.

`tellsimp` is used when it is important to modify the expression before the simplifier works on it, for instance if the simplifier "knows" something about the expression, but what it returns is not to your liking. If the simplifier "knows" something about the main operator of the expression, but is simply not doing enough for you, you probably want to use `tellsimpafter`.

The pattern may not be a sum, product, single variable, or number.

The system variable `rules` is the list of rules defined by `defrule`, `defmatch`, `tellsimp`, and `tellsimpafter`.

Examples:

```
(%i1) matchdeclare (x, freeof (%i));
(%o1)                         done
(%i2) %iargs: false$
(%i3) tellsimp (sin(%i*x), %i*sinh(x));
(%o3)                 [sinrule1, simp-%sin]
(%i4) trigexpand (sin (%i*y + x));
(%o4)         sin(x) cos(%i y) + %i cos(x) sinh(y)
(%i5) %iargs:true$
(%i6) errcatch(0^0);
                  0
expt: undefined: 0
(%o6)                           []
(%i7) ev (tellsimp (0^0, 1), simp: false);
(%o7)                  [^rule1, simpexpt]
(%i8) 0^0;
(%o8)                           1
(%i9) remrule ("^", %th(2)[1]);
(%o9)                           ^
(%i10) tellsimp (sin(x)^2, 1 - cos(x)^2);
(%o10)                 [^rule2, simpexpt]
(%i11) (1 + sin(x))^2;
                                       2
(%o11)                    (sin(x) + 1)
(%i12) expand (%);
                                    2
(%o12)               2 sin(x) - cos (x) + 2
(%i13) sin(x)^2;
                                  2
(%o13)                     1 - cos (x)
(%i14) kill (rules);
(%o14)                         done
(%i15) matchdeclare (a, true);
(%o15)                         done
(%i16) tellsimp (sin(a)^2, 1 - cos(a)^2);
(%o16)                 [^rule3, simpexpt]
```

```
(%i17) sin(y)^2;
                                      2
(%o17)                         1 - cos (y)
```

`tellsimpafter` (*pattern*, *replacement*) [Function]

Defines a simplification rule which the Maxima simplifier applies after built-in simplification rules. *pattern* is an expression, comprising pattern variables (declared by `matchdeclare`) and other atoms and operators, considered literals for the purpose of pattern matching. *replacement* is substituted for an actual expression which matches *pattern*; pattern variables in *replacement* are assigned the values matched in the actual expression.

pattern may be any nonatomic expression in which the main operator is not a pattern variable; the simplification rule is associated with the main operator. The names of functions (with one exception, described below), lists, and arrays may appear in *pattern* as the main operator only as literals (not pattern variables); this rules out expressions such as `aa(x)` and `bb[y]` as patterns, if `aa` and `bb` are pattern variables. Names of functions, lists, and arrays which are pattern variables may appear as operators other than the main operator in *pattern*.

There is one exception to the above rule concerning names of functions. The name of a subscripted function in an expression such as `aa[x](y)` may be a pattern variable, because the main operator is not `aa` but rather the Lisp atom `mqapply`. This is a consequence of the representation of expressions involving subscripted functions.

Simplification rules are applied after evaluation (if not suppressed through quotation or the flag `noeval`). Rules established by `tellsimpafter` are applied in the order they were defined, and after any built-in rules. Rules are applied bottom-up, that is, applied first to subexpressions before application to the whole expression. It may be necessary to repeatedly simplify a result (for example, via the quote-quote operator `''` or the flag `infeval`) to ensure that all rules are applied.

Pattern variables are treated as local variables in simplification rules. Once a rule is defined, the value of a pattern variable does not affect the rule, and is not affected by the rule. An assignment to a pattern variable which results from a successful rule match does not affect the current assignment (or lack of it) of the pattern variable. However, as with all atoms in Maxima, the properties of pattern variables (as declared by `put` and related functions) are global.

The rule constructed by `tellsimpafter` is named after the main operator of *pattern*. Rules for built-in operators, and user-defined operators defined by `infix`, `prefix`, `postfix`, `matchfix`, and `nofix`, have names which are Lisp identifiers. Rules for other functions have names which are Maxima identifiers.

The treatment of noun and verb forms is slightly confused. If a rule is defined for a noun (or verb) form and a rule for the corresponding verb (or noun) form already exists, the newly-defined rule applies to both forms (noun and verb). If a rule for the corresponding verb (or noun) form does not exist, the newly-defined rule applies only to the noun (or verb) form.

The rule constructed by `tellsimpafter` is an ordinary Lisp function. If the name of the rule is `$foorule1`, the construct `:lisp (trace $foorule1)` traces the function, and `:lisp (symbol-function '$foorule1)` displays its definition.

`tellsimpafter` quotes its arguments. `tellsimpafter` returns the list of rules for the main operator of *pattern*, including the newly established rule.

See also `matchdeclare`, `defmatch`, `defrule`, `tellsimp`, `let`, `kill`, `remrule` and `clear_rules`.

Examples:

pattern may be any nonatomic expression in which the main operator is not a pattern variable.

```
(%i1) matchdeclare (aa, atom, [ll, mm], listp, xx, true)$
(%i2) tellsimpafter (sin (ll), map (sin, ll));
(%o2)                  [sinrule1, simp-%sin]
(%i3) sin ([1/6, 1/4, 1/3, 1/2, 1]*%pi);
                     1     1     sqrt(3)
(%o3)               [-, -------, -------, 1, 0]
                     2  sqrt(2)     2
(%i4) tellsimpafter (ll^mm, map ("^", ll, mm));
(%o4)                    [^rule1, simpexpt]
(%i5) [a, b, c]^[1, 2, 3];
                                2   3
(%o5)                       [a, b , c ]
(%i6) tellsimpafter (foo (aa (xx)), aa (foo (xx)));
(%o6)                    [foorule1, false]
(%i7) foo (bar (u - v));
(%o7)                     bar(foo(u - v))
```

Rules are applied in the order they were defined. If two rules can match an expression, the rule which was defined first is applied.

```
(%i1) matchdeclare (aa, integerp);
(%o1)                           done
(%i2) tellsimpafter (foo (aa), bar_1 (aa));
(%o2)                    [foorule1, false]
(%i3) tellsimpafter (foo (aa), bar_2 (aa));
(%o3)               [foorule2, foorule1, false]
(%i4) foo (42);
(%o4)                         bar_1(42)
```

Pattern variables are treated as local variables in simplification rules. (Compare to `defmatch`, which treats pattern variables as global variables.)

```
(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1)                           done
(%i2) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb));
(%o2)                    [foorule1, false]
(%i3) bb: 12345;
(%o3)                           12345
(%i4) foo (42, %e);
(%o4)                  bar(aa = 42, bb = %e)
(%i5) bb;
(%o5)                           12345
```

As with all atoms, properties of pattern variables are global even though values are local. In this example, an assignment property is declared via `define_variable`. This is a property of the atom `bb` throughout Maxima.

translator: bb was declared with mode boolean, but it has value: %e – an error. To debug this try: debugmode(true);

```
(%i1) matchdeclare (aa, integerp, bb, atom);
(%o1)                         done
(%i2) tellsimpafter (foo(aa, bb), bar('aa=aa, 'bb=bb));
(%o2)                    [foorule1, false]
(%i3) foo (42, %e);
(%o3)                 bar(aa = 42, bb = %e)
(%i4) define_variable (bb, true, boolean);
(%o4)                         true
(%i5) foo (42, %e);
```

Rules are named after main operators. Names of rules for built-in and user-defined operators are Lisp identifiers, while names for other functions are Maxima identifiers.

```
(%i1) tellsimpafter (foo (%pi + %e), 3*%pi);
(%o1)                    [foorule1, false]
(%i2) tellsimpafter (foo (%pi * %e), 17*%e);
(%o2)              [foorule2, foorule1, false]
(%i3) tellsimpafter (foo (%i ^ %e), -42*%i);
(%o3)         [foorule3, foorule2, foorule1, false]
(%i4) tellsimpafter (foo (9) + foo (13), quux (22));
(%o4)                    [+rule1, simplus]
(%i5) tellsimpafter (foo (9) * foo (13), blurf (22));
(%o5)                   [*rule1, simptimes]
(%i6) tellsimpafter (foo (9) ^ foo (13), mumble (22));
(%o6)                   [^rule1, simpexpt]
(%i7) rules;
(%o7) [foorule1, foorule2, foorule3, +rule1, *rule1, ^rule1]
(%i8) foorule_name: first (%o1);
(%o8)                        foorule1
(%i9) plusrule_name: first (%o4);
(%o9)                         +rule1
(%i10) remrule (foo, foorule1);
(%o10)                         foo
(%i11) remrule ("^", ?\^rule1);
(%o11)                          ^
(%i12) rules;
(%o12)        [foorule2, foorule3, +rule1, *rule1]
```

A worked example: anticommutative multiplication.

```
(%i1) gt (i, j) := integerp(j) and i < j;
(%o1)          gt(i, j) := integerp(j) and (i < j)
(%i2) matchdeclare (i, integerp, j, gt(i));
(%o2)                         done
```

```
(%i3) tellsimpafter (s[i]^^2, 1);
(%o3)                 [^^rule1, simpncexpt]
(%i4) tellsimpafter (s[i] . s[j], -s[j] . s[i]);
(%o4)                   [.rule1, simpnct]
(%i5) s[1] . (s[1] + s[2]);
(%o5)                    s  . (s  + s )
                          1     2    1
(%i6) expand (%);
(%o6)                     1 - s  . s
                                2    1
(%i7) factor (expand (sum (s[i], i, 0, 9)^^5));
(%o7) 100 (s  + s  + s  + s  + s  + s  + s  + s  + s  + s )
           9    8    7    6    5    4    3    2    1    0
```

`clear_rules` () [Function]

Executes `kill (rules)` and then resets the next rule number to 1 for addition `+`, multiplication `*`, and exponentiation `^`.

31 Sets

31.1 Introduction to Sets

Maxima provides set functions, such as intersection and union, for finite sets that are defined by explicit enumeration. Maxima treats lists and sets as distinct objects. This feature makes it possible to work with sets that have members that are either lists or sets.

In addition to functions for finite sets, Maxima provides some functions related to combinatorics; these include the Stirling numbers of the first and second kind, the Bell numbers, multinomial coefficients, partitions of nonnegative integers, and a few others. Maxima also defines a Kronecker delta function.

31.1.1 Usage

To construct a set with members `a_1, ..., a_n`, write `set(a_1, ..., a_n)` or `{a_1, ..., a_n}`; to construct the empty set, write `set()` or `{}`. In input, `set(...)` and `{ ... }` are equivalent. Sets are always displayed with curly braces.

If a member is listed more than once, simplification eliminates the redundant member.

```
(%i1) set();
(%o1)                           {}
(%i2) set(a, b, a);
(%o2)                         {a, b}
(%i3) set(a, set(b));
(%o3)                        {a, {b}}
(%i4) set(a, [b]);
(%o4)                        {a, [b]}
(%i5) {};
(%o5)                           {}
(%i6) {a, b, a};
(%o6)                         {a, b}
(%i7) {a, {b}};
(%o7)                        {a, {b}}
(%i8) {a, [b]};
(%o8)                        {a, [b]}
```

Two would-be elements x and y are redundant (i.e., considered the same for the purpose of set construction) if and only if `is(`x `=` y`)` yields `true`. Note that `is(equal(`x`,` y`))` can yield `true` while `is(`x `=` y`)` yields `false`; in that case the elements x and y are considered distinct.

```
(%i1) x: a/c + b/c;
                                b   a
(%o1)                           - + -
                                c   c
(%i2) y: a/c + b/c;
                                b   a
(%o2)                           - + -
                                c   c
```

```
(%i3) z: (a + b)/c;
                                b + a
(%o3)                           -----
                                  c
(%i4) is (x = y);
(%o4)                           true
(%i5) is (y = z);
(%o5)                           false
(%i6) is (equal (y, z));
(%o6)                           true
(%i7) y - z;
                             b + a   b   a
(%o7)                      - ----- + - + -
                               c     c   c
(%i8) ratsimp (%);
(%o8)                             0
(%i9) {x, y, z};
                            b + a  b   a
(%o9)                      {-----, - + -}
                              c    c   c
```

To construct a set from the elements of a list, use `setify`.

```
(%i1) setify ([b, a]);
(%o1)                          {a, b}
```

Set members `x` and `y` are equal provided `is(x = y)` evaluates to `true`. Thus `rat(x)` and `x` are equal as set members; consequently,

```
(%i1) {x, rat(x)};
(%o1)                            {x}
```

Further, since `is((x - 1)*(x + 1) = x^2 - 1)` evaluates to `false`, `(x - 1)*(x + 1)` and `x^2 - 1` are distinct set members; thus

```
(%i1) {(x - 1)*(x + 1), x^2 - 1};
                                          2
(%o1)                {(x - 1) (x + 1), x  - 1}
```

To reduce this set to a singleton set, apply `rat` to each set member:

```
(%i1) {(x - 1)*(x + 1), x^2 - 1};
                                          2
(%o1)                {(x - 1) (x + 1), x  - 1}
(%i2) map (rat, %);
                                 2
(%o2)/R/                        {x  - 1}
```

To remove redundancies from other sets, you may need to use other simplification functions. Here is an example that uses `trigsimp`:

```
(%i1) {1, cos(x)^2 + sin(x)^2};
                               2         2
(%o1)                {1, sin (x) + cos (x)}
```

```
(%i2) map (trigsimp, %);
(%o2)                          {1}
```

A set is simplified when its members are non-redundant and sorted. The current version of the set functions uses the Maxima function `orderlessp` to order sets; however, *future versions of the set functions might use a different ordering function.*

Some operations on sets, such as substitution, automatically force a re-simplification; for example,

```
(%i1) s: {a, b, c}$
(%i2) subst (c=a, s);
(%o2)                         {a, b}
(%i3) subst ([a=x, b=x, c=x], s);
(%o3)                          {x}
(%i4) map (lambda ([x], x^2), set (-1, 0, 1));
(%o4)                         {0, 1}
```

Maxima treats lists and sets as distinct objects; functions such as `union` and `intersection` complain if any argument is not a set. If you need to apply a set function to a list, use the `setify` function to convert it to a set. Thus

union: argument must be a set; found: [1,2] – an error. To debug this try: debugmode(true);

```
(%i1) union ([1, 2], {a, b});
(%i2) union (setify ([1, 2]), {a, b});
(%o2)                      {1, 2, a, b}
```

To extract all set elements of a set `s` that satisfy a predicate `f`, use `subset(s, f)`. (A *predicate* is a boolean-valued function.) For example, to find the equations in a given set that do not depend on a variable `z`, use

```
(%i1) subset ({x + y + z, x - y + 4, x + y - 5},
                                    lambda ([e], freeof (z, e)));
(%o1)                {- y + x + 4, y + x - 5}
```

The section Section 31.2 [Functions and Variables for Sets], page 568, has a complete list of the set functions in Maxima.

31.1.2 Set Member Iteration

There two ways to to iterate over set members. One way is the use `map`; for example:

```
(%i1) map (f, {a, b, c});
(%o1)                   {f(a), f(b), f(c)}
```

The other way is to use `for` *x* `in` *s* `do`

```
(%i1) s: {a, b, c};
(%o1)                         {a, b, c}
(%i2) for si in s do print (concat (si, 1));
a1
b1
c1
(%o2)                          done
```

The Maxima functions `first` and `rest` work correctly on sets. Applied to a set, `first` returns the first displayed element of a set; which element that is may be implementation-dependent. If `s` is a set, then `rest(s)` is equivalent to `disjoin(first(s), s)`. Currently, there are other Maxima functions that work correctly on sets. In future versions of the set functions, `first` and `rest` may function differently or not at all.

Maxima's `orderless` and `ordergreat` mechanisms are incompatible with the set functions. If you need to use either `orderless` or `ordergreat`, call those functions before constructing any sets, and do not call `unorder`.

31.1.3 Authors

Stavros Macrakis of Cambridge, Massachusetts and Barton Willis of the University of Nebraska at Kearney (UNK) wrote the Maxima set functions and their documentation.

31.2 Functions and Variables for Sets

`adjoin` (*x*, *a*) [Function]

Returns the union of the set *a* with `{`*x*`}`.

`adjoin` complains if *a* is not a literal set.

`adjoin(`*x*`, a)` and `union(set(`*x*`), a)` are equivalent; however, `adjoin` may be somewhat faster than `union`.

See also `disjoin`.

Examples:

```
(%i1) adjoin (c, {a, b});
(%o1)                        {a, b, c}
(%i2) adjoin (a, {a, b});
(%o2)                         {a, b}
```

`belln` (*n*) [Function]

Represents the *n*-th Bell number. `belln(n)` is the number of partitions of a set with *n* members.

For nonnegative integers *n*, `belln(`*n*`)` simplifies to the *n*-th Bell number. `belln` does not simplify for any other arguments.

`belln` distributes over equations, lists, matrices, and sets.

Examples:

`belln` applied to nonnegative integers.

```
(%i1) makelist (belln (i), i, 0, 6);
(%o1)                [1, 1, 2, 5, 15, 52, 203]
(%i2) is (cardinality (set_partitions ({})) = belln (0));
(%o2)                           true
(%i3) is (cardinality (set_partitions ({1, 2, 3, 4, 5, 6})) =
                          belln (6));
(%o3)                           true
```

`belln` applied to arguments which are not nonnegative integers.

```
(%i1) [belln (x), belln (sqrt(3)), belln (-9)];
(%o1)         [belln(x), belln(sqrt(3)), belln(- 9)]
```

`cardinality (a)` [Function]

Returns the number of distinct elements of the set *a*.

`cardinality` ignores redundant elements even when simplification is disabled.

Examples:

```
(%i1) cardinality ({});
(%o1)                          0
(%i2) cardinality ({a, a, b, c});
(%o2)                          3
(%i3) simp : false;
(%o3)                        false
(%i4) cardinality ({a, a, b, c});
(%o4)                          3
```

`cartesian_product (b_1, ... , b_n)` [Function]

Returns a set of lists of the form `[x_1, ..., x_n]`, where *x_1*, ..., *x_n* are elements of the sets *b_1*, ... , *b_n*, respectively.

`cartesian_product` complains if any argument is not a literal set.

See also `cartesian_product_list`.

Examples:

```
(%i1) cartesian_product ({0, 1});
(%o1)                     {[0], [1]}
(%i2) cartesian_product ({0, 1}, {0, 1});
(%o2)          {[0, 0], [0, 1], [1, 0], [1, 1]}
(%i3) cartesian_product ({x}, {y}, {z});
(%o3)                     {[x, y, z]}
(%i4) cartesian_product ({x}, {-1, 0, 1});
(%o4)             {[x, - 1], [x, 0], [x, 1]}
```

`cartesian_product_list (b_1, ... , b_n)` [Function]

Returns a list of lists of the form `[x_1, ..., x_n]`, where *x_1*, ..., *x_n* are elements of the lists *b_1*, ... , *b_n*, respectively, comprising all possible combinations of the elements of *b_1*, ... , *b_n*.

The list returned by `cartesian_product_list` is equivalent to the following recursive definition. Let L be the list returned by `cartesian_product_list(b_2, ..., b_n)`. Then `cartesian_product_list(b_1, b_2, ..., b_n)` (i.e., *b_1* in addition to *b_2*, ..., *b_n*) returns a list comprising each element of L appended to the first element of *b_1*, each element of L appended to the second element of *b_1*, each element of L appended to the third element of *b_1*, etc. The order of the list returned by `cartesian_product_list(b_1, b_2, ..., b_n)` may therefore be summarized by saying the lesser indices (1, 2, 3, ...) vary more slowly than the greater indices.

The list returned by `cartesian_product_list` contains duplicate elements if any argument *b_1*, ..., *b_n* contains duplicates. In this respect, `cartesian_product_list` differs from `cartesian_product`, which returns no duplicates. Also, the ordering of the list returned `cartesian_product_list` is determined by the order of the elements of *b_1*, ..., *b_n*. Again, this differs from `cartesian_product`, which returns a set (with order determined by `orderlessp`).

The length of the list returned by `cartesian_product_list` is equal to the product of the lengths of the arguments *b_1*, ..., *b_n*.

See also `cartesian_product`.

`cartesian_product_list` complains if any argument is not a list.

Examples:

`cartesian_product_list` returns a list of lists comprising all possible combinations.

```
(%i1) cartesian_product_list ([0, 1]);
(%o1)                      [[0], [1]]
(%i2) cartesian_product_list ([0, 1], [0, 1]);
(%o2)           [[0, 0], [0, 1], [1, 0], [1, 1]]
(%i3) cartesian_product_list ([x], [y], [z]);
(%o3)                      [[x, y, z]]
(%i4) cartesian_product_list ([x], [-1, 0, 1]);
(%o4)              [[x, - 1], [x, 0], [x, 1]]
(%i5) cartesian_product_list ([a, h, e], [c, b, 4]);
(%o5) [[a, c], [a, b], [a, 4], [h, c], [h, b], [h, 4], [e, c],
                                                [e, b], [e, 4]]
```

The order of the list returned by `cartesian_product_list` may be summarized by saying the lesser indices vary more slowly than the greater indices.

```
(%i1) cartesian_product_list ([1, 2, 3], [a, b], [i, ii]);
(%o1) [[1, a, i], [1, a, ii], [1, b, i], [1, b, ii], [2, a, i],
[2, a, ii], [2, b, i], [2, b, ii], [3, a, i], [3, a, ii],
[3, b, i], [3, b, ii]]
```

The list returned by `cartesian_product_list` contains duplicate elements if any argument contains duplicates.

```
(%i1) cartesian_product_list ([e, h], [3, 7, 3]);
(%o1)   [[e, 3], [e, 7], [e, 3], [h, 3], [h, 7], [h, 3]]
```

The length of the list returned by `cartesian_product_list` is equal to the product of the lengths of the arguments.

```
(%i1) foo: cartesian_product_list ([1, 1, 2, 2, 3], [h, z, h]);
(%o1) [[1, h], [1, z], [1, h], [1, h], [1, z], [1, h], [2, h],
  [2, z], [2, h], [2, h], [2, z], [2, h], [3, h], [3, z], [3, h]]
(%i2) is (length (foo) = 5*3);
(%o2)                         true
```

`disjoin` (*x*, *a*) [Function]

Returns the set *a* without the member *x*. If *x* is not a member of *a*, return *a* unchanged.

`disjoin` complains if *a* is not a literal set.

`disjoin(x, a)`, `delete(x, a)`, and `setdifference(a, set(x))` are all equivalent. Of these, `disjoin` is generally faster than the others.

Examples:

```
(%i1) disjoin (a, {a, b, c, d});
(%o1)                       {b, c, d}
```

```
(%i2) disjoin (a + b, {5, z, a + b, %pi});
(%o2)                       {5, %pi, z}
(%i3) disjoin (a - b, {5, z, a + b, %pi});
(%o3)                    {5, %pi, b + a, z}
```

`disjointp` (*a*, *b*) [Function]

Returns `true` if and only if the sets *a* and *b* are disjoint.

`disjointp` complains if either *a* or *b* is not a literal set.

Examples:

```
(%i1) disjointp ({a, b, c}, {1, 2, 3});
(%o1)                           true
(%i2) disjointp ({a, b, 3}, {1, 2, 3});
(%o2)                           false
```

`divisors` (*n*) [Function]

Represents the set of divisors of *n*.

`divisors(n)` simplifies to a set of integers when *n* is a nonzero integer. The set of divisors includes the members 1 and *n*. The divisors of a negative integer are the divisors of its absolute value.

`divisors` distributes over equations, lists, matrices, and sets.

Examples:

We can verify that 28 is a perfect number: the sum of its divisors (except for itself) is 28.

```
(%i1) s: divisors(28);
(%o1)                  {1, 2, 4, 7, 14, 28}
(%i2) lreduce ("+", args(s)) - 28;
(%o2)                           28
```

`divisors` is a simplifying function. Substituting 8 for `a` in `divisors(a)` yields the divisors without reevaluating `divisors(8)`.

```
(%i1) divisors (a);
(%o1)                        divisors(a)
(%i2) subst (8, a, %);
(%o2)                       {1, 2, 4, 8}
```

`divisors` distributes over equations, lists, matrices, and sets.

```
(%i1) divisors (a = b);
(%o1)                divisors(a) = divisors(b)
(%i2) divisors ([a, b, c]);
(%o2)         [divisors(a), divisors(b), divisors(c)]
(%i3) divisors (matrix ([a, b], [c, d]));
                     [ divisors(a)  divisors(b) ]
(%o3)                [                          ]
                     [ divisors(c)  divisors(d) ]
(%i4) divisors ({a, b, c});
(%o4)         {divisors(a), divisors(b), divisors(c)}
```

`elementp` (*x*, *a*) [Function]

Returns `true` if and only if *x* is a member of the set *a*.

`elementp` complains if *a* is not a literal set.

Examples:

```
(%i1) elementp (sin(1), {sin(1), sin(2), sin(3)});
(%o1)                          true
(%i2) elementp (sin(1), {cos(1), cos(2), cos(3)});
(%o2)                          false
```

`emptyp` (*a*) [Function]

Return `true` if and only if *a* is the empty set or the empty list.

Examples:

```
(%i1) map (emptyp, [{}, []]);
(%o1)                       [true, true]
(%i2) map (emptyp, [a + b, {{}}, %pi]);
(%o2)                 [false, false, false]
```

`equiv_classes` (*s*, *F*) [Function]

Returns a set of the equivalence classes of the set *s* with respect to the equivalence relation *F*.

F is a function of two variables defined on the Cartesian product of *s* with *s*. The return value of *F* is either `true` or `false`, or an expression *expr* such that `is(`*expr*`)` is either `true` or `false`.

When *F* is not an equivalence relation, `equiv_classes` accepts it without complaint, but the result is generally incorrect in that case.

Examples:

The equivalence relation is a lambda expression which returns `true` or `false`.

```
(%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0},
                        lambda ([x, y], is (equal (x, y))));
(%o1)            {{1, 1.0}, {2, 2.0}, {3, 3.0}}
```

The equivalence relation is the name of a relational function which `is` evaluates to `true` or `false`.

```
(%i1) equiv_classes ({1, 1.0, 2, 2.0, 3, 3.0}, equal);
(%o1)            {{1, 1.0}, {2, 2.0}, {3, 3.0}}
```

The equivalence classes are numbers which differ by a multiple of 3.

```
(%i1) equiv_classes ({1, 2, 3, 4, 5, 6, 7},
                      lambda ([x, y], remainder (x - y, 3) = 0));
(%o1)              {{1, 4, 7}, {2, 5}, {3, 6}}
```

`every` [Function]

`every` (*f*, *s*)
`every` (*f*, *L_1*, ..., *L_n*)

Returns `true` if the predicate *f* is `true` for all given arguments.

Given one set as the second argument, `every(f, s)` returns `true` if `is(f(a_i))` returns `true` for all *a_i* in *s*. `every` may or may not evaluate *f* for all *a_i* in *s*. Since sets are unordered, `every` may evaluate `f(a_i)` in any order.

Given one or more lists as arguments, `every(f, L_1, ..., L_n)` returns `true` if `is(f(x_1, ..., x_n))` returns `true` for all *x_1*, ..., *x_n* in *L_1*, ..., *L_n*, respectively. `every` may or may not evaluate *f* for every combination *x_1*, ..., *x_n*. `every` evaluates lists in the order of increasing index.

Given an empty set `{}` or empty lists `[]` as arguments, `every` returns `true`.

When the global flag `maperror` is `true`, all lists *L_1*, ..., *L_n* must have equal lengths. When `maperror` is `false`, list arguments are effectively truncated to the length of the shortest list.

Return values of the predicate *f* which evaluate (via `is`) to something other than `true` or `false` are governed by the global flag `prederror`. When `prederror` is `true`, such values are treated as `false`, and the return value from `every` is `false`. When `prederror` is `false`, such values are treated as `unknown`, and the return value from `every` is `unknown`.

Examples:

`every` applied to a single set. The predicate is a function of one argument.

```
(%i1) every (integerp, {1, 2, 3, 4, 5, 6});
(%o1)                         true
(%i2) every (atom, {1, 2, sin(3), 4, 5 + y, 6});
(%o2)                         false
```

`every` applied to two lists. The predicate is a function of two arguments.

```
(%i1) every ("=", [a, b, c], [a, b, c]);
(%o1)                         true
(%i2) every ("#", [a, b, c], [a, b, c]);
(%o2)                         false
```

Return values of the predicate *f* which evaluate to something other than `true` or `false` are governed by the global flag `prederror`.

```
(%i1) prederror : false;
(%o1)                         false
(%i2) map (lambda ([a, b], is (a < b)), [x, y, z],
                  [x^2, y^2, z^2]);
(%o2)                [unknown, unknown, unknown]
(%i3) every ("<", [x, y, z], [x^2, y^2, z^2]);
(%o3)                        unknown
(%i4) prederror : true;
(%o4)                         true
(%i5) every ("<", [x, y, z], [x^2, y^2, z^2]);
(%o5)                         false
```

`extremal_subset` [Function]

`extremal_subset` (*s*, *f*, *max*)
`extremal_subset` (*s*, *f*, *min*)

Returns the subset of *s* for which the function *f* takes on maximum or minimum values.

`extremal_subset(s, f, max)` returns the subset of the set or list *s* for which the real-valued function *f* takes on its maximum value.

`extremal_subset(s, f, min)` returns the subset of the set or list *s* for which the real-valued function *f* takes on its minimum value.

Examples:

```
(%i1) extremal_subset ({-2, -1, 0, 1, 2}, abs, max);
(%o1)                        {- 2, 2}
(%i2) extremal_subset ({sqrt(2), 1.57, %pi/2}, sin, min);
(%o2)                        {sqrt(2)}
```

`flatten` (*expr*) [Function]

Collects arguments of subexpressions which have the same operator as *expr* and constructs an expression from these collected arguments.

Subexpressions in which the operator is different from the main operator of `expr` are copied without modification, even if they, in turn, contain some subexpressions in which the operator is the same as for `expr`.

It may be possible for `flatten` to construct expressions in which the number of arguments differs from the declared arguments for an operator; this may provoke an error message from the simplifier or evaluator. `flatten` does not try to detect such situations.

Expressions with special representations, for example, canonical rational expressions (CRE), cannot be flattened; in such cases, `flatten` returns its argument unchanged.

Examples:

Applied to a list, `flatten` gathers all list elements that are lists.

```
(%i1) flatten ([a, b, [c, [d, e], f], [[g, h]], i, j]);
(%o1)            [a, b, c, d, e, f, g, h, i, j]
```

Applied to a set, `flatten` gathers all members of set elements that are sets.

```
(%i1) flatten ({a, {b}, {{c}}});
(%o1)                        {a, b, c}
(%i2) flatten ({a, {[a], {a}}});
(%o2)                        {a, [a]}
```

`flatten` is similar to the effect of declaring the main operator n-ary. However, `flatten` has no effect on subexpressions which have an operator different from the main operator, while an n-ary declaration affects those.

```
(%i1) expr: flatten (f (g (f (f (x)))));
(%o1)                      f(g(f(f(x))))
(%i2) declare (f, nary);
(%o2)                         done
```

```
(%i3) ev (expr);
(%o3)                      f(g(f(f(x))))
```

`flatten` treats subscripted functions the same as any other operator.

```
(%i1) flatten (f[5] (f[5] (x, y), z));
(%o1)                       f (x, y, z)
                             5
```

It may be possible for `flatten` to construct expressions in which the number of arguments differs from the declared arguments for an operator;

mod: expected exactly 2 arguments but got 3: [5, 7, 4] – an error. To debug this try: debugmode(true);

```
(%i1) 'mod (5, 'mod (7, 4));
(%o1)                    mod(5, mod(7, 4))
(%i2) flatten (%);
(%o2)                      mod(5, 7, 4)
(%i3) ''%, nouns;
```

`full_listify (a)` [Function]

Replaces every set operator in *a* by a list operator, and returns the result. `full_listify` replaces set operators in nested subexpressions, even if the main operator is not `set`.

`listify` replaces only the main operator.

Examples:

```
(%i1) full_listify ({a, b, {c, {d, e, f}, g}});
(%o1)                [a, b, [c, [d, e, f], g]]
(%i2) full_listify (F (G ({a, b, H({c, d, e})})));
(%o2)              F(G([a, b, H([c, d, e])]))
```

`fullsetify (a)` [Function]

When *a* is a list, replaces the list operator with a set operator, and applies `fullsetify` to each member which is a set. When *a* is not a list, it is returned unchanged.

`setify` replaces only the main operator.

Examples:

In line `(%o2)`, the argument of `f` isn't converted to a set because the main operator of `f([b])` isn't a list.

```
(%i1) fullsetify ([a, [a]]);
(%o1)                         {a, {a}}
(%i2) fullsetify ([a, f([b])]);
(%o2)                        {a, f([b])}
```

`identity (x)` [Function]

Returns *x* for any argument *x*.

Examples:

`identity` may be used as a predicate when the arguments are already Boolean values.

```
(%i1) every (identity, [true, true]);
(%o1)                          true
```

integer_partitions [Function]

integer_partitions (*n*)

integer_partitions (*n*, *len*)

Returns integer partitions of n, that is, lists of integers which sum to n.

`integer_partitions(n)` returns the set of all partitions of the integer n. Each partition is a list sorted from greatest to least.

`integer_partitions(n, len)` returns all partitions that have length *len* or less; in this case, zeros are appended to each partition with fewer than *len* terms to make each partition have exactly *len* terms. Each partition is a list sorted from greatest to least.

A list $[a_1, ..., a_m]$ is a partition of a nonnegative integer n when (1) each a_i is a nonzero integer, and (2) $a_1 + ... + a_m = n$. Thus 0 has no partitions.

Examples:

```
(%i1) integer_partitions (3);
(%o1)               {[1, 1, 1], [2, 1], [3]}
(%i2) s: integer_partitions (25)$
(%i3) cardinality (s);
(%o3)                          1958
(%i4) map (lambda ([x], apply ("+", x)), s);
(%o4)                          {25}
(%i5) integer_partitions (5, 3);
(%o5) {[2, 2, 1], [3, 1, 1], [3, 2, 0], [4, 1, 0], [5, 0, 0]}
(%i6) integer_partitions (5, 2);
(%o6)               {[3, 2], [4, 1], [5, 0]}
```

To find all partitions that satisfy a condition, use the function `subset`; here is an example that finds all partitions of 10 that consist of prime numbers.

```
(%i1) s: integer_partitions (10)$
(%i2) cardinality (s);
(%o2)                           42
(%i3) xprimep(x) := integerp(x) and (x > 1) and primep(x)$
(%i4) subset (s, lambda ([x], every (xprimep, x)));
(%o4) {[2, 2, 2, 2, 2], [3, 3, 2, 2], [5, 3, 2], [5, 5], [7, 3]}
```

intersect (*a_1*, ..., *a_n*) [Function]

`intersect` is the same as `intersection`, which see.

intersection (*a_1*, ..., *a_n*) [Function]

Returns a set containing the elements that are common to the sets *a_1* through *a_n*.

`intersection` complains if any argument is not a literal set.

Examples:

```
(%i1) S_1 : {a, b, c, d};
(%o1)                       {a, b, c, d}
(%i2) S_2 : {d, e, f, g};
(%o2)                       {d, e, f, g}
(%i3) S_3 : {c, d, e, f};
(%o3)                       {c, d, e, f}
```

```
(%i4) S_4 : {u, v, w};
(%o4)                        {u, v, w}
(%i5) intersection (S_1, S_2);
(%o5)                            {d}
(%i6) intersection (S_2, S_3);
(%o6)                         {d, e, f}
(%i7) intersection (S_1, S_2, S_3);
(%o7)                            {d}
(%i8) intersection (S_1, S_2, S_3, S_4);
(%o8)                            {}
```

`kron_delta` (*x1*, *x2*, ..., *xp*) [Function]

Represents the Kronecker delta function.

`kron_delta` simplifies to 1 when *xi* and *yj* are equal for all pairs of arguments, and it simplifies to 0 when *xi* and *yj* are not equal for some pair of arguments. Equality is determined using `is(equal(xi,xj))` and inequality by `is(notequal(xi,xj))`. For exactly one argument, `kron_delta` signals an error.

Examples:

```
(%i1) kron_delta(a,a);
(%o1)                             1
(%i2) kron_delta(a,b,a,b);
(%o2)                    kron_delta(a, b)
(%i3) kron_delta(a,a,b,a+1);
(%o3)                             0
(%i4) assume(equal(x,y));
(%o4)                       [equal(x, y)]
(%i5) kron_delta(x,y);
(%o5)                             1
```

`listify` (*a*) [Function]

Returns a list containing the members of *a* when *a* is a set. Otherwise, `listify` returns *a*.

`full_listify` replaces all set operators in *a* by list operators.

Examples:

```
(%i1) listify ({a, b, c, d});
(%o1)                        [a, b, c, d]
(%i2) listify (F ({a, b, c, d}));
(%o2)                      F([a, b, c, d])
```

`makeset` (*expr*, *x*, *s*) [Function]

Returns a set with members generated from the expression *expr*, where *x* is a list of variables in *expr*, and *s* is a set or list of lists. To generate each set member, *expr* is evaluated with the variables *x* bound in parallel to a member of *s*.

Each member of *s* must have the same length as *x*. The list of variables *x* must be a list of symbols, without subscripts. Even if there is only one symbol, *x* must be a list of one element, and each member of *s* must be a list of one element.

See also makelist.

Examples:

```
(%i1) makeset (i/j, [i, j], [[1, a], [2, b], [3, c], [4, d]]);
                            1  2  3  4
(%o1)                      {-, -, -, -}
                            a  b  c  d
(%i2) S : {x, y, z}$
(%i3) S3 : cartesian_product (S, S, S);
(%o3) {[x, x, x], [x, x, y], [x, x, z], [x, y, x], [x, y, y],
[x, y, z], [x, z, x], [x, z, y], [x, z, z], [y, x, x],
[y, x, y], [y, x, z], [y, y, x], [y, y, y], [y, y, z],
[y, z, x], [y, z, y], [y, z, z], [z, x, x], [z, x, y],
[z, x, z], [z, y, x], [z, y, y], [z, y, z], [z, z, x],
[z, z, y], [z, z, z]}
(%i4) makeset (i + j + k, [i, j, k], S3);
(%o4) {3 x, 3 y, y + 2 x, 2 y + x, 3 z, z + 2 x, z + y + x,
                                        z + 2 y, 2 z + x, 2 z + y}
(%i5) makeset (sin(x), [x], {[1], [2], [3]});
(%o5)               {sin(1), sin(2), sin(3)}
```

moebius (*n*) [Function]

Represents the Moebius function.

When n is product of k distinct primes, moebius(*n*) simplifies to $(-1)^k$; when $n = 1$, it simplifies to 1; and it simplifies to 0 for all other positive integers.

moebius distributes over equations, lists, matrices, and sets.

Examples:

```
(%i1) moebius (1);
(%o1)                           1
(%i2) moebius (2 * 3 * 5);
(%o2)                          - 1
(%i3) moebius (11 * 17 * 29 * 31);
(%o3)                           1
(%i4) moebius (2^32);
(%o4)                           0
(%i5) moebius (n);
(%o5)                      moebius(n)
(%i6) moebius (n = 12);
(%o6)                    moebius(n) = 0
(%i7) moebius ([11, 11 * 13, 11 * 13 * 15]);
(%o7)                      [- 1, 1, 1]
(%i8) moebius (matrix ([11, 12], [13, 14]));
                           [ - 1  0 ]
(%o8)                      [        ]
                           [ - 1  1 ]
(%i9) moebius ({21, 22, 23, 24});
(%o9)                     {- 1, 0, 1}
```

multinomial_coeff [Function]

multinomial_coeff (*a_1*, ..., *a_n*)
multinomial_coeff ()

Returns the multinomial coefficient.

When each *a_k* is a nonnegative integer, the multinomial coefficient gives the number of ways of placing `a_1 + ... + a_n` distinct objects into n boxes with *a_k* elements in the k'th box. In general, `multinomial_coeff (a_1, ..., a_n)` evaluates to `(a_1 + ... + a_n)!/(a_1! ... a_n!)`.

`multinomial_coeff()` (with no arguments) evaluates to 1.

`minfactorial` may be able to simplify the value returned by `multinomial_coeff`.

Examples:

factorial: factorial of negative integer -6 not defined. – an error. To debug this try: debugmode(true);

```
(%i1) multinomial_coeff (1, 2, x);
                              (x + 3)!
(%o1)                         --------
                                2 x!
(%i2) minfactorial (%);
                        (x + 1) (x + 2) (x + 3)
(%o2)                   -----------------------
                                    2
(%i3) multinomial_coeff (-6, 2);
(%i4) minfactorial (%);
                        (x + 1) (x + 2) (x + 3)
(%o4)                   -----------------------
                                    2
```

num_distinct_partitions [Function]

num_distinct_partitions (*n*)
num_distinct_partitions (*n*, *list*)

Returns the number of distinct integer partitions of n when n is a nonnegative integer. Otherwise, `num_distinct_partitions` returns a noun expression.

`num_distinct_partitions(n, list)` returns a list of the number of distinct partitions of 1, 2, 3, ..., n.

A distinct partition of n is a list of distinct positive integers k_1, ..., k_m such that $n = k_1 + ... + k_m$.

Examples:

```
(%i1) num_distinct_partitions (12);
(%o1)                          15
(%i2) num_distinct_partitions (12, list);
(%o2)        [1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15]
(%i3) num_distinct_partitions (n);
(%o3)              num_distinct_partitions(n)
```

num_partitions [Function]

num_partitions (*n*)
num_partitions (*n*, *list*)

Returns the number of integer partitions of *n* when *n* is a nonnegative integer. Otherwise, num_partitions returns a noun expression.

num_partitions(*n*, list) returns a list of the number of integer partitions of 1, 2, 3, ..., *n*.

For a nonnegative integer *n*, num_partitions(*n*) is equal to cardinality(integer_partitions(*n*)); however, num_partitions does not actually construct the set of partitions, so it is much faster.

Examples:

```
(%i1) num_partitions (5) = cardinality (integer_partitions (5));
(%o1)                          7 = 7
(%i2) num_partitions (8, list);
(%o2)              [1, 2, 3, 5, 7, 11, 15, 22]
(%i3) num_partitions (n);
(%o3)                    num_partitions(n)
```

partition_set (a, *f*) [Function]

Partitions the set *a* according to the predicate *f*.

partition_set returns a list of two sets. The first set comprises the elements of *a* for which *f* evaluates to false, and the second comprises any other elements of *a*. partition_set does not apply is to the return value of *f*.

partition_set complains if *a* is not a literal set.

See also subset.

Examples:

```
(%i1) partition_set ({2, 7, 1, 8, 2, 8}, evenp);
(%o1)                    [{1, 7}, {2, 8}]
(%i2) partition_set ({x, rat(y), rat(y) + z, 1},
                      lambda ([x], ratp(x)));
(%o2)/R/              [{1, x}, {y, y + z}]
```

permutations (a) [Function]

Returns a set of all distinct permutations of the members of the list or set *a*. Each permutation is a list, not a set.

When *a* is a list, duplicate members of *a* are included in the permutations.

permutations complains if *a* is not a literal list or set.

See also random_permutation.

Examples:

```
(%i1) permutations ([a, a]);
(%o1)                        {[a, a]}
(%i2) permutations ([a, a, b]);
(%o2)           {[a, a, b], [a, b, a], [b, a, a]}
```

powerset [Function]

powerset (a)
powerset (a, n)

Returns the set of all subsets of *a*, or a subset of that set.

`powerset(a)` returns the set of all subsets of the set *a*. `powerset(a)` has `2^cardinality(a)` members.

`powerset(a, n)` returns the set of all subsets of *a* that have cardinality *n*.

`powerset` complains if *a* is not a literal set, or if *n* is not a nonnegative integer.

Examples:

```
(%i1) powerset ({a, b, c});
(%o1) {{}, {a}, {a, b}, {a, b, c}, {a, c}, {b}, {b, c}, {c}}
(%i2) powerset ({w, x, y, z}, 4);
(%o2)                     {{w, x, y, z}}
(%i3) powerset ({w, x, y, z}, 3);
(%o3)     {{w, x, y}, {w, x, z}, {w, y, z}, {x, y, z}}
(%i4) powerset ({w, x, y, z}, 2);
(%o4)   {{w, x}, {w, y}, {w, z}, {x, y}, {x, z}, {y, z}}
(%i5) powerset ({w, x, y, z}, 1);
(%o5)                 {{w}, {x}, {y}, {z}}
(%i6) powerset ({w, x, y, z}, 0);
(%o6)                          {{}}
```

random_permutation (a) [Function]

Returns a random permutation of the set or list *a*, as constructed by the Knuth shuffle algorithm.

The return value is a new list, which is distinct from the argument even if all elements happen to be the same. However, the elements of the argument are not copied.

Examples:

```
(%i1) random_permutation ([a, b, c, 1, 2, 3]);
(%o1)                   [c, 1, 2, 3, a, b]
(%i2) random_permutation ([a, b, c, 1, 2, 3]);
(%o2)                   [b, 3, 1, c, a, 2]
(%i3) random_permutation ({x + 1, y + 2, z + 3});
(%o3)                 [y + 2, z + 3, x + 1]
(%i4) random_permutation ({x + 1, y + 2, z + 3});
(%o4)                 [x + 1, y + 2, z + 3]
```

setdifference (a, b) [Function]

Returns a set containing the elements in the set *a* that are not in the set *b*.

`setdifference` complains if either *a* or *b* is not a literal set.

Examples:

```
(%i1) S_1 : {a, b, c, x, y, z};
(%o1)                   {a, b, c, x, y, z}
(%i2) S_2 : {aa, bb, c, x, y, zz};
(%o2)                 {aa, bb, c, x, y, zz}
```

```
(%i3) setdifference (S_1, S_2);
(%o3)                        {a, b, z}
(%i4) setdifference (S_2, S_1);
(%o4)                       {aa, bb, zz}
(%i5) setdifference (S_1, S_1);
(%o5)                             {}
(%i6) setdifference (S_1, {});
(%o6)                    {a, b, c, x, y, z}
(%i7) setdifference ({}, S_1);
(%o7)                             {}
```

`setequalp` (*a*, *b*) [Function]

Returns `true` if sets *a* and *b* have the same number of elements and `is(`*x* `=` *y*`)` is `true` for `x` in the elements of *a* and `y` in the elements of *b*, considered in the order determined by `listify`. Otherwise, `setequalp` returns `false`.

Examples:

```
(%i1) setequalp ({1, 2, 3}, {1, 2, 3});
(%o1)                           true
(%i2) setequalp ({a, b, c}, {1, 2, 3});
(%o2)                           false
(%i3) setequalp ({x^2 - y^2}, {(x + y) * (x - y)});
(%o3)                           false
```

`setify` (*a*) [Function]

Constructs a set from the elements of the list *a*. Duplicate elements of the list *a* are deleted and the elements are sorted according to the predicate `orderlessp`.

`setify` complains if *a* is not a literal list.

Examples:

```
(%i1) setify ([1, 2, 3, a, b, c]);
(%o1)                    {1, 2, 3, a, b, c}
(%i2) setify ([a, b, c, a, b, c]);
(%o2)                         {a, b, c}
(%i3) setify ([7, 13, 11, 1, 3, 9, 5]);
(%o3)                 {1, 3, 5, 7, 9, 11, 13}
```

`setp` (*a*) [Function]

Returns `true` if and only if *a* is a Maxima set.

`setp` returns `true` for unsimplified sets (that is, sets with redundant members) as well as simplified sets.

`setp` is equivalent to the Maxima function `setp(a) := not atom(a) and op(a) = 'set`.

Examples:

```
(%i1) simp : false;
(%o1)                           false
(%i2) {a, a, a};
(%o2)                         {a, a, a}
```

```
(%i3) setp (%);
(%o3)                          true
```

set_partitions [Function]

set_partitions (a)
set_partitions (a, n)

Returns the set of all partitions of a, or a subset of that set.

set_partitions(a, n) returns a set of all decompositions of a into n nonempty disjoint subsets.

set_partitions(a) returns the set of all partitions.

stirling2 returns the cardinality of the set of partitions of a set.

A set of sets P is a partition of a set S when

1. each member of P is a nonempty set,
2. distinct members of P are disjoint,
3. the union of the members of P equals S.

Examples:

The empty set is a partition of itself, the conditions 1 and 2 being vacuously true.

```
(%i1) set_partitions ({});
(%o1)                          {{}}
```

The cardinality of the set of partitions of a set can be found using stirling2.

```
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) cardinality(p) = stirling2 (6, 3);
(%o3)                        90 = 90
```

Each member of p should have $n = 3$ members; let's check.

```
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (cardinality, p);
(%o3)                           {3}
```

Finally, for each member of p, the union of its members should equal s; again let's check.

```
(%i1) s: {0, 1, 2, 3, 4, 5}$
(%i2) p: set_partitions (s, 3)$
(%i3) map (lambda ([x], apply (union, listify (x))), p);
(%o3)                {{0, 1, 2, 3, 4, 5}}
```

some [Function]

some (f, a)
some (f, L_1, ..., L_n)

Returns true if the predicate f is true for one or more given arguments.

Given one set as the second argument, some(f, s) returns true if is(f(a_i)) returns true for one or more a_i in s. some may or may not evaluate f for all a_i in s. Since sets are unordered, some may evaluate f(a_i) in any order.

Given one or more lists as arguments, `some(f, L_1, ..., L_n)` returns `true` if `is(f(x_1, ..., x_n))` returns `true` for one or more *x_1*, ..., *x_n* in *L_1*, ..., *L_n*, respectively. `some` may or may not evaluate *f* for some combinations *x_1*, ..., *x_n*. `some` evaluates lists in the order of increasing index.

Given an empty set `{}` or empty lists `[]` as arguments, `some` returns `false`.

When the global flag `maperror` is `true`, all lists *L_1*, ..., *L_n* must have equal lengths. When `maperror` is `false`, list arguments are effectively truncated to the length of the shortest list.

Return values of the predicate *f* which evaluate (via `is`) to something other than `true` or `false` are governed by the global flag `prederror`. When `prederror` is `true`, such values are treated as `false`. When `prederror` is `false`, such values are treated as `unknown`.

Examples:

`some` applied to a single set. The predicate is a function of one argument.

```
(%i1) some (integerp, {1, 2, 3, 4, 5, 6});
(%o1)                          true
(%i2) some (atom, {1, 2, sin(3), 4, 5 + y, 6});
(%o2)                          true
```

`some` applied to two lists. The predicate is a function of two arguments.

```
(%i1) some ("=", [a, b, c], [a, b, c]);
(%o1)                          true
(%i2) some ("#", [a, b, c], [a, b, c]);
(%o2)                          false
```

Return values of the predicate *f* which evaluate to something other than `true` or `false` are governed by the global flag `prederror`.

```
(%i1) prederror : false;
(%o1)                          false
(%i2) map (lambda ([a, b], is (a < b)), [x, y, z],
           [x^2, y^2, z^2]);
(%o2)               [unknown, unknown, unknown]
(%i3) some ("<", [x, y, z], [x^2, y^2, z^2]);
(%o3)                          unknown
(%i4) some ("<", [x, y, z], [x^2, y^2, z + 1]);
(%o4)                          true
(%i5) prederror : true;
(%o5)                          true
(%i6) some ("<", [x, y, z], [x^2, y^2, z^2]);
(%o6)                          false
(%i7) some ("<", [x, y, z], [x^2, y^2, z + 1]);
(%o7)                          true
```

`stirling1 (n, m)` [Function]

Represents the Stirling number of the first kind.

When *n* and *m* are nonnegative integers, the magnitude of `stirling1 (n, m)` is the number of permutations of a set with *n* members that have *m* cycles.

`stirling1` is a simplifying function. Maxima knows the following identities:

1. $stirling1(1,k) = kron_delta(1,k), k >= 0$,(see https://dlmf.nist.gov/26.8.E2)
2. $stirling1(n,n) = 1, n >= 0$ (see https://dlmf.nist.gov/26.8.E1)
3. $stirling1(n,n-1) = -binomial(n,2), n >= 1$, (see https://dlmf.nist.gov/26.8.E16)
4. $stirling1(n,0) = kron_delta(n,0), n >= 0$ (see https://dlmf.nist.gov/26.8.E14 and https://dlmf.nist.gov/26.8.E1)
5. $stirling1(n,1) = (-1)^(n-1)(n-1)!, n >= 1$ (see https://dlmf.nist.gov/26.8.E14)
6. $stirling1(n,k) = 0, n >= 0$ and $k > n$.

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. `stirling1` does not simplify for non-integer arguments.

Examples:

```
(%i1) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling1 (n, n);
(%o3)                                1
```

`stirling2 (n, m)` [Function]

Represents the Stirling number of the second kind.

When n and m are nonnegative integers, `stirling2 (n, m)` is the number of ways a set with cardinality n can be partitioned into m disjoint subsets.

`stirling2` is a simplifying function. Maxima knows the following identities.

1. $stirling2(n,0) = 1, n >= 1$ (see https://dlmf.nist.gov/26.8.E17 and stirling2(0,0) = 1)
2. $stirling2(n,n) = 1, n >= 0$, (see https://dlmf.nist.gov/26.8.E4)
3. $stirling2(n,1) = 1, n >= 1$, (see https://dlmf.nist.gov/26.8.E17 and stirling2(0,1) = 0)
4. $stirling2(n,2) = 2^(n-1) - 1, n >= 1$, (see https://dlmf.nist.gov/26.8.E17)
5. $stirling2(n,n-1) = binomial(n,2), n >= 1$ (see https://dlmf.nist.gov/26.8.E16)
6. $stirling2(n,k) = 0, n >= 0$ and $k > n$.

These identities are applied when the arguments are literal integers or symbols declared as integers, and the first argument is nonnegative. `stirling2` does not simplify for non-integer arguments.

Examples:

```
(%i1) declare (n, integer)$
(%i2) assume (n >= 0)$
(%i3) stirling2 (n, n);
(%o3)                                1
```

`stirling2` does not simplify for non-integer arguments.

```
(%i1) stirling2 (%pi, %pi);
(%o1)                  stirling2(%pi, %pi)
```

`subset` (*a*, *f*) [Function]

Returns the subset of the set *a* that satisfies the predicate *f*.

`subset` returns a set which comprises the elements of *a* for which *f* returns anything other than `false`. `subset` does not apply `is` to the return value of *f*.

`subset` complains if *a* is not a literal set.

See also `partition_set`.

Examples:

```
(%i1) subset ({1, 2, x, x + y, z, x + y + z}, atom);
(%o1)                      {1, 2, x, z}
(%i2) subset ({1, 2, 7, 8, 9, 14}, evenp);
(%o2)                       {2, 8, 14}
```

`subsetp` (*a*, *b*) [Function]

Returns `true` if and only if the set *a* is a subset of *b*.

`subsetp` complains if either *a* or *b* is not a literal set.

Examples:

```
(%i1) subsetp ({1, 2, 3}, {a, 1, b, 2, c, 3});
(%o1)                           true
(%i2) subsetp ({a, 1, b, 2, c, 3}, {1, 2, 3});
(%o2)                           false
```

`symmdifference` (*a_1*, ..., *a_n*) [Function]

Returns the symmetric difference of sets *a_1*, ..., *a_n*.

Given two arguments, `symmdifference (a, b)` is the same as `union (setdifference (a, b), setdifference (b, a))`.

`symmdifference` complains if any argument is not a literal set.

Examples:

```
(%i1) S_1 : {a, b, c};
(%o1)                         {a, b, c}
(%i2) S_2 : {1, b, c};
(%o2)                         {1, b, c}
(%i3) S_3 : {a, b, z};
(%o3)                         {a, b, z}
(%i4) symmdifference ();
(%o4)                            {}
(%i5) symmdifference (S_1);
(%o5)                         {a, b, c}
(%i6) symmdifference (S_1, S_2);
(%o6)                          {1, a}
(%i7) symmdifference (S_1, S_2, S_3);
(%o7)                         {1, b, z}
```

```
(%i8) symmdifference ({}, S_1, S_2, S_3);
(%o8)                        {1, b, z}
```

`union` (`a_1`, ..., `a_n`) [Function]

Returns the union of the sets *a_1* through *a_n*.

`union()` (with no arguments) returns the empty set.

`union` complains if any argument is not a literal set.

Examples:

```
(%i1) S_1 : {a, b, c + d, %e};
(%o1)                     {%e, a, b, d + c}
(%i2) S_2 : {%pi, %i, %e, c + d};
(%o2)                  {%e, %i, %pi, d + c}
(%i3) S_3 : {17, 29, 1729, %pi, %i};
(%o3)                 {17, 29, 1729, %i, %pi}
(%i4) union ();
(%o4)                             {}
(%i5) union (S_1);
(%o5)                     {%e, a, b, d + c}
(%i6) union (S_1, S_2);
(%o6)               {%e, %i, %pi, a, b, d + c}
(%i7) union (S_1, S_2, S_3);
(%o7)       {17, 29, 1729, %e, %i, %pi, a, b, d + c}
(%i8) union ({}, S_1, S_2, S_3);
(%o8)       {17, 29, 1729, %e, %i, %pi, a, b, d + c}
```

32 Function Definition

32.1 Introduction to Function Definition

32.2 Function

32.2.1 Ordinary functions

To define a function in Maxima you use the `:=` operator. E.g.

```
f(x) := sin(x)
```

defines a function `f`. Anonymous functions may also be created using `lambda`. For example

```
lambda ([i, j], ...)
```

can be used instead of `f` where

```
f(i,j) := block ([], ...);
map (lambda ([i], i+1), l)
```

would return a list with 1 added to each term.

You may also define a function with a variable number of arguments, by having a final argument which is assigned to a list of the extra arguments:

```
(%i1) f ([u]) := u;
(%o1)                         f([u]) := u
(%i2) f (1, 2, 3, 4);
(%o2)                        [1, 2, 3, 4]
(%i3) f (a, b, [u]) := [a, b, u];
(%o3)                f(a, b, [u]) := [a, b, u]
(%i4) f (1, 2, 3, 4, 5, 6);
(%o4)                   [1, 2, [3, 4, 5, 6]]
```

The right hand side of a function is an expression. Thus if you want a sequence of expressions, you do

```
f(x) := (expr1, expr2, ...., exprn);
```

and the value of *exprn* is what is returned by the function.

If you wish to make a `return` from some expression inside the function then you must use `block` and `return`.

```
block ([], expr1, ..., if (a > 10) then return(a), ..., exprn)
```

is itself an expression, and so could take the place of the right hand side of a function definition. Here it may happen that the return happens earlier than the last expression.

The first `[]` in the block, may contain a list of variables and variable assignments, such as `[a: 3, b, c: []]`, which would cause the three variables `a`,`b`,and `c` to not refer to their global values, but rather have these special values for as long as the code executes inside the `block`, or inside functions called from inside the `block`. This is called *dynamic* binding, since the variables last from the start of the block to the time it exits. Once you return from the `block`, or throw out of it, the old values (if any) of the variables will be restored. It is certainly a good idea to protect your variables in this way. Note that the assignments in the block variables, are done in parallel. This means, that if you had used `c: a` in the

above, the value of `c` would have been the value of `a` at the time you just entered the block, but before `a` was bound. Thus doing something like

```
block ([a: a], expr1, ... a: a+3, ..., exprn)
```

will protect the external value of `a` from being altered, but would let you access what that value was. Thus the right hand side of the assignments, is evaluated in the entering context, before any binding occurs. Using just `block ([x], ...)` would cause the `x` to have itself as value, just as if it would have if you entered a fresh Maxima session.

The actual arguments to a function are treated in exactly same way as the variables in a block. Thus in

```
f(x) := (expr1, ..., exprn);
```

and

```
f(1);
```

we would have a similar context for evaluation of the expressions as if we had done

```
block ([x: 1], expr1, ..., exprn)
```

Inside functions, when the right hand side of a definition, may be computed at runtime, it is useful to use `define` and possibly `buildq`.

32.2.2 Memoizing Functions

A *memoizing function* caches the result the first time it is called with a given argument, and returns the stored value, without recomputing it, when that same argument is given. Memoizing functions are often called *array function* and are in fact handled like arrays in many ways:

The names of memoizing functions are appended to the global list `arrays` (not the global list `functions`). `arrayinfo` returns the list of arguments for which there are stored values, and `listarray` returns the stored values. `dispfun` and `fundef` return the array function definition.

`arraymake` constructs an array function call, analogous to `funmake` for ordinary functions. `arrayapply` applies an array function to its arguments, analogous to `apply` for ordinary functions. There is nothing exactly analogous to `map` for array functions, although `map(lambda([x], a[x]), L)` or `makelist(a[x], x, L)`, where L is a list, are not too far off the mark.

`remarray` removes an array function definition (including any stored function values), analogous to `remfunction` for ordinary functions.

`kill(a[x])` removes the value of the array function a stored for the argument x; the next time a is called with argument x, the function value is recomputed. However, there is no way to remove all of the stored values at once, except for `kill(a)` or `remarray(a)`, which also remove the function definition.

Examples

If evaluating the function needs much time and only a limited number of points is ever evaluated (which means not much time is spent looking up results in a long list of cached results) Memoizing functions can speed up calculations considerably.

```
(%i1) showtime:true$
Evaluation took 0.0000 seconds (0.0000 elapsed) using 32 bytes.
```

```
(%i2) a[x]:=float(sum(sin(x*t),t,1,10000));
Evaluation took 0.0000 seconds (0.0000 elapsed) using 288 bytes.
(%o2)        a  := float(sum(sin(x t), t, 1, 10000))
              x
(%i3) a[1];
Evaluation took 4.6100 seconds (4.6300 elapsed) using 455.060 MB.
(%o3)                  1.6338910217924467
(%i4) a[1];
Evaluation took 0.0000 seconds (0.0000 elapsed) using 24 bytes.
(%o4)                  1.6338910217924467
```

As the memoizing function is only evaluated once for each input value changes in variables the memoizing function uses are not considered for values that are already cached:

```
(%i1) a[x]:=b*x;
(%o1)                        a  := b x
                              x
(%i2) b:1;
(%o2)                           1
(%i3) a[2];
(%o3)                           2
(%i4) b:2;
(%o4)                           2
(%i5) a[1];
(%o5)                           2
(%i6) a[2];
(%o6)                           2
```

32.3 Macros

`buildq` (*L*, *expr*) [Function]

Substitutes variables named by the list *L* into the expression *expr*, in parallel, without evaluating *expr*. The resulting expression is simplified, but not evaluated, after `buildq` carries out the substitution.

The elements of *L* are symbols or assignment expressions *`symbol`*`:` *`value`*, evaluated in parallel. That is, the binding of a variable on the right-hand side of an assignment is the binding of that variable in the context from which `buildq` was called, not the binding of that variable in the variable list *L*. If some variable in *L* is not given an explicit assignment, its binding in `buildq` is the same as in the context from which `buildq` was called.

Then the variables named by *L* are substituted into *expr* in parallel. That is, the substitution for every variable is determined before any substitution is made, so the substitution for one variable has no effect on any other.

If any variable *x* appears as `splice (x)` in *expr*, then *x* must be bound to a list, and the list is spliced (interpolated) into *expr* instead of substituted.

Any variables in *expr* not appearing in *L* are carried into the result verbatim, even if they have bindings in the context from which `buildq` was called.

Examples

`a` is explicitly bound to `x`, while `b` has the same binding (namely 29) as in the calling context, and `c` is carried through verbatim. The resulting expression is not evaluated until the explicit evaluation `''%`.

```
(%i1) (a: 17, b: 29, c: 1729)$
(%i2) buildq ([a: x, b], a + b + c);
(%o2)                       x + c + 29
(%i3) ''%;
(%o3)                        x + 1758
```

`e` is bound to a list, which appears as such in the arguments of `foo`, and interpolated into the arguments of `bar`.

```
(%i1) buildq ([e: [a, b, c]], foo (x, e, y));
(%o1)                  foo(x, [a, b, c], y)
(%i2) buildq ([e: [a, b, c]], bar (x, splice (e), y));
(%o2)                   bar(x, a, b, c, y)
```

The result is simplified after substitution. If simplification were applied before substitution, these two results would be the same.

```
(%i1) buildq ([e: [a, b, c]], splice (e) + splice (e));
(%o1)                     2 c + 2 b + 2 a
(%i2) buildq ([e: [a, b, c]], 2 * splice (e));
(%o2)                         2 a b c
```

The variables in L are bound in parallel; if bound sequentially, the first result would be `foo (b, b)`. Substitutions are carried out in parallel; compare the second result with the result of `subst`, which carries out substitutions sequentially.

```
(%i1) buildq ([a: b, b: a], foo (a, b));
(%o1)                         foo(b, a)
(%i2) buildq ([u: v, v: w, w: x, x: y, y: z, z: u],
               bar (u, v, w, x, y, z));
(%o2)                  bar(v, w, x, y, z, u)
(%i3) subst ([u=v, v=w, w=x, x=y, y=z, z=u],
             bar (u, v, w, x, y, z));
(%o3)                  bar(u, u, u, u, u, u)
```

Construct a list of equations with some variables or expressions on the left-hand side and their values on the right-hand side. `macroexpand` shows the expression returned by `show_values`.

```
(%i1) show_values ([L]) ::= buildq ([L], map ("=", 'L, L));
(%o1)   show_values([L]) ::= buildq([L], map("=", 'L, L))
(%i2) (a: 17, b: 29, c: 1729)$
(%i3) show_values (a, b, c - a - b);
(%o3)          [a = 17, b = 29, c - b - a = 1683]
(%i4) macroexpand (show_values (a, b, c - a - b));
(%o4)    map(=, '([a, b, c - b - a]), [a, b, c - b - a])
```

Given a function of several arguments, create another function for which some of the arguments are fixed.

```
(%i1) curry (f, [a]) :=
        buildq ([f, a], lambda ([[x]], apply (f, append (a, x))))$
(%i2) by3 : curry ("*", 3);
(%o2)        lambda([[x]], apply(*, append([3], x)))
(%i3) by3 (a + b);
(%o3)                        3 (b + a)
```

`macroexpand` (*expr*) [Function]

Returns the macro expansion of *expr* without evaluating it, when `expr` is a macro function call. Otherwise, `macroexpand` returns *expr*.

If the expansion of *expr* yields another macro function call, that macro function call is also expanded.

`macroexpand` quotes its argument. However, if the expansion of a macro function call has side effects, those side effects are executed.

See also `::=`, `macros`, and `macroexpand1`..

Examples

```
(%i1) g (x) ::= x / 99;
                                      x
(%o1)                       g(x) ::= --
                                      99
(%i2) h (x) ::= buildq ([x], g (x - a));
(%o2)            h(x) ::= buildq([x], g(x - a))
(%i3) a: 1234;
(%o3)                           1234
(%i4) macroexpand (h (y));
                                  y - a
(%o4)                             -----
                                   99
(%i5) h (y);
                                y - 1234
(%o5)                           --------
                                   99
```

`macroexpand1` (*expr*) [Function]

Returns the macro expansion of *expr* without evaluating it, when `expr` is a macro function call. Otherwise, `macroexpand1` returns *expr*.

`macroexpand1` quotes its argument. However, if the expansion of a macro function call has side effects, those side effects are executed.

If the expansion of *expr* yields another macro function call, that macro function call is not expanded.

See also `::=`, `macros`, and `macroexpand`.

Examples

```
(%i1) g (x) ::= x / 99;
                                      x
(%o1)                       g(x) ::= --
                                      99
```

```
(%i2) h (x) ::= buildq ([x], g (x - a));
(%o2)            h(x) ::= buildq([x], g(x - a))
(%i3) a: 1234;
(%o3)                          1234
(%i4) macroexpand1 (h (y));
(%o4)                        g(y - a)
(%i5) h (y);
                             y - 1234
(%o5)                        --------
                                99
```

`macros` [Global variable]

Default value: `[]`

`macros` is the list of user-defined macro functions. The macro function definition operator `::=` puts a new macro function onto this list, and `kill`, `remove`, and `remfunction` remove macro functions from the list.

See also `infolists`.

`splice (a)` [Function]

Splices (interpolates) the list named by the atom *a* into an expression, but only if `splice` appears within `buildq`; otherwise, `splice` is treated as an undefined function. If appearing within `buildq` as *a* alone (without `splice`), *a* is substituted (not interpolated) as a list into the result. The argument of `splice` can only be an atom; it cannot be a literal list or an expression which yields a list.

Typically `splice` supplies the arguments for a function or operator. For a function `f`, the expression `f (splice (a))` within `buildq` expands to `f (a[1], a[2], a[3], ...)`. For an operator `o`, the expression `"o" (splice (a))` within `buildq` expands to `"o" (a[1], a[2], a[3], ...)`, where `o` may be any type of operator (typically one which takes multiple arguments). Note that the operator must be enclosed in double quotes `"`.

Examples

```
(%i1) buildq ([x: [1, %pi, z - y]], foo (splice (x)) / length (x));
                        foo(1, %pi, z - y)
(%o1)                  ----------------------
                       length([1, %pi, z - y])
(%i2) buildq ([x: [1, %pi]], "/" (splice (x)));
                                1
(%o2)                           ---
                                %pi
(%i3) matchfix ("<>", "<>");
(%o3)                           <>
(%i4) buildq ([x: [1, %pi, z - y]], "<>" (splice (x)));
(%o4)                    <>1, %pi, z - y<>
```

32.4 Functions and Variables for Function Definition

apply (*F*, [*arg_1*, ..., *arg_n*]) [Function]

Constructs and evaluates an expression *F*(*arg_1*, ..., *arg_n*). The function arguments [*arg_1*, ..., *arg_n*] may be of any length and comprise any expressions. apply evaluates all of its arguments, *F* and *arg_1*, ..., *arg_n* alike, unless evaluation is prevented by quotation.

apply does not attempt to distinguish a memoizing function from an ordinary function; when *F* is the name of a memoizing function, apply evaluates *F*(...) (that is, a function call with parentheses instead of square brackets). arrayapply evaluates a function call with square brackets in this case.

See also funmake and args.

Examples:

The function arguments [*arg_1*, ..., *arg_n*] may be of any length. Here min and "+" are applied to a list L.

```
(%i1) L : [1, 5, -10.2, 4, 3];
(%o1)                 [1, 5, - 10.2, 4, 3]
(%i2) apply (min, L);
(%o2)                         - 10.2
(%i3) apply ("+", L);
(%o3)                  2.8000000000000007
```

apply evaluates all of its arguments, unless evaluation is prevented by quotation. First example: dispfun ordinarily does not evaluate its argument, but we can ensure the evaluation of the argument via apply.

fundef: no such function: fname – an error. To debug this try: debugmode(true);

```
(%i1) F (x) := x / 1729;
                                      x
(%o1)                      F(x) := ----
                                    1729
(%i2) fname : F;
(%o2)                               F
(%i3) dispfun (F);
                                      x
(%t3)                      F(x) := ----
                                    1729

(%o3)                           [%t3]
(%i4) dispfun (fname);
(%i5) apply (dispfun, [fname]);
                                      x
(%t5)                      F(x) := ----
                                    1729

(%o5)                           [%t5]
```

`apply` evaluates all of its arguments, unless evaluation is prevented by quotation. Second example: create a function that declares all of its arguments to be complex.

```
(%i1) g([u]):=apply('declare,[u,complex])$
(%i2) g(a,b,c)$
(%i3) facts();
(%o3) [kind(a, complex), kind(b, complex), kind(c, complex)]
```

`apply` evaluates all of its arguments, unless evaluation is prevented by quotation. Third example: `apply` ordinarily evaluates its first argument, but single quote `'` prevents evaluation. Note that `demoivre` is the name of a global variable and also a function.

apply: found false where a function was expected. – an error. To debug this try: debugmode(true);

```
(%i1) demoivre;
(%o1)                          false
(%i2) demoivre (exp (%i * x));
(%o2)                   %i sin(x) + cos(x)
(%i3) apply (demoivre, [exp (%i * x)]);
(%i4) apply ('demoivre, [exp (%i * x)]);
(%o4)                   %i sin(x) + cos(x)
```

The function arguments `[`*arg_1*`, ..., `*arg_n*`]` may be of any length and comprise any expressions. Convert a nested list into a matrix by calling `apply`.

```
(%i1) a:[[1,2],[3,4]];
(%o1)                     [[1, 2], [3, 4]]
(%i2) apply(matrix,a);
                              [ 1  2 ]
(%o2)                         [      ]
                              [ 3  4 ]
```

`block` [Function]

`block` (`[`*v_1*, ..., *v_m*`]`, *expr_1*, ..., *expr_n*)
`block` (*expr_1*, ..., *expr_n*)

The function `block` allows to make the variables *v_1*, ..., *v_m* to be local for a sequence of commands. If these variables are already bound `block` saves the current values of the variables *v_1*, ..., *v_m* (if any) upon entry to the block, then unbinds the variables so that they evaluate to themselves; The local variables may be bound to arbitrary values within the block but when the block is exited the saved values are restored, and the values assigned within the block are lost.

If there is no need to define local variables then the list at the beginning of the `block` command may be omitted. In this case if neither `return` nor `go` are used `block` behaves similar to the following construct:

```
( expr_1, expr_2,... , expr_n );
```

expr_1, ..., *expr_n* will be evaluated in sequence and the value of the last expression will be returned. The sequence can be modified by the `go`, `throw`, and `return` functions. The last expression is *expr_n* unless `return` or an expression containing `throw` is evaluated.

The declaration `local(v_1, ..., v_m)` within `block` saves the properties associated with the symbols *v_1*, ..., *v_m*, removes any properties before evaluating other expressions, and restores any saved properties on exit from the block. Some declarations are implemented as properties of a symbol, including `:=`, `array`, `dependencies`, `atvalue`, `matchdeclare`, `atomgrad`, `constant`, `nonscalar`, `assume`, and some others. The effect of `local` is to make such declarations effective only within the block; otherwise declarations within a block are actually global declarations.

`block` may appear within another `block`. Local variables are established each time a new `block` is evaluated. Local variables appear to be global to any enclosed blocks. If a variable is non-local in a block, its value is the value most recently assigned by an enclosing block, if any, otherwise, it is the value of the variable in the global environment. This policy may coincide with the usual understanding of "dynamic scope".

The value of the block is the value of the last statement or the value of the argument to the function `return` which may be used to exit explicitly from the block. The function `go` may be used to transfer control to the statement of the block that is tagged with the argument to `go`. To tag a statement, precede it by an atomic argument as another statement in the block. For example: `block ([x], x:1, loop, x: x+1, ..., go(loop), ...)`. The argument to `go` must be the name of a tag appearing within the block. One cannot use `go` to transfer to a tag in a block other than the one containing the `go`.

Blocks typically appear on the right side of a function definition but can be used in other places as well.

See also `return` and `go`.

`break` (*expr_1*, ..., *expr_n*) [Function]

Evaluates and prints *expr_1*, ..., *expr_n* and then causes a Maxima break at which point the user can examine and change his environment. Upon typing `exit;` the computation resumes.

`catch` (*expr_1*, ..., *expr_n*) [Function]

Evaluates *expr_1*, ..., *expr_n* one by one; if any leads to the evaluation of an expression of the form `throw (arg)`, then the value of the `catch` is the value of `throw (arg)`, and no further expressions are evaluated. This "non-local return" thus goes through any depth of nesting to the nearest enclosing `catch`. If there is no `catch` enclosing a `throw`, an error message is printed.

If the evaluation of the arguments does not lead to the evaluation of any `throw` then the value of `catch` is the value of *expr_n*.

```
(%i1) lambda ([x], if x < 0 then throw(x) else f(x))$
(%i2) g(l) := catch (map (''%, l))$
(%i3) g ([1, 2, 3, 7]);
(%o3)               [f(1), f(2), f(3), f(7)]
(%i4) g ([1, 2, -3, 7]);
(%o4)                          - 3
```

The function `g` returns a list of `f` of each element of `l` if `l` consists only of non-negative numbers; otherwise, `g` "catches" the first negative element of `l` and "throws" it up.

compfile [Function]

compfile (*filename*, *f_1*, ..., *f_n*)
compfile (*filename*, *functions*)
compfile (*filename*, *all*)

Translates Maxima functions into Lisp and writes the translated code into the file *filename*.

compfile(*filename*, *f_1*, ..., *f_n*) translates the specified functions. compfile (*filename*, functions) and compfile (*filename*, all) translate all user-defined functions.

The Lisp translations are not evaluated, nor is the output file processed by the Lisp compiler. translate creates and evaluates Lisp translations. compile_file translates Maxima into Lisp, and then executes the Lisp compiler.

See also translate, translate_file, and compile_file.

compile [Function]

compile (*f_1*, ..., *f_n*)
compile (*functions*)
compile (*all*)

Translates Maxima functions *f_1*, ..., *f_n* into Lisp, evaluates the Lisp translations, and calls the Lisp function COMPILE on each translated function. compile returns a list of the names of the compiled functions.

compile (all) or compile (functions) compiles all user-defined functions.

compile quotes its arguments; the quote-quote operator '' defeats quotation.

Compiling a function to native code can mean a big increase in speed and might cause the memory footprint to reduce drastically. Code tends to be especially effective when the flexibility it needs to provide is limited. If compilation doesn't provide the speed that is needed a few ways to limit the code's functionality are the following:

- If the function accesses global variables the complexity of the function can be drastically be reduced by limiting these variables to one data type, for example using mode_declare or a statement like the following one: put(x_1, bigfloat, numerical_type)
- The compiler might warn about undeclared variables if text could either be a named option to a command or (if they are assigned a value to) the name of a variable. Prepending the option with a single quote ' tells the compiler that the text is meant as an option.

define [Function]

define (*f*(*x_1*, ..., *x_n*), *expr*)
define (*f*[*x_1*, ..., *x_n*], *expr*)
define (*f*[*x_1*, ..., *x_n*](*y_1*, ..., *y_m*), *expr*)
define (*funmake* (*f*, [*x_1*, ..., *x_n*]), *expr*)
define (*arraymake* (*f*, [*x_1*, ..., *x_n*]), *expr*)
define (*ev* (*expr_1*), *expr_2*)

Defines a function named *f* with arguments *x_1*, ..., *x_n* and function body *expr*. define always evaluates its second argument (unless explicitly quoted). The function

so defined may be an ordinary Maxima function (with arguments enclosed in parentheses) or a memoizing function (with arguments enclosed in square brackets).

When the last or only function argument *x_n* is a list of one element, the function defined by `define` accepts a variable number of arguments. Actual arguments are assigned one-to-one to formal arguments *x_1*, ..., *x_(n - 1)*, and any further actual arguments, if present, are assigned to *x_n* as a list.

When the first argument of `define` is an expression of the form `f(x_1, ..., x_n)` or `f[x_1, ..., x_n]`, the function arguments are evaluated but *f* is not evaluated, even if there is already a function or variable by that name.

When the first argument is an expression with operator `funmake`, `arraymake`, or `ev`, the first argument is evaluated; this allows for the function name to be computed, as well as the body.

All function definitions appear in the same namespace; defining a function `f` within another function `g` does not automatically limit the scope of `f` to `g`. However, `local(f)` makes the definition of function `f` effective only within the block or other compound expression in which `local` appears.

If some formal argument *x_k* is a quoted symbol (after evaluation), the function defined by `define` does not evaluate the corresponding actual argument. Otherwise all actual arguments are evaluated.

See also `:=` and `::=`.

Examples:

`define` always evaluates its second argument (unless explicitly quoted).

```
(%i1) expr : cos(y) - sin(x);
(%o1)                     cos(y) - sin(x)
(%i2) define (F1 (x, y), expr);
(%o2)              F1(x, y) := cos(y) - sin(x)
(%i3) F1 (a, b);
(%o3)                     cos(b) - sin(a)
(%i4) F2 (x, y) := expr;
(%o4)                    F2(x, y) := expr
(%i5) F2 (a, b);
(%o5)                     cos(y) - sin(x)
```

The function defined by `define` may be an ordinary Maxima function or a memoizing function.

```
(%i1) define (G1 (x, y), x.y - y.x);
(%o1)                G1(x, y) := x . y - y . x
(%i2) define (G2 [x, y], x.y - y.x);
(%o2)                 G2     := x . y - y . x
                       x, y
```

When the last or only function argument *x_n* is a list of one element, the function defined by `define` accepts a variable number of arguments.

```
(%i1) define (H ([L]), '(apply ("+", L)));
(%o1)                H([L]) := apply("+", L)
```

```
(%i2) H (a, b, c);
(%o2)                          c + b + a
```

When the first argument is an expression with operator `funmake`, `arraymake`, or `ev`, the first argument is evaluated.

```
(%i1) [F : I, u : x];
(%o1)                           [I, x]
(%i2) funmake (F, [u]);
(%o2)                            I(x)
(%i3) define (funmake (F, [u]), cos(u) + 1);
(%o3)                    I(x) := cos(x) + 1
(%i4) define (arraymake (F, [u]), cos(u) + 1);
(%o4)                     I  := cos(x) + 1
                           x
(%i5) define (foo (x, y), bar (y, x));
(%o5)                 foo(x, y) := bar(y, x)
(%i6) define (ev (foo (x, y)), sin(x) - cos(y));
(%o6)              bar(y, x) := sin(x) - cos(y)
```

`define_variable` (*name*, *default_value*, *mode*) [Function]

Introduces a global variable into the Maxima environment. `define_variable` is useful in user-written packages, which are often translated or compiled as it gives the compiler hints of the type ("mode") of a variable and therefore avoids requiring it to generate generic code that can deal with every variable being an integer, float, maxima object, array etc.

`define_variable` carries out the following steps:

1. `mode_declare (`*name*`, `*mode*`)` declares the mode ("type") of *name* to the translator which can considerably speed up compiled code as it allows having to create generic code. See `mode_declare` for a list of the possible modes.
2. If the variable is unbound, *default_value* is assigned to *name*.
3. Associates *name* with a test function to ensure that *name* is only assigned values of the declared mode.

The `value_check` property can be assigned to any variable which has been defined via `define_variable` with a mode other than `any`. The `value_check` property is a lambda expression or the name of a function of one variable, which is called when an attempt is made to assign a value to the variable. The argument of the `value_check` function is the would-be assigned value.

`define_variable` evaluates `default_value`, and quotes `name` and `mode`. `define_variable` returns the current value of `name`, which is `default_value` if `name` was unbound before, and otherwise it is the previous value of `name`.

Examples:

`foo` is a Boolean variable, with the initial value `true`.

translator: foo was declared with mode boolean , but it has value: %pi – an error. To debug this try: debugmode(true);

```
(%i1) define_variable (foo, true, boolean);
(%o1)                           true
```

```
(%i2) foo;
(%o2)                          true
(%i3) foo: false;
(%o3)                          false
(%i4) foo: %pi;
(%i5) foo;
(%o5)                          false
```

`bar` is an integer variable, which must be prime.

1440 is not prime. – an error. To debug this try: debugmode(true);

```
(%i1) define_variable (bar, 2, integer);
(%o1)                           2
(%i2) qput (bar, prime_test, value_check);
(%o2)                       prime_test
(%i3) prime_test (y) := if not primep(y) then
                        error (y, "is not prime.");
(%o3) prime_test(y) := if not primep(y)
                                    then error(y, "is not prime.")
(%i4) bar: 1439;
(%o4)                          1439
(%i5) bar: 1440;
#0: prime_test(y=1440)
(%i6) bar;
(%o6)                          1439
```

`baz_quux` is a variable which cannot be assigned a value. The mode `any_check` is like `any`, but `any_check` enables the `value_check` mechanism, and `any` does not.

Cannot assign to ‘baz_quux’. – an error. To debug this try: debugmode(true);

```
(%i1) define_variable (baz_quux, 'baz_quux, any_check);
(%o1)                         baz_quux
(%i2) F: lambda ([y], if y # 'baz_quux then
                 error ("Cannot assign to ‘baz_quux'."));
(%o2) lambda([y], if y # 'baz_quux
                        then error(Cannot assign to ‘baz_quux'.))
(%i3) qput (baz_quux, ''F, value_check);
(%o3) lambda([y], if y # 'baz_quux
                        then error(Cannot assign to ‘baz_quux'.))
(%i4) baz_quux: 'baz_quux;
(%o4)                         baz_quux
(%i5) baz_quux: sqrt(2);
#0: lambda([y],if y # 'baz_quux then error("Cannot assign to ‘baz_quux'."))(y=sqr
(%i6) baz_quux;
(%o6)                         baz_quux
```

`dispfun` [Function]

`dispfun` (*f_1*, ..., *f_n*)
`dispfun` (*all*)

Displays the definition of the user-defined functions *f_1*, ..., *f_n*. Each argument may be the name of a macro (defined with `::=`), an ordinary function (defined with `:=` or `define`), an array function (defined with `:=` or `define`, but enclosing arguments in square brackets `[ ]`), a subscripted function (defined with `:=` or `define`, but enclosing some arguments in square brackets and others in parentheses `( )`), one of a family of subscripted functions selected by a particular subscript value, or a subscripted function defined with a constant subscript.

`dispfun (all)` displays all user-defined functions as given by the `functions`, `arrays`, and `macros` lists, omitting subscripted functions defined with constant subscripts.

`dispfun` creates an intermediate expression label (`%t1`, `%t2`, etc.) for each displayed function, and assigns the function definition to the label. In contrast, `fundef` returns the function definition.

`dispfun` quotes its arguments; the quote-quote operator `''` defeats quotation. `dispfun` returns the list of intermediate expression labels corresponding to the displayed functions.

Examples:

```
(%i1) m(x, y) ::= x^(-y);
                                        - y
(%o1)                    m(x, y) ::= x
(%i2) f(x, y) :=  x^(-y);
                                        - y
(%o2)                     f(x, y) := x
(%i3) g[x, y] :=  x^(-y);
                                      - y
(%o3)                      g     := x
                            x, y
(%i4) h[x](y) :=  x^(-y);
                                      - y
(%o4)                      h (y) := x
                            x
(%i5) i[8](y) :=  8^(-y);
                                      - y
(%o5)                      i (y) := 8
                            8
```

```
(%i6) dispfun (m, f, g, h, h[5], h[10], i[8]);
                                        - y
(%t6)                    m(x, y) ::= x

                                         - y
(%t7)                     f(x, y) := x

                                        - y
(%t8)                      g     := x
                            x, y

                                        - y
(%t9)                      h (y) := x
                            x

                                       1
(%t10)                     h (y) := --
                            5        y
                                     5

                                        1
(%t11)                     h  (y) := ---
                            10         y
                                      10

                                        - y
(%t12)                     i (y) := 8
                            8

(%o12)       [%t6, %t7, %t8, %t9, %t10, %t11, %t12]
(%i13) ''%;
                       - y              - y            - y
(%o13) [m(x, y) ::= x   , f(x, y) := x   , g     := x   ,
                                              x, y
                  - y           1              1             - y
        h (y) := x   , h (y) := --, h  (y) := ---, i (y) := 8   ]
         x             5        y   10         y   8
                                5             10
```

`fullmap` (*f*, *expr_1*, ...) [Function]

Similar to `map`, but `fullmap` keeps mapping down all subexpressions until the main operators are no longer the same.

`fullmap` is used by the Maxima simplifier for certain matrix manipulations; thus, Maxima sometimes generates an error message concerning `fullmap` even though `fullmap` was not explicitly called by the user.

Examples:

```
(%i1) a + b * c;
(%o1)                          b c + a
(%i2) fullmap (g, %);
(%o2)                    g(b) g(c) + g(a)
(%i3) map (g, %th(2));
(%o3)                      g(b c) + g(a)
```

`fullmapl` (*f*, *list_1*, ...) [Function]

Similar to `fullmap`, but `fullmapl` only maps onto lists and matrices.

Example:

```
(%i1) fullmapl ("+", [3, [4, 5]], [[a, 1], [0, -1.5]]);
(%o1)                [[a + 3, 4], [4, 3.5]]
```

`functions` [System variable]

Default value: `[]`

`functions` is the list of ordinary Maxima functions in the current session. An ordinary function is a function constructed by `define` or `:=` and called with parentheses `()`. A function may be defined at the Maxima prompt or in a Maxima file loaded by `load` or `batch`.

`Memoizing functions` (called with square brackets, e.g., `F[x]`) and subscripted functions (called with square brackets and parentheses, e.g., `F[x](y)`) are listed by the global variable `arrays`, and not by `functions`.

Lisp functions are not kept on any list.

Examples:

```
(%i1) F_1 (x) := x - 100;
(%o1)                    F_1(x) := x - 100
(%i2) F_2 (x, y) := x / y;
                                        x
(%o2)                     F_2(x, y) := -
                                        y
(%i3) define (F_3 (x), sqrt (x));
(%o3)                    F_3(x) := sqrt(x)
(%i4) G_1 [x] := x - 100;
(%o4)                     G_1  := x - 100
                             x
(%i5) G_2 [x, y] := x / y;
                                       x
(%o5)                      G_2     := -
                              x, y    y
(%i6) define (G_3 [x], sqrt (x));
(%o6)                     G_3  := sqrt(x)
                             x
(%i7) H_1 [x] (y) := x^y;
                                        y
(%o7)                     H_1 (y) := x
                             x
```

```
(%i8) functions;
(%o8)              [F_1(x), F_2(x, y), F_3(x)]
(%i9) arrays;
(%o9)                  [G_1, G_2, G_3, H_1]
```

`fundef` (*f*) [Function]

Returns the definition of the function *f*.

The argument may be

- the name of a macro (defined with `::=`),
- an ordinary function (defined with `:=` or `define`),
- a memoizing function (defined with `:=` or `define`, but enclosing arguments in square brackets `[ ]`),
- a subscripted function (defined with `:=` or `define`, but enclosing some arguments in square brackets and others in parentheses `( )`),
- one of a family of subscripted functions selected by a particular subscript value,
- or a subscripted function defined with a constant subscript.

`fundef` quotes its argument; the quote-quote operator `''` defeats quotation.

`fundef (f)` returns the definition of *f*. In contrast, `dispfun (f)` creates an intermediate expression label and assigns the definition to the label.

`funmake` (*F*, [*arg_1*, ..., *arg_n*]) [Function]

Returns an expression *F*(*arg_1*, ..., *arg_n*). The return value is simplified, but not evaluated, so the function *F* is not called, even if it exists.

`funmake` does not attempt to distinguish memoizing functions from ordinary functions; when *F* is the name of a memoizing function, `funmake` returns *F*(...) (that is, a function call with parentheses instead of square brackets). `arraymake` returns a function call with square brackets in this case.

`funmake` evaluates its arguments.

See also `apply` and `args`.

Examples:

`funmake` applied to an ordinary Maxima function.

```
(%i1) F (x, y) := y^2 - x^2;
                                      2    2
(%o1)                  F(x, y) := y  - x
(%i2) funmake (F, [a + 1, b + 1]);
(%o2)                    F(a + 1, b + 1)
(%i3) ''%;
                                 2          2
(%o3)                   (b + 1)  - (a + 1)
```

`funmake` applied to a macro.

```
(%i1) G (x) ::= (x - 1)/2;
                                     x - 1
(%o1)                    G(x) ::= -----
                                       2
```

```
(%i2) funmake (G, [u]);
(%o2)                           G(u)
(%i3) ''%;
                                u - 1
(%o3)                           -----
                                  2
```

`funmake` applied to a subscripted function.

```
(%i1) H [a] (x) := (x - 1)^a;
                                                a
(%o1)                    H (x) := (x - 1)
                          a
(%i2) funmake (H [n], [%e]);
                                               n
(%o2)                lambda([x], (x - 1) )(%e)
(%i3) ''%;
                                           n
(%o3)                          (%e - 1)
(%i4) funmake ('(H [n]), [%e]);
(%o4)                           H (%e)
                                 n
(%i5) ''%;
                                           n
(%o5)                          (%e - 1)
```

`funmake` applied to a symbol which is not a defined function of any kind.

```
(%i1) funmake (A, [u]);
(%o1)                           A(u)
(%i2) ''%;
(%o2)                           A(u)
```

`funmake` evaluates its arguments, but not the return value.

```
(%i1) det(a,b,c) := b^2 -4*a*c;
                                           2
(%o1)               det(a, b, c) := b  - 4 a c
(%i2) (x : 8, y : 10, z : 12);
(%o2)                            12
(%i3) f : det;
(%o3)                            det
(%i4) funmake (f, [x, y, z]);
(%o4)                      det(8, 10, 12)
(%i5) ''%;
(%o5)                           - 284
```

Maxima simplifies `funmake`'s return value.

```
(%i1) funmake (sin, [%pi / 2]);
(%o1)                             1
```

`lambda` [Function]

`lambda` ([*x_1*, ..., *x_m*], *expr_1*, ..., *expr_n*)
`lambda` ([[*L*]], *expr_1*, ..., *expr_n*)
`lambda` ([*x_1*, ..., *x_m*, [*L*]], *expr_1*, ..., *expr_n*)

Defines and returns a lambda expression (that is, an anonymous function). The function may have required arguments *x_1*, ..., *x_m* and/or optional arguments *L*, which appear within the function body as a list. The return value of the function is *expr_n*. A lambda expression can be assigned to a variable and evaluated like an ordinary function. A lambda expression may appear in some contexts in which a function name is expected.

When the function is evaluated, unbound local variables *x_1*, ..., *x_m* are created. `lambda` may appear within `block` or another `lambda`; local variables are established each time another `block` or `lambda` is evaluated. Local variables appear to be global to any enclosed `block` or `lambda`. If a variable is not local, its value is the value most recently assigned in an enclosing `block` or `lambda`, if any, otherwise, it is the value of the variable in the global environment. This policy may coincide with the usual understanding of "dynamic scope".

After local variables are established, *expr_1* through *expr_n* are evaluated in turn. The special variable `%%`, representing the value of the preceding expression, is recognized. `throw` and `catch` may also appear in the list of expressions.

`return` cannot appear in a lambda expression unless enclosed by `block`, in which case `return` defines the return value of the block and not of the lambda expression, unless the block happens to be *expr_n*. Likewise, `go` cannot appear in a lambda expression unless enclosed by `block`.

`lambda` quotes its arguments; the quote-quote operator `''` defeats quotation.

Examples:

- A lambda expression can be assigned to a variable and evaluated like an ordinary function.

```
(%i1) f: lambda ([x], x^2);
                                      2
(%o1)                     lambda([x], x )
(%i2) f(a);
                                    2
(%o2)                              a
```

- A lambda expression may appear in contexts in which a function evaluation is expected.

```
(%i1) lambda ([x], x^2) (a);
                                    2
(%o1)                              a
(%i2) apply (lambda ([x], x^2), [a]);
                                    2
(%o2)                              a
(%i3) map (lambda ([x], x^2), [a, b, c, d, e]);
                          2   2   2   2   2
(%o3)                [a , b , c , d , e ]
```

- Argument variables are local variables. Other variables appear to be global variables. Global variables are evaluated at the time the lambda expression is evaluated, unless some special evaluation is forced by some means, such as `''`.

```
(%i1) a: %pi$
(%i2) b: %e$
(%i3) g: lambda ([a], a*b);
(%o3)                    lambda([a], a b)
(%i4) b: %gamma$
(%i5) g(1/2);
                              %gamma
(%o5)                         ------
                                2
(%i6) g2: lambda ([a], a*''b);
(%o6)                  lambda([a], a %gamma)
(%i7) b: %e$
(%i8) g2(1/2);
                              %gamma
(%o8)                         ------
                                2
```

- Lambda expressions may be nested. Local variables within the outer lambda expression appear to be global to the inner expression unless masked by local variables of the same names.

```
(%i1) h: lambda ([a, b], h2: lambda ([a], a*b), h2(1/2));
                                                      1
(%o1)     lambda([a, b], h2 : lambda([a], a b), h2(-))
                                                      2
(%i2) h(%pi, %gamma);
                              %gamma
(%o2)                         ------
                                2
```

- Since `lambda` quotes its arguments, lambda expression `i` below does not define a "multiply by `a`" function. Such a function can be defined via `buildq`, as in lambda expression `i2` below.

```
(%i1) i: lambda ([a], lambda ([x], a*x));
(%o1)              lambda([a], lambda([x], a x))
(%i2) i(1/2);
(%o2)                     lambda([x], a x)
(%i3) i2: lambda([a], buildq([a: a], lambda([x], a*x)));
(%o3)    lambda([a], buildq([a : a], lambda([x], a x)))
(%i4) i2(1/2);
                                      1
(%o4)                  lambda([x], (-) x)
                                      2
```

```
(%i5) i2(1/2)(%pi);
                                %pi
(%o5)                           ---
                                 2
```

- A lambda expression may take a variable number of arguments, which are indicated by `[L]` as the sole or final argument. The arguments appear within the function body as a list.

```
(%i1) f : lambda ([aa, bb, [cc]], aa * cc + bb);
(%o1)          lambda([aa, bb, [cc]], aa cc + bb)
(%i2) f (foo, %i, 17, 29, 256);
(%o2)       [17 foo + %i, 29 foo + %i, 256 foo + %i]
(%i3) g : lambda ([[aa]], apply ("+", aa));
(%o3)             lambda([[aa]], apply(+, aa))
(%i4) g (17, 29, x, y, z, %e);
(%o4)                   z + y + x + %e + 46
```

`local` (*v_1*, ..., *v_n*) [Function]

Saves the properties associated with the symbols *v_1*, ..., *v_n*, removes any properties before evaluating other expressions, and restores any saved properties on exit from the block or other compound expression in which `local` appears.

Some declarations are implemented as properties of a symbol, including `:=`, `array`, `dependencies`, `atvalue`, `matchdeclare`, `atomgrad`, `constant`, `nonscalar`, `assume`, and some others. The effect of `local` is to make such declarations effective only within the block or other compound expression in which `local` appears; otherwise such declarations are global declarations.

`local` can only appear in `block` or in the body of a function definition or `lambda` expression, and only one occurrence is permitted in each.

`local` quotes its arguments. `local` returns `done`.

Example:

A local function definition.

```
(%i1) foo (x) := 1 - x;
(%o1)                    foo(x) := 1 - x
(%i2) foo (100);
(%o2)                           - 99
(%i3) block (local (foo), foo (x) := 2 * x, foo (100));
(%o3)                            200
(%i4) foo (100);
(%o4)                           - 99
```

`macroexpansion` [Option variable]

Default value: `false`

`macroexpansion` controls whether the expansion (that is, the return value) of a macro function is substituted for the macro function call. A substitution may speed up subsequent expression evaluations, at the cost of storing the expansion.

`false` The expansion of a macro function is not substituted for the macro function call.

`expand` The first time a macro function call is evaluated, the expansion is stored. The expansion is not recomputed on subsequent calls; any side effects (such as `print` or assignment to global variables) happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

`displace` The first time a macro function call is evaluated, the expansion is substituted for the call, thus modifying the expression from which the macro function was called. The expansion is not recomputed on subsequent calls; any side effects happen only when the macro function call is first evaluated. Expansion in an expression does not affect other expressions which have the same macro function call.

Examples

When `macroexpansion` is `false`, a macro function is called every time the calling expression is evaluated, and the calling expression is not modified.

```
(%i1) f (x) := h (x) / g (x);
                                     h(x)
(%o1)                       f(x) := ----
                                     g(x)
(%i2) g (x) ::= block (print ("x + 99 is equal to", x),
                       return (x + 99));
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
                                                  return(x + 99))
(%i3) h (x) ::= block (print ("x - 99 is equal to", x),
                       return (x - 99));
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
                                                  return(x - 99))
(%i4) macroexpansion: false;
(%o4)                          false
(%i5) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
                             a b - 99
(%o5)                        --------
                             a b + 99
(%i6) dispfun (f);
                                     h(x)
(%t6)                       f(x) := ----
                                     g(x)

(%o6)                           [%t6]
(%i7) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
                             a b - 99
(%o7)                        --------
                             a b + 99
```

When `macroexpansion` is `expand`, a macro function is called once, and the calling expression is not modified.

```
(%i1) f (x) := h (x) / g (x);
                                      h(x)
(%o1)                      f(x) := ----
                                      g(x)
(%i2) g (x) ::= block (print ("x + 99 is equal to", x),
                       return (x + 99));
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
                                                  return(x + 99))
(%i3) h (x) ::= block (print ("x - 99 is equal to", x),
                       return (x - 99));
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
                                                  return(x - 99))
(%i4) macroexpansion: expand;
(%o4)                          expand
(%i5) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
                              a b - 99
(%o5)                         --------
                              a b + 99
(%i6) dispfun (f);
                         mmacroexpanded(x - 99, h(x))
(%t6)          f(x) := ----------------------------
                         mmacroexpanded(x + 99, g(x))

(%o6)                            [%t6]
(%i7) f (a * b);
                              a b - 99
(%o7)                         --------
                              a b + 99
```

When `macroexpansion` is `displace`, a macro function is called once, and the calling expression is modified.

```
(%i1) f (x) := h (x) / g (x);
                                      h(x)
(%o1)                      f(x) := ----
                                      g(x)
(%i2) g (x) ::= block (print ("x + 99 is equal to", x),
                       return (x + 99));
(%o2) g(x) ::= block(print("x + 99 is equal to", x),
                                                  return(x + 99))
(%i3) h (x) ::= block (print ("x - 99 is equal to", x),
                       return (x - 99));
(%o3) h(x) ::= block(print("x - 99 is equal to", x),
                                                  return(x - 99))
```

```
(%i4) macroexpansion: displace;
(%o4)                         displace
(%i5) f (a * b);
x - 99 is equal to x
x + 99 is equal to x
                              a b - 99
(%o5)                         --------
                              a b + 99
(%i6) dispfun (f);
                                   x - 99
(%t6)                     f(x) := ------
                                   x + 99

(%o6)                            [%t6]
(%i7) f (a * b);
                              a b - 99
(%o7)                         --------
                              a b + 99
```

mode_declare (*y_1*, *mode_1*, ..., *y_n*, *mode_n*) [Function]
modedeclare (*y_1*, *mode_1*, ..., *y_n*, *mode_n*) [Function]

A `mode_declare` informs the compiler which type (lisp programmers name the type: "mode") a function parameter or its return value will be of. This can greatly boost the efficiency of the code the compiler generates: Without knowing the type of all variables and knowing the return value of all functions a function uses in advance very generic (and thus potentially slow) code needs to be generated.

The arguments of `mode_declare` are pairs consisting of a variable (or a list of variables all having the same mode) and a mode. Available modes ("types") are:

`array`	`an declared array (see the detailed description below)`
`boolean`	`true or false`
`integer`	`integers (including arbitrary-size integers)`
`fixnum`	`integers (excluding arbitrary-size integers)`
`float`	`machine-size floating-point numbers`
`real`	`machine-size floating-point or integer`
`number`	`Numbers`
`any`	`any kind of object (useful for arrays of any)`

A function parameter named `a` can be declared as an array filled with elements of the type `t` the following way:

```
mode_declare (a, array(t, dim1, dim2, ...))
```

If none of the elements of the array `a` needs to be checked if it still doesn't contain a value additional code can be omitted by declaring this fact, too:

```
mode_declare (a, array (t, complete, dim1, dim2, ...))
```

The `complete` has no effect if all array elements are of the type `fixnum` or `float`: Machine-sized numbers inevitably contain a value (and will automatically be initialized to 0 in most lisp implementations).

Another way to tell that all entries of the array `a` are of the type ("mode") `m` and have been assigned a value to would be:

```
mode_declare (completearray (a), m))
```

Numeric code using arrays might run faster still if the size of the array is known at compile time, as well, as in:

```
mode_declare (completearray (a [10, 10]), float)
```

for a floating point number array named `a` which is 10 x 10.

`mode_declare` also can be used in order to declare the type of the result of a function. In this case the function compilation needs to be preceded by another `mode_declare` statement. For example the expression,

```
mode_declare ([function (f_1, f_2, ...)], fixnum)
```

declares that the values returned by `f_1`, `f_2`, ... are single-word integers.

`modedeclare` is a synonym for `mode_declare`.

If the type of function parameters and results doesn't match the declaration by `mode_declare` the function may misbehave or a warning or an error might occur, see `mode_checkp`, `mode_check_errorp` and `mode_check_warnp`.

See `mode_identity` for declaring the type of lists and `define_variable` for declaring the type of all global variables compiled code uses, as well.

Example:

```
(%i1) square_float(f):=(
     mode_declare(f,float),
     f*f
 );
(%o1)   square_float(f) := (mode_declare(f, float), f f)
(%i2) mode_declare([function(f)],float);
(%o2)                     [[function(f)]]
(%i3) compile(square_float);
(%o3)                     [square_float]
(%i4) square_float(100.0);
(%o4)                          10000.0
```

`mode_checkp` [Option variable]

Default value: `true`

When `mode_checkp` is `true`, `mode_declare` does not only define which type a variable will be of so the compiler can generate more efficient code, but will also create a runtime warning if the variable isn't of the variable type the code was compiled to deal with.

```
(%i1) mode_checkp:true;
(%o1)                           true
```

```
(%i2) square(f):=(
         mode_declare(f,float),
         f^2);
      compile(square);
      square(2.3);
      square(4);
                                                          2
(%o2)       square(f) := (mode_declare(f, float), f )
```

`mode_check_errorp` [Option variable]

Default value: `false`

When `mode_check_errorp` is `true`, `mode_declare` calls error.

`mode_check_warnp` [Option variable]

Default value: `true`

When `mode_check_warnp` is `true`, mode errors are described.

`mode_identity` (*arg_1*, *arg_2*) [Function]

`mode_identity` works similar to `mode_declare`, but is used for informing the compiler that a thing like a `macro` or a list operation will only return a specific type of object. The purpose of doing so is that maxima supports many objects: Machine integers, arbitrary length integers, equations, machine floats, big floats, which means that for everything that deals with return values of operations that can result in any object the compiler needs to output generic (and therefore potentially slow) code.

The first argument to `mode_identity` is the type of return value something will return (for possible types see `mode_declare`). (i.e., one of `float`, `fixnum`, `number`, The second argument is the expression that will return an object of this type.

If the the return value of this expression is of a type the code was not compiled for error or warning is signalled.

If you knew that `first (l)` returned a number then you could write

```
mode_identity (number, first (l)).
```

However, if you need this construct more often it would be more efficient to define a function that returns a number fist:

```
firstnumb (x) ::= buildq ([x], mode_identity (number, first(x)));
compile(firstnumb)
```

`firstnumb` now can be used every time you need the first element of a list that is guaranteed to be filled with numbers.

`remfunction` [Function]

`remfunction` (*f_1*, ..., *f_n*)
`remfunction` (*all*)

Unbinds the function definitions of the symbols *f_1*, ..., *f_n*. The arguments may be the names of ordinary functions (created by `:=` or `define`) or macro functions (created by `::=`).

`remfunction (all)` unbinds all function definitions.

`remfunction` quotes its arguments.

`remfunction` returns a list of the symbols for which the function definition was unbound. `false` is returned in place of any symbol for which there is no function definition.

`remfunction` does not apply to memoizing functions or subscripted functions. `remarray` applies to those types of functions.

`savedef` [Option variable]

Default value: `true`

When `savedef` is `true`, the Maxima version of a user function is preserved when the function is translated. This permits the definition to be displayed by `dispfun` and allows the function to be edited.

When `savedef` is `false`, the names of translated functions are removed from the `functions` list.

`translate` [Function]

`translate` (*f_1*, ..., *f_n*)
`translate` (*functions*)
`translate` (*all*)

Translates the user-defined functions *f_1*, ..., *f_n* from the Maxima language into Lisp and evaluates the Lisp translations. Typically the translated functions run faster than the originals.

`translate (all)` or `translate (functions)` translates all user-defined functions.

Functions to be translated should include a call to `mode_declare` at the beginning when possible in order to produce more efficient code. For example:

```
f (x_1, x_2, ...) := block ([v_1, v_2, ...],
    mode_declare (v_1, mode_1, v_2, mode_2, ...), ...)
```

where the *x_1*, *x_2*, ... are the parameters to the function and the *v_1*, *v_2*, ... are the local variables.

The names of translated functions are removed from the `functions` list if `savedef` is `false` (see below) and are added to the `props` lists.

Functions should not be translated unless they are fully debugged.

Expressions are assumed simplified; if they are not, correct but non-optimal code gets generated. Thus, the user should not set the `simp` switch to `false` which inhibits simplification of the expressions to be translated.

The switch `translate`, if `true`, causes automatic translation of a user's function to Lisp.

Note that translated functions may not run identically to the way they did before translation as certain incompatibilities may exist between the Lisp and Maxima versions. Principally, the `rat` function with more than one argument and the `ratvars` function should not be used if any variables are `mode_declare`'d canonical rational expressions (CRE). Also the `prederror: false` setting will not translate.

`savedef` - if `true` will cause the Maxima version of a user function to remain when the function is `translate`'d. This permits the definition to be displayed by `dispfun` and allows the function to be edited.

`transrun` - if `false` will cause the interpreted version of all functions to be run (provided they are still around) rather than the translated version.

The result returned by `translate` is a list of the names of the functions translated.

`translate_file` [Function]

`translate_file` (*maxima_filename*)
`translate_file` (*maxima_filename*, *lisp_filename*)

Translates a file of Maxima code into a file of Lisp code. `translate_file` returns a list of three filenames: the name of the Maxima file, the name of the Lisp file, and the name of file containing additional information about the translation. `translate_file` evaluates its arguments.

`translate_file ("foo.mac"); load("foo.LISP")` is the same as the command `batch ("foo.mac")` except for certain restrictions, the use of `''` and `%`, for example.

`translate_file (`*maxima_filename*`)` translates a Maxima file *maxima_filename* into a similarly-named Lisp file. For example, `foo.mac` is translated into `foo.LISP`. The Maxima filename may include a directory name or names, in which case the Lisp output file is written to the same directory from which the Maxima input comes.

`translate_file (`*maxima_filename*`, `*lisp_filename*`)` translates a Maxima file *maxima_filename* into a Lisp file *lisp_filename*. `translate_file` ignores the filename extension, if any, of `lisp_filename`; the filename extension of the Lisp output file is always `LISP`. The Lisp filename may include a directory name or names, in which case the Lisp output file is written to the specified directory.

`translate_file` also writes a file of translator warning messages of various degrees of severity. The filename extension of this file is `UNLISP`. This file may contain valuable information, though possibly obscure, for tracking down bugs in translated code. The `UNLISP` file is always written to the same directory from which the Maxima input comes.

`translate_file` emits Lisp code which causes some declarations and definitions to take effect as soon as the Lisp code is compiled. See `compile_file` for more on this topic.

See also

`tr_array_as_ref`
⟨undefined⟩ `tr_bound_function_applyp`,
`tr_exponent`
⟨undefined⟩ `tr_file_tty_messagesp`,
⟨undefined⟩ `tr_float_can_branch_complex`,
⟨undefined⟩ `tr_function_call_default`,
⟨undefined⟩ `tr_numer`,
⟨undefined⟩ `tr_optimize_max_loop`,
⟨undefined⟩ `tr_state_vars`,
⟨undefined⟩ `tr_warnings_get`,
`tr_warn_bad_function_calls`
⟨undefined⟩ `tr_warn_fexpr`,
⟨undefined⟩ `tr_warn_meval`,
⟨undefined⟩ `tr_warn_mode`,

⟨undefined⟩ tr_warn_undeclared,
and ⟨undefined⟩ tr_warn_undefined_variable.

transrun [Option variable]

Default value: true

When transrun is false will cause the interpreted version of all functions to be run (provided they are still around) rather than the translated version.

tr_array_as_ref [Option variable]

Default value: true

If translate_fast_arrays is false, array references in Lisp code emitted by translate_file are affected by tr_array_as_ref. When tr_array_as_ref is true, array names are evaluated, otherwise array names appear as literal symbols in translated code.

tr_array_as_ref has no effect if translate_fast_arrays is true.

tr_bound_function_applyp [Option variable]

Default value: true

When tr_bound_function_applyp is true and tr_function_call_default is general, if a bound variable (such as a function argument) is found being used as a function then Maxima will rewrite that function call using apply and print a warning message.

For example, if g is defined by g(f,x) := f(x+1) then translating g will cause Maxima to print a warning and rewrite f(x+1) as apply(f,[x+1]).

```
(%i1) f (x) := x^2$
(%i2) g (f) := f (3)$
(%i3) tr_bound_function_applyp : true$
(%i4) translate (g)$
warning: f is a bound variable in f(3), but it is used as a function.
note: instead I'll translate it as: apply(f,[3])
(%i5) g (lambda ([x], x));
(%o5)                          3
(%i6) tr_bound_function_applyp : false$
(%i7) translate (g)$
(%i8) g (lambda ([x], x));
(%o8)                          9
```

tr_file_tty_messagesp [Option variable]

Default value: false

When tr_file_tty_messagesp is true, messages generated by translate_file during translation of a file are displayed on the console and inserted into the UNLISP file. When false, messages about translation of the file are only inserted into the UNLISP file.

tr_float_can_branch_complex [Option variable]

Default value: true

Tells the Maxima-to-Lisp translator to assume that the functions `acos`, `asin`, `asec`, `acsc`, `acosh`, `asech`, `atanh`, `acoth`, `log` and `sqrt` can return complex results.

When it is `true` then `acos(x)` is of mode `any` even if `x` is of mode `float` (as set by `mode_declare`). When `false` then `acos(x)` is of mode `float` if and only if `x` is of mode `float`.

`tr_function_call_default` [Option variable]

Default value: `general`

`false` means give up and call `meval`, `expr` means assume Lisp fixed arg function. `general`, the default gives code good for `mexprs` and `mlexprs` but not `macros`. `general` assures variable bindings are correct in compiled code. In `general` mode, when translating F(X), if F is a bound variable, then it assumes that `apply (f, [x])` is meant, and translates a such, with appropriate warning. There is no need to turn this off. With the default settings, no warning messages implies full compatibility of translated and compiled code with the Maxima interpreter.

`tr_numer` [Option variable]

Default value: `false`

When `tr_numer` is `true`, `numer` properties are used for atoms which have them, e.g. `%pi`.

`tr_optimize_max_loop` [Option variable]

Default value: 100

`tr_optimize_max_loop` is the maximum number of times the macro-expansion and optimization pass of the translator will loop in considering a form. This is to catch macro expansion errors, and non-terminating optimization properties.

`tr_state_vars` [System variable]

Default value:

```
[translate_fast_arrays, tr_function_call_default, tr_bound_function_applyp,
tr_array_as_ref, tr_numer, tr_float_can_branch_complex, define_variable]
```

The list of the switches that affect the form of the translated output. This information is useful to system people when trying to debug the translator. By comparing the translated product to what should have been produced for a given state, it is possible to track down bugs.

`tr_warnings_get ()` [Function]

Prints a list of warnings which have been given by the translator during the current translation.

`tr_warn_bad_function_calls` [Option variable]

Default value: `true`

- Gives a warning when when function calls are being made which may not be correct due to improper declarations that were made at translate time.

`tr_warn_fexpr` [Option variable]

Default value: `compfile`

- Gives a warning if any FEXPRs are encountered. FEXPRs should not normally be output in translated code, all legitimate special program forms are translated.

`tr_warn_meval` [Option variable]

Default value: `compfile`

- Gives a warning if the function `meval` gets called. If `meval` is called that indicates problems in the translation.

`tr_warn_mode` [Option variable]

Default value: `all`

- Gives a warning when variables are assigned values inappropriate for their mode.

`tr_warn_undeclared` [Option variable]

Default value: `compile`

- Determines when to send warnings about undeclared variables to the TTY.

`tr_warn_undefined_variable` [Option variable]

Default value: `all`

- Gives a warning when undefined global variables are seen.

`compile_file` [Function]

`compile_file` (*filename*)
`compile_file` (*filename*, *compiled_filename*)
`compile_file` (*filename*, *compiled_filename*, *lisp_filename*)

Translates the Maxima file *filename* into Lisp, and executes the Lisp compiler. The compiled code is not loaded into Maxima.

`compile_file` returns a list of the names of four files: the original Maxima file, the Lisp translation, notes on translation, and the compiled code. If the compilation fails, the fourth item is `false`.

Some declarations and definitions take effect as soon as the Lisp code is compiled (without loading the compiled code). These include functions defined with the `:=` operator, macros define with the `::=` operator, `alias`, `declare`, `define_variable`, `mode_declare`, and `infix`, `matchfix`, `nofix`, `postfix`, `prefix`, and `compfile`.

Assignments and function calls are not evaluated until the compiled code is loaded. In particular, within the Maxima file, assignments to the translation flags (`tr_numer`, etc.) have no effect on the translation.

filename may not contain `:lisp` statements.

`compile_file` evaluates its arguments.

`declare_translated` (*f_1*, *f_2*, ...) [Function]

When translating a file of Maxima code to Lisp, it is important for the translator to know which functions it sees in the file are to be called as translated or compiled functions, and which ones are just Maxima functions or undefined. Putting this declaration at the top of the file, lets it know that although a symbol does which does not yet have a Lisp function value, will have one at call time. `(MFUNCTION-CALL fn arg1 arg2 ...)` is generated when the translator does not know `fn` is going to be a Lisp function.

33 Program Flow

33.1 Lisp and Maxima

Maxima is a fairly complete programming language. But since it is written in Lisp, it additionally can provide easy access to Lisp functions and variables from Maxima and vice versa. Lisp and Maxima symbols are distinguished by a naming convention. A Lisp symbol which begins with a dollar sign `$` corresponds to a Maxima symbol without the dollar sign.

A Maxima symbol which begins with a question mark `?` corresponds to a Lisp symbol without the question mark. For example, the Maxima symbol `foo` corresponds to the Lisp symbol `$FOO`, while the Maxima symbol `?foo` corresponds to the Lisp symbol `FOO`. Note that `?foo` is written without a space between `?` and `foo`; otherwise it might be mistaken for `describe ("foo")`.

Hyphen `-`, asterisk `*`, or other special characters in Lisp symbols must be escaped by backslash `\` where they appear in Maxima code. For example, the Lisp identifier `*foo-bar*` is written `?\*foo\-bar\*` in Maxima.

Lisp code may be executed from within a Maxima session. A single line of Lisp (containing one or more forms) may be executed by the special command `:lisp`. For example,

```
(%i1) :lisp (foo $x $y)
```

calls the Lisp function `foo` with Maxima variables `x` and `y` as arguments. The `:lisp` construct can appear at the interactive prompt or in a file processed by `batch` or `demo`, but not in a file processed by `load`, `batchload`, `translate_file`, or `compile_file`.

The function `to_lisp` opens an interactive Lisp session. Entering `(to-maxima)` closes the Lisp session and returns to Maxima.

Lisp functions and variables which are to be visible in Maxima as functions and variables with ordinary names (no special punctuation) must have Lisp names beginning with the dollar sign `$`.

Maxima is case-sensitive, distinguishing between lowercase and uppercase letters in identifiers. There are some rules governing the translation of names between Lisp and Maxima.

1. A Lisp identifier not enclosed in vertical bars corresponds to a Maxima identifier in lowercase. Whether the Lisp identifier is uppercase, lowercase, or mixed case, is ignored. E.g., Lisp `$foo`, `$FOO`, and `$Foo` all correspond to Maxima `foo`. But this is because `$foo`, `$FOO` and `$Foo` are converted by the Lisp reader by default to the Lisp symbol `$FOO`.
2. A Lisp identifier which is all uppercase or all lowercase and enclosed in vertical bars corresponds to a Maxima identifier with case reversed. That is, uppercase is changed to lowercase and lowercase to uppercase. E.g., Lisp `|$FOO|` and `|$foo|` correspond to Maxima `foo` and `FOO`, respectively.
3. A Lisp identifier which is mixed uppercase and lowercase and enclosed in vertical bars corresponds to a Maxima identifier with the same case. E.g., Lisp `|$Foo|` corresponds to Maxima `Foo`.

The `#$` Lisp macro allows the use of Maxima expressions in Lisp code. `#$`*expr*`$` expands to a Lisp expression equivalent to the Maxima expression *expr*.

```
(msetq $foo #$[x, y]$)
```

This has the same effect as entering

```
(%i1) foo: [x, y];
```

The Lisp function `displa` prints an expression in Maxima format.

```
(%i1) :lisp #$[x, y, z]$
((MLIST SIMP) $X $Y $Z)
(%i1) :lisp (displa '((MLIST SIMP) $X $Y $Z))
[x, y, z]
NIL
```

Functions defined in Maxima are not ordinary Lisp functions. The Lisp function `mfuncall` calls a Maxima function. For example:

```
(%i1) foo(x,y) := x*y$
(%i2) :lisp (mfuncall '$foo 'a 'b)
((MTIMES SIMP) A B)
```

Some Lisp functions are shadowed in the Maxima package, namely the following.

```
complement      continue      //
float           functionp     array
exp             listen        signum
atan            asin          acos
asinh           acosh         atanh
tanh            cosh          sinh
tan             break         gcd
```

33.2 Garbage Collection

One of the advantages of using lisp is that it uses "Garbage Collection". In other words it automatically takes care of freeing memory occupied for example of intermediate results that were used during symbolic computation.

Garbage Collection avoids many errors frequently found in C programs (memory being freed too early, multiple times or not at all).

`garbage_collect ()` [Function]

Tries to manually trigger the lisp's garbage collection. This rarely is necessary as the lisp will employ an excellent algorithm for determining when to start garbage collection.

If maxima knows how to do manually trigger the garbage collection for the current lisp `garbage_collect` returns `true`, else `false`.

33.3 Introduction to Program Flow

Maxima provides a `do` loop for iteration, as well as more primitive constructs such as `go`.

33.4 Functions and Variables for Program Flow

backtrace [Function]

backtrace ()
backtrace (*n*)

Prints the call stack, that is, the list of functions which called the currently active function.

`backtrace ()` prints the entire call stack.

`backtrace (`*n*`)` prints the *n* most recent functions, including the currently active function.

`backtrace` can be called from a script, a function, or the interactive prompt (not only in a debugging context).

Examples:

- `backtrace ()` prints the entire call stack.

```
(%i1) h(x) := g(x/7)$
(%i2) g(x) := f(x-11)$
(%i3) f(x) := e(x^2)$
(%i4) e(x) := (backtrace(), 2*x + 13)$
(%i5) h(10);
#0: e(x=4489/49)
#1: f(x=-(67/7))
#2: g(x=10/7)
#3: h(x=10)
                                 9615
(%o5)                            ----
                                  49
```

- `backtrace (`*n*`)` prints the *n* most recent functions, including the currently active function.

```
(%i1) h(x) := (backtrace(1), g(x/7))$
(%i2) g(x) := (backtrace(1), f(x-11))$
(%i3) f(x) := (backtrace(1), e(x^2))$
(%i4) e(x) := (backtrace(1), 2*x + 13)$
(%i5) h(10);
#0: h(x=10)
#0: g(x=10/7)
#0: f(x=-(67/7))
#0: e(x=4489/49)
                                 9615
(%o5)                            ----
                                  49
```

do [Special operator]
while [Special operator]
unless [Special operator]
for [Special operator]
from [Special operator]

`thru` [Special operator]
`step` [Special operator]
`next` [Special operator]
`in` [Special operator]

The `do` statement is used for performing iteration. The general form of the `do` statements maxima supports is:

- `for` *variable*`:` *initial_value* `step` *increment* `thru` *limit* `do` *body*
- `for` *variable*`:` *initial_value* `step` *increment* `while` *condition* `do` *body*
- `for` *variable*`:` *initial_value* `step` *increment* `unless` *condition* `do` *body*
- `for` *variable* `in` *list* `do` *body*

If the loop is expected to generate a list as output the command `makelist` may be the appropriate command to use instead, See Section 7.4.3 [Performance considerations for Lists], page 94.

initial_value, *increment*, *limit*, and *body* can be any expression. *list* is a list. If the increment is 1 then `"step 1"` may be omitted; As always, if `body` needs to contain more than one command these commands can be specified as a comma-separated list surrounded by parenthesis or as a `block`. Due to its great generality the `do` statement will be described in two parts. The first form of the `do` statement (which is shown in the first three items above) is analogous to that used in several other programming languages (Fortran, Algol, PL/I, etc.); then the other features will be mentioned.

The execution of the `do` statement proceeds by first assigning the *initial_value* to the *variable* (henceforth called the control-variable). Then: (1) If the control-variable has exceeded the limit of a `thru` specification, or if the condition of the `unless` is `true`, or if the condition of the `while` is `false` then the `do` terminates. (2) The *body* is evaluated. (3) The increment is added to the control-variable. The process from (1) to (3) is performed repeatedly until the termination condition is satisfied. One may also give several termination conditions in which case the `do` terminates when any of them is satisfied.

In general the `thru` test is satisfied when the control-variable is greater than the *limit* if the *increment* was non-negative, or when the control-variable is less than the *limit* if the *increment* was negative. The *increment* and *limit* may be non-numeric expressions as long as this inequality can be determined. However, unless the *increment* is syntactically negative (e.g. is a negative number) at the time the `do` statement is input, Maxima assumes it will be positive when the `do` is executed. If it is not positive, then the `do` may not terminate properly.

Note that the *limit*, *increment*, and termination condition are evaluated each time through the loop. Thus if any of these involve much computation, and yield a result that does not change during all the executions of the *body*, then it is more efficient to set a variable to their value prior to the `do` and use this variable in the `do` form.

The value normally returned by a `do` statement is the atom `done`. However, the function `return` may be used inside the *body* to exit the `do` prematurely and give it any desired value. Note however that a `return` within a `do` that occurs in a `block` will exit only the `do` and not the `block`. Note also that the `go` function may not be used to exit from a `do` into a surrounding `block`.

The control-variable is always local to the `do` and thus any variable may be used without affecting the value of a variable with the same name outside of the `do`. The control-variable is unbound after the `do` terminates.

```
(%i1) for a:-3 thru 26 step 7 do display(a)$
                              a = - 3

                               a = 4

                              a = 11

                              a = 18

                              a = 25


(%i1) s: 0$
(%i2) for i: 1 while i <= 10 do s: s+i;
(%o2)                          done
(%i3) s;
(%o3)                           55
```

Note that the condition `while i <= 10` is equivalent to `unless i > 10` and also `thru 10`.

```
(%i1) series: 1$
(%i2) term: exp (sin (x))$
(%i3) for p: 1 unless p > 7 do
    (term: diff (term, x)/p,
     series: series + subst (x=0, term)*x^p)$
(%i4) series;
                   7    6     5    4    2
                  x    x     x    x    x
(%o4)             -- - --- - -- - -- + -- + x + 1
                  90   240   15   8    2
```

which gives 8 terms of the Taylor series for `e^sin(x)`.

```
(%i1) poly: 0$
(%i2) for i: 1 thru 5 do
    for j: i step -1 thru 1 do
        poly: poly + i*x^j$
(%i3) poly;
                    5      4       3       2
(%o3)           5 x  + 9 x  + 12 x  + 14 x  + 15 x
(%i4) guess: -3.0$
(%i5) for i: 1 thru 10 do
    (guess: subst (guess, x, 0.5*(x + 10/x)),
    if abs (guess^2 - 10) < 0.00005 then return (guess));
(%o5)                   - 3.162280701754386
```

This example computes the negative square root of 10 using the Newton- Raphson iteration a maximum of 10 times. Had the convergence criterion not been met the value returned would have been `done`.

Instead of always adding a quantity to the control-variable one may sometimes wish to change it in some other way for each iteration. In this case one may use `next` *`expression`* instead of `step` *`increment`*. This will cause the control-variable to be set to the result of evaluating *expression* each time through the loop.

```
(%i1) for count: 2 next 3*count thru 20 do display (count)$
                              count = 2

                              count = 6

                             count = 18
```

As an alternative to `for` *`variable`*`:` *`value`* `...do...` the syntax `for` *`variable`* `from` *`value`* `...do...` may be used. This permits the `from` *`value`* to be placed after the `step` or `next` value or after the termination condition. If `from` *`value`* is omitted then 1 is used as the initial value.

Sometimes one may be interested in performing an iteration where the control-variable is never actually used. It is thus permissible to give only the termination conditions omitting the initialization and updating information as in the following example to compute the square-root of 5 using a poor initial guess.

```
(%i1) x: 1000$
(%i2) thru 20 do x: 0.5*(x + 5.0/x)$
(%i3) x;
(%o3)                    2.23606797749979
(%i4) sqrt(5), numer;
(%o4)                    2.23606797749979
```

If it is desired one may even omit the termination conditions entirely and just give `do` *`body`* which will continue to evaluate the *body* indefinitely. In this case the function `return` should be used to terminate execution of the `do`.

```
(%i1) newton (f, x):= ([y, df, dfx], df: diff (f ('x), 'x),
    do (y: ev(df), x: x - f(x)/y,
        if abs (f (x)) < 5e-6 then return (x)))$
warning: parser: I'll let it stand, but (...) doesn't recognize local variables.
warning: parser: did you mean to say: block([y, df, dfx], ...) ?
(%i2) sqr (x) := x^2 - 5.0$
(%i3) newton (sqr, 1000);
(%o3)                   2.2360680270621947
```

(Note that `return`, when executed, causes the current value of `x` to be returned as the value of the `do`. The `block` is exited and this value of the `do` is returned as the value of the `block` because the `do` is the last statement in the block.)

One other form of the `do` is available in Maxima. The syntax is:

`for` *`variable`* `in` *`list`* *`end_tests`* `do` *`body`*

The elements of *list* are any expressions which will successively be assigned to the `variable` on each iteration of the *body*. The optional termination tests *end_tests* can be used to terminate execution of the `do`; otherwise it will terminate when the *list* is exhausted or when a `return` is executed in the *body*. (In fact, `list` may be any non-atomic expression, and successive parts are taken.)

```
(%i1) for f in [log, rho, atan] do ldisp(f(1))$
(%t1)                                0

(%t2)                           rho(1)

                                  %pi
(%t3)                             ---
                                   4

(%i4) ev(%t3,numer);
(%o4)                    0.7853981633974483
```

`errcatch` (*expr_1*, ..., *expr_n*) [Function]

Evaluates *expr_1*, ..., *expr_n* one by one and returns `[`*expr_n*`]` (a list) if no error occurs. If an error occurs in the evaluation of any argument, `errcatch` prevents the error from propagating and returns the empty list `[]` without evaluating any more arguments.

`errcatch` is useful in `batch` files where one suspects an error might occur which would terminate the `batch` if the error weren't caught.

See also `errormsg`.

`error` (*expr_1*, ..., *expr_n*) [Function]

`error` [System variable]

Evaluates and prints *expr_1*, ..., *expr_n*, and then causes an error return to top level Maxima or to the nearest enclosing `errcatch`.

The variable `error` is set to a list describing the error. The first element of `error` is a format string, which merges all the strings among the arguments *expr_1*, ..., *expr_n*, and the remaining elements are the values of any non-string arguments.

`errormsg()` formats and prints `error`. This is effectively reprinting the most recent error message.

`warning` (*expr_1*, ..., *expr_n*) [Function]

Evaluates and prints *expr_1*, ..., *expr_n*, as a warning message that is formatted in a standard way so a maxima front-end may be able to recognize the warning and to format it accordingly.

The function `warning` always returns false.

`error_size` [Option variable]

Default value: 60

`error_size` modifies error messages according to the size of expressions which appear in them. If the size of an expression (as determined by the Lisp function `ERROR-SIZE`) is greater than `error_size`, the expression is replaced in the message by a symbol,

and the symbol is assigned the expression. The symbols are taken from the list error_syms.

Otherwise, the expression is smaller than error_size, and the expression is displayed in the message.

See also error and error_syms.

Example:

The size of U, as determined by ERROR-SIZE, is 24.

```
(%i1) U: (C^D^E + B + A)/(cos(X-1) + 1)$
(%i2) error_size: 20$
(%i3) error ("Example expression is", U);
Example expression is errexp1
 -- an error. To debug this try: debugmode(true);
(%i4) errexp1;
                              E
                             D
                            C   + B + A
(%o4)                       --------------
                            cos(X - 1) + 1
(%i5) error_size: 30$
(%i6) error ("Example expression is", U);

                            E
                           D
                          C   + B + A
Example expression is --------------
                         cos(X - 1) + 1
 -- an error.  Quitting.  To debug this try debugmode(true);
```

error_syms [Option variable]

Default value: [errexp1, errexp2, errexp3]

In error messages, expressions larger than error_size are replaced by symbols, and the symbols are set to the expressions. The symbols are taken from the list error_syms. The first too-large expression is replaced by error_syms[1], the second by error_syms[2], and so on.

If there are more too-large expressions than there are elements of error_syms, symbols are constructed automatically, with the *n*-th symbol equivalent to concat ('errexp, *n*).

See also error and error_size.

errormsg () [Function]

Reprints the most recent error message. The variable error holds the message, and errormsg formats and prints it.

errormsg [Option variable]

Default value: true

When false the output of error messages is suppressed.

The option variable errormsg can not be set in a block to a local value. The global value of errormsg is always present.

sin(y): expected exactly 1 arguments but got 2: [a, b] – an error. To debug this try: debugmode(true); – an error. To debug this try: debugmode(true);

```
(%i1) errormsg;
(%o1)                          true
(%i2) sin(a,b);
(%i3) errormsg: false;
(%o3)                          false
(%i4) sin(a,b);
```

The option variable errormsg can not be set in a block to a local value.

```
(%i1) f(bool):=block([errormsg:bool],
                     print ("value of errormsg is",errormsg))$
(%i2) errormsg:true;
(%o2)                          true
(%i3) f(false);
value of errormsg is false
(%o3)                          false
(%i4) errormsg:false;
(%o4)                          false
(%i5) f(true);
value of errormsg is true
(%o5)                          true
```

go (*tag*) [Function]

is used within a block to transfer control to the statement of the block which is tagged with the argument to go. To tag a statement, precede it by an atomic argument as another statement in the block. For example:

```
block ([x], x:1, loop, x+1, ..., go(loop), ...)
```

The argument to go must be the name of a tag appearing in the same block. One cannot use go to transfer to tag in a block other than the one containing the go.

if [Special operator]

Represents conditional evaluation. Various forms of if expressions are recognized.

if *cond_1* then *expr_1* else *expr_0* evaluates to *expr_1* if *cond_1* evaluates to true, otherwise the expression evaluates to *expr_0*.

The command if *cond_1* then *expr_1* elseif *cond_2* then *expr_2* elseif ... else *expr_0* evaluates to *expr_k* if *cond_k* is true and all preceding conditions are false. If none of the conditions are true, the expression evaluates to expr_0.

A trailing else false is assumed if else is missing. That is, the command if *cond_1* then *expr_1* is equivalent to if *cond_1* then *expr_1* else false, and the command if *cond_1* then *expr_1* elseif ... elseif *cond_n* then *expr_n* is equivalent to if *cond_1* then *expr_1* elseif ... elseif *cond_n* then *expr_n* else false.

The alternatives *expr_0*, ..., *expr_n* may be any Maxima expressions, including nested if expressions. The alternatives are neither simplified nor evaluated unless the corresponding condition is true.

The conditions *cond_1*, ..., *cond_n* are expressions which potentially or actually evaluate to `true` or `false`. When a condition does not actually evaluate to `true` or `false`, the behavior of `if` is governed by the global flag `prederror`. When `prederror` is `true`, it is an error if any evaluated condition does not evaluate to `true` or `false`. Otherwise, conditions which do not evaluate to `true` or `false` are accepted, and the result is a conditional expression.

Among other elements, conditions may comprise relational and logical operators as follows.

Operation	**Symbol**	**Type**
less than	`<`	relational infix
less than or equal to	`<=`	relational infix
equality (syntactic)	`=`	relational infix
equality (value)	`equal`	relational function
negation of equal	`notequal`	relational function
greater than or equal to	`>=`	relational infix
greater than	`>`	relational infix
and	`and`	logical infix
or	`or`	logical infix
not	`not`	logical infix

`map` (*f*, *expr_1*, ..., *expr_n*) [Function]

Returns an expression whose leading operator is the same as that of the expressions *expr_1*, ..., *expr_n* but whose subparts are the results of applying *f* to the corresponding subparts of the expressions. *f* is either the name of a function of *n* arguments or is a `lambda` form of *n* arguments.

`maperror` - if `false` will cause all of the mapping functions to (1) stop when they finish going down the shortest *expr_i* if not all of the *expr_i* are of the same length and (2) apply *f* to [*expr_1*, *expr_2*, ...] if the *expr_i* are not all the same type of object. If `maperror` is `true` then an error message will be given in the above two instances.

One of the uses of this function is to `map` a function (e.g. `partfrac`) onto each term of a very large expression where it ordinarily wouldn't be possible to use the function on the entire expression due to an exhaustion of list storage space in the course of the computation.

See also `scanmap`, `maplist`, `outermap`, `matrixmap` and `apply`.

```
(%i1) map(f,x+a*y+b*z);
(%o1)                 f(b z) + f(a y) + f(x)
(%i2) map(lambda([u],partfrac(u,x)),x+1/(x^3+4*x^2+5*x+2));
                      1       1         1
(%o2)              ----- - ----- + -------- + x
                   x + 2   x + 1           2
                                    (x + 1)
```

```
(%i3) map(ratsimp, x/(x^2+x)+(y^2+y)/y);
                                    1
(%o3)                        y + ----- + 1
                                  x + 1
(%i4) map("=",[a,b],[-0.5,3]);
(%o4)                     [a = - 0.5, b = 3]
```

`mapatom` (*expr*) [Function]

Returns `true` if and only if *expr* is treated by the mapping routines as an atom. "Mapatoms" are atoms, numbers (including rational numbers), subscripted variables and structure references.

`maperror` [Option variable]

Default value: `true`

When `maperror` is `false`, causes all of the mapping functions, for example

```
map (f, expr_1, expr_2, ...)
```

to (1) stop when they finish going down the shortest *expr_i* if not all of the *expr_i* are of the same length and (2) apply *f* to [*expr_1*, *expr_2*, ...] if the *expr_i* are not all the same type of object.

If `maperror` is `true` then an error message is displayed in the above two instances.

`mapprint` [Option variable]

Default value: `true`

When `mapprint` is `true`, various information messages from `map`, `maplist`, and `fullmap` are produced in certain situations. These include situations where `map` would use `apply`, or `map` is truncating on the shortest list.

If `mapprint` is `false`, these messages are suppressed.

`maplist` (*f*, *expr_1*, ..., *expr_n*) [Function]

Returns a list of the applications of *f* to the parts of the expressions *expr_1*, ..., *expr_n*. *f* is the name of a function, or a lambda expression.

`maplist` differs from `map(f, expr_1, ..., expr_n)` which returns an expression with the same main operator as *expr_i* has (except for simplifications and the case where `map` does an `apply`).

`prederror` [Option variable]

Default value: `false`

When `prederror` is `true`, an error message is displayed whenever the predicate of an `if` statement or an `is` function fails to evaluate to either `true` or `false`.

If `false`, `unknown` is returned instead in this case. The `prederror: false` mode is not supported in translated code; however, `maybe` is supported in translated code.

See also `is` and `maybe`.

`return` (*value*) [Function]

May be used to exit explicitly from the current `block`, `while`, `for` or `do` loop bringing its argument. It therefore can be compared with the `return` statement found in other programming languages but it yields one difference: In maxima only returns from the

current block, not from the entire function it was called in. In this aspect it more closely resembles the `break` statement from C.

```
(%i1) for i:1 thru 10 do o:i;
(%o1)                          done
(%i2) for i:1 thru 10 do if i=3 then return(i);
(%o2)                           3
(%i3) for i:1 thru 10 do
    (
        block([i],
            i:3,
            return(i)
        ),
        return(8)
    );
(%o3)                           8
(%i4) block([i],
    i:4,
    block([o],
        o:5,
        return(o)
    ),
    return(i),
    return(10)
 );
(%o4)                           4
```

See also `for`, `while`, `do` and `block`.

`scanmap` [Function]

`scanmap` (*f*, *expr*)
`scanmap` (*f*, *expr*, *bottomup*)

`scanmap` (*f*, *expr*, topdown)

Recursively applies *f* to *expr*, in a top down manner. This is most useful when complete factorization is desired, for example:

```
(%i1) exp:(a^2+2*a+1)*y + x^2$
(%i2) scanmap(factor,exp);
                                  2      2
(%o2)                      (a + 1)  y + x
```

Note the way in which `scanmap` applies the given function `factor` to the constituent subexpressions of *expr*; if another form of *expr* is presented to `scanmap` then the result may be different. Thus, `%o2` is not recovered when `scanmap` is applied to the expanded form of `exp`:

```
(%i1) scanmap(factor,expand(exp));
(%o1)                          exp
```

Here is another example of the way in which `scanmap` recursively applies a given function to all subexpressions, including exponents:

```
(%i1) expr : u*v^(a*x+b) + c$
(%i2) scanmap('f, expr);
                          f(f(f(a) f(x)) + f(b))
(%o2)    f(f(f(u) f(f(v                        )) + f(c))
```

`scanmap (f, expr, bottomup)` applies f to *expr* in a bottom-up manner. E.g., for undefined `f`,

```
scanmap(f,a*x+b) ->
   f(a*x+b) -> f(f(a*x)+f(b)) -> f(f(f(a)*f(x))+f(b))
scanmap(f,a*x+b,bottomup) -> f(a)*f(x)+f(b)
    -> f(f(a)*f(x))+f(b) ->
     f(f(f(a)*f(x))+f(b))
```

In this case, you get the same answer both ways.

`scanmap (f, expr, topdown)` has the same effect as calling `scanmap (f, expr)`.

`throw (expr)` [Function]

Evaluates *expr* and throws the value back to the most recent `catch`. `throw` is used with `catch` as a nonlocal return mechanism.

`outermap (f, a_1, ..., a_n)` [Function]

Applies the function f to each one of the elements of the outer product *a_1* cross *a_2* ... cross *a_n*.

f is the name of a function of n arguments or a lambda expression of n arguments. Each argument *a_k* may be a list or nested list, or a matrix, or any other kind of expression.

The `outermap` return value is a nested structure. Let x be the return value. Then x has the same structure as the first list, nested list, or matrix argument, `x[i_1]...[i_m]` has the same structure as the second list, nested list, or matrix argument, `x[i_1]...[i_m][j_1]...[j_n]` has the same structure as the third list, nested list, or matrix argument, and so on, where m, n, ... are the numbers of indices required to access the elements of each argument (one for a list, two for a matrix, one or more for a nested list). Arguments which are not lists or matrices have no effect on the structure of the return value.

Note that the effect of `outermap` is different from that of applying f to each one of the elements of the outer product returned by `cartesian_product`. `outermap` preserves the structure of the arguments in the return value, while `cartesian_product` does not.

`outermap` evaluates its arguments.

See also `map`, `maplist`, and `apply`.

Examples:

Elementary examples of `outermap`. To show the argument combinations more clearly, `F` is left undefined.

```
(%i1) outermap (F, [a, b, c], [1, 2, 3]);
(%o1) [[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)],
                                    [F(c, 1), F(c, 2), F(c, 3)]]
```

```
(%i2) outermap (F, matrix ([a, b], [c, d]), matrix ([1, 2], [3, 4]));
        [ [ F(a, 1)  F(a, 2) ]  [ F(b, 1)  F(b, 2) ] ]
        [ [                  ]  [                  ] ]
        [ [ F(a, 3)  F(a, 4) ]  [ F(b, 3)  F(b, 4) ] ]
(%o2)   [                                            ]
        [ [ F(c, 1)  F(c, 2) ]  [ F(d, 1)  F(d, 2) ] ]
        [ [                  ]  [                  ] ]
        [ [ F(c, 3)  F(c, 4) ]  [ F(d, 3)  F(d, 4) ] ]
(%i3) outermap (F, [a, b], x, matrix ([1, 2], [3, 4]));
       [ F(a, x, 1)  F(a, x, 2) ]  [ F(b, x, 1)  F(b, x, 2) ]
(%o3) [[                        ], [                        ]]
       [ F(a, x, 3)  F(a, x, 4) ]  [ F(b, x, 3)  F(b, x, 4) ]
(%i4) outermap (F, [a, b], matrix ([1, 2]), matrix ([x], [y]));
       [ [ F(a, 1, x) ]  [ F(a, 2, x) ] ]
(%o4) [[ [            ]  [            ] ],
       [ [ F(a, 1, y) ]  [ F(a, 2, y) ] ]
                           [ [ F(b, 1, x) ]  [ F(b, 2, x) ] ]
                           [ [            ]  [            ] ]]
                           [ [ F(b, 1, y) ]  [ F(b, 2, y) ] ]
(%i5) outermap ("+", [a, b, c], [1, 2, 3]);
(%o5) [[a + 1, a + 2, a + 3], [b + 1, b + 2, b + 3],
                                        [c + 1, c + 2, c + 3]]
```

A closer examination of the `outermap` return value. The first, second, and third arguments are a matrix, a list, and a matrix, respectively. The return value is a matrix. Each element of that matrix is a list, and each element of each list is a matrix.

```
(%i1) arg_1 :  matrix ([a, b], [c, d]);
                           [ a  b ]
(%o1)                      [      ]
                           [ c  d ]
(%i2) arg_2 : [11, 22];
(%o2)                      [11, 22]
(%i3) arg_3 : matrix ([xx, yy]);
(%o3)                     [ xx  yy ]
```

```
(%i4) xx_0 : outermap (lambda ([x, y, z], x / y + z), arg_1,
                                                    arg_2, arg_3);
                [  [      a        a  ]  [      a        a  ]  ]
                [ [[ xx + --  yy + -- ], [ xx + --  yy + -- ]] ]
                [  [      11       11 ]  [      22       22 ]  ]
(%o4)  Col 1 = [                                               ]
                [  [      c        c  ]  [      c        c  ]  ]
                [ [[ xx + --  yy + -- ], [ xx + --  yy + -- ]] ]
                [  [      11       11 ]  [      22       22 ]  ]
                  [  [      b        b  ]  [      b        b  ]  ]
                  [ [[ xx + --  yy + -- ], [ xx + --  yy + -- ]] ]
                  [  [      11       11 ]  [      22       22 ]  ]
          Col 2 = [                                               ]
                  [  [      d        d  ]  [      d        d  ]  ]
                  [ [[ xx + --  yy + -- ], [ xx + --  yy + -- ]] ]
                  [  [      11       11 ]  [      22       22 ]  ]
(%i5) xx_1 : xx_0 [1][1];
            [      a        a  ]  [      a        a  ]
(%o5)      [[ xx + --  yy + -- ], [ xx + --  yy + -- ]]
            [      11       11 ]  [      22       22 ]
(%i6) xx_2 : xx_0 [1][1] [1];
                       [      a        a  ]
(%o6)                  [ xx + --  yy + -- ]
                       [      11       11 ]
(%i7) xx_3 : xx_0 [1][1] [1] [1][1];
                                  a
(%o7)                        xx + --
                                  11
(%i8) [op (arg_1), op (arg_2), op (arg_3)];
(%o8)                    [matrix, [, matrix]
(%i9) [op (xx_0), op (xx_1), op (xx_2)];
(%o9)                    [matrix, [, matrix]
```

`outermap` preserves the structure of the arguments in the return value, while `cartesian_product` does not.

```
(%i1) outermap (F, [a, b, c], [1, 2, 3]);
(%o1) [[F(a, 1), F(a, 2), F(a, 3)], [F(b, 1), F(b, 2), F(b, 3)],
                                     [F(c, 1), F(c, 2), F(c, 3)]]
(%i2) setify (flatten (%));
(%o2) {F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3),
                                       F(c, 1), F(c, 2), F(c, 3)}
(%i3) map (lambda ([L], apply (F, L)),
                       cartesian_product ({a, b, c}, {1, 2, 3}));
(%o3) {F(a, 1), F(a, 2), F(a, 3), F(b, 1), F(b, 2), F(b, 3),
                                       F(c, 1), F(c, 2), F(c, 3)}
(%i4) is (equal (%, %th (2)));
(%o4)                          true
```

34 Debugging

34.1 Source Level Debugging

Maxima has a built-in source level debugger. The user can set a breakpoint at a function, and then step line by line from there. The call stack may be examined, together with the variables bound at that level.

The command `:help` or `:h` shows the list of debugger commands. (In general, commands may be abbreviated if the abbreviation is unique. If not unique, the alternatives will be listed.) Within the debugger, the user can also use any ordinary Maxima functions to examine, define, and manipulate variables and expressions.

A breakpoint is set by the `:br` command at the Maxima prompt. Within the debugger, the user can advance one line at a time using the `:n` ("next") command. The `:bt` ("backtrace") command shows a list of stack frames. The `:r` ("resume") command exits the debugger and continues with execution. These commands are demonstrated in the example below.

```
(%i1) load ("/tmp/foobar.mac");

(%o1)                           /tmp/foobar.mac

(%i2) :br foo
Turning on debugging debugmode(true)
Bkpt 0 for foo (in /tmp/foobar.mac line 1)

(%i2) bar (2,3);
Bkpt 0:(foobar.mac 1)
/tmp/foobar.mac:1::

(dbm:1) :bt                          <-- :bt typed here gives a backtrace
#0: foo(y=5)(foobar.mac line 1)
#1: bar(x=2,y=3)(foobar.mac line 9)

(dbm:1) :n                           <-- Here type :n to advance line
(foobar.mac 2)
/tmp/foobar.mac:2::

(dbm:1) :n                           <-- Here type :n to advance line
(foobar.mac 3)
/tmp/foobar.mac:3::

(dbm:1) u;                           <-- Investigate value of u
28

(dbm:1) u: 33;                       <-- Change u to be 33
33
```

```
(dbm:1) :r                          <-- Type :r to resume the computation

(%o2)                                 1094
```

The file `/tmp/foobar.mac` is the following:

```
foo(y) := block ([u:y^2],
  u: u+3,
  u: u^2,
  u);

bar(x,y) := (
  x: x+2,
  y: y+2,
  x: foo(y),
  x+y);
```

USE OF THE DEBUGGER THROUGH EMACS

If the user is running the code under GNU emacs in a shell window (dbl shell), or is running the graphical interface version, Xmaxima, then if he stops at a break point, he will see his current position in the source file which will be displayed in the other half of the window, either highlighted in red, or with a little arrow pointing at the right line. He can advance single lines at a time by typing M-n (Alt-n).

Under Emacs you should run in a `dbl` shell, which requires the `dbl.el` file in the elisp directory. Make sure you install the elisp files or add the Maxima elisp directory to your path: e.g., add the following to your `.emacs` file or the `site-init.el`

```
(setq load-path (cons "/usr/share/maxima/5.9.1/emacs" load-path))
(autoload 'dbl "dbl")
```

then in emacs

```
M-x dbl
```

should start a shell window in which you can run programs, for example Maxima, gcl, gdb etc. This shell window also knows about source level debugging, and display of source code in the other window.

The user may set a break point at a certain line of the file by typing `C-x space`. This figures out which function the cursor is in, and then it sees which line of that function the cursor is on. If the cursor is on, say, line 2 of `foo`, then it will insert in the other window the command, "`:br foo 2`", to break `foo` at its second line. To have this enabled, the user must have maxima-mode.el turned on in the window in which the file `foobar.mac` is visiting. There are additional commands available in that file window, such as evaluating the function into the Maxima, by typing `Alt-Control-x`.

34.2 Keyword Commands

Keyword commands are special keywords which are not interpreted as Maxima expressions. A keyword command can be entered at the Maxima prompt or the debugger prompt, although not at the break prompt. Keyword commands start with a colon, '`:`'. For example, to evaluate a Lisp form you may type `:lisp` followed by the form to be evaluated.

```
(%i1) :lisp (+ 2 3)
```

```
5
```

The number of arguments taken depends on the particular command. Also, you need not type the whole command, just enough to be unique among the break keywords. Thus `:br` would suffice for `:break`.

The keyword commands are listed below.

`:break F n`
: Set a breakpoint in function `F` at line offset `n` from the beginning of the function. If `F` is given as a string, then it is assumed to be a file, and `n` is the offset from the beginning of the file. The offset is optional. If not given, it is assumed to be zero (first line of the function or file).

`:bt` Print a backtrace of the stack frames

`:continue`
: Continue the computation

`:delete` Delete the specified breakpoints, or all if none are specified

`:disable` Disable the specified breakpoints, or all if none are specified

`:enable` Enable the specified breakpoints, or all if none are specified

`:frame n` Print stack frame `n`, or the current frame if none is specified

`:help` Print help on a debugger command, or all commands if none is specified

`:info` Print information about item

`:lisp some-form`
: Evaluate `some-form` as a Lisp form

`:lisp-quiet some-form`
: Evaluate Lisp form `some-form` without any output

`:next` Like `:step`, except `:next` steps over function calls

`:quit` Quit the current debugger level without completing the computation

`:resume` Continue the computation

`:step` Continue the computation until it reaches a new source line

`:top` Return to the Maxima prompt (from any debugger level) without completing the computation

Note: Keyword commands must (currently) start at the beginning of a line. Not even a single space character is allowed before the colon.

34.3 Functions and Variables for Debugging

`debugmode` [Option variable]

Default value: `false`

When `debugmode` is `true`, Maxima will start the Maxima debugger when a Maxima error occurs. At this point the user may enter commands to examine the call stack,

set breakpoints, step through Maxima code, and so on. See `debugging` for a list of Maxima debugger commands.

When `debugmode` is `lisp`, Maxima will start the Lisp debugger when a Maxima error occurs.

In either case, enabling `debugmode` will not catch Lisp errors.

`refcheck` [Option variable]

Default value: `false`

When `refcheck` is `true`, Maxima prints a message each time a bound variable is used for the first time in a computation.

`setcheck` [Option variable]

Default value: `false`

If `setcheck` is set to a list of variables (which can be subscripted), Maxima prints a message whenever the variables, or subscripted occurrences of them, are bound with the ordinary assignment operator `:`, the `::` assignment operator, or function argument binding, but not the function assignment `:=` nor the macro assignment `::=` operators. The message comprises the name of the variable and the value it is bound to.

`setcheck` may be set to `all` or `true` thereby including all variables.

Each new assignment of `setcheck` establishes a new list of variables to check, and any variables previously assigned to `setcheck` are forgotten.

The names assigned to `setcheck` must be quoted if they would otherwise evaluate to something other than themselves. For example, if `x`, `y`, and `z` are already bound, then enter

```
setcheck: ['x, 'y, 'z]$
```

to put them on the list of variables to check.

No printout is generated when a variable on the `setcheck` list is assigned to itself, e.g., `X: 'X`.

`setcheckbreak` [Option variable]

Default value: `false`

When `setcheckbreak` is `true`, Maxima will present a break prompt whenever a variable on the `setcheck` list is assigned a new value. The break occurs before the assignment is carried out. At this point, `setval` holds the value to which the variable is about to be assigned. Hence, one may assign a different value by assigning to `setval`.

See also `setcheck` and `setval`.

`setval` [System variable]

Holds the value to which a variable is about to be set when a `setcheckbreak` occurs. Hence, one may assign a different value by assigning to `setval`.

See also `setcheck` and `setcheckbreak`.

`timer (f_1, ..., f_n)` [Function]
`timer (all)`
`timer ()`

Given functions *f_1*, ..., *f_n*, `timer` puts each one on the list of functions for which timing statistics are collected. `timer(f)$ timer(g)$` puts `f` and then `g` onto the list; the list accumulates from one call to the next.

`timer(all)` puts all user-defined functions (as named by the global variable `functions`) on the list of timed functions.

With no arguments, `timer` returns the list of timed functions.

Maxima records how much time is spent executing each function on the list of timed functions. `timer_info` returns the timing statistics, including the average time elapsed per function call, the number of calls, and the total time elapsed. `untimer` removes functions from the list of timed functions.

`timer` quotes its arguments. `f(x) := x^2$ g:f$ timer(g)$` does not put `f` on the timer list.

If `trace(f)` is in effect, then `timer(f)` has no effect; `trace` and `timer` cannot both be in effect at the same time.

See also `timer_reset` and `timer_devalue`.

`timer_reset` [Function]
`timer_reset (f_1, ..., f_n)`
`timer_reset ()`

Given functions *f_1*, ..., *f_n*, `timer_reset` sets the accumulated elapsed time for each function to zero.

With no arguments, `timer_reset` sets the accumulated elapsed time for each function on the global timer list to zero.

`untimer (f_1, ..., f_n)` [Function]
`untimer ()`

Given functions *f_1*, ..., *f_n*, `untimer` removes each function from the timer list.

With no arguments, `untimer` removes all functions currently on the timer list.

After `untimer (f)` is executed, `timer_info (f)` still returns previously collected timing statistics, although `timer_info()` (with no arguments) does not return information about any function not currently on the timer list. `timer (f)` resets all timing statistics to zero and puts `f` on the timer list again.

`timer_devalue` [Option variable]

Default value: `false`

When `timer_devalue` is `true`, Maxima subtracts from each timed function the time spent in other timed functions. Otherwise, the time reported for each function includes the time spent in other functions. Note that time spent in untimed functions is not subtracted from the total time.

See also `timer` and `timer_info`.

`timer_info` (*f_1*, ..., *f_n*) [Function]
`timer_info` ()

Given functions *f_1*, ..., *f_n*, `timer_info` returns a matrix containing timing information for each function. With no arguments, `timer_info` returns timing information for all functions currently on the timer list.

The matrix returned by `timer_info` contains the function name, time per function call, number of function calls, total time, and `gctime`, which meant "garbage collection time" in the original Macsyma but is now always zero.

The data from which `timer_info` constructs its return value can also be obtained by the `get` function:

```
get(f, 'calls);  get(f, 'runtime);  get(f, 'gctime);
```

See also `timer` and `timer_reset`.

`trace` (*f_1*, ..., *f_n*) [Function]
`trace` (*all*)
`trace` ()

Given functions *f_1*, ..., *f_n*, `trace` instructs Maxima to print out debugging information whenever those functions are called. `trace(f)$ trace(g)$` puts `f` and then `g` onto the list of functions to be traced; the list accumulates from one call to the next.

`trace(all)` puts all user-defined functions (as named by the global variable `functions`) on the list of functions to be traced.

With no arguments, `trace` returns a list of all the functions currently being traced.

The `untrace` function disables tracing. See also `trace_options`.

`trace` quotes its arguments. Thus, `f(x) := x^2$ g:f$ trace(g)$` does not put `f` on the trace list.

When a function is redefined, it is removed from the timer list. Thus after `timer(f)$ f(x) := x^2$`, function `f` is no longer on the timer list.

If `timer (f)` is in effect, then `trace (f)` has no effect; `trace` and `timer` can't both be in effect for the same function.

`trace_options` [Function]
`trace_options` (*f*, *option_1*, ..., *option_n*)
`trace_options` (*f*)

Sets the trace options for function *f*. Any previous options are superseded. `trace_options (f, ...)` has no effect unless `trace (f)` is also called (either before or after `trace_options`).

`trace_options (f)` resets all options to their default values.

The following option keywords are recognized. The presence of the option keyword alone enables the option unconditionally. Specifying an option keyword with a predicate function *p* as its argument makes the option conditional on the predicate.

- `noprint`, `noprint(p)` Do not print a message at function entry and exit.
- `break`, `break(p)` Stop execution before the function is entered, and after the function is exited. See also `break`. The arguments of the function and its return value are available as `trace_break_arg` when entering and exiting the function, respectively.

- `lisp_print`, `lisp_print(`*p*`)` Display arguments and return values as Lisp objects.
- `info`, `info(`*p*`)` Display the return value of *p* at function entry and exit. The function *p* may also have side effects, such as displaying output or modifying global variables.
- `errorcatch`, `errorcatch(`*p*`)` Catch errors, giving the option to signal an error, retry the function call, or specify a return value.

The predicate function, if supplied, is called with four arguments.

- The recursion level for the function, an integer.
- Whether the function is being entered or exited, indicated by a symbol, either `enter` or `exit`, respectively.
- The name of the function, a symbol.
- The arguments of the traced function (on entering) or the function return value (on exiting).

If the predicate function returns `false`, the corresponding trace option is disabled; if any value other than `false` value is returned, the trace option is enabled.

`trace_options` quotes (does not evaluate) its arguments.

Examples:

The presence of the option keyword alone enables the option unconditionally.

```
(%i1) ff(n) := if equal(n, 0) then 1 else n * ff(n - 1);
(%o1)    ff(n) := if equal(n, 0) then 1 else n ff(n - 1)
(%i2) trace (ff);
(%o2)                          [ff]
(%i3) trace_options (ff, lisp_print, break);
(%o3)                   [lisp_print, break]
(%i4) ff(3);
Trace entering ff level 1

Entering a Maxima break point. Type 'exit;' to resume.
_trace_break_arg;
[3]
_exit;
(1 ENTER $FF (3))
Trace entering ff level 2

Entering a Maxima break point. Type 'exit;' to resume.
_exit;
 (2 ENTER $FF (2))
Trace entering ff level 3

Entering a Maxima break point. Type 'exit;' to resume.
_exit;
  (3 ENTER $FF (1))
Trace entering ff level 4
```

```
Entering a Maxima break point. Type 'exit;' to resume.
_exit;
   (4 ENTER $FF (0))
Trace exiting ff level 4

Entering a Maxima break point. Type 'exit;' to resume.
_exit;
   (4 EXIT $FF 1)
Trace exiting ff level 3

Entering a Maxima break point. Type 'exit;' to resume.
_exit;
  (3 EXIT $FF 1)
Trace exiting ff level 2

Entering a Maxima break point. Type 'exit;' to resume.
_exit;
 (2 EXIT $FF 2)
Trace exiting ff level 1

Entering a Maxima break point. Type 'exit;' to resume.
_trace_break_arg;
6
_exit;
(1 EXIT $FF 6)
(%o4)                          6
```

Specifying an option keyword with a predicate function p as its argument makes the option conditional on the predicate.

```
(%i1) ff(n) := if equal(n, 0) then 1 else n * ff(n - 1);
(%o1)    ff(n) := if equal(n, 0) then 1 else n ff(n - 1)
(%i2) trace (ff);
(%o2)                          [ff]
(%i3) trace_options (ff, break(pp));
(%o3)                      [break(pp)]
(%i4) pp (level, direction, fnname, item) := (print (item), fnname = 'ff and leve
(%o4) pp(level, direction, fnname, item) :=
(print(item), (fnname = 'ff) and (level = 3)
 and (direction = 'exit))
(%i5) ff(6);
[6]
1 Enter ff [6]
[5]
 2 Enter ff [5]
[4]
  3 Enter ff [4]
```

```
[3]
   4 Enter ff [3]
[2]
    5 Enter ff [2]
[1]
     6 Enter ff [1]
[0]
      7 Enter ff [0]
1
      7 Exit  ff 1
1
     6 Exit  ff 1
2
    5 Exit  ff 2
6
   4 Exit  ff 6
24
Trace exiting ff level 3

Entering a Maxima break point. Type 'exit;' to resume.
_trace_break_arg;
24
_exit;
  3 Exit  ff 24
120
 2 Exit  ff 120
720
1 Exit  ff 720
(%o5)                          720
```

`trace_break_arg` [Debugging variable]

When a traced function stops at a breakpoint, `trace_break_arg` is bound to the value of function arguments when entering the function, or the return value of the function, when exiting.

Breakpoints for traced functions are specified by the option keyword `break` of the function `trace_options`, which see.

`untrace` [Function]

`untrace` (*f_1*, ..., *f_n*)
`untrace` ()

Given functions *f_1*, ..., *f_n*, `untrace` disables tracing enabled by the `trace` function. With no arguments, `untrace` disables tracing for all functions.

`untrace` returns a list of the functions for which it disabled tracing.

35 Error and warning messages

This chapter provides detailed information about the meaning of some error messages or on how to recover from errors.

35.1 Error Messages

35.1.1 apply: no such "list" element

One common cause for this error message is that square brackets operator (`[ ]`) was used trying to access a list element that whose element number was `< 1` or `> length(list)`.

35.1.2 argument must be a non-atomic expression

This normally means that a list, a set or something else that consists of more than one element was expected. One possible cause for this error message is a construct of the following type:

```
(%i1) l:[1,2,3];
(%o1)                         [1, 2, 3]
(%i2) append(l,4);
append: argument must be a non-atomic expression; found 4
 -- an error. To debug this try: debugmode(true);
```

The correct way to append variables or numbers to a list is to wrap them in a single-element list first:

```
(%i1) l:[1,2,3];
(%o1)                         [1, 2, 3]
(%i2) append(l,[4]);
(%o2)                      [1, 2, 3, 4]
```

35.1.3 assignment: cannot assign to <function name>

Maxima supports several assignment operators. When trying to define a function `:=` has to be used.

35.1.4 expt: undefined: 0 to a negative exponent.

This message notifies about a classical division by zero error.

35.1.5 incorrect syntax: , is not a prefix operator

This might be caused by a command starting with a comma (`,`) or by one comma being directly followed by another one..

35.1.6 incorrect syntax: Illegal use of delimiter)

Common reasons for this error appearing are a closing parenthesis without an opening one or a closing parenthesis directly preceded by a comma.

35.1.7 loadfile: failed to load <filename>

This error message normally indicates that the file exists, but can not be read. If the file is present and readable there is another possible for this error message: Maxima can compile packages to native binary files in order to make them run faster. If after compiling the file something in the system has changed in a way that makes it incompatible with the binary the binary the file cannot be loaded any more. Maxima normally puts binary files it creates from its own packages in a folder named `binary` within the folder whose name it is printed after typing:

```
(%i1) maxima_userdir;
(%o1)                  /home/gunter/.maxima
```

If this directory is missing maxima will recreate it again as soon as it has to compile a package.

35.1.8 makelist: second argument must evaluate to a number

`makelist` expects the second argument to be the name of the variable whose value is to be stepped. This time instead of the name of a still-undefined variable maxima has found something else, possibly a list or the name of a list.

35.1.9 Only symbols can be bound

The most probable cause for this error is that there was an attempt to either use a number or a variable whose numerical value is known as a loop counter.

35.1.10 operators of arguments must all be the same

One possible reason for this error message to appear is a try to use `append` in order to add an equation to a list:

```
(%i1) l:[a=1,b=2,c=3];
(%o1)                  [a = 1, b = 2, c = 3]
(%i2) append(l,d=5);
append: operators of arguments must all be the same.
 -- an error. To debug this try: debugmode(true);
```

In order to add an equation to a list it has to be wrapped in a single-element list first:

```
(%i1) l:[a=1,b=2,c=3];
(%o1)                  [a = 1, b = 2, c = 3]
(%i2) append(l,[d=5]);
(%o2)             [a = 1, b = 2, c = 3, d = 5]
```

35.1.11 Out of memory

Lisp typically handles several types of memory containing at least one stack and a heap that contains user objects. To avoid running out of memory several approaches might be useful:

- If possible, the best solution normally is to use an algorithm that is more memory-efficient.
- Compiling a function might drastically reduce the amount of memory it needs.
- Arrays of a fixed type might be more memory-efficient than lists.

- If maxima is run by sbcl sbcl's memory limit might be set to a value that is too low to solve the current problem. In this case the command-line option `--dynamic-space-size <n>` allows to tell sbcl to reserve `n` megabytes for the heap. It is to note, though, that sbcl has to handle several distinct types of memory and therefore might be able to only reserve about half of the available physical memory. Also note that 32-bit processes might only be able to access 2GB of physical memory.

35.1.12 part: fell off the end

`part()` was used to access the `n`th item in something that has less than `n` items.

35.1.13 undefined variable (draw or plot)

A function could not be plotted since it still contained a variable maxima doesn't know the value of.

In order to find out which variable this could be it is sometimes helpful to temporarily replace the name of the drawing command (`draw2d`, `plot2d` or similar) by a random name (for example `ddraw2d`) that doesn't coincide with the name of an existing function to make maxima print out what parameters the drawing command sees.

```
(%i1) load("draw")$
(%i2) f(x):=sin(omega*t);
(%o2) f(x) := sin(omega t)
(%i3) draw2d(
        explicit(
          f(x),
          x,1,10
        )
      );
draw2d (explicit): non defined variable
 -- an error. To debug this try: debugmode(true);
(%i4) ddraw2d(
        explicit(
          f(x),
          x,1,10
        )
      );
(%o4) ddraw2d(explicit(sin(omega t), x, 1, 10))
```

35.1.14 VTK is not installed, which is required for Scene

This might either mean that VTK is actually not installed - or cannot be found by maxima - or that Maxima has no write access to the temporary directory whose name is output if the following maxima command is entered:

```
(%i1) maxima_tempdir;
(%o1)                          /tmp
```

Note: The `scene()` command requires VTK with TCL/TK bindings.

35.2 Warning Messages

35.2.1 Encountered undefined variable <x> in translation

A function was compiled but the type of the variable `x` was not known. This means that the compiled command contains additional code that makes it retain all the flexibility maxima provides in respect to this variable. If `x` isn't meant as a variable name but just a named option to a command prepending the named option by a single quote (`'`) should resolve this issue.

35.2.2 Rat: replaced <x> by <y> = <z>

`rat` was called on an expression containing floating point numbers (including big floats) and `keepfloat` was false. This means the number was replaced by a rational number approximating the floating-point number.

See also `ratprint`, `ratepsilon`, `bftorat`, `fpprintprec` and `rationalize`.

36 Package affine

36.1 Introduction to Affine

`affine` is a package to work with groups of polynomials.

36.2 Functions and Variables for Affine

`fast_linsolve` ([*expr_1*, ..., *expr_m*], [*x_1*, ..., *x_n*]) [Function]

Solves the simultaneous linear equations *expr_1*, ..., *expr_m* for the variables *x_1*, ..., *x_n*. Each *expr_i* may be an equation or a general expression; if given as a general expression, it is treated as an equation of the form `expr_i = 0`.

The return value is a list of equations of the form `[x_1 = a_1, ..., x_n = a_n]` where *a_1*, ..., *a_n* are all free of *x_1*, ..., *x_n*.

`fast_linsolve` is faster than `linsolve` for system of equations which are sparse.

`load("affine")` loads this function.

`grobner_basis` ([*expr_1*, ..., *expr_m*]) [Function]

Returns a Groebner basis for the equations *expr_1*, ..., *expr_m*. The function `polysimp` can then be used to simplify other functions relative to the equations.

```
grobner_basis ([3*x^2+1, y*x])$

polysimp (y^2*x + x^3*9 + 2) ==> -3*x + 2
```

`polysimp(f)` yields 0 if and only if *f* is in the ideal generated by *expr_1*, ..., *expr_m*, that is, if and only if *f* is a polynomial combination of the elements of *expr_1*, ..., *expr_m*.

`load("affine")` loads this function.

`set_up_dot_simplifications` [Function]

`set_up_dot_simplifications` (*eqns*, *check_through_degree*)

`set_up_dot_simplifications` (*eqns*)

The *eqns* are polynomial equations in non commutative variables. The value of `current_variables` is the list of variables used for computing degrees. The equations must be homogeneous, in order for the procedure to terminate.

If you have checked overlapping simplifications in `dot_simplifications` above the degree of *f*, then the following is true: `dotsimp (f)` yields 0 if and only if *f* is in the ideal generated by the equations, i.e., if and only if *f* is a polynomial combination of the elements of the equations.

The degree is that returned by `nc_degree`. This in turn is influenced by the weights of individual variables.

`load("affine")` loads this function.

`declare_weights` (*x_1*, *w_1*, ..., *x_n*, *w_n*) [Function]

Assigns weights *w_1*, ..., *w_n* to *x_1*, ..., *x_n*, respectively. These are the weights used in computing `nc_degree`.

`load("affine")` loads this function.

`nc_degree` (*p*) [Function]

Returns the degree of a noncommutative polynomial *p*. See `declare_weights`.

`load("affine")` loads this function.

`dotsimp` (*f*) [Function]

Returns 0 if and only if *f* is in the ideal generated by the equations, i.e., if and only if *f* is a polynomial combination of the elements of the equations.

`load("affine")` loads this function.

`fast_central_elements` ([*x_1*, ..., *x_n*], *n*) [Function]

If `set_up_dot_simplifications` has been previously done, finds the central polynomials in the variables *x_1*, ..., *x_n* in the given degree, *n*.

For example:

```
set_up_dot_simplifications ([y.x + x.y], 3);
fast_central_elements ([x, y], 2);
[y.y, x.x];
```

`load("affine")` loads this function.

`check_overlaps` (*n*, *add_to_simps*) [Function]

Checks the overlaps thru degree *n*, making sure that you have sufficient simplification rules in each degree, for `dotsimp` to work correctly. This process can be speeded up if you know before hand what the dimension of the space of monomials is. If it is of finite global dimension, then `hilbert` should be used. If you don't know the monomial dimensions, do not specify a `rank_function`. An optional third argument `reset`, `false` says don't bother to query about resetting things.

`load("affine")` loads this function.

`mono` ([*x_1*, ..., *x_n*], *n*) [Function]

Returns the list of independent monomials relative to the current dot simplifications of degree *n* in the variables *x_1*, ..., *x_n*.

`load("affine")` loads this function.

`monomial_dimensions` (*n*) [Function]

Compute the Hilbert series through degree *n* for the current algebra.

`load("affine")` loads this function.

`extract_linear_equations` ([*p_1*, ..., *p_n*], [*m_1*, ..., *m_n*]) [Function]

Makes a list of the coefficients of the noncommutative polynomials *p_1*, ..., *p_n* of the noncommutative monomials *m_1*, ..., *m_n*. The coefficients should be scalars. Use `list_nc_monomials` to build the list of monomials.

`load("affine")` loads this function.

`list_nc_monomials` [Function]

`list_nc_monomials` ([*p_1*, ..., *p_n*])
`list_nc_monomials` (*p*)

Returns a list of the non commutative monomials occurring in a polynomial *p* or a list of polynomials *p_1*, ..., *p_n*.

`load("affine")` loads this function.

`all_dotsimp_denoms` [Option variable]

Default value: `false`

When `all_dotsimp_denoms` is a list, the denominators encountered by `dotsimp` are appended to the list. `all_dotsimp_denoms` may be initialized to an empty list `[]` before calling `dotsimp`.

By default, denominators are not collected by `dotsimp`.

37 Package alt-display

37.1 Introduction to alt-display

The *alt-display* package provides a means to change the way that Maxima displays its output. The **alt-display1d** and **alt-display2d** Lisp hooks were introduced to Maxima in 2002, but were not easily accessible from the Maxima REPL until the introduction of this package.

The package provides a general purpose function to define alternative display functions, and a separate function to set the display function. The package also provides customized display functions to produce output in TEX, Texinfo, XML and all three output formats within Texinfo.

Here is a sample session:

```
(%i1) load("alt-display.mac")$
(%i2) set_alt_display(2,tex_display)$

(%i3) x/(x^2+y^2) = 1;
\mbox{\tt\red({\it \%o_3}) \black}$${{x}\over{y^2+x^2}}=1$$

(%i4) set_alt_display(2,mathml_display)$

(%i5) x/(x^2+y^2) = 1;
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi>
<mfenced separators=""><msub><mi>%o</mi> <mn>5</mn></msub>
<mo>,</mo><mfrac><mrow><mi>x</mi> </mrow> <mrow><msup><mrow>
<mi>y</mi> </mrow> <mn>2</mn> </msup> <mo>+</mo> <msup><mrow>
<mi>x</mi> </mrow> <mn>2</mn> </msup> </mrow></mfrac> <mo>=</mo>
<mn>1</mn> </mfenced> </math>

(%i6) set_alt_display(2,multi_display_for_texinfo)$

(%i7) x/(x^2+y^2) = 1;

@iftex
@tex
\mbox{\tt\red({\it \%o_7}) \black}$${{x}\over{y^2+x^2}}=1$$
@end tex
@end iftex
@ifhtml
@html

<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi>
<mfenced separators=""><msub><mi>%o</mi> <mn>7</mn></msub>
<mo>,</mo><mfrac><mrow><mi>x</mi> </mrow> <mrow><msup><mrow>
<mi>y</mi> </mrow> <mn>2</mn> </msup> <mo>+</mo> <msup><mrow>
<mi>x</mi> </mrow> <mn>2</mn> </msup> </mrow></mfrac> <mo>=</mo>
```

```
<mn>1</mn> </mfenced> </math>
@end html
@end ifhtml
@ifinfo
@example
(%o7) x/(y^2+x^2) = 1
@end example
@end ifinfo
```

If the alternative display function causes an error, the error is trapped and the display function is reset to the default display. In the following example, the `error` function is set to display the output. This throws an error, which is handled by resetting the 2d-display to the default.

```
(%i8) set_alt_display(2,?error)$

(%i9) x;

Error in *alt-display2d*.
Message: Condition designator ((MLABEL) $%O9 $X) is not of type
        (OR SYMBOL STRING FUNCTION).
*alt-display2d* reset to nil.
 -- an error. To debug this try: debugmode(true);

(%i10) x;
(%o10)                                  x
```

37.2 Functions and Variables for alt-display

`alt_display_output_type` (*form*) [Function]

Determine the type of output to be printed. *Form* must be a lisp form suitable for printing via Maxima's built-in `displa` function. At present, this function returns one of three values: *text*, *label* or *unknown*.

An example where `alt_display_output_type` is used. In `my_display`, a text form is printed between a pair of tags `TEXT;>>` and `<<TEXT;` while a label form is printed between a pair tags `OUT;>>` and `<<OUT;` in addition to the usual output label.

The function `set_prompt` also ensures that input labels are printed between matching `PROMPT;>>` and `<<PROMPT;` tags.

Thanks to Eric Stemmler.

```
(%i1) (load("mactex-utilities"), load("alt-display.mac")) $

(%i2) define_alt_display(my_display(form),
block([type,txttmplt,labtmplt],
txttmplt:"~%TEXT;>>~%~a~%<<TEXT;~%",
labtmplt:"~%OUT;>>~%(~a) ~a~a~a~%<<OUT;~%",
type:alt_display_output_type(form),
if type='text then
```

```
   printf(true,txttmplt,first(form))
else if type='label then
     printf(true,labtmplt,first(form),"$$",tex1(second(form)),"$$")
else
     block([alt_display1d:false, alt_display2d:false], displa(form))) $

(%i3) (set_prompt('prefix, "PROMPT;>>",'suffix, "<<PROMPT;"),
      set_alt_display(1,my_display)) $

PROMPT;>>(%i4) <<PROMPT;integrate(x^n,x);
PROMPT;>>
TEXT;>>
Is n equal to -1?
<<TEXT;
<<PROMPT;
n;

OUT;>>
(%o4) $$\frac{x^{n+1}}{n+1}$$
<<OUT;
PROMPT;>>(%i5) <<PROMPT;
```

`define_alt_display` (*function*(*input*), *expr*) [Function]

This function is similar to `define`: it evaluates its arguments and expands into a function definition. The *function* is a function of a single input *input*. For convenience, a substitution is applied to *expr* after evaluation, to provide easy access to Lisp variable names.

Set a time-stamp on each prompt:

```
(%i1) load("alt-display.mac")$

(%i2) display2d: false$

(%i3) define_alt_display(time_stamp(x),
                block([alt_display1d:false,alt_display2d:false],
                      prompt_prefix:printf(false,"~a~%",timedate()),
                      displa(x)));

(%o3) time_stamp(x):=block(
                 [\*alt\-display1d\*:false,
                  \*alt\-display2d\*:false],
                 \*prompt\-prefix\*
                  :printf(false,"~a~%",timedate()),displa(x))
(%i4) set_alt_display(1,time_stamp);

(%o4) done
2017-11-27 16:15:58-06:00
(%i5)
```

The input line `%i3` defines `time_stamp` using `define_alt_display`. The output line `%o3` shows that the Maxima variable names `alt_display1d`, `alt_display2d` and `prompt_prefix` have been replaced by their Lisp translations, as has `displa` been replaced by `?displa` (the display function).

The display variables `alt_display1d` and `alt_display2d` are both bound to `false` in the body of `time_stamp` to prevent an infinite recursion in `displa`.

`info_display` (*form*) [Function]

This is an alias for the default 1-d display function. It may be used as an alternative 1-d or 2-d display function.

```
(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2,info_display);

(%o2) done
(%i3) x/y;

(%o3) x/y
```

`mathml_display` (*form*) [Function]

Produces MathML output.

```
(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2,mathml_display);
<math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi>
 <mfenced separators=""><msub><mi>%o</mi> <mn>2</mn></msub>
 <mo>,</mo><mi>done</mi> </mfenced> </math>
```

`tex_display` (*form*) [Function]

Produces TeX output.

```
(%i2) set_alt_display(2,tex_display);
\mbox{\tt\red({\it \%o_2}) \black}$$\mathbf{done}$$
(%i3) x/(x^2+y^2);
\mbox{\tt\red({\it \%o_3}) \black}$${{x}\over{y^2+x^2}}$$
```

`multi_display_for_texinfo` (*form*) [Function]

Produces Texinfo output using all three display functions.

```
(%i2) set_alt_display(2,multi_display_for_texinfo)$

(%i3) x/(x^2+y^2);

@iftex
@tex
\mbox{\tt\red({\it \%o_3}) \black}$${{x}\over{y^2+x^2}}$$
@end tex
@end iftex
@ifhtml
```

```
@html

   <math xmlns="http://www.w3.org/1998/Math/MathML"> <mi>mlabel</mi>
   <mfenced separators=""><msub><mi>%o</mi> <mn>3</mn></msub>
   <mo>,</mo><mfrac><mrow><mi>x</mi> </mrow> <mrow><msup><mrow>
   <mi>y</mi> </mrow> <mn>2</mn> </msup> <mo>+</mo> <msup><mrow>
   <mi>x</mi> </mrow> <mn>2</mn> </msup> </mrow></mfrac> </mfenced> </math>
@end html
@end ifhtml
@ifinfo
@example
(%o3) x/(y^2+x^2)
@end example
@end ifinfo
```

`reset_displays ()` [Functions]

Resets the prompt prefix and suffix to the empty string, and sets both 1-d and 2-d display functions to the default.

`set_alt_display (`*num*, *display-function*`)` [Function]

The input *num* is the display to set; it may be either 1 or 2. The second input *display-function* is the display function to use. The display function may be either a Maxima function or a `lambda` expression.

Here is an example where the display function is a `lambda` expression; it just displays the result as TEX.

```
(%i1) load("alt-display.mac")$

(%i2) set_alt_display(2, lambda([form], tex(?caddr(form))))$

(%i3) integrate(exp(-t^2),t,0,inf);
$${{\sqrt{\pi}}\over{2}}$$
```

A user-defined display function should take care that it *prints* its output. A display function that returns a string will appear to display nothing, nor cause any errors.

`set_prompt (`*fix*, *expr*`)` [Function]

Set the prompt prefix or suffix to *expr*. The input *fix* must evaluate to one of `prefix`, `suffix`, `general`, `prolog` or `epilog`. The input *expr* must evaluate to either a string or `false`; if `false`, the *fix* is reset to the default value.

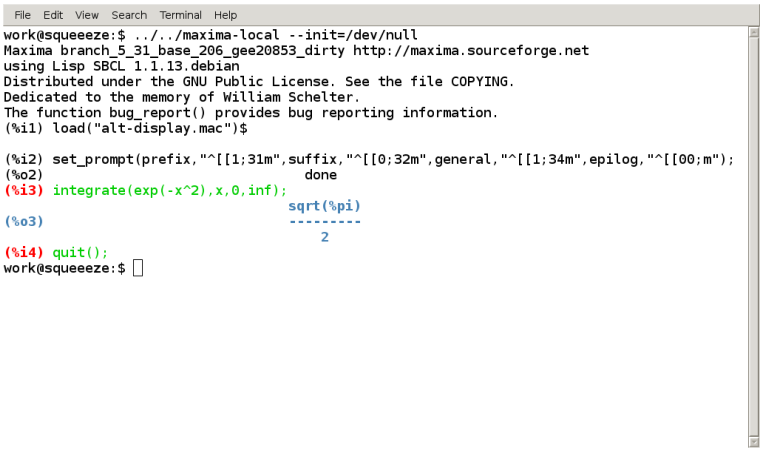
```
(%i1) load("alt-display.mac")$
(%i2) set_prompt('prefix,printf(false,"It is now: ~a~%",timedate()))$

It is now: 2014-01-07 15:23:23-05:00
(%i3)
```

The following example shows the effect of each option, except `prolog`. Note that the `epilog` prompt is printed as Maxima closes down. The `general` is printed between the end of input and the output, unless the input line ends in `$`.

Here is an example to show where the prompt strings are placed.

```
(%i1) load("alt-display.mac")$

(%i2) set_prompt(prefix, "<<prefix>> ", suffix, "<<suffix>> ",
                 general, printf(false,"<<general>>~%"),
                 epilog, printf(false,"<<epilog>>~%"));

(%o2)                                    done
<<prefix>> (%i3) <<suffix>> x/y;
<<general>>
                                           x
(%o3)                                      -
                                           y
<<prefix>> (%i4) <<suffix>> quit();
<<general>>
<<epilog>>
```

Here is an example that shows how to colorize the input and output when Maxima is running in a terminal or terminal emulator like Emacs[1].



Each prompt string starts with the ASCII escape character (27) followed by an open square bracket (91); each string ends with a lower-case m (109). The webpages https://misc.flogisoft.com/bash/tip_colors_and_formatting and https://www.tldp.org/HOWTO/Bash-Prompt-HOWTO/x329.html provide information on how to use control strings to set the terminal colors.

[1] Readers using the `info` reader in `Emacs` will see the actual prompt strings; other readers will see the colorized output

38 Package asympa

38.1 Introduction to asympa

asympa [Function]

`asympa` is a package for asymptotic analysis. The package contains simplification functions for asymptotic analysis, including the "big O" and "little o" functions that are widely used in complexity analysis and numerical analysis.

`load ("asympa")` loads this package.

38.2 Functions and variables for asympa

39 Package atensor

39.1 Introduction to atensor

`atensor` is an algebraic tensor manipulation package. To use `atensor`, type `load("atensor")`, followed by a call to the `init_atensor` function.

For alternative packages for handling tensors see Chapter 51 [Package ctensor], page 723, and Chapter 71 [Package itensor], page 1091.

The essence of `atensor` is a set of simplification rules for the noncommutative (dot) product operator (`"."`). `atensor` recognizes several algebra types; the corresponding simplification rules are put into effect when the `init_atensor` function is called.

The capabilities of `atensor` can be demonstrated by defining the algebra of quaternions as a Clifford-algebra Cl(0,2) with two basis vectors. The three quaternionic imaginary units are then the two basis vectors and their product, i.e.:

```
i = v     j = v     k = v  . v
     1         2         1    2
```

Although the `atensor` package has a built-in definition for the quaternion algebra, it is not used in this example, in which we endeavour to build the quaternion multiplication table as a matrix:

```
(%i1) load("atensor");
(%o1)       /share/tensor/atensor.mac
(%i2) init_atensor(clifford,0,0,2);
(%o2)                                done
(%i3) atensimp(v[1].v[1]);
(%o3)                                 - 1
(%i4) atensimp((v[1].v[2]).(v[1].v[2]));
(%o4)                                 - 1
(%i5) q:zeromatrix(4,4);
                                [ 0  0  0  0 ]
                                [            ]
                                [ 0  0  0  0 ]
(%o5)                           [            ]
                                [ 0  0  0  0 ]
                                [            ]
                                [ 0  0  0  0 ]
(%i6) q[1,1]:1;
(%o6)                                 1
(%i7) for i thru adim do q[1,i+1]:q[i+1,1]:v[i];
(%o7)                                done
(%i8) q[1,4]:q[4,1]:v[1].v[2];
(%o8)                               v  . v
                                     1    2
(%i9) for i from 2 thru 4 do for j from 2 thru 4 do
      q[i,j]:atensimp(q[i,1].q[1,j]);
(%o9)                                done
```

```
(%i10) q;
                    [    1        v         v      v  . v  ]
                    [              1         2      1    2 ]
                    [                                     ]
                    [   v         - 1     v  . v    - v    ]
                    [    1                 1    2      2   ]
(%o10)              [                                     ]
                    [   v      - v  . v     - 1      v     ]
                    [    2        1    2              1    ]
                    [                                     ]
                    [ v  . v      v        - v       - 1   ]
                    [  1    2      2          1            ]
```

`atensor` recognizes as base vectors indexed symbols, where the symbol is that stored in `asymbol` and the index runs between 1 and `adim`. For indexed symbols, and indexed symbols only, the bilinear forms `sf`, `af`, and `av` are evaluated. The evaluation substitutes the value of `aform[i,j]` in place of `fun(v[i],v[j])` where `v` represents the value of `asymbol` and `fun` is either `af` or `sf`; or, it substitutes `v[aform[i,j]]` in place of `av(v[i],v[j])`.

Needless to say, the functions `sf`, `af` and `av` can be redefined.

When the `atensor` package is loaded, the following flags are set:

```
dotscrules:true;
dotdistrib:true;
dotexptsimp:false;
```

If you wish to experiment with a nonassociative algebra, you may also consider setting `dotassoc` to `false`. In this case, however, `atensimp` will not always be able to obtain the desired simplifications.

39.2 Functions and Variables for atensor

`init_atensor` [Function]

`init_atensor` (*alg_type*, *opt_dims*)
`init_atensor` (*alg_type*)

Initializes the `atensor` package with the specified algebra type. *alg_type* can be one of the following:

`universal`: The universal algebra has no commutation rules.

`grassmann`: The Grassman algebra is defined by the commutation relation `u.v+v.u=0`.

`clifford`: The Clifford algebra is defined by the commutation relation `u.v+v.u=-2*sf(u,v)` where `sf` is a symmetric scalar-valued function. For this algebra, *opt_dims* can be up to three nonnegative integers, representing the number of positive, degenerate, and negative dimensions of the algebra, respectively. If any *opt_dims* values are supplied, `atensor` will configure the values of `adim` and `aform` appropriately. Otherwise, `adim` will default to 0 and `aform` will not be defined.

`symmetric`: The symmetric algebra is defined by the commutation relation `u.v-v.u=0`.

`symplectic`: The symplectic algebra is defined by the commutation relation `u.v-v.u=2*af(u,v)` where `af` is an antisymmetric scalar-valued function. For the

symplectic algebra, *opt_dims* can be up to two nonnegative integers, representing the nondegenerate and degenerate dimensions, respectively. If any *opt_dims* values are supplied, `atensor` will configure the values of `adim` and `aform` appropriately. Otherwise, `adim` will default to 0 and `aform` will not be defined.

`lie_envelop`: The algebra of the Lie envelope is defined by the commutation relation `u.v-v.u=2*av(u,v)` where `av` is an antisymmetric function.

The `init_atensor` function also recognizes several predefined algebra types:

`complex` implements the algebra of complex numbers as the Clifford algebra Cl(0,1). The call `init_atensor(complex)` is equivalent to `init_atensor(clifford,0,0,1)`.

`quaternion` implements the algebra of quaternions. The call `init_atensor (quaternion)` is equivalent to `init_atensor (clifford,0,0,2)`.

`pauli` implements the algebra of Pauli-spinors as the Clifford-algebra Cl(3,0). A call to `init_atensor(pauli)` is equivalent to `init_atensor(clifford,3)`.

`dirac` implements the algebra of Dirac-spinors as the Clifford-algebra Cl(3,1). A call to `init_atensor(dirac)` is equivalent to `init_atensor(clifford,3,0,1)`.

`atensimp` (*expr*) [Function]

Simplifies an algebraic tensor expression *expr* according to the rules configured by a call to `init_atensor`. Simplification includes recursive application of commutation relations and resolving calls to `sf`, `af`, and `av` where applicable. A safeguard is used to ensure that the function always terminates, even for complex expressions.

`alg_type` [Function]

The algebra type. Valid values are `universal`, `grassmann`, `clifford`, `symmetric`, `symplectic` and `lie_envelop`.

`adim` [Variable]

Default value: 0

The dimensionality of the algebra. `atensor` uses the value of `adim` to determine if an indexed object is a valid base vector. See `abasep`.

`aform` [Variable]

Default value: `ident(3)`

Default values for the bilinear forms `sf`, `af`, and `av`. The default is the identity matrix `ident(3)`.

`asymbol` [Variable]

Default value: `v`

The symbol for base vectors.

`sf` (*u*, *v*) [Function]

A symmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using `abasep` and if that is the case, substitutes the corresponding value from the matrix `aform`.

`af` (*u*, *v*) [Function]

An antisymmetric scalar function that is used in commutation relations. The default implementation checks if both arguments are base vectors using `abasep` and if that is the case, substitutes the corresponding value from the matrix `aform`.

`av` (*u*, *v*) [Function]

An antisymmetric function that is used in commutation relations. The default implementation checks if both arguments are base vectors using `abasep` and if that is the case, substitutes the corresponding value from the matrix `aform`.

For instance:

```
(%i1) load("atensor");
(%o1)       /share/tensor/atensor.mac
(%i2) adim:3;
(%o2)                                3
(%i3) aform:matrix([0,3,-2],[-3,0,1],[2,-1,0]);
                                [  0    3   - 2 ]
                                [               ]
(%o3)                           [ - 3   0    1  ]
                                [               ]
                                [  2   - 1   0  ]
(%i4) asymbol:x;
(%o4)                                 x
(%i5) av(x[1],x[2]);
(%o5)                                x
                                      3
```

`abasep` (*v*) [Function]

Checks if its argument is an `atensor` base vector. That is, if it is an indexed symbol, with the symbol being the same as the value of `asymbol`, and the index having a numeric value between 1 and `adim`.

40 Package augmented_lagrangian

40.1 Functions and Variables for augmented_lagrangian

augmented_lagrangian_method [Function]

augmented_lagrangian_method (*FOM*, *xx*, *C*, *yy*)
augmented_lagrangian_method (*FOM*, *xx*, *C*, *yy*, *optional_args*)
augmented_lagrangian_method ([*FOM*, *grad*], *xx*, *C*, *yy*)
augmented_lagrangian_method ([*FOM*, *grad*], *xx*, *C*, *yy*, *optional_args*)

Returns an approximate minimum of the expression *FOM* with respect to the variables *xx*, holding the constraints *C* equal to zero. *yy* is a list of initial guesses for *xx*. The method employed is the augmented Lagrangian method (see Refs [1] and [2]).

grad, if present, is the gradient of *FOM* with respect to *xx*, represented as a list of expressions, one for each variable in *xx*. If not present, the gradient is constructed automatically.

FOM and each element of *grad*, if present, must be ordinary expressions, not names of functions or lambda expressions.

`optional_args` represents additional arguments, specified as *`symbol`* `=` *`value`*. The optional arguments recognized are:

`niter` Number of iterations of the augmented Lagrangian algorithm

`lbfgs_tolerance`
Tolerance supplied to LBFGS

`iprint` IPRINT parameter (a list of two integers which controls verbosity) supplied to LBFGS

`%lambda` Initial value of `%lambda` to be used for calculating the augmented Lagrangian

This implementation minimizes the augmented Lagrangian by applying the limited-memory BFGS (LBFGS) algorithm, which is a quasi-Newton algorithm.

`load("augmented_lagrangian")` loads this function.

See also Chapter 73 [Package lbfgs], page 1137,

References:

[1] http://www-fp.mcs.anl.gov/otc/Guide/OptWeb/continuous/constrained/nonlinearcon/auglag.html

[2] http://www.cs.ubc.ca/spider/ascher/542/chap10.pdf

Examples:

```
(%i1) load ("lbfgs");
(%o1)  /home/gunter/src/maxima-code/share/lbfgs/lbfgs.mac
(%i2) load ("augmented_lagrangian");
(%o2) /home/gunter/src/maxima-code/share/contrib/augmented_lagra\
ngian.mac
```

```
(%i3) FOM: x^2 + 2*y^2;
                                 2    2
(%o3)                          2 y  + x
(%i4) xx: [x, y];
(%o4)                           [x, y]
(%i5) C: [x + y - 1];
(%o5)                         [y + x - 1]
(%i6) yy: [1, 1];
(%o6)                           [1, 1]
(%i7) augmented_lagrangian_method(FOM, xx, C, yy, iprint=[-1,0]);
(%o7) [[x = 0.666659841080023, y = 0.333340272455448],
                                  %lambda = [- 1.333337940892518]]
```

Same example as before, but this time the gradient is supplied as an argument.

```
(%i1) load ("lbfgs")$
(%i2) load ("augmented_lagrangian")$
(%i3) FOM: x^2 + 2*y^2;
                                 2    2
(%o3)                          2 y  + x
(%i4) xx: [x, y];
(%o4)                           [x, y]
(%i5) grad : [2*x, 4*y];
(%o5)                         [2 x, 4 y]
(%i6) C: [x + y - 1];
(%o6)                         [y + x - 1]
(%i7) yy: [1, 1];
(%o7)                           [1, 1]
(%i8) augmented_lagrangian_method ([FOM, grad], xx, C, yy,
                                iprint = [-1, 0]);
(%o8) [[x = 0.6666598410800247, y = 0.3333402724554464],
                                  %lambda = [- 1.333337940892525]]
```

41 Package bernstein

41.1 Functions and Variables for Bernstein

`bernstein_poly` (*k*, *n*, *x*) [Function]

Provided `k` is not a negative integer, the Bernstein polynomials are defined by `bernstein_poly(k,n,x) = binomial(n,k) x^k (1-x)^(n-k)`; for a negative integer `k`, the Bernstein polynomial `bernstein_poly(k,n,x)` vanishes. When either `k` or `n` are non integers, the option variable `bernstein_explicit` controls the expansion of the Bernstein polynomials into its explicit form; example:

```
(%i1) load("bernstein")$

(%i2) bernstein_poly(k,n,x);
(%o2)                bernstein_poly(k, n, x)
(%i3) bernstein_poly(k,n,x), bernstein_explicit : true;
                                          n - k  k
(%o3)            binomial(n, k) (1 - x)      x
```

The Bernstein polynomials have both a gradef property and an integrate property:

```
(%i4) diff(bernstein_poly(k,n,x),x);
(%o4) (bernstein_poly(k - 1, n - 1, x)
                                  - bernstein_poly(k, n - 1, x)) n
(%i5) integrate(bernstein_poly(k,n,x),x);
(%o5)
                                                                 k + 1
 hypergeometric([k + 1, k - n], [k + 2], x) binomial(n, k) x
 ------------------------------------------------------------------
                               k + 1
```

For numeric inputs, both real and complex, the Bernstein polynomials evaluate to a numeric result:

```
(%i6) bernstein_poly(5,9, 1/2 + %i);
                        39375 %i   39375
(%o6)                   -------- + -----
                          128       256
(%i7) bernstein_poly(5,9, 0.5b0 + %i);
(%o7)           3.076171875b2 %i + 1.5380859375b2
```

To use `bernstein_poly`, first `load("bernstein")`.

`bernstein_explicit` [Variable]

Default value: `false`

When either `k` or `n` are non integers, the option variable `bernstein_explicit` controls the expansion of `bernstein(k,n,x)` into its explicit form; example:

```
(%i1) bernstein_poly(k,n,x);
(%o1)                bernstein_poly(k, n, x)
(%i2) bernstein_poly(k,n,x), bernstein_explicit : true;
```

```
                                              n - k  k
(%o2)                 binomial(n, k) (1 - x)      x
```

When both `k` and `n` are explicitly integers, `bernstein(k,n,x)` *always* expands to its explicit form.

`multibernstein_poly` (*[k1,k2,..., kp]*, *[n1,n2,..., np]*, *[x1,x2,..., xp]*) [Function]

The multibernstein polynomial `multibernstein_poly` (*[k1, k2, ..., kp]*, *[n1, n2, ..., np]*, *[x1, x2, ..., xp]*) is the product of bernstein polynomials `bernstein_poly(k1, n1, x1) bernstein_poly(k2, n2, x2) ... bernstein_poly(kp, np, xp)`.

To use `multibernstein_poly`, first `load("bernstein")`.

`bernstein_approx` (*f*, *[x1, x1, ..., xn]*, *n*) [Function]

Return the `n`-th order uniform Bernstein polynomial approximation for the function `(x1, x2, ..., xn) |-->` `f`. Examples

```
(%i1) bernstein_approx(f(x),[x], 2);
                    2       1                          2
(%o1)       f(1) x  + 2 f(-) (1 - x) x + f(0) (1 - x)
                            2
(%i2) bernstein_approx(f(x,y),[x,y], 2);
                  2  2       1                2
(%o2) f(1, 1) x  y  + 2 f(-, 1) (1 - x) x y
                             2
                   2  2          1   2
 + f(0, 1) (1 - x)  y  + 2 f(1, -) x  (1 - y) y
                                2
       1  1                                1         2
 + 4 f(-, -) (1 - x) x (1 - y) y + 2 f(0, -) (1 - x)  (1 - y) y
       2  2                                2
            2        2       1                       2
 + f(1, 0) x  (1 - y)  + 2 f(-, 0) (1 - x) x (1 - y)
                               2
                   2        2
 + f(0, 0) (1 - x)  (1 - y)
```

To use `bernstein_approx`, first `load("bernstein")`.

`bernstein_expand` (*e*, *[x1, x1, ..., xn]*) [Function]

Express the *polynomial* `e` exactly as a linear combination of multi-variable Bernstein polynomials.

```
(%i1) bernstein_expand(x*y+1,[x,y]);
(%o1)    2 x y + (1 - x) y + x (1 - y) + (1 - x) (1 - y)
(%i2) expand(%);
(%o2)                         x y + 1
```

Maxima signals an error when the first argument isn't a polynomial.

To use `bernstein_expand`, first `load("bernstein")`.

42 Package bitwise

The package `bitwise` provides functions that allow to manipulate bits of integer constants. As always maxima attempts to simplify the result of the operation if the actual value of a constant isn't known considering attributes that might be known for the variables, see the `declare` mechanism.

42.1 Functions and Variables for bitwise

`bit_not` (*int*) [Function]

Inverts all bits of a signed integer. The result of this action reads `-int - 1`.

```
(%i1) load("bitwise")$
(%i2) bit_not(i);
(%o2)                        bit_not(i)
(%i3) bit_not(bit_not(i));
(%o3)                                i
(%i4) bit_not(3);
(%o4)                              - 4
(%i5) bit_not(100);
(%o5)                             - 101
(%i6) bit_not(-101);
(%o6)                              100
```

`bit_and` (*int1*, ...) [Function]

This function calculates a bitwise `and` of two or more signed integers.

```
(%i1) load("bitwise")$
(%i2) bit_and(i,i);
(%o2)                                i
(%i3) bit_and(i,i,i);
(%o3)                                i
(%i4) bit_and(1,3);
(%o4)                                1
(%i5) bit_and(-7,7);
(%o5)                                1
```

If it is known if one of the parameters to `bit_and` is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")$
(%i2) declare(e,even,o,odd);
(%o2)                             done
(%i3) bit_and(1,e);
(%o3)                                0
(%i4) bit_and(1,o);
(%o4)                                1
```

`bit_or` (*int1*, ...) [Function]

This function calculates a bitwise `or` of two or more signed integers.

```
(%i1) load("bitwise")$
```

```
(%i2) bit_or(i,i);
(%o2)                           i
(%i3) bit_or(i,i,i);
(%o3)                           i
(%i4) bit_or(1,3);
(%o4)                           3
(%i5) bit_or(-7,7);
(%o5)                          - 1
```

If it is known if one of the parameters to `bit_or` is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")$
(%i2) declare(e,even,o,odd);
(%o2)                         done
(%i3) bit_or(1,e);
(%o3)                         e + 1
(%i4) bit_or(1,o);
(%o4)                           o
```

`bit_xor` (*int1*, ...) [Function]

This function calculates a bitwise `or` of two or more signed integers.

```
(%i1) load("bitwise")$
(%i2) bit_xor(i,i);
(%o2)                           0
(%i3) bit_xor(i,i,i);
(%o3)                           i
(%i4) bit_xor(1,3);
(%o4)                           2
(%i5) bit_xor(-7,7);
(%o5)                          - 2
```

If it is known if one of the parameters to `bit_xor` is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")$
(%i2) declare(e,even,o,odd);
(%o2)                         done
(%i3) bit_xor(1,e);
(%o3)                         e + 1
(%i4) bit_xor(1,o);
(%o4)                         o - 1
```

`bit_lsh` (*int*, *nBits*) [Function]

This function shifts all bits of the signed integer `int` to the left by `nBits` bits. The width of the integer is extended by `nBits` for this process. The result of `bit_lsh` therefore is `int * 2`.

```
(%i1) load("bitwise")$
(%i2) bit_lsh(0,1);
(%o2)                           0
```

```
(%i3) bit_lsh(1,0);
(%o3)                           1
(%i4) bit_lsh(1,1);
(%o4)                           2
(%i5) bit_lsh(1,i);
(%o5)                     bit_lsh(1, i)
(%i6) bit_lsh(-3,1);
(%o6)                          - 6
(%i7) bit_lsh(-2,1);
(%o7)                          - 4
```

bit_rsh (*int*, *nBits*) [Function]

This function shifts all bits of the signed integer `int` to the right by `nBits` bits. The width of the integer is reduced by `nBits` for this process.

```
(%i1) load("bitwise")$
(%i2) bit_rsh(0,1);
(%o2)                           0
(%i3) bit_rsh(2,0);
(%o3)                           2
(%i4) bit_rsh(2,1);
(%o4)                           1
(%i5) bit_rsh(2,2);
(%o5)                           0
(%i6) bit_rsh(-3,1);
(%o6)                          - 2
(%i7) bit_rsh(-2,1);
(%o7)                          - 1
(%i8) bit_rsh(-2,2);
(%o8)                          - 1
```

bit_length (*int*) [Function]

determines how many bits a variable needs to be long in order to store the number `int`. This function only operates on positive numbers.

```
(%i1) load("bitwise")$
(%i2) bit_length(0);
(%o2)                           0
(%i3) bit_length(1);
(%o3)                           1
(%i4) bit_length(7);
(%o4)                           3
(%i5) bit_length(8);
(%o5)                           4
```

bit_onep (*int*, *nBit*) [Function]

determines if bits `nBit` is set in the signed integer `int`.

```
(%i1) load("bitwise")$
(%i2) bit_onep(85,0);
(%o2)                          true
```

```
(%i3) bit_onep(85,1);
(%o3)                          false
(%i4) bit_onep(85,2);
(%o4)                          true
(%i5) bit_onep(85,3);
(%o5)                          false
(%i6) bit_onep(85,100);
(%o6)                          false
(%i7) bit_onep(i,100);
(%o7)                    bit_onep(i, 100)
```

For signed numbers the sign bit is interpreted to be more than `nBit` to the left of the leftmost bit of `int` that reads `1`.

```
(%i1) load("bitwise")$
(%i2) bit_onep(-2,0);
(%o2)                          false
(%i3) bit_onep(-2,1);
(%o3)                          true
(%i4) bit_onep(-2,2);
(%o4)                          true
(%i5) bit_onep(-2,3);
(%o5)                          true
(%i6) bit_onep(-2,4);
(%o6)                          true
```

If it is known if the number to be tested is even this information is taken into consideration by the function.

```
(%i1) load("bitwise")$
(%i2) declare(e,even,o,odd);
(%o2)                          done
(%i3) bit_onep(e,0);
(%o3)                          false
(%i4) bit_onep(o,0);
(%o4)                          true
```

43 Package bode

43.1 Functions and Variables for bode

bode_gain (*H*, *range*, ...*plot_opts*...) [Function]

Function to draw Bode gain plots.

Examples (1 through 7 from

http://www.swarthmore.edu/NatSci/echeeve1/Ref/Bode/BodeHow.html,

8 from Ron Crummett):

```
(%i1) load("bode")$

(%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$

(%i3) bode_gain (H1 (s), [w, 1/1000, 1000])$

(%i4) H2 (s) := 1 / (1 + s/omega0)$

(%i5) bode_gain (H2 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i6) H3 (s) := 1 / (1 + s/omega0)^2$

(%i7) bode_gain (H3 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i8) H4 (s) := 1 + s/omega0$

(%i9) bode_gain (H4 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i10) H5 (s) := 1/s$

(%i11) bode_gain (H5 (s), [w, 1/1000, 1000])$

(%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$

(%i13) bode_gain (H6 (s), [w, 1/1000, 1000]),
                  omega0 = 10, zeta = 1/10$

(%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$

(%i15) bode_gain (H7 (s), [w, 1/1000, 1000]),
                  omega0 = 10, zeta = 1/10$

(%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$

(%i17) bode_gain (H8 (s), [w, 1/1000, 1000])$
```

To use this function write first `load("bode")`. See also bode_phase.

bode_phase (*H*, *range*, *...plot_opts...*) [Function]

Function to draw Bode phase plots.

Examples (1 through 7 from

http://www.swarthmore.edu/NatSci/echeeve1/Ref/Bode/BodeHow.html,

8 from Ron Crummett):

```
(%i1) load("bode")$

(%i2) H1 (s) := 100 * (1 + s) / ((s + 10) * (s + 100))$

(%i3) bode_phase (H1 (s), [w, 1/1000, 1000])$

(%i4) H2 (s) := 1 / (1 + s/omega0)$

(%i5) bode_phase (H2 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i6) H3 (s) := 1 / (1 + s/omega0)^2$

(%i7) bode_phase (H3 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i8) H4 (s) := 1 + s/omega0$

(%i9) bode_phase (H4 (s), [w, 1/1000, 1000]), omega0 = 10$

(%i10) H5 (s) := 1/s$

(%i11) bode_phase (H5 (s), [w, 1/1000, 1000])$

(%i12) H6 (s) := 1/((s/omega0)^2 + 2 * zeta * (s/omega0) + 1)$

(%i13) bode_phase (H6 (s), [w, 1/1000, 1000]),
                   omega0 = 10, zeta = 1/10$

(%i14) H7 (s) := (s/omega0)^2 + 2 * zeta * (s/omega0) + 1$

(%i15) bode_phase (H7 (s), [w, 1/1000, 1000]),
                   omega0 = 10, zeta = 1/10$

(%i16) H8 (s) := 0.5 / (0.0001 * s^3 + 0.002 * s^2 + 0.01 * s)$

(%i17) bode_phase (H8 (s), [w, 1/1000, 1000])$

(%i18) block ([bode_phase_unwrap : false],
              bode_phase (H8 (s), [w, 1/1000, 1000]));

(%i19) block ([bode_phase_unwrap : true],
              bode_phase (H8 (s), [w, 1/1000, 1000]));
```

To use this function write first `load("bode")`. See also `bode_gain`.

44 Package cartan

44.1 Introduction to cartan

`cartan` is a package which implements the calculus of differential forms. Included are facilities for computing exterior products, contraction of vectors and forms, exterior derivatives, and Lie derivatives. The differential forms are given in terms of the natural basis forms derived from the coordinates on the manifold. Type `load(cartan);` to load this package. For a demonstration, type `demo(cartan);`

`cartan` was implemented by F.B. Estabrook and H.D. Wahlquist.

44.2 Functions and Variables for cartan

`init_cartan` (`[`*x_1*, ..., *x_m*`]`) [Function]

Initializes some global variables for the package. The argument, a Maxima list, is an ordered list of coordinates. The coordinate list is stored as the value of the variable `cartan_coords`. The value of the variable `cartan_basis` is a list of the basis 1-forms. The dimension is stored as the value of the variable `cartan_dim`.

```
(%i1) load("cartan")$
(%i2) init_cartan([x,y,z]);
(%o2)                              [dx, dy, dz]
(%i3) cartan_basis;
(%o3)                              [dx, dy, dz]
(%i4) cartan_dim;
(%o4)                                   3
```

`~` [Infix Operator]

The tilde "~" is an infix operator which denotes the exterior (wedge) product. The canonical ordering of the products is determined by the order in which the coordinates were specified. For example, if the coordinate list is specified as `[x,y,z]`, then the two-form `dy~dx` simplifies to `-dx~dy`.

```
(%i1) init_cartan([x,y,z]);
(%o1)                              [dx, dy, dz]
(%i2) vr:[v1,v2,v3];
(%o2)                              [v1, v2, v3]
(%i3) f1:x~dy+z~dz;
(%o3)                             z dz + x dy
(%i4) f2:dz~f1;
(%o4)                              - x dy dz
```

`|` [Infix Operator]

The bar "|" is an infix operator which defines the contraction of a vector on a form. The vector should be given on the left.

`ext_diff` (*form*) [Function]

The exterior derivative operator. It takes one argument, which should be a differential form.

`lie_diff` (*vector1*,[*vector2*,*form*]) [Function]

The Lie derivative operator. The first argument is a vector field. The second argument may be either a vector field or a differential form.

(Note: display2d is set to false in the following example.)

```
(%i1) init_cartan([x,y,z]);
(%o1) [dx,dy,dz]
(%i2) vr:[v1,v2,v3];
(%o2) [v1,v2,v3]
(%i3) f1:x ~ dy+z ~ dz;
(%o3) z*dz+x*dy
(%i4) vr | f1;
(%o4) v3*z+v2*x
(%i5) f2:dz ~ f1;
(%o5) -(x*dy*dz)
(%i6) ext_diff(f1);
(%o6) dx*dy
(%i7) ext_diff(f3(x,y,z) ~ dx);
(%o7) -('diff(f3(x,y,z),z,1)*dx*dz)-'diff(f3(x,y,z),y,1)*dx*dy
(%i8) depends([v1,v2,v3],[x,y,z]);
(%o8) [v1(x,y,z),v2(x,y,z),v3(x,y,z)]
(%i9) lie_diff(vr,f2);
(%o9) -('diff(v3,z,1)*x*dy*dz)-'diff(v2,y,1)*x*dy*dz-v1*dy*dz
                                  -'diff(v2,x,1)*x*dx*dz+'diff(v3,x,1)*x*dx*dy
(%i10) depends([w1,w2,w3],[x,y,z]);
(%o10) [w1(x,y,z),w2(x,y,z),w3(x,y,z)]
(%i11) wr:[w1,w2,w3];
(%o11) [w1,w2,w3]
(%i12) lie_diff(vr,wr);
(%o12) [-('diff(v1,z,1)*w3)-'diff(v1,y,1)*w2+v3*'diff(w1,z,1)+v2*'diff(w1,y,1)
                           +v1*'diff(w1,x,1)-'diff(v1,x,1)*w1,
        -('diff(v2,z,1)*w3)+v3*'diff(w2,z,1)+v2*'diff(w2,y,1)+v1*'diff(w2,x,1)
                           -'diff(v2,y,1)*w2-'diff(v2,x,1)*w1,
        v3*'diff(w3,z,1)+v2*'diff(w3,y,1)+v1*'diff(w3,x,1)-'diff(v3,z,1)*w3
                        -'diff(v3,y,1)*w2-'diff(v3,x,1)*w1]
```

45 Package celine

45.1 Introduction to celine

Maxima implementation of Sister Celine's method. Barton Willis wrote this code. It is released under the Creative Commons CC0 license.

Celine's method is described in Sections 4.1–4.4 of the book "A=B", by Marko Petkovsek, Herbert S. Wilf, and Doron Zeilberger. This book is available at http://www.math.rutgers.edu/~zeilberg/AeqB.pdf

Let f = F(n,k). The function celine returns a set of recursion relations for F of the form

p_0(n) * fff(n,k) + p_1(n) * fff(n+1,k) + ... + p_p(n) * fff(n+p,k+q),

where p_0 through p_p are polynomials. If Maxima is unable to determine that sum(sum(a(i,j) * F(n+i,k+j),i,0,p),j,0,q) / F(n,k) is a rational function of n and k, celine returns the empty set. When f involves parameters (variables other than n or k), celine might make assumptions about these parameters. Using 'put' with a key of 'proviso,' Maxima saves these assumptions on the input label.

To use this function, first load the package integer_sequence, opsubst, and to_poly_solve.

Examples:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) celine(n!,n,k,1,0);
(%o5)       {fff(n + 1, k) - n fff(n, k) - fff(n, k)}
```

Verification that this result is correct:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) g1:{fff(n+1,k)-n*fff(n,k)-fff(n,k)};
(%o5)       {fff(n + 1, k) - n fff(n, k) - fff(n, k)}
(%i6) ratsimp(minfactorial(first(g1))),fff(n,k) := n!;
(%o6)                           0
```

An example with parameters including the test that the result of the example is correct:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
                             (a)  (- k)
                                k      n
(%o5)                       -----------
                                (b)
                                   k
```

```
(%i6) recur : celine(e,n,k,2,1);
(%o6) {fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
 + n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
 - fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
    2
 - n  fff(n, k)}
(%i7) /* Test this result for correctness */
(%i8) first(%), fff(n,k) := ''(e)$
(%i9) makefact(makegamma(%))$
(%o9)                           0
(%i10) minfactorial(factor(minfactorial(factor(%))));
```

The proviso data suggests that setting a = b may result in a lower order recursion which is shown by the following example:

```
(%i1) load("integer_sequence")$
(%i2) load("opsubst")$
(%i3) load("to_poly_solve")$
(%i4) load("celine")$
(%i5) e : pochhammer(a,k) * pochhammer(-k,n) / (pochhammer(b,k));
                           (a)  (- k)
                              k      n
(%o5)                         -----------
                              (b)
                                 k
(%i6) recur : celine(e,n,k,2,1);
(%o6) {fff(n + 2, k + 1) - fff(n + 2, k) - b fff(n + 1, k + 1)
 + n ((- fff(n + 1, k + 1)) + 2 fff(n + 1, k) - a fff(n, k)
 - fff(n, k)) + a (fff(n + 1, k) - fff(n, k)) + 2 fff(n + 1, k)
    2
 - n  fff(n, k)}
(%i7) get('%,'proviso);
(%o7)                          false
(%i8) celine(subst(b=a,e),n,k,1,1);
(%o8) {fff(n + 1, k + 1) - fff(n + 1, k) + n fff(n, k)
                                                    + fff(n, k)}
```

46 Package clebsch_gordan

46.1 Functions and Variables for clebsch_gordan

`clebsch_gordan` (*j1*, *j2*, *m1*, *m2*, *j*, *m*) [Function]
Compute the Clebsch-Gordan coefficient `<j1, j2, m1, m2 | j, m>`.

`racah_v` (*a*, *b*, *c*, *a1*, *b1*, *c1*) [Function]
Compute Racah's V coefficient (computed in terms of a related Clebsch-Gordan coefficient).

`racah_w` (*j1*, *j2*, *j5*, *j4*, *j3*, *j6*) [Function]
Compute Racah's W coefficient (computed in terms of a Wigner `6j` symbol)

`wigner_3j` (*j1*, *j2*, *j3*, *m1*, *m2*, *m3*) [Function]
Compute Wigner's `3j` symbol (computed in terms of a related Clebsch-Gordan coefficient).

`wigner_6j` (*j1*, *j2*, *j3*, *j4*, *j5*, *j6*) [Function]
Compute Wigner's `6j` symbol.

`wigner_9j` (*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *j*,) [Function]
Compute Wigner's `9j` symbol.

47 Package cobyla

47.1 Introduction to cobyla

`fmin_cobyla` is a Common Lisp translation (via `f2cl`) of the Fortran constrained optimization routine COBYLA by Powell[1][2][3].

COBYLA minimizes an objective function $F(X)$ subject to M inequality constraints of the form $g(X) \geq 0$ on X, where X is a vector of variables that has N components.

Equality constraints $g(X) = 0$ can often be implemented by a pair of inequality constraints $g(X) \geq 0$ and $-g(X) \geq 0$. Maxima's interface to COBYLA allows equality constraints and internally converts the equality constraints to a pair of inequality constraints.

The algorithm employs linear approximations to the objective and constraint functions, the approximations being formed by linear interpolation at $N+1$ points in the space of the variables. The interpolation points are regarded as vertices of a simplex. The parameter *RHO* controls the size of the simplex and it is reduced automatically from *RHOBEG* to *RHOEND*. For each *RHO* the subroutine tries to achieve a good vector of variables for the current size, and then *RHO* is reduced until the value *RHOEND* is reached. Therefore, *RHOBEG* and *RHOEND* should be set to reasonable initial changes to and the required accuracy in the variables respectively, but this accuracy should be viewed as a subject for experimentation because it is not guaranteed. The routine treats each constraint individually when calculating a change to the variables, rather than lumping the constraints together into a single penalty function. The name of the subroutine is derived from the phrase Constrained Optimization BY Linear Approximations.

References:

[1] Fortran Code is from http://plato.asu.edu/sub/nlores.html#general

[2] M. J. D. Powell, "A direct search optimization method that models the objective and constraint functions by linear interpolation," in Advances in Optimization and Numerical Analysis, eds. S. Gomez and J.-P. Hennart (Kluwer Academic: Dordrecht, 1994), p. 51-67. doi:10.1007/978-94-015-8330-5_4

[3] M. J. D. Powell, "Direct search algorithms for optimization calculations," Acta Numerica 7, 287-336 (1998). doi:10.1017/S0962492900002841 Also available as University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Numerical Analysis Group, Report NA1998/04 from https://web.archive.org/web/20160607190705/http://www.damtp.cam.ac.uk:80/user/na/reports.html

47.2 Functions and Variables for cobyla

`fmin_cobyla` [Function]

`fmin_cobyla` (*F*, *X*, *Y*)
`fmin_cobyla` (*F*, *X*, *Y*, *optional_args*)

Returns an approximate minimum of the expression F with respect to the variables X, subject to an optional set of constraints. Y is a list of initial guesses for X.

F must be ordinary expressions, not names of functions or lambda expressions.

`optional_args` represents additional arguments, specified as *`symbol`* `=` *`value`*. The optional arguments recognized are:

`constraints`
: List of inequality and equality constraints that must be satisfied by X. The inequality constraints must be actual inequalities of the form $g(X) \geq h(X)$ or $g(X) \leq h(X)$. The equality constraints must be of the form $g(X) = h(X)$.

`rhobeg`
: Initial value of the internal *RHO* variable which controls the size of simplex. (Defaults to 1.0)

`rhoend`
: The desired final value rho parameter. It is approximately the accuracy in the variables. (Defaults to 1d-6.)

`iprint`
: Verbose output level. (Defaults to 0)
  - 0 - No output
  - 1 - Summary at the end of the calculation
  - 2 - Each new value of RHO and SIGMA is printed, including the vector of variables, some function information when RHO is reduced.
  - 3 - Like 2, but information is printed when F(X) is computed.

`maxfun`
: The maximum number of function evaluations. (Defaults to 1000).

On return, a vector is given:

1. The value of the variables giving the minimum. This is a list of elements of the form *`var`* `=` *`value`* for each of the variables listed in X.
2. The minimized function value
3. The number of function evaluations.
4. Return code with the following meanings

   0 - No errors.

   1 - Limit on maximum number of function evaluations reached.

   2 - Rounding errors inhibiting progress.

   -1 - *MAXCV* value exceeds *RHOEND*. This indicates that the constraints were probably not satisfied. User should investigate the value of the constraints.

MAXCV stands for "MAXimum Constraint Violation" and is the value of $max(0.0, -c1(x), -c2(x), ... - cm(x))$ where $ck(x)$ denotes the k'th constraint function. (Note that maxima allows constraints of the form $f(x) = g(x)$, which are internally converted to $f(x) - g(x) >= 0$ and $g(x) - f(x) >= 0$ which is required by COBYLA).

`load("fmin_cobyla")` loads this function.

`bf_fmin_cobyla` [Function]

`bf_fmin_cobyla` (*`F`*, *`X`*, *`Y`*)

`bf_fmin_cobyla` (*`F`*, *`X`*, *`Y`*, *optional_args*)

This function is identical to `fmin_cobyla`, except that bigfloat operations are used, and the default value for *rhoend* is `10^(fpprec/2)`.

See fmin_cobyla for more information.

load("bf_fmin_cobyla") loads this function.

47.3 Examples for cobyla

Minimize $x_1 x_2$ with $1 - x_1^2 - x_2^2 \geq 0$. The theoretical solution is

$$x_1 = \frac{1}{\sqrt{2}}$$
$$x_2 = -\frac{1}{\sqrt{2}}$$

```
(%i1) load("fmin_cobyla")$
(%i2) fmin_cobyla(x1*x2, [x1, x2], [1,1],
            constraints = [x1^2+x2^2<=1], iprint=1);
   Normal return from subroutine COBYLA

   NFVALS =   64   F =-5.000000E-01    MAXCV = 2.000122E-12
   X = 7.071076E-01  -7.071060E-01
(%o2) [[x1 = 0.707107555323284, x2 = - 0.7071060070503778],
                                     - 0.4999999999998015, 64, 0]
```

Here is the same example but the constraint is $x_1^2 + x_2^2 \leq -1$ which is impossible over the reals.

```
(%i1) fmin_cobyla(x1*x2, [x1, x2], [1,1],
         constraints = [x1^2+x2^2 <= -1], iprint=1);
   Normal return from subroutine COBYLA

   NFVALS =   65   F = 3.016417E-13    MAXCV = 1.000000E+00
   X =-3.375179E-07  -8.937057E-07
(%o1) [[x1 = - 3.375178983064622e-7, x2 = - 8.937056510780022e-7],
                                                3.016416530564557e-13, 65, - 1]
(%i2) subst(%o1[2], [x1^2+x2^2 <= -1]);
(%o2)                  [- 6.847914590915444e-13 <= - 1]
```

We see the return code (%o1[4]) is -1 indicating that the constraints may not be satisfied. Substituting the solution into the constraint equation as shown in %o2 shows that the constraint is, of course, violated.

There are additional examples in the share/cobyla/ex directory and in share/cobyla/rtest_cobyla.mac.

48 Package colnew

48.1 Introduction to colnew

colnew solves mixed-order systems of boundary-value problems (BVPs) in ordinary differential equations(ODEs). It is a Common Lisp translation (via *f2cl*) of the Fortran routine COLNEW (see [Bader&Ascher 1987], page 708).

The method uses collocation at Gaussian points and interpolation using basis functions. Approximate solutions are computed on a sequence of automatically selected meshes until a user-specified set of tolerances is satisfied. A damped Newton's method is used for the nonlinear iteration.

COLNEW has some advanced features:

- **Continuation:** Most iterative numerical solvers require an initial guess. There are parameter dependent problems that have an obvious solution for a special value of the parameter (for example a coupling constant vanishes). One can then slowly vary the parameter, solving the problem numerically using as a guess the previous solution, until one gets into a strong coupling regime, hence the name continuation. Sometimes an artificial parameter is introduced to allow the use of continuation.
- **Singularities:** Another interesting feature of COLNEW is that one can manage singularities in the differential equation by setting interior "boundary points" on the singularities. This is because the differential equation is only imposed in the interior of the intervals, but the solution is checked for continuity and continuity of the derivative on the boundary of the intervals.
- **Stiff differential equations:** COLNEW can be used for solving some stiff differential equations due to the use of adaptive meshes.

The maxima interface to COLNEW exposes the full power and complexity of the Fortran 77 implementation.

COLNEW is a modification of the package COLSYS (see [Ascher 1981a], page 708, and [Ascher 1981b], page 708). It incorporates a new basis representation replacing B-splines, and improvements for the linear and nonlinear algebraic equation solvers. The package can be referenced as either COLNEW or COLSYS.

Many practical problems that are not in the standard form accepted by COLNEW can be converted into this form. See [Asher&Russell 1981], page 708.

48.2 Functions and Variables for colnew

`colnew_expert` [Function]

`colnew_expert` (*`ncomp`*, *`m`*, *`aleft`*, *`aright`*, *`zeta`*, *`ipar`*, *`ltol`*, *`tol`*, *`fixpnt`*, *`ispace`*, *`fspace`*, *`iflag`*, *`fsub`*, *`dfsub`*, *`gsub`*, *`dgsub`*, *`guess`*)

colnew_expert solves mixed-order systems of boundary-value problems (BVPs) in ordinary differential equations (ODEs) using a numerical collocation method.

colnew_expert returns the list [*iflag*, *fspace*, *ispace*]. *iflag* is an error flag. Lists *fspace* and *ispace* contain the state of the solution and can be: used by `colnew_appsln` to

calculate solution values at arbitrary points in the solution domain; and passed back to *colnew_expert* to restart the solution process with different arguments.

The function arguments are:

`ncomp` Number of differential equations (ncomp ≤ 20)

`m` Integer list of length *ncomp*. m_j is the order of the j-th differential equation, with $1 \leq m_j \leq 4$ and $m^* = \sum_j m_j \leq 40$.

`aleft` Left end of interval

`aright` Right end of interval

`zeta` *zeta* (ζ) is a real list of length m^*. ζ_j is the j-th boundary or side condition point. The list ζ must be ordered, with $\zeta_j \leq \zeta_{j+1}$. All side condition points must be mesh points in all meshes used, see description of ipar[11] and fixpnt below.

`ipar` A integer list of length 11. The parameters in ipar are:

- *ipar[1]*
  - 0, if the problem is linear
  - 1, if the problem is nonlinear
- *ipar[2]* (= k)
  Number of collocation points per subinterval , where $\max_j m_j \leq k \leq 7$. If *ipar[2]*=0 then colnew sets $k = \max\left(\max_i m_i + 5, 5 - \max_i m_i\right)$.
- *ipar[3]* (= n)
  Number of subintervals in the initial mesh.
  If *ipar[3]* = 0 then colnew arbitrarily sets $n = 5$.
- *ipar[4]* (= ntol)
  Number of solution and derivative tolerances.
  Require $0 <$ ntol $< m^*$.
- *ipar[5]* (= ndimf)
  The length of list *fspace*. Its size provides a constraint on *nmax*. Choose ipar[5] according to the formula ipar[5] $\geq$ nmax$\cdot$nsizef where nsizef $= 4 + 3m^* + (5 + \text{kd}) \times \text{kdm} + (2m^* - \text{nrec}) \times 2m^*$.
- *ipar[6]* (= ndimi)
  The length of list *ispace*. Its size provides a constraint on *nmax*, the maximum number of subintervals. Choose *ipar[6]* according to the formula ipar[6] = nmax $\times$ nsizei where nsizei $= 3 +$ kdm with

  $$\text{kdm} = \text{kd} + m^*$$
  $$\text{kd} = k + \text{ncomp}$$
  $$\text{nrec} = \textit{number of right end boundary conditions}$$

  .
- *ipar[7]* (= iprint)
  output control
  - -1, for full diagnostic printout

- 0, for selected printout
- 1, for no printout

- *ipar[8]* (= iread)
  - 0, causes colnew to generate a uniform initial mesh.
  - 1, if the initial mesh is provided by the user. it is defined in fspace as follows: the mesh aleft=x[1]<x[2]< ... <x[n]<x[n+1]=aright will occupy fspace[1], ..., fspace[n+1]. the user needs to supply only the interior mesh points fspace[j] = x[j], j = 2, ..., n.
  - 2, if the initial mesh is supplied by the user as with *ipar[8]*=1, and in addition no adaptive mesh selection is to be done.
- *ipar[9]* (= iguess)
  - 0, if no initial guess for the solution is provided
  - 1, if an initial guess is provided by the user in subroutine guess.
  - 2, if an initial mesh and approximate solution coefficients are provided by the user in *fspace*. (the former and new mesh are the same).
  - 3, if a former mesh and approximate solution coefficients are provided by the user in *fspace*, and the new mesh is to be taken twice as coarse; i.e.,every second point from the former mesh.
  - 4, if in addition to a former initial mesh and approximate solution coefficients, a new mesh is provided in *fspace* as well. (see description of output for further details on iguess = 2, 3, and 4.)
- *ipar[10]*
  - 0, if the problem is regular
  - 1, if the first relax factor is =rstart, and the nonlinear iteration does not rely on past convergence (use for an extra sensitive nonlinear problem only).
  - 2, if we are to return immediately upon (a) two successive non-convergences, or (b) after obtaining error estimate for the first time.
- *ipar[11]* (= nfxpnt , the dimension of fixpnt)
  The number of fixed points in the mesh other than *aleft* and *aright*. The code requires that all side condition points other than *aleft* and *aright* (see description of *zeta*) must be included as fixed points in *fixpnt*.

`ltol` A list of length *ntol=ipar[4]*. *ltol[j]=k* specifies that the j-th tolerance in *tol* controls the error in the k-th component of z(u).

The list *ltol* must be ordered with $1 \leq ltol[1] < ltol[2] < ... < ltol[ntol] \leq mstar$.

`tol` An list of length *ntol=ipar[4]*. *tol[j]* is the error tolerance on the ltol[j]-th component of z(u).

Thus, the code attempts to satisfy for j=1,...,ntol on each subinterval $abs(z(v)-z(u))[k] \leq tol(j)*abs(z(u))[k]+tol(j)$ if v(x) is the approximate solution vector.

`fixpnt` An list of length *ipar[11]*. It contains the points, other than *aleft* and *aright*, which are to be included in every mesh. All side condition points other than *aleft* and *aright* (see *zeta*) be included as fixed points in *fixpnt*.

`ispace` An integer work list of length *ipar[6]*.

`fspace` A real work list of length *ipar[5]*.

`fsub` *fsub* is a function f(x,z1,...,z[mstar]) which realizes the system of ODEs.

It returns a list of *ncomp* values, one for each ODE. Each value is the highest order derivative in each ode in terms of of x,z1,...,z[mstar] .

`dfsub` *dfsub* is a function df(x,z1,...,z[mstar]) for evaluating the Jacobian of f.

`gsub` Name of subroutine gsub(i,z1,z2,...,z[mstar]) for evaluating the i-th component of the boundary value function g(z1,...,z[mstar]). The independent variable x is not an argument of g. The value *x=zeta[i]* must be substituted in advance.

`dgsub` Name of subroutine dgsub(i,z1,...,z[mstar]) for evaluating the i-th row of the Jacobian of g(z1,...,z[mstar]).

`guess` Name of subroutine to evaluate the initial approximation for (u(x)) and for dmval(u(x))= vector of the mj-th derivatives of u(x). This subroutine is needed only if using *ipar(9)* = 1.

The return value of *colnew_expert* is the list *[iflag, fspace, ispace]*, where:

`iflag` The mode of return from *colnew_expert*.

- = 1 for normal return
- = 0 if the collocation matrix is singular.
- = -1 if the expected no. of subintervals exceeds storage specifications.
- = -2 if the nonlinear iteration has not converged.
- = -3 if there is an input data error.

`fspace` A list of floats of length *ipar[5]*.

`ispace` A list of integers of length *ipar[6]*.

`colnew_appsln` uses *fspace* and *ispace* to calculate solution values at arbitrary points. The lists can also be used to restart the solution process with modified meshes and parameters.

`colnew_appsln` [Function]

`colnew_appsln` (*x*, *zlen*, *fspace*, *ispace*)

Return a list of solution values from `colnew_expert` results.

The function arguments are:

`x` List of x-coordinates to calculate solution.

`zlen`	*mstar*, the length of the solution list z
`fspace`	List *fspace* returned from `colnew_expert`.
`ispace`	List *ispace* returned from `colnew_expert`.

48.3 Examples for colnew

COLNEW is best learned by example.

48.3.1 Example 1: A uniformly loaded beam of variable stiffness

The problem describes a uniformly loaded beam of variable stiffness, simply supported at both ends.

The problem from [Gawain&Ball 1978], page 708, and is Example 1 from [Ascher 1981a], page 708. The maxima code is in file share/colnew/prob1.mac and a Fortran implementation is in share/colnew/ex1.

The differential equation is

$$(x^3u'')'' = x^3u'''' + 6x^2u''' + 6xu'' = 1 \quad \text{over} \quad 1 \le x \le 2$$

with boundary conditions

$$\begin{aligned} u(1) &= 0 \\ u''(1) &= 0 \\ u(2) &= 0 \\ u''(2) &= 0 \end{aligned}$$

The exact solution is

$$u(x) = \frac{10\log 2 - 3}{4}(1-x) + \frac{1}{2}\left(\frac{1}{x} + (3+x)\log x - x\right)$$

There is $nconc = 1$ differential equation of fourth order. The list of orders $m = [4]$ and $m^* = \sum_j m_j = 4$.

The unknown vector of length m^* is

$$\begin{aligned} z(x) &= [z_1(x), z_2(x), z_3(x), z_4(x)] \\ &= [u(x), u'(x), u''(x), u'''(x)] \end{aligned}$$

The differential equation is expressed as

$$u''''(x) = F(x, z_1, z_2, z_3, z_4) = 1 - 6x^2z_3 - 6xz_2$$

There are $m^* = 4$ boundary conditions. They are given by a function $G(z_1, z_2, z_3, z_4)$ that returns a list of length m^*. The j-th boundary condition applies at $x = \zeta_j$ and is satisfied when $g[j] = 0$. We have

j	zeta[j]	Condition	g[j]
1	1.0	$u = 0$	z_1
2	1.0	$u'' = 0$	z_3
3	2.0	$u = 0$	z_1
4	2.0	$u'' = 0$	z_3

giving $\zeta = [1.0, 1, 0, 2.0, 2.0]$ and $G(z_1, z_2, z_3, z_4) = [z_1, z_3, z_1, z_3]$.

The Jacobians df and dg of f and g respectively are determined symbolically.

The solution uses the default five collocation points per subinterval and the first mesh contains only one subinterval. The maximum error magnitude in the approximate solution is evaluated at 100 equidistant points by function `colnew_appsln` using the results returned by COLNEW and compared to the estimates from the code.

```
(%i1) load("colnew")$

(%i2) /* One differential equation of order 4 */
 m : [4];

(%o2)                           [4]
(%i3) /* Location of boundary conditions */
 zeta : float([1,1,2,2]);

(%o3)                 [1.0, 1.0, 2.0, 2.0]
(%i4) /* Set up parameter array.  Use defaults for all except as shown */
 ipar : makelist(0,k,1,11);
(%o4)           [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
(%i5) /* initial mesh size */
 ipar[3] : 1$
(%i6) /* number of tolerances */
 ipar[4] : 2$
(%i7) /* size of real work array */
 ipar[5] : 2000$
(%i8) /* size of integer work array */
 ipar[6] : 200$

(%i9) /* Two error tolerances (on u and its second derivative) */
 ltol : [1, 3];
(%o9)                         [1, 3]
(%i10) tol : [1d-7, 1d-7];

(%o10)                  [1.0e-7, 1.0e-7]
(%i11) /* Real work array */
 fspace : makelist(0d0, k, 1, ipar[5])$
(%i12) /* Integer work array */
 ispace : makelist(0, k, 1, ipar[6])$
(%i13) /* no internal fixed points */
 fixpnt : []$
```

```
(%i14) /* Define the equations */
 fsub(x, z1, z2, z3, z4) := [(1-6*x^2*z4-6*x*z3)/x^3];
                                        2
                                  1 - 6 x  z4 + (- 6) x z3
(%o14) fsub(x, z1, z2, z3, z4) := [-----------------------]
                                              3
                                             x
(%i15) df : jacobian(fsub(x,z1, z2, z3, z4),[z1,z2,z3,z4]);
                       [         6     6 ]
(%o15)                 [ 0  0  - --  - - ]
                       [          2    x ]
                       [         x       ]
(%i16) define(dfsub(x, z1, z2, z3, z4), df);

                                    [         6     6 ]
(%o16)   dfsub(x, z1, z2, z3, z4) := [ 0  0  - --  - - ]
                                    [          2    x ]
                                    [         x       ]
(%i17) g(z1, z2, z3, z4) := [z1, z3, z1, z3];
(%o17)        g(z1, z2, z3, z4) := [z1, z3, z1, z3]
(%i18) gsub(i, z1, z2, z3, z4) :=
 subst(['z1=z1,'z2=z2,'z3=z3,'z4=z4], g(z1, z2, z3, z4)[i]);

(%o18) gsub(i, z1, z2, z3, z4) :=
subst(['z1 = z1, 'z2 = z2, 'z3 = z3, 'z4 = z4],
g(z1, z2, z3, z4) )
                  i
(%i19) dg:jacobian(g(z1, z2, z3, z4), [z1,z2,z3,z4]);
                        [ 1  0  0  0 ]
                        [            ]
                        [ 0  0  1  0 ]
(%o19)                  [            ]
                        [ 1  0  0  0 ]
                        [            ]
                        [ 0  0  1  0 ]
(%i20) dgsub(i, z1, z2, z3, z4) :=
 subst(['z1=z1,'z2=z2,'z3=z3,'z4=z4], row(dg, i)[1]);

(%o20) dgsub(i, z1, z2, z3, z4) :=
     subst(['z1 = z1, 'z2 = z2, 'z3 = z3, 'z4 = z4], row(dg, i) )
                                                                 1
(%i21) /* Exact solution */
  exact(x) := [.25*(10.*log(2.)-3.)*(1.-x) + .5*(1./x+(3.+x)*log(x)-x),
             -.25*(10.*log(2.)-3.) + .5*(-1./x/x+log(x)+(3.+x)/x-1.),
             .5*(2./x**3+1./x-3./x/x),
             .5*(-6./x**4-1./x/x+6./x**3)]$
```

```
(%i22) [iflag, fspace, ispace] :
 colnew_expert(1, m, 1d0, 2d0, zeta, ipar, ltol, tol, fixpnt, ispace, fspace,
             0, fsub, dfsub, gsub, dgsub, dummy)$


 VERSION *COLNEW* OF COLSYS .


 THE MAXIMUM NUMBER OF SUBINTERVALS IS MIN (  12 (ALLOWED FROM FSPACE),  16 (ALLOWED F

 THE NEW MESH (OF    1 SUBINTERVALS),
    1.000000    2.000000

 THE NEW MESH (OF    2 SUBINTERVALS),
    1.000000    1.500000    2.000000

 THE NEW MESH (OF    4 SUBINTERVALS),
    1.000000    1.250000    1.500000    1.750000    2.000000
(%i23) /* Calculate the error at 101 points using the known exact solution */
 block([x : 1,
       err : makelist(0d0, k, 1, 4),
       j],
  for j : 1 thru 101 do
    block([],
      zval : colnew_appsln([x], 4, fspace, ispace)[1],
      u : float(exact(x)),
      err : map(lambda([a,b], max(a,b)), err, abs(u-zval)),
      x : x + 0.01),
  print("The exact errors are:"),
  printf(true, "   ~{ ~11,4e~}~%", err));
The exact errors are:
     1.7389e-10   6.2679e-9   2.1843e-7   9.5743e-6
(%o23)                       false
```

48.3.2 Example 2: Deformation of a spherical cap

These equations describe the small finite deformation of a thin shallow spherical cap of constant thickness subject to a quadratically varying axisymmetric external pressure distribution superimposed on a uniform internal pressure distribution. The problem is described in [Parker&Wan 1984], page 708, and is Example 2 from [Ascher 1981a], page 708. The maxima code is in file share/colnew/prob2.mac and a Fortran implementation is in share/colnew/ex2.

There are two nonlinear differential equations for ϕ and ψ over $0 < x < 1$.

$$\frac{\epsilon^4}{\mu}\left[\phi'' + \frac{1}{x}\phi' - \frac{1}{x^2}\phi\right] + \psi\left(1 - \frac{1}{x}\phi\right) - \phi = -\gamma x\left(1 - \frac{x^2}{2}\right)$$
$$\mu\left[\psi'' + \frac{1}{x}\psi' - \frac{1}{x^2}\psi\right] - \phi\left(1 - \frac{\phi}{2x}\right) = 0$$

subject to boundary conditions $\phi = 0$ and $x\psi' - 0.3\psi + 0.7x = 0$ at $x = 0$ and $x = 1$.

For $\epsilon = \mu = 0.01$, two solutions exists. These are obtained by starting the nonlinear iteration from two different guesses to the solution: initially with the default initial guess; and secondly, with the initial conditions given by the function *solutn*.

There are $nconc = 2$ differential equations of second order. The list of orders $m = [2, 2]$ and $m^* = \sum_i m_i = 4$.

The vector of unknowns of length $m^* = 4$ is $z(x) = [\phi(x), \phi'(x), \psi(x), \psi'(x)]$.

The differential equation is expressed as

$$F(x, z_1, z_2, z_3, z_4) = \begin{bmatrix} \phi''(x) \\ \psi''(x) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{z_1}{x^2} - \frac{z_2}{x} + \left[z_1 - z_3\left(1 - \frac{z_1}{x}\right) - \gamma x\left(1 - \frac{x^2}{2}\right)\right]\frac{\mu}{\epsilon^4} \\ \frac{z_3}{x^2} - \frac{z_4}{x} + \frac{z_1}{\mu}\left(1 - \frac{z_1}{2x}\right) \end{bmatrix}$$

There are four boundary conditions given by list *zeta* and function $G(z_1, z_2, z_3, z_4)$.

j	**zeta[j]**	**Condition**	**g[j]**
1	0.0	$\phi = 0$	z_1
2	0.0	$x\psi' - 0.3\psi + 0.7x = 0$	z_3
3	1.0	$\phi = 0$	z_1
4	1.0	$x\psi' - 0.3\psi + 0.7x = 0$	$z_4 - 0.3\ z_3 + 0.7$

giving $\zeta = [0.0, 0.0, 1.0, 1.0]$ and $G(z_1, z_2, z_3, z_4) = [z_1, z_3, z_1, z_4 - 0.3z_3 + 0.7]$.

Note that x is not an argument of function G. The value of $x = \zeta_j$ must be substituted.

```
(%i1) load("colnew")$

(%i2) /* Define constants */
 gamma : 1.1;
(%o2)                          1.1
(%i3) eps : 0.01;
(%o3)                         0.01
(%i4) dmu : eps;
(%o4)                         0.01
(%i5) eps4mu : eps^4/dmu;
(%o5)                        1.0e-6
(%i6) xt : sqrt(2*(gamma-1)/gamma);
(%o6)                0.42640143271122105
(%i7) /* Number of differential equations */
 ncomp : 2;
(%o7)                          2
```

```
(%i8) /* Orders */
 m : [2, 2];

(%o8)                         [2, 2]
(%i9) /* Interval ends */
 aleft : 0.0;
(%o9)                          0.0
(%i10) aright : 1.0;

(%o10)                         1.0
(%i11) /* Locations of side conditions */
 zeta : float([0, 0, 1, 1])$
(%i12) /* Set up parameter array.  */
 ipar : makelist(0,k,1,11);
(%o12)          [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
(%i13) /* Non-linear prob */
 ipar[1] : 1;
(%o13)                          1
(%i14) /* 4 collocation points per subinterval */
 ipar[2] : 4;
(%o14)                          4
(%i15) /* Initial uniform mesh of 10 subintervals */
 ipar[3] : 10;
(%o15)                         10
(%i16) ipar[8] : 0;
(%o16)                          0
(%i17) /* Size of fspace, ispace */
 ipar[5] : 40000;
(%o17)                        40000
(%i18) ipar[6] : 2500;
(%o18)                        2500
(%i19) /* No output */
 ipar[7] : 1;
(%o19)                          1
(%i20) /* No initial approx is provided */
 ipar[9] : 0;
(%o20)                          0
(%i21) /* Regular problem */
 ipar[10] : 0;
(%o21)                          0
(%i22) /* No fixed points in mesh */
 ipar[11] : 0;
(%o22)                          0
(%i23) /* Tolerances on all components */
 ipar[4] : 4;

(%o23)                          4
```

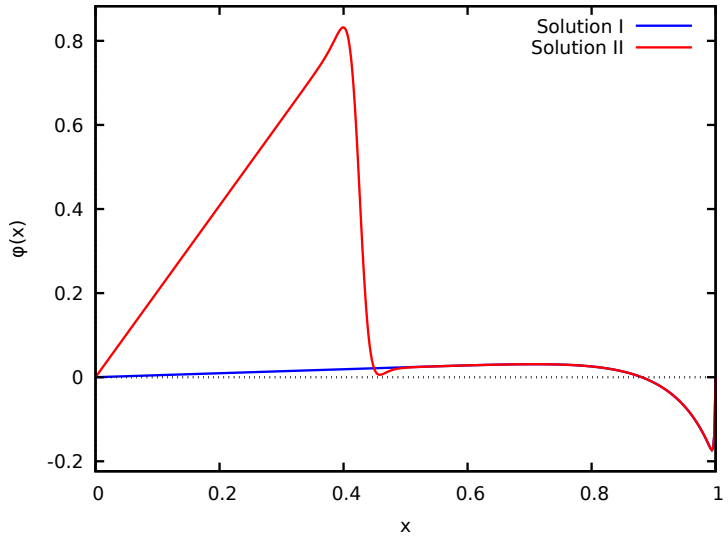
```
(%i24) /* Tolerances on all four components */
 ltol : [1, 2, 3, 4];
(%o24)                    [1, 2, 3, 4]
(%i25) tol : [1d-5, 1d-5, 1d-5, 1d-5];

(%o25)          [1.0e-5, 1.0e-5, 1.0e-5, 1.0e-5]
(%i26) fspace : makelist(0d0, k, 1, ipar[5])$
(%i27) ispace : makelist(0, k, 1, ipar[6])$
(%i28) fixpnt : []$

(%i29) /* Define the equations */
 fsub(x, z1, z2, z3, z4) :=
 [z1/x/x - z2/x + (z1-z3*(1-z1/x) - gamma*x*(1-x*x/2))/eps4mu,
  z3/x/x - z4/x + z1*(1-z1/2/x)/dmu];

(%o29) fsub(x, z1, z2, z3, z4) :=
 z1                     z1                      x x
 --        z1 - z3 (1 - --) + (- gamma) x (1 - ---)
 x    z2                x                        2
[-- - -- + ----------------------------------------,
 x    x                     eps4mu
                  z1
                  --
z3                2
--        z1 (1 - --)
x    z4           x
-- - -- + -----------]
x    x        dmu
(%i30) df : jacobian(fsub(x,z1, z2, z3, z4),[z1,z2,z3,z4]);
(%o30)
      [             z3        1      1             z1           ]
      [  1000000.0 (-- + 1) + --   - -  1000000.0 (-- - 1)   0  ]
      [             x          2     x             x            ]
      [                       x                                 ]
      [                                                         ]
      [            z1     50.0 z1               1             1 ]
      [ 100.0 (1 - ---) - -------   0           --          - - ]
      [            2 x       x                 2            x ]
      [                                        x               ]
(%i31) dfsub(x, z1, z2, z3, z4) :=
  subst(['x=x,'z1=z1,'z2=z2,'z3=z3,'z4=z4], df);

(%o31) dfsub(x, z1, z2, z3, z4) :=
      subst(['x = x, 'z1 = z1, 'z2 = z2, 'z3 = z3, 'z4 = z4], df)
(%i32) g(z1, z2, z3, z4) := [z1, z3, z1, z4 - 0.3*z3 + .7];
(%o32) g(z1, z2, z3, z4) := [z1, z3, z1, z4 - 0.3 z3 + 0.7]
```

```
(%i33) gsub(i, z1, z2, z3, z4) :=
    subst(['z1=z1,'z2=z2,'z3=z3,'z4=z4], g(z1, z2, z3, z4)[i]);

(%o33) gsub(i, z1, z2, z3, z4) :=
subst(['z1 = z1, 'z2 = z2, 'z3 = z3, 'z4 = z4],
g(z1, z2, z3, z4) )
                  i
(%i34) dg:jacobian(g(z1, z2, z3, z4), [z1,z2,z3,z4]);
                        [ 1  0    0    0 ]
                        [                 ]
                        [ 0  0    1    0 ]
(%o34)                  [                 ]
                        [ 1  0    0    0 ]
                        [                 ]
                        [ 0  0  - 0.3  1 ]
(%i35) dgsub(i, z1, z2, z3, z4) := subst(['z1=z1,'z2=z2,'z3=z3,'z4=z4], row(dg, i)[1])

(%o35) dgsub(i, z1, z2, z3, z4) :=
     subst(['z1 = z1, 'z2 = z2, 'z3 = z3, 'z4 = z4], row(dg, i) )
                                                                 1
(%i36) /* Initial approximation function for second run */
 solutn(x) :=
 block([cons : gamma*x*(1-0.5*x*x),
       dcons : gamma*(1-1.5*x*x),
       d2cons : -3*gamma*x],
  if is(x > xt) then
    [[0, 0, -cons, -dcons],
     [0, -d2cons]]
  else
    [[2*x, 2, -2*x + cons, -2 + dcons],
     [0, d2cons]]);

(%o36) solutn(x) := block([cons : gamma x (1 - 0.5 x x),
dcons : gamma (1 - 1.5 x x), d2cons : - 3 gamma x],
if is(x > xt) then [[0, 0, - cons, - dcons], [0, - d2cons]]
 else [[2 x, 2, - 2 x + cons, - 2 + dcons], [0, d2cons]])
(%i37) /* First run with default initial guess */
 [iflag, fspace, ispace] :
 colnew_expert(ncomp, m, aleft, aright, zeta, ipar, ltol, tol, fixpnt, ispace, fspace,
 0, fsub, dfsub, gsub, dgsub, dummy)$

(%i38) /* Check return status iflag, 1 = success */
 iflag;

(%o38)                          1
```

```
(%i39) /* Print values of solution at x = 0,.05,...,1 */
 x : 0;
(%o39)                          0
(%i40) for j : 1 thru 21 do
 block([],
   zval : colnew_appsln([x], 4, fspace, ispace)[1],
   printf(true, "~5,2f  ~{~15,5e~}~%", x, zval),
   x : x + 0.05);

 0.00       0.00000e+0     4.73042e-2   -2.72134e-31    -1.10497e+0
 0.05       2.36520e-3     4.73037e-2    -5.51761e-2    -1.10064e+0
 0.10       4.73037e-3     4.73030e-2    -1.09919e-1    -1.08765e+0
 0.15       7.09551e-3     4.73030e-2    -1.63796e-1    -1.06600e+0
 0.20       9.46069e-3     4.73039e-2    -2.16375e-1    -1.03569e+0
 0.25       1.18259e-2     4.73040e-2    -2.67221e-1    -9.96720e-1
 0.30       1.41911e-2     4.73020e-2    -3.15902e-1    -9.49092e-1
 0.35       1.65562e-2     4.72980e-2    -3.61986e-1    -8.92804e-1
 0.40       1.89215e-2     4.72993e-2    -4.05038e-1    -8.27857e-1
 0.45       2.12850e-2     4.72138e-2    -4.44627e-1    -7.54252e-1
 0.50       2.36370e-2     4.67629e-2    -4.80320e-1    -6.72014e-1
 0.55       2.59431e-2     4.51902e-2    -5.11686e-1    -5.81260e-1
 0.60       2.81093e-2     4.07535e-2    -5.38310e-1    -4.82374e-1
 0.65       2.99126e-2     2.98538e-2    -5.59805e-1    -3.76416e-1
 0.70       3.08743e-2     5.53985e-3    -5.75875e-1    -2.65952e-1
 0.75       3.00326e-2    -4.51680e-2    -5.86417e-1    -1.56670e-1
 0.80       2.55239e-2    -1.46617e-1    -5.91753e-1    -6.04539e-2
 0.85       1.37512e-2    -3.46952e-1    -5.93069e-1    -1.40102e-3
 0.90      -1.25155e-2    -7.52826e-1    -5.93303e-1    -2.86234e-2
 0.95      -6.94274e-2    -1.65084e+0    -5.99062e-1    -2.48115e-1
 1.00      2.63400e-14     1.19263e+2    -6.25420e-1    -8.87626e-1
(%o40)                        done
(%i41) /* Second run with initial guess */
 ipar[9] : 1;
(%o41)                          1
(%i42) [iflag, fspace, ispace] :
 colnew_expert(ncomp, m, aleft, aright, zeta, ipar, ltol, tol, fixpnt, ispace, fspace,
 0, fsub, dfsub, gsub, dgsub, solutn)$

(%i43) /* Check return status iflag, 1 = success */
 iflag;

(%o43)                          1
(%i44) /* Print values of solution at x = 0,.05,...,1 */
 x : 0;
(%o44)                          0
```

```
(%i45) for j : 1 thru 21 do
 block([],
   zval : colnew_appsln([x], 4, fspace, ispace)[1],
   printf(true, "~5,2f  ~{~15,5e~}~%", x, zval),
   x : x + 0.05);
 0.00        0.00000e+0      2.04139e+0     5.11228e-33     -9.03975e-1
 0.05        1.02070e-1      2.04139e+0     -4.52648e-2     -9.07936e-1
 0.10        2.04139e-1      2.04139e+0     -9.09256e-2     -9.19819e-1
 0.15        3.06209e-1      2.04140e+0     -1.37379e-1     -9.39624e-1
 0.20        4.08279e-1      2.04141e+0     -1.85020e-1     -9.67352e-1
 0.25        5.10351e-1      2.04152e+0     -2.34246e-1     -1.00301e+0
 0.30        6.12448e-1      2.04303e+0     -2.85454e-1     -1.04663e+0
 0.35        7.15276e-1      2.10661e+0     -3.39053e-1     -1.09916e+0
 0.40        8.32131e-1     -2.45181e-1     -3.96124e-1     -1.20544e+0
 0.45        1.77510e-2     -3.57554e+0     -4.45400e-1     -7.13543e-1
 0.50        2.25122e-2      1.23608e-1     -4.80360e-1     -6.70074e-1
 0.55        2.58693e-2      4.85257e-2     -5.11692e-1     -5.81075e-1
 0.60        2.80994e-2      4.11112e-2     -5.38311e-1     -4.82343e-1
 0.65        2.99107e-2      2.99116e-2     -5.59805e-1     -3.76409e-1
 0.70        3.08739e-2      5.55200e-3     -5.75875e-1     -2.65950e-1
 0.75        3.00325e-2     -4.51649e-2     -5.86417e-1     -1.56669e-1
 0.80        2.55239e-2     -1.46616e-1     -5.91753e-1     -6.04538e-2
 0.85        1.37512e-2     -3.46952e-1     -5.93069e-1     -1.40094e-3
 0.90       -1.25155e-2     -7.52826e-1     -5.93303e-1     -2.86233e-2
 0.95       -6.94274e-2     -1.65084e+0     -5.99062e-1     -2.48115e-1
 1.00        2.64996e-14     1.19263e+2     -6.25420e-1     -8.87626e-1
(%o45)                          done
```

Columns 1 (x) and 2 (ϕ) of the two sets of results above, and the figure below, can be compared with Figure 1 in [Ascher 1981a], page 708.



48.3.3 Example 3: Rotating flow of viscous incompressible fluid

Example 3 from [Ascher 1981a], page 708, describes the velocities in the boundary layer produced by the rotating flow of a viscous incompressible fluid over a stationary infinite disk (see [Gawain&Ball 1978], page 708).

The differential equations are:

$$G'' + \frac{3-n}{2} HG' + (n-1)H'G - s(G-1) = 0$$
$$H''' + \frac{3-n}{2} HH'' + n\left(H'\right)^2 - 1 + G^2 - sH' = 0$$

with the boundary conditions

$$G(0) = H(0) = H'(0) = 0$$
$$G(\infty) = 1$$
$$H'(\infty) = 0$$

The solution uses a number of techniques to obtain convergence. Refer to [Ascher 1981a], page 708, for details.

The maxima code is in file share/colnew/prob3.mac. The reference Fortran implementation is in share/colnew/ex3.

48.3.4 Example 4: Quantum Neumann equation

A more sophisticated example is [Bellon&Talon 2005], page 709, which deals with singularities in the solution domain, provides an initial guess to the solution and uses continuation to solve the system of non-linear differential equations.

The maxima code is in file share/colnew/prob4.mac. The Fortran implementation is in share/colnew/ex4.

48.3.5 Example 5: Simple example of continuation

This example (see [Ascher et al, 1995, Example 9.2], page 709) solves a numerically difficult boundary value problem using continuation. The maxima code is in file share/colnew/prob5.mac and a Fortran implementation is in share/colnew/ex5. The linear differential equation is

$$\epsilon u'' + xu' = -\epsilon\pi^2 \cos \pi x - \pi x \sin \pi x \quad -1 < x < 1$$

with boundary conditions

$$u(-1) = -2$$
$$u(1) = 0$$

The exact solution is

$$u(x) = \cos \pi x + \frac{\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)}{\operatorname{erf}\left(\frac{1}{\sqrt{2\epsilon}}\right)}$$

When ϵ is small the solution has a rapid transition near $x = 0$ and is difficult to solve numerically. COLNEW is able to solve the problem for directly for $\epsilon = 10^{-6}$, but here we will use continuation to solve it succesively for $\epsilon = [10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$.

There is *nconc* $= 1$ differential equation of second order. The list of orders $m = [2]$ and $m^* = \sum_j m[j] = 2$.

The unknown vector of length m^* is $z(x) = [z_1(x), z_2(x)] = [u(x), u'(x)]$.

The differential equation is expressed as

$$[u''(x)] = F(x, z_1, z_2) = \left[-\frac{x}{\epsilon} z_2 - \pi^2 \cos \pi x - \frac{\pi x}{\epsilon} \sin \pi x\right]$$

There are $m^* = 2$ boundary conditions. They are given by a function $G(z_1, z_2)$ that returns a list of length m^*. The j-th boundary condition applies at $x = \zeta_j$ and is satisfied when $g[j] = 0$. We have

j	**zeta[j]**	**Condition**	**g[j]**
1	-1.0	$u = -2$	$z_1 + 2$
2	1.0	$u = 0$	z_1

giving $\zeta = [-1.0, 1, 0]$ and $G(z_1, z_2) = [z_1 + 2, z_1]$.

The Jacobians *df* and *dg* of f and g respectively are determined symbolically.

The ODE will be solved for multiple values of ϵ. The functions *fsub*, *dfsub*, *gsub* and *dgsub* are defined before *e* is set, so that it can be changed in the program.

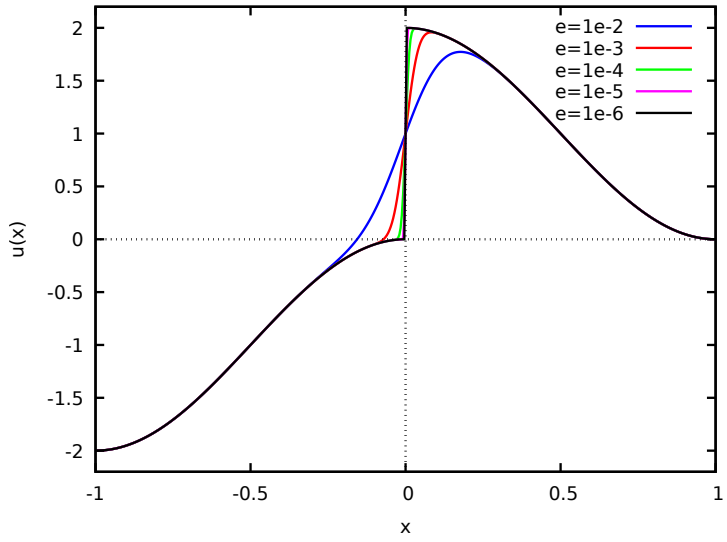
```
(%i1) load("colnew")$
(%i2) kill(e,x,z1,z2)$
(%i3) /* Exact solution */
 exact(x):=cos(%pi*x)+erf(x/sqrt(2*e))/erf(1/sqrt(2*e))$
(%i4) /* Define the equations.  Do this before e is defined */
 f: [-(x/e)*z2 - %pi^2*cos(%pi*x) - (%pi*x/e)*sin(%pi*x)];
             x z2   %pi x sin(%pi x)      2
(%o4)     [- ---- - ---------------- - %pi  cos(%pi x)]
              e            e
(%i5) define(fsub(x,z1,z2),f);
                             x z2   %pi x sin(%pi x)
(%o5) fsub(x, z1, z2) := [- ---- - ----------------
                              e            e
                                                    2
                                                - %pi  cos(%pi x)]
(%i6) df: jacobian(f,[z1,z2]);
                            [      x ]
(%o6)                       [ 0  - - ]
                            [      e ]
(%i7) define(dfsub(x,z1,z2),df);
                                      [      x ]
(%o7)            dfsub(x, z1, z2) := [ 0  - - ]
                                      [      e ]
```

```
(%i8) /* Build the functions gsub and dgsub
   Use define and buildq to remove dependence on g and dg */
 g: [z1+2,z1];
(%o8)                      [z1 + 2, z1]
(%i9) define(gsub(i,z1,z2),buildq([g],g[i]));
(%o9)           gsub(i, z1, z2) := [z1 + 2, z1]
                                                    i
(%i10) dg: jacobian(g,[z1,z2]);
                             [ 1  0 ]
(%o10)                       [      ]
                             [ 1  0 ]
(%i11) define(
 dgsub(i,z1,z2),
 buildq([val:makelist(dg[i],i,1,length(dg))],block([dg:val],dg[i])));
(%o11) dgsub(i, z1, z2) := block([dg : [[1, 0], [1, 0]]], dg )
                                                          i
(%i12) /* Define constant epsilon */
 e : 0.01$
(%i13) /* Number of differential equations */
 ncomp : 1$
(%i14) /* Orders */
 m : [2]$
(%i15) /* Interval ends */
 aleft:-1.0$
(%i16) aright:1.0$
(%i17) /* Locations of side conditions */
 zeta : float([-1, 1])$
(%i18) /* Set up parameter array.  */
 ipar : makelist(0,k,1,11)$
(%i19) /* linear prob */
 ipar[1] : 0$
(%i20) /* 5 collocation points per subinterval */
 ipar[2] : 5$
(%i21) /* Initial uniform mesh of 1 subintervals */
 ipar[3] : 1$
(%i22) ipar[8] : 0$
(%i23) /* Size of fspace, ispace */
 ipar[5] : 10000$
(%i24) ipar[6] :  1000$
(%i25) /* No output.  Don't do this for development. */
 ipar[7]:1$
(%i26) /* No initial guess is provided */
 ipar[9] : 0$
(%i27) /* Regular problem */
 ipar[10] : 0$
(%i28) /* No fixed points in mesh */
 ipar[11] : 0$
```

```
(%i29) /* Tolerances on two components */
 ipar[4] : 2$
(%i30) /* Two error tolerances (on u and its derivative)
   Relatively large tolerances to keep the example small */
 ltol : [1, 2]$
(%i31) tol : [1e-4, 1e-4]$
(%i32) fspace : makelist(0e0, k, 1, ipar[5])$
(%i33) ispace : makelist(0, k, 1, ipar[6])$
(%i34) fixpnt : []$
(%i35) /* First run with default initial guess.
   Returns iflag. 1 = success */
 ([iflag, fspace, ispace] :
  colnew_expert(ncomp, m, aleft, aright, zeta, ipar, ltol, tol,
  fixpnt, ispace, fspace,
  0, fsub, dfsub, gsub, dgsub, dummy),
  if (iflag#1) then error("On return from colnew_expert: iflag = ",iflag),
  iflag);
(%o35)                          1
(%i36) /* Function to generate equally spaced list of values */
 xlist(xmin,xmax,n):=block([dx:(xmax-xmin)/n],makelist(i,i,0,n)*dx+xmin)$
(%i37) /* x values for solution.  Cluster around x=0 */
 X: xlist(aleft,aright,500)$
(%i38) /* Generate solution values for z1=u(x) */
 ans:colnew_appsln(X,2,fspace,ispace)$
(%i39) z:maplist(first,ans)$
(%i40) Z:[z]$
(%i41) /* Compare with exact solution and report */
 y:float(map(exact,X))$
(%i42) maxerror:apply(max,abs(y-z));
(%o42)                6.881499912125832e-7
(%i43) printf(true," e: ~8,3e  iflag ~3d  Mesh size ~3d  max error ~8,3e~%",
  e,iflag,ispace[1],maxerror);
 e: 1.000e-2  iflag   1  Mesh size  16  max error 6.881e-7
(%o43)                        false
(%i44) /* Now use continuation to solve for progressively smaller e
   Use previous solution as initial guess and every second point
   from previous mesh as initial mesh */
 ipar[9] : 3$
```

```
(%i45) /* Run COLNEW using continuation for new value of e
   Set new mesh size ipar[3] from previous size ispace[1]
   Push list of values of z1=u(x) on to list Z */
 run_it(e_):=block(
  e:e_,
  ipar[3]:ispace[1],
  [iflag, fspace, ispace]:
     colnew_expert(ncomp,m,aleft,aright,zeta,ipar,ltol,tol,fixpnt,
     ispace,fspace,0,fsub,dfsub,gsub,dgsub,dummy),
  if (iflag#1) then error("On return from colnew_expert: iflag =",iflag),
  ans:colnew_appsln(X,2,fspace,ispace),
  z:maplist(first,ans),
  push(z,Z),
  y:float(map(exact,X)),
  maxerror:apply(max,abs(y-z)),
  printf(true," e: ~8,3e  iflag ~3d  Mesh size ~3d  max error ~8,3e~%",
    e,iflag,ispace[1],maxerror),
  iflag
 )$
(%i46) for e_ in [1e-3,1e-4,1e-5,1e-6] do run_it(e_)$
 e: 1.000e-3  iflag   1  Mesh size  20  max error 3.217e-7
 e: 1.000e-4  iflag   1  Mesh size  40  max error 3.835e-7
 e: 1.000e-5  iflag   1  Mesh size  38  max error 8.689e-9
 e: 1.000e-6  iflag   1  Mesh size  60  max error 6.333e-7
(%i47) /* Z is list of solutions z1 = u(x).  Restore order. */
 Z:reverse(Z)$
(%i48) /* Plot z1=u(x) for each value of e */
 plot2d([
  [discrete,X,Z[1]], [discrete,X,Z[2]], [discrete,X,Z[3]],
  [discrete,X,Z[4]], [discrete,X,Z[5]]],
  [legend,"e=1e-2","e=1e-3","e=1e-4","e=1e-5","e=1e-6"],
  [xlabel,"x"],[ylabel,"u(x)"],
  [png_file,"./colnew-ex5.png"]); */
 done$
(%o48)  [/tmp/5zscumjmw4lichu2.gnuplot, ./colnew-ex5.png]
```

The figure below shows the solution for $\epsilon = [10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$.



48.4 References for colnew

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49 Package combinatorics

49.1 Package combinatorics

The `combinatorics` package provides several functions to work with permutations and to permute elements of a list. The permutations of degree n are all the $n!$ possible orderings of the first n positive integers, 1, 2, ..., n. The functions in this packages expect a permutation to be represented by a list of those integers.

Cycles are represented as a list of two or more integers *i_1*, *i_2*, ..., *i_m*, all different. Such a list represents a permutation where the integer *i_2* appears in the *i_1*th position, the integer *i_3* appears in the *i_2*th position and so on, until the integer *i_1*, which appears in the *i_m*th position.

For instance, [4, 2, 1, 3] is one of the 24 permutations of degree four, which can also be represented by the cycle [1, 4, 3]. The functions where cycles are used to represent permutations also require the order of the permutation to avoid ambiguity. For instance, the same cycle [1, 4, 3] could refer to the permutation of order 6: [4, 2, 1, 3, 5, 6]. A product of cycles must be represented by a list of cycles; the cycles at the end of the list are applied first. For example, [[2, 4], [1, 3, 6, 5]] is equivalent to the permutation [3, 4, 6, 2, 1, 5].

A cycle can be written in several ways. for instance, [1, 3, 6, 5], [3, 6, 5, 1] and [6, 5, 1, 3] are all equivalent. The canonical form used in the package is the one that places the lowest index in the first place. A cycle with only two indices is also called a transposition and if the two indices are consecutive, it is called an adjacent transposition.

To run an interactive tutorial, use the command `demo (combinatorics)`. Since this is an additional package, it must be loaded with the command `load("combinatorics")`.

49.2 Functions and Variables for Combinatorics

`apply_cycles` (*cl*,*l*) [Function]

Permutes the list or set *l* applying to it the list of cycles *cl*. The cycles at the end of the list are applied first and the first cycle in the list *cl* is the last one to be applied.

See also `permute`.

Example:

```
(%i1) load("combinatorics")$
(%i2) lis1:[a,b*c^2,4,z,x/y,1/2,ff23(x),0];
                          2        x  1
(%o2)            [a, b c , 4, z, -, -, ff23(x), 0]
                                   y  2
(%i3) apply_cycles ([[1, 6], [2, 6, 5, 7]], lis1);
                   x  1                          2
(%o3)              [-, -, 4, z, ff23(x), a, b c , 0]
                    y  2
```

`cyclep` (*c*, *n*) [Function]

Returns true if *c* is a valid cycle of order *n* namely, a list of non-repeated positive integers less or equal to *n*. Otherwise, it returns false.

See also permp.

Examples:

```
(%i1) load("combinatorics")$
(%i2) cyclep ([-2,3,4], 5);
(%o2)                          false
(%i3) cyclep ([2,3,4,2], 5);
(%o3)                          false
(%i4) cyclep ([6,3,4], 5);
(%o4)                          false
(%i5) cyclep ([6,3,4], 6);
(%o5)                          true
```

perm_cycles (*p*) [Function]

Returns permutation *p* as a product of cycles. The cycles are written in a canonical form, in which the lowest index in the cycle is placed in the first position.

See also perm_decomp.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_cycles ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                 [[1, 4], [2, 6, 5, 7]]
```

perm_decomp (*p*) [Function]

Returns the minimum set of adjacent transpositions whose product equals the given permutation *p*.

See also perm_cycles.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_decomp ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2) [[6, 7], [5, 6], [6, 7], [3, 4], [4, 5], [2, 3], [3, 4],
                              [4, 5], [5, 6], [1, 2], [2, 3], [3, 4]]
```

perm_inverse (*p*) [Function]

Returns the inverse of a permutation of *p*, namely, a permutation *q* such that the products *pq* and *qp* are equal to the identity permutation: [1, 2, 3, ..., *n*], where *n* is the length of *p*.

See also permult.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_inverse ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                [4, 7, 3, 1, 6, 2, 5, 8]
```

perm_length (*p*) [Function]

Determines the minimum number of adjacent transpositions necessary to write permutation *p* as a product of adjacent transpositions. An adjacent transposition is a cycle with only two numbers, which are consecutive integers.

See also perm_decomp.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_length ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                          12
```

`perm_lex_next` (*p*) [Function]

Returns the permutation that comes after the given permutation *p*, in the sequence of permutations in lexicographic order.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_next ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                [4, 6, 3, 1, 7, 5, 8, 2]
```

`perm_lex_rank` (*p*) [Function]

Finds the position of permutation *p*, an integer from 1 to the degree *n* of the permutation, in the sequence of permutations in lexicographic order.

See also `perm_lex_unrank` and `perms_lex`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_rank ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                          18255
```

`perm_lex_unrank` (*n*, *i*) [Function]

Returns the *n*-degree permutation at position *i* (from 1 to *n*!) in the lexicographic ordering of permutations.

See also `perm_lex_rank` and `perms_lex`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_lex_unrank (8, 18255);
(%o2)                [4, 6, 3, 1, 7, 5, 2, 8]
```

`perm_next` (*p*) [Function]

Returns the permutation that comes after the given permutation *p*, in the sequence of permutations in Trotter-Johnson order.

See also `perms`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_next ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                [4, 6, 3, 1, 7, 5, 8, 2]
```

`perm_parity` (*p*) [Function]

Finds the parity of permutation *p*: 0 if the minimum number of adjacent transpositions necessary to write permutation *p* as a product of adjacent transpositions is even, or 1 if that number is odd.

See also `perm_decomp`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_parity ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                           0
```

`perm_rank` (*p*) [Function]

Finds the position of permutation *p*, an integer from 1 to the degree *n* of the permutation, in the sequence of permutations in Trotter-Johnson order.

See also `perm_unrank` and `perms`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_rank ([4, 6, 3, 1, 7, 5, 2, 8]);
(%o2)                          19729
```

`perm_undecomp` (*cl*, *n*) [Function]

Converts the list of cycles *cl* of degree *n* into an *n* degree permutation, equal to their product.

See also `perm_decomp`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_undecomp ([[1,6],[2,6,5,7]], 8);
(%o2)                [5, 6, 3, 4, 7, 1, 2, 8]
```

`perm_unrank` (*n*, *i*) [Function]

Returns the *n*-degree permutation at position *i* (from 1 to *n*!) in the Trotter-Johnson ordering of permutations.

See also `perm_rank` and `perms`.

Example:

```
(%i1) load("combinatorics")$
(%i2) perm_unrank (8, 19729);
(%o2)                [4, 6, 3, 1, 7, 5, 2, 8]
```

`permp` (*p*) [Function]

Returns true if *p* is a valid permutation namely, a list of length *n*, whose elements are all the positive integers from 1 to *n*, without repetitions. Otherwise, it returns false.

Examples:

```
(%i1) load("combinatorics")$
(%i2) permp ([2,0,3,1]);
(%o2)                          false
(%i3) permp ([2,1,4,3]);
(%o3)                          true
```

perms [Function]

perms (*n*)
perms (*n*, *i*)
perms (*n*, *i*, *j*)

`perms(n)` returns a list of all n-degree permutations in the so-called Trotter-Johnson order.

`perms(n, i)` returns the n-degree permutation which is at the ith position (from 1 to $n!$) in the Trotter-Johnson ordering of the permutations.

`perms(n, i, j)` returns a list of the n-degree permutations between positions i and j in the Trotter-Johnson ordering of the permutations.

The sequence of permutations in Trotter-Johnson order starts with the identity permutation and each consecutive permutation can be obtained from the previous one a by single adjacent transposition.

See also perm_next, perm_rank and perm_unrank.

Examples:

```
(%i1) load("combinatorics")$
(%i2) perms (4);
(%o2) [[1, 2, 3, 4], [1, 2, 4, 3], [1, 4, 2, 3], [4, 1, 2, 3],
[4, 1, 3, 2], [1, 4, 3, 2], [1, 3, 4, 2], [1, 3, 2, 4],
[3, 1, 2, 4], [3, 1, 4, 2], [3, 4, 1, 2], [4, 3, 1, 2],
[4, 3, 2, 1], [3, 4, 2, 1], [3, 2, 4, 1], [3, 2, 1, 4],
[2, 3, 1, 4], [2, 3, 4, 1], [2, 4, 3, 1], [4, 2, 3, 1],
[4, 2, 1, 3], [2, 4, 1, 3], [2, 1, 4, 3], [2, 1, 3, 4]]
(%i3) perms (4, 12);
(%o3)                      [[4, 3, 1, 2]]
(%i4) perms (4, 12, 14);
(%o4)       [[4, 3, 1, 2], [4, 3, 2, 1], [3, 4, 2, 1]]
```

perms_lex [Function]

perms_lex (*n*)
perms_lex (*n*, *i*)
perms_lex (*n*, *i*, *j*)

`perms_lex(n)` returns a list of all n-degree permutations in the so-called lexicographic order.

`perms_lex(n, i)` returns the n-degree permutation which is at the ith position (from 1 to $n!$) in the lexicographic ordering of the permutations.

`perms_lex(n, i, j)` returns a list of the n-degree permutations between positions i and j in the lexicographic ordering of the permutations.

The sequence of permutations in lexicographic order starts with all the permutations with the lowest index, 1, followed by all permutations starting with the following index, 2, and so on. The permutations starting by an index i are the permutations of the first n integers different from i and they are also placed in lexicographic order, where the permutations with the lowest of those integers are placed first and so on.

See also perm_lex_next, perm_lex_rank and perm_lex_unrank.

Examples:

```
(%i1) load("combinatorics")$
(%i2) perms_lex (4);
(%o2) [[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2],
[1, 4, 2, 3], [1, 4, 3, 2], [2, 1, 3, 4], [2, 1, 4, 3],
[2, 3, 1, 4], [2, 3, 4, 1], [2, 4, 1, 3], [2, 4, 3, 1],
[3, 1, 2, 4], [3, 1, 4, 2], [3, 2, 1, 4], [3, 2, 4, 1],
[3, 4, 1, 2], [3, 4, 2, 1], [4, 1, 2, 3], [4, 1, 3, 2],
[4, 2, 1, 3], [4, 2, 3, 1], [4, 3, 1, 2], [4, 3, 2, 1]]
(%i3) perms_lex (4, 12);
(%o3)                     [[2, 4, 3, 1]]
(%i4) perms_lex (4, 12, 14);
(%o4)       [[2, 4, 3, 1], [3, 1, 2, 4], [3, 1, 4, 2]]
```

`permult` (*p_1*, ..., *p_m*) [Function]

Returns the product of two or more permutations *p_1*, ..., *p_m*.

Example:

```
(%i1) load("combinatorics")$
(%i2) permult ([2,3,1], [3,1,2], [2,1,3]);
(%o2)                        [2, 1, 3]
```

`permute` (*p*, *l*) [Function]

Applies the permutation *p* to the elements of the list (or set) *l*.

Example:

```
(%i1) load("combinatorics")$
(%i2) lis1: [a,b*c^2,4,z,x/y,1/2,ff23(x),0];
                          2        x  1
(%o2)           [a, b c , 4, z, -, -, ff23(x), 0]
                                   y  2
(%i3) permute ([4, 6, 3, 1, 7, 5, 2, 8], lis1);
                      1                 x     2
(%o3)           [z, -, 4, a, ff23(x), -, b c , 0]
                      2                 y
```

`random_perm` (*n*) [Function]

Returns a random permutation of degree *n*.

See also `random_permutation`.

Example:

```
(%i1) load("combinatorics")$
(%i2) random_perm (7);
(%o2)                  [6, 3, 4, 7, 5, 1, 2]
```

50 Package contrib_ode

50.1 Introduction to contrib_ode

Maxima's ordinary differential equation (ODE) solver `ode2` solves elementary linear ODEs of first and second order. The function `contrib_ode` extends `ode2` with additional methods for linear and non-linear first order ODEs and linear homogeneous second order ODEs. The code is still under development and the calling sequence may change in future releases. Once the code has stabilized it may be moved from the contrib directory and integrated into Maxima.

This package must be loaded with the command `load("contrib_ode")` before use.

The calling convention for `contrib_ode` is identical to `ode2`. It takes three arguments: an ODE (only the left hand side need be given if the right hand side is 0), the dependent variable, and the independent variable. When successful, it returns a list of solutions.

The form of the solution differs from `ode2`. As non-linear equations can have multiple solutions, `contrib_ode` returns a list of solutions. Each solution can have a number of forms:

- an explicit solution for the dependent variable,
- an implicit solution for the dependent variable,
- a parametric solution in terms of variable `%t`, or
- a transformation into another ODE in variable `%u`.

`contrib_ode` uses the global variables `%c`, `%k1`, `%k2`, `method` and `yp` similarly to `ode2`.

If `contrib_ode` cannot obtain a solution for whatever reason, it returns `false`, after perhaps printing out an error message.

It is necessary to return a list of solutions, as even first order non-linear ODEs can have multiple solutions. For example:

```
(%i1) load("contrib_ode")$
(%i2) eqn:x*'diff(y,x)^2-(1+x*y)*'diff(y,x)+y=0;
                     dy 2             dy
(%o2)             x (--)  - (1 + x y) -- + y = 0
                     dx               dx
(%i3) contrib_ode(eqn,y,x);
                     dy 2             dy
(%t3)             x (--)  - (1 + x y) -- + y = 0
                     dx               dx

                first order equation not linear in y'

                                                 x
(%o3)              [y = log(x) + %c, y = %c %e ]
(%i4) method;
(%o4)                         factor
```

Nonlinear ODEs can have singular solutions without constants of integration, as in the second solution of the following example:

```
(%i1) load("contrib_ode")$
(%i2) eqn:'diff(y,x)^2+x*'diff(y,x)-y=0;
                        dy 2     dy
(%o2)                  (--)  + x -- - y = 0
                        dx       dx
(%i3) contrib_ode(eqn,y,x);
                        dy 2     dy
(%t3)                  (--)  + x -- - y = 0
                        dx       dx

              first order equation not linear in y'

                                              2
                                   2         x
(%o3)              [y = %c x + %c , y = - --]
                                             4
(%i4) method;
(%o4)                         clairaut
```

The following ODE has two parametric solutions in terms of the dummy variable `%t`. In this case the parametric solutions can be manipulated to give explicit solutions.

```
(%i1) load("contrib_ode")$
(%i2) eqn:'diff(y,x)=(x+y)^2;
                           dy          2
(%o2)                      -- = (x + y)
                           dx
(%i3) contrib_ode(eqn,y,x);
(%o3) [[x = %c - atan(sqrt(%t)), y = (- x) - sqrt(%t)],
                     [x = atan(sqrt(%t)) + %c, y = sqrt(%t) - x]]
(%i4) method;
(%o4)                          lagrange
```

The following example (Kamke 1.112) demonstrates an implicit solution.

```
(%i1) load("contrib_ode")$
(%i2) assume(x>0,y>0);
(%o2)                       [x > 0, y > 0]
(%i3) eqn:x*'diff(y,x)-x*sqrt(y^2+x^2)-y;
                       dy           2    2
(%o3)                x -- - x sqrt(y  + x ) - y
                       dx
(%i4) contrib_ode(eqn,y,x);
                                    y
(%o4)                    [x - asinh(-) = %c]
                                    x
(%i5) method;
(%o5)                           lie
```

The following Riccati equation is transformed into a linear second order ODE in the variable %u. Maxima is unable to solve the new ODE, so it is returned unevaluated.

```
(%i1) load("contrib_ode")$
(%i2) eqn:x^2*'diff(y,x)=a+b*x^n+c*x^2*y^2;
                     2 dy      2  2      n
(%o2)               x  -- = c x  y  + b x  + a
                       dx
(%i3) contrib_ode(eqn,y,x);
                d%u
                ---                                2
                dx         2  a        n - 2    d %u
(%o3)  [[y = - ----, %u c  (-- + b x     ) + ---- c = 0]]
                %u c           2                  2
                              x                  dx
(%i4) method;
(%o4)                          riccati
```

For first order ODEs `contrib_ode` calls `ode2`. It then tries the following methods: factorization, Clairaut, Lagrange, Riccati, Abel and Lie symmetry methods. The Lie method is not attempted on Abel equations if the Abel method fails, but it is tried if the Riccati method returns an unsolved second order ODE.

For second order ODEs `contrib_ode` calls `ode2` then `odelin`.

Extensive debugging traces and messages are displayed if the command `put('contrib_ode,true,'verbose)` is executed.

50.2 Functions and Variables for contrib_ode

`contrib_ode` (*eqn*, *y*, *x*) [Function]

Returns a list of solutions of the ODE *eqn* with independent variable *x* and dependent variable *y*.

`odelin` (*eqn*, *y*, *x*) [Function]

`odelin` solves linear homogeneous ODEs of first and second order with independent variable *x* and dependent variable *y*. It returns a fundamental solution set of the ODE.

For second order ODEs, `odelin` uses a method, due to Bronstein and Lafaille, that searches for solutions in terms of given special functions.

```
(%i1) load("contrib_ode")$
(%i2) odelin(x*(x+1)*'diff(y,x,2)+(x+5)*'diff(y,x,1)+(-4)*y,y,x);
       gauss_a(- 6, - 2, - 3, - x)  gauss_b(- 6, - 2, - 3, - x)
(%o2) {--------------------------, --------------------------}
                    4                            4
                   x                            x
```

`ode_check` (*eqn*, *soln*) [Function]

Returns the value of ODE *eqn* after substituting a possible solution *soln*. The value is equivalent to zero if *soln* is a solution of *eqn*.

```
(%i1) load("contrib_ode")$
```

```
(%i2) eqn:'diff(y,x,2)+(a*x+b)*y;
                            2
                           d y
(%o2)                      --- + (b + a x) y
                             2
                           dx
(%i3) ans:[y = bessel_y(1/3,2*(a*x+b)^(3/2)/(3*a))*%k2*sqrt(a*x+b)
         +bessel_j(1/3,2*(a*x+b)^(3/2)/(3*a))*%k1*sqrt(a*x+b)];
                                    3/2
                        1  2 (b + a x)
(%o3) [y = bessel_y(-, --------------) %k2 sqrt(a x + b)
                        3       3 a
                                              3/2
                                1  2 (b + a x)
                    + bessel_j(-, --------------) %k1 sqrt(a x + b)]
                                3       3 a
(%i4) ode_check(eqn,ans[1]);
(%o4)                               0
```

gauss_a (*a*, *b*, *c*, *x*) [Function]

gauss_a(a,b,c,x) and gauss_b(a,b,c,x) are 2F1 hypergeometric functions. They represent any two independent solutions of the hypergeometric differential equation x*(1-x) diff(y,x,2) + [c-(a+b+1)x] diff(y,x) - a*b*y = 0 See A&S eqn 15.5.1 and DLMF 15.10.

The only use of these functions is in solutions of ODEs returned by odelin and contrib_ode. The definition and use of these functions may change in future releases of Maxima.

See also gauss_b, dgauss_a and gauss_b.

gauss_b (*a*, *b*, *c*, *x*) [Function]

See gauss_a.

dgauss_a (*a*, *b*, *c*, *x*) [Function]

The derivative with respect to *x* of gauss_a(*a*, *b*, *c*, *x*).

dgauss_b (*a*, *b*, *c*, *x*) [Function]

The derivative with respect to *x* of gauss_b(*a*, *b*, *c*, *x*).

kummer_m (*a*, *b*, *x*) [Function]

Kummer's M function, see A&S eqn 13.1.2 and DLMF 13.2.

The only use of this function is in solutions of ODEs returned by odelin and contrib_ode. The definition and use of this function may change in future releases of Maxima.

See also kummer_u, dkummer_m, and dkummer_u.

kummer_u (*a*, *b*, *x*) [Function]

Kummer's U function, see A&S eqn 13.1.3 and DLMF 13.2.6.

See kummer_m.

`dkummer_m` (*a*, *b*, *x*) [Function]
The derivative with respect to *x* of `kummer_m`(*a*, *b*, *x*).

`dkummer_u` (*a*, *b*, *x*) [Function]
The derivative with respect to *x* of `kummer_u`(*a*, *b*, *x*).

`bessel_simplify` (*expr*) [Function]
Simplifies expressions containing Bessel functions `bessel_j`, `bessel_y`, `bessel_i`, `bessel_k`, `hankel_1`, `hankel_2`, `struve_h` and `struve_l`. Recurrence relations A&S eqn 9.1.27 and DLMF 10.6#i are used to replace functions of highest order n by functions of order n-1 and n-2.

This process is repeated until all the orders differ by less than 2.

```
(%i1) load("contrib_ode")$
(%i2) bessel_simplify(4*bessel_j(n,x^2)*(x^2-n^2/x^2)
  +x*((bessel_j(n-2,x^2)-bessel_j(n,x^2))*x
  -(bessel_j(n,x^2)-bessel_j(n+2,x^2))*x)
  -2*bessel_j(n+1,x^2)+2*bessel_j(n-1,x^2));
(%o2)                          0
(%i3) bessel_simplify( -2*bessel_j(1,z)*z^3 - 10*bessel_j(2,z)*z^2
 + 15*%pi*bessel_j(1,z)*struve_h(3,z)*z - 15*%pi*struve_h(1,z)
   *bessel_j(3,z)*z - 15*%pi*bessel_j(0,z)*struve_h(2,z)*z
 + 15*%pi*struve_h(0,z)*bessel_j(2,z)*z - 30*%pi*bessel_j(1,z)
   *struve_h(2,z) + 30*%pi*struve_h(1,z)*bessel_j(2,z));
(%o3)                          0
```

`expintegral_e_simplify` (*expr*) [Function]
Simplify expressions containing exponential integral `expintegral_e` using the recurrence A&S eqn 5.1.14.

expintegral_e(n+1,z) = (1/n) * (exp(-z)-z*expintegral_e(n,z)) n = 1,2,3 ....

50.3 Possible improvements to contrib_ode

These routines are work in progress. I still need to:

- Extend the FACTOR method `ode1_factor` to work for multiple roots.
- Extend the FACTOR method `ode1_factor` to attempt to solve higher order factors. At present it only attempts to solve linear factors.
- Fix the LAGRANGE routine `ode1_lagrange` to prefer real roots over complex roots.
- Add additional methods for Riccati equations.
- Improve the detection of Abel equations of second kind. The existing pattern matching is weak.
- Work on the Lie symmetry group routine `ode1_lie`. There are quite a few problems with it: some parts are unimplemented; some test cases seem to run forever; other test cases crash; yet others return very complex "solutions". I wonder if it really ready for release yet.
- Add more test cases.

50.4 Test cases for contrib_ode

The routines have been tested on a approximately one thousand test cases from Murphy, Kamke, Zwillinger and elsewhere. These are included in the tests subdirectory.

- The Clairaut routine `ode1_clairaut` finds all known solutions, including singular solutions, of the Clairaut equations in Murphy and Kamke.
- The other routines often return a single solution when multiple solutions exist.
- Some of the "solutions" from `ode1_lie` are overly complex and impossible to check.
- There are some crashes.

50.5 References for contrib_ode

1. E. Kamke, Differentialgleichungen Lösungsmethoden und Lösungen, Vol 1, Geest & Portig, Leipzig, 1961
2. G. M. Murphy, Ordinary Differential Equations and Their Solutions, Van Nostrand, New York, 1960
3. D. Zwillinger, Handbook of Differential Equations, 3rd edition, Academic Press, 1998
4. F. Schwarz, Symmetry Analysis of Abel's Equation, Studies in Applied Mathematics, 100:269-294 (1998)
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6. E. S. Cheb-Terrab, A. D. Roche, Symmetries and First Order ODE Patterns, Computer Physics Communications 113 (1998), p 239. (http://lie.uwaterloo.ca/papers/ode_vii.pdf)
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8. G. W. Bluman, S. C. Anco, Symmetry and Integration Methods for Differential Equations, Springer, (2002)
9. M. Bronstein, S. Lafaille, Solutions of linear ordinary differential equations in terms of special functions, Proceedings of ISSAC 2002, Lille, ACM Press, 23-28. (http://www-sop.inria.fr/cafe/Manuel.Bronstein/publications/issac2002.pdf)

51 Package ctensor

51.1 Introduction to ctensor

`ctensor` is a component tensor manipulation package. To use the `ctensor` package, type `load("ctensor")`.

For alternative packages for handling tensors see Chapter 39 [Package atensor], page 663, and Chapter 71 [Package itensor], page 1091, .

To begin an interactive session with `ctensor`, type `csetup()`. You are first asked to specify the dimension of the manifold. If the dimension is 2, 3 or 4 then the list of coordinates defaults to `[x,y]`, `[x,y,z]` or `[x,y,z,t]` respectively. These names may be changed by assigning a new list of coordinates to the variable `ct_coords` (described below) and the user is queried about this. Care must be taken to avoid the coordinate names conflicting with other object definitions.

Next, the user enters the metric either directly or from a file by specifying its ordinal position. The metric is stored in the matrix `lg`. Finally, the metric inverse is computed and stored in the matrix `ug`. One has the option of carrying out all calculations in a power series.

A sample protocol is begun below for the static, spherically symmetric metric (standard coordinates) which will be applied to the problem of deriving Einstein's vacuum equations (which lead to the Schwarzschild solution) as an example. Many of the functions in `ctensor` will be displayed for the standard metric as examples.

```
(%i1) load("ctensor");
(%o1)      /share/tensor/ctensor.mac
(%i2) csetup();
Enter the dimension of the coordinate system:
4;
Do you wish to change the coordinate names?
n;
Do you want to
1. Enter a new metric?

2. Enter a metric from a file?

3. Approximate a metric with a Taylor series?
1;

Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4
1;
Row 1 Column 1:
a;
Row 2 Column 2:
x^2;
Row 3 Column 3:
```

```
x^2*sin(y)^2;
Row 4 Column 4:
-d;

Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
depends([a,d],x);
Do you wish to see the metric?
y;
                          [ a  0       0        0  ]
                          [                        ]
                          [     2                  ]
                          [ 0  x       0        0  ]
                          [                        ]
                          [         2    2         ]
                          [ 0  0   x  sin (y)   0  ]
                          [                        ]
                          [ 0  0       0       - d ]
(%o2)                                done
(%i3) christof(mcs);
                                            a
                                           x
(%t3)                          mcs        = ---
                                  1, 1, 1   2 a

                                            1
(%t4)                           mcs        = -
                                   1, 2, 2   x

                                            1
(%t5)                           mcs        = -
                                   1, 3, 3   x

                                           d
                                          x
(%t6)                          mcs        = ---
                                  1, 4, 4   2 d

                                             x
(%t7)                          mcs        = - -
                                  2, 2, 1     a

                                          cos(y)
(%t8)                         mcs        = ------
                                 2, 3, 3   sin(y)

                                              2
```

```
                                          x sin (y)
(%t9)                       mcs        = - ---------
                               3, 3, 1         a

(%t10)                     mcs        = - cos(y) sin(y)
                              3, 3, 2

                                           d
                                            x
(%t11)                          mcs        = ---
                                   4, 4, 1   2 a
(%o11)                             done
```

51.2 Functions and Variables for ctensor

51.2.1 Initialization and setup

`csetup` () [Function]

A function in the `ctensor` (component tensor) package which initializes the package and allows the user to enter a metric interactively. See `ctensor` for more details.

`cmetric` [Function]

`cmetric` (*dis*)
`cmetric` ()

A function in the `ctensor` (component tensor) package that computes the metric inverse and sets up the package for further calculations.

If `cframe_flag` is `false`, the function computes the inverse metric `ug` from the (user-defined) matrix `lg`. The metric determinant is also computed and stored in the variable `gdet`. Furthermore, the package determines if the metric is diagonal and sets the value of `diagmetric` accordingly. If the optional argument *dis* is present and not equal to `false`, the user is prompted to see the metric inverse.

If `cframe_flag` is `true`, the function expects that the values of `fri` (the inverse frame matrix) and `lfg` (the frame metric) are defined. From these, the frame matrix `fr` and the inverse frame metric `ufg` are computed.

`ct_coordsys` [Function]

`ct_coordsys` (*coordinate_system*, *extra_arg*)
`ct_coordsys` (*coordinate_system*)

Sets up a predefined coordinate system and metric. The argument *coordinate_system* can be one of the following symbols:

SYMBOL	Dim	Coordinates	Description/comments
cartesian2d	2	[x,y]	Cartesian 2D coordinate system
polar	2	[r,phi]	Polar coordinate system

elliptic	2	[u,v]	Elliptic coord. system
confocalelliptic	2	[u,v]	Confocal elliptic coordinates
bipolar	2	[u,v]	Bipolar coord. system
parabolic	2	[u,v]	Parabolic coord. system
cartesian3d	3	[x,y,z]	Cartesian 3D coordinate system
polarcylindrical	3	[r,theta,z]	Polar 2D with cylindrical z
ellipticcylindrical	3	[u,v,z]	Elliptic 2D with cylindrical z
confocalellipsoidal	3	[u,v,w]	Confocal ellipsoidal
bipolarcylindrical	3	[u,v,z]	Bipolar 2D with cylindrical z
paraboliccylindrical	3	[u,v,z]	Parabolic 2D with cylindrical z
paraboloidal	3	[u,v,phi]	Paraboloidal coords.
conical	3	[u,v,w]	Conical coordinates
toroidal	3	[phi,u,v]	Toroidal coordinates
spherical	3	[r,theta,phi]	Spherical coord. system
oblatespheroidal	3	[u,v,phi]	Oblate spheroidal coordinates
oblatespheroidalsqrt	3	[u,v,phi]	
prolatespheroidal	3	[u,v,phi]	Prolate spheroidal coordinates
prolatespheroidalsqrt	3	[u,v,phi]	
ellipsoidal	3	[r,theta,phi]	Ellipsoidal coordinates
cartesian4d	4	[x,y,z,t]	Cartesian 4D coordinate system
spherical4d	4	[r,theta,eta,phi]	Spherical 4D coordinate system
exteriorschwarzschild	4	[t,r,theta,phi]	Schwarzschild metric
interiorschwarzschild	4	[t,z,u,v]	Interior Schwarzschild metric
kerr_newman	4	[t,r,theta,phi]	Charged axially symmetric metric

`coordinate_system` can also be a list of transformation functions, followed by a list containing the coordinate variables. For instance, you can specify a spherical metric as follows:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
     r*sin(theta),[r,theta,phi]]);
(%o2)                                done
(%i3) lg:trigsimp(lg);
```

```
                              [ 1  0         0        ]
                              [                       ]
                              [     2                 ]
(%o3)                         [ 0  r         0        ]
                              [                       ]
                              [         2    2        ]
                              [ 0  0   r  cos (theta) ]
(%i4) ct_coords;
(%o4)                              [r, theta, phi]
(%i5) dim;
(%o5)                                      3
```

Transformation functions can also be used when `cframe_flag` is `true`:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) cframe_flag:true;
(%o2)                                    true
(%i3) ct_coordsys([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
      r*sin(theta),[r,theta,phi]]);
(%o3)                                    done
(%i4) fri;
(%o4)
 [cos(phi)cos(theta) -cos(phi) r sin(theta) -sin(phi) r cos(theta)]
 [                                                                 ]
 [sin(phi)cos(theta) -sin(phi) r sin(theta)  cos(phi) r cos(theta)]
 [                                                                 ]
 [    sin(theta)           r cos(theta)                0          ]

(%i5) cmetric();
(%o5)                                    false
(%i6) lg:trigsimp(lg);
                              [ 1  0         0        ]
                              [                       ]
                              [     2                 ]
(%o6)                         [ 0  r         0        ]
                              [                       ]
                              [         2    2        ]
                              [ 0  0   r  cos (theta) ]
```

The optional argument *extra_arg* can be any one of the following:

`cylindrical` tells `ct_coordsys` to attach an additional cylindrical coordinate.

`minkowski` tells `ct_coordsys` to attach an additional coordinate with negative metric signature.

`all` tells `ct_coordsys` to call `cmetric` and `christof(false)` after setting up the metric.

If the global variable `verbose` is set to `true`, `ct_coordsys` displays the values of `dim`, `ct_coords`, and either `lg` or `lfg` and `fri`, depending on the value of `cframe_flag`.

`init_ctensor ()` [Function]

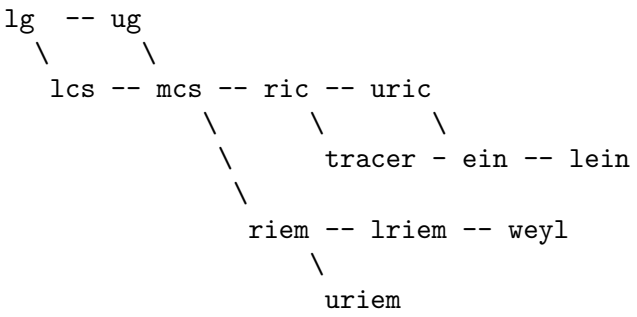
Initializes the `ctensor` package.

The `init_ctensor` function reinitializes the `ctensor` package. It removes all arrays and matrices used by `ctensor`, resets all flags, resets `dim` to 4, and resets the frame metric to the Lorentz-frame.

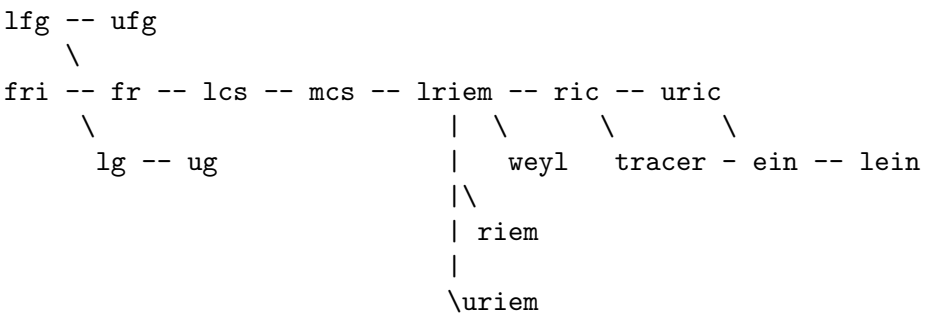
51.2.2 The tensors of curved space

The main purpose of the `ctensor` package is to compute the tensors of curved space(time), most notably the tensors used in general relativity.

When a metric base is used, `ctensor` can compute the following tensors:



`ctensor` can also work using moving frames. When `cframe_flag` is set to `true`, the following tensors can be calculated:



`christof (`*dis*`)` [Function]

A function in the `ctensor` (component tensor) package. It computes the Christoffel symbols of both kinds. The argument *dis* determines which results are to be immediately displayed. The Christoffel symbols of the first and second kinds are stored in the arrays `lcs[i,j,k]` and `mcs[i,j,k]` respectively and defined to be symmetric in the first two indices. If the argument to `christof` is `lcs` or `mcs` then the unique non-zero values of `lcs[i,j,k]` or `mcs[i,j,k]`, respectively, will be displayed. If the

argument is `all` then the unique non-zero values of `lcs[i,j,k]` and `mcs[i,j,k]` will be displayed. If the argument is `false` then the display of the elements will not occur. The array elements `mcs[i,j,k]` are defined in such a manner that the final index is contravariant.

`ricci` (*dis*) [Function]

A function in the `ctensor` (component tensor) package. `ricci` computes the covariant (symmetric) components `ric[i,j]` of the Ricci tensor. If the argument *dis* is `true`, then the non-zero components are displayed.

`uricci` (*dis*) [Function]

This function first computes the covariant components `ric[i,j]` of the Ricci tensor. Then the mixed Ricci tensor is computed using the contravariant metric tensor. If the value of the argument *dis* is `true`, then these mixed components, `uric[i,j]` (the index `i` is covariant and the index `j` is contravariant), will be displayed directly. Otherwise, `ricci(false)` will simply compute the entries of the array `uric[i,j]` without displaying the results.

`scurvature` () [Function]

Returns the scalar curvature (obtained by contracting the Ricci tensor) of the Riemannian manifold with the given metric.

`einstein` (*dis*) [Function]

A function in the `ctensor` (component tensor) package. `einstein` computes the mixed Einstein tensor after the Christoffel symbols and Ricci tensor have been obtained (with the functions `christof` and `ricci`). If the argument *dis* is `true`, then the non-zero values of the mixed Einstein tensor `ein[i,j]` will be displayed where `j` is the contravariant index. The variable `rateinstein` will cause the rational simplification on these components. If `ratfac` is `true` then the components will also be factored.

`leinstein` (*dis*) [Function]

Covariant Einstein-tensor. `leinstein` stores the values of the covariant Einstein tensor in the array `lein`. The covariant Einstein-tensor is computed from the mixed Einstein tensor `ein` by multiplying it with the metric tensor. If the argument *dis* is `true`, then the non-zero values of the covariant Einstein tensor are displayed.

`riemann` (*dis*) [Function]

A function in the `ctensor` (component tensor) package. `riemann` computes the Riemann curvature tensor from the given metric and the corresponding Christoffel symbols. The following index conventions are used:

$$R[i,j,k,l] = R^{l}_{ijk} = \left| \right._{ij,k} - \left| \right._{ik,j} + \left| \right.^{l}_{mk} \left| \right._{ij} - \left| \right.^{m}_{ij} \left| \right._{mk}$$

This notation is consistent with the notation used by the `itensor` package and its `icurvature` function. If the optional argument *dis* is `true`, the unique non-zero components `riem[i,j,k,l]` will be displayed. As with the Einstein tensor, various switches set by the user control the simplification of the components of the Riemann

tensor. If `ratriemann` is `true`, then rational simplification will be done. If `ratfac` is `true` then each of the components will also be factored.

If the variable `cframe_flag` is `false`, the Riemann tensor is computed directly from the Christoffel-symbols. If `cframe_flag` is `true`, the covariant Riemann-tensor is computed first from the frame field coefficients.

`lriemann` (*dis*) [Function]

Covariant Riemann-tensor (`lriem[]`).

Computes the covariant Riemann-tensor as the array `lriem`. If the argument *dis* is `true`, unique non-zero values are displayed.

If the variable `cframe_flag` is `true`, the covariant Riemann tensor is computed directly from the frame field coefficients. Otherwise, the (3,1) Riemann tensor is computed first.

For information on index ordering, see `riemann`.

`uriemann` (*dis*) [Function]

Computes the contravariant components of the Riemann curvature tensor as array elements `uriem[i,j,k,l]`. These are displayed if *dis* is `true`.

`rinvariant` () [Function]

Forms the Kretschmann-invariant (`kinvariant`) obtained by contracting the tensors

```
lriem[i,j,k,l]*uriem[i,j,k,l].
```

This object is not automatically simplified since it can be very large.

`weyl` (*dis*) [Function]

Computes the Weyl conformal tensor. If the argument *dis* is `true`, the non-zero components `weyl[i,j,k,l]` will be displayed to the user. Otherwise, these components will simply be computed and stored. If the switch `ratweyl` is set to `true`, then the components will be rationally simplified; if `ratfac` is `true` then the results will be factored as well.

51.2.3 Taylor series expansion

The `ctensor` package has the ability to truncate results by assuming that they are Taylor-series approximations. This behavior is controlled by the `ctayswitch` variable; when set to true, `ctensor` makes use internally of the function `ctaylor` when simplifying results.

The `ctaylor` function is invoked by the following `ctensor` functions:

```
Function     Comments
--------------------------------
christof()   For mcs only
ricci()
uricci()
einstein()
riemann()
weyl()
checkdiv()
```

`ctaylor ()` [Function]

The `ctaylor` function truncates its argument by converting it to a Taylor-series using `taylor`, and then calling `ratdisrep`. This has the combined effect of dropping terms higher order in the expansion variable `ctayvar`. The order of terms that should be dropped is defined by `ctaypov`; the point around which the series expansion is carried out is specified in `ctaypt`.

As an example, consider a simple metric that is a perturbation of the Minkowski metric. Without further restrictions, even a diagonal metric produces expressions for the Einstein tensor that are far too complex:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) ratfac:true;
(%o2)                                true
(%i3) derivabbrev:true;
(%o3)                                true
(%i4) ct_coords:[t,r,theta,phi];
(%o4)                        [t, r, theta, phi]
(%i5) lg:matrix([-1,0,0,0],[0,1,0,0],[0,0,r^2,0],
               [0,0,0,r^2*sin(theta)^2]);
                       [ - 1  0  0         0        ]
                       [                            ]
                       [  0   1  0         0        ]
                       [                            ]
(%o5)                  [          2                 ]
                       [  0   0  r         0        ]
                       [                            ]
                       [              2    2        ]
                       [  0   0  0   r  sin (theta) ]
(%i6) h:matrix([h11,0,0,0],[0,h22,0,0],[0,0,h33,0],[0,0,0,h44]);
                           [ h11   0    0    0  ]
                           [                    ]
                           [  0   h22   0    0  ]
(%o6)                      [                    ]
                           [  0    0   h33   0  ]
                           [                    ]
                           [  0    0    0   h44 ]
(%i7) depends(l,r);
(%o7)                              [l(r)]
(%i8) lg:lg+l*h;
      [ h11 l - 1      0          0                0           ]
      [                                                       ]
      [     0      h22 l + 1      0                0           ]
      [                                                       ]
(%o8) [                         2                             ]
      [     0          0      r  + h33 l           0           ]
```

```
      [                                                          ]
      [                                   2    2                 ]
      [     0          0          0       r  sin (theta) + h44 l ]
(%i9) cmetric(false);
(%o9)                                done
(%i10) einstein(false);
(%o10)                               done
(%i11) ntermst(ein);
[[1, 1], 62]
[[1, 2], 0]
[[1, 3], 0]
[[1, 4], 0]
[[2, 1], 0]
[[2, 2], 24]
[[2, 3], 0]
[[2, 4], 0]
[[3, 1], 0]
[[3, 2], 0]
[[3, 3], 46]
[[3, 4], 0]
[[4, 1], 0]
[[4, 2], 0]
[[4, 3], 0]
[[4, 4], 46]
(%o12)                               done
```

However, if we recompute this example as an approximation that is linear in the variable `l`, we get much simpler expressions:

```
(%i14) ctayswitch:true;
(%o14)                               true
(%i15) ctayvar:l;
(%o15)                                 l
(%i16) ctaypov:1;
(%o16)                                 1
(%i17) ctaypt:0;
(%o17)                                 0
(%i18) christof(false);
(%o18)                               done
(%i19) ricci(false);
(%o19)                               done
(%i20) einstein(false);
(%o20)                               done
(%i21) ntermst(ein);
[[1, 1], 6]
[[1, 2], 0]
```

```
[[1, 3], 0]
[[1, 4], 0]
[[2, 1], 0]
[[2, 2], 13]
[[2, 3], 2]
[[2, 4], 0]
[[3, 1], 0]
[[3, 2], 2]
[[3, 3], 9]
[[3, 4], 0]
[[4, 1], 0]
[[4, 2], 0]
[[4, 3], 0]
[[4, 4], 9]
(%o21)                                done
(%i22) ratsimp(ein[1,1]);
                         2      2  4               2     2
(%o22) - (((h11 h22 - h11 ) (l )  r  - 2 h33 l    r ) sin (theta)
                              r               r r

                            2               2      4    2
                - 2 h44 l    r  - h33 h44 (l ) )/(4 r  sin (theta))
                         r r                r
```

This capability can be useful, for instance, when working in the weak field limit far from a gravitational source.

51.2.4 Frame fields

When the variable `cframe_flag` is set to true, the `ctensor` package performs its calculations using a moving frame.

`frame_bracket` (*fr*, *fri*, *diagframe*) [Function]

The frame bracket (`fb[]`).

Computes the frame bracket according to the following definition:

$$ifb^{c}_{ab} = ifri^{c}_{d,e} - ifri^{c}_{e,d}\ )\ ifr^{d}_{a}\ ifr^{e}_{b}$$

51.2.5 Algebraic classification

A new feature (as of November, 2004) of `ctensor` is its ability to compute the Petrov classification of a 4-dimensional spacetime metric. For a demonstration of this capability, see the file `share/tensor/petrov.dem`.

nptetrad () [Function]

Computes a Newman-Penrose null tetrad (np) and its raised-index counterpart (npi). See petrov for an example.

The null tetrad is constructed on the assumption that a four-dimensional orthonormal frame metric with metric signature (-,+,+,+) is being used. The components of the null tetrad are related to the inverse frame matrix as follows:

```
np  = (fri  + fri ) / sqrt(2)
  1       1      2

np  = (fri  - fri ) / sqrt(2)
  2       1      2

np  = (fri  + %i fri ) / sqrt(2)
  3       3         4

np  = (fri  - %i fri ) / sqrt(2)
  4       3         4
```

psi (*dis*) [Function]

Computes the five Newman-Penrose coefficients psi[0]...psi[4]. If dis is set to true, the coefficients are displayed. See petrov for an example.

These coefficients are computed from the Weyl-tensor in a coordinate base. If a frame base is used, the Weyl-tensor is first converted to a coordinate base, which can be a computationally expensive procedure. For this reason, in some cases it may be more advantageous to use a coordinate base in the first place before the Weyl tensor is computed. Note however, that constructing a Newman-Penrose null tetrad requires a frame base. Therefore, a meaningful computation sequence may begin with a frame base, which is then used to compute lg (computed automatically by cmetric) and then ug. See petrov for an example. At this point, you can switch back to a coordinate base by setting cframe_flag to false before beginning to compute the Christoffel symbols. Changing to a frame base at a later stage could yield inconsistent results, as you may end up with a mixed bag of tensors, some computed in a frame base, some in a coordinate base, with no means to distinguish between the two.

petrov () [Function]

Computes the Petrov classification of the metric characterized by psi[0]...psi[4].

For example, the following demonstrates how to obtain the Petrov-classification of the Kerr metric:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) (cframe_flag:true,gcd:spmod,ctrgsimp:true,ratfac:true);
(%o2)                                true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3)                                done
```

```
(%i4) ug:invert(lg)$
(%i5) weyl(false);
(%o5)                                done
(%i6) nptetrad(true);
(%t6) np =

[ sqrt(r - 2 m)           sqrt(r)                                  ]
[---------------   ----------------------    0            0       ]
[sqrt(2) sqrt(r)   sqrt(2) sqrt(r - 2 m)                           ]
[                                                                  ]
[ sqrt(r - 2 m)            sqrt(r)                                 ]
[---------------  - ----------------------   0            0       ]
[sqrt(2) sqrt(r)    sqrt(2) sqrt(r - 2 m)                          ]
[                                                                  ]
[                                          r      %i r sin(theta) ]
[       0                   0          -------   --------------- ]
[                                      sqrt(2)       sqrt(2)     ]
[                                                                  ]
[                                         r       %i r sin(theta)]
[       0                   0          -------  - ---------------]
[                                      sqrt(2)        sqrt(2)    ]

                             sqrt(r)         sqrt(r - 2 m)
(%t7) npi = matrix([- ----------------------,---------------, 0, 0],
                     sqrt(2) sqrt(r - 2 m) sqrt(2) sqrt(r)

          sqrt(r)            sqrt(r - 2 m)
[- ----------------------, - ---------------, 0, 0],
   sqrt(2) sqrt(r - 2 m)    sqrt(2) sqrt(r)

           1               %i
[0, 0, ---------, --------------------],
       sqrt(2) r  sqrt(2) r sin(theta)

           1                %i
[0, 0, ---------, - --------------------])
       sqrt(2) r    sqrt(2) r sin(theta)

(%o7)                                done
(%i7) psi(true);
(%t8)                              psi  = 0
                                     0

(%t9)                              psi  = 0
                                     1

                                         m
```

```
(%t10)                                  psi  = --
                                           2    3
                                                r

(%t11)                                  psi  = 0
                                           3

(%t12)                                  psi  = 0
                                           4
(%o12)                                    done
(%i12) petrov();
(%o12)                                     D
```

The Petrov classification function is based on the algorithm published in "Classifying geometries in general relativity: III Classification in practice" by Pollney, Skea, and d'Inverno, Class. Quant. Grav. 17 2885-2902 (2000). Except for some simple test cases, the implementation is untested as of December 19, 2004, and is likely to contain errors.

51.2.6 Torsion and nonmetricity

`ctensor` has the ability to compute and include torsion and nonmetricity coefficients in the connection coefficients.

The torsion coefficients are calculated from a user-supplied tensor `tr`, which should be a rank (2,1) tensor. From this, the torsion coefficients `kt` are computed according to the following formulae:

```
                m          m      m
        - g  tr   - g   tr   - tr   g
           im  kj    jm   ki     ij  km
kt   = -------------------------------
  ijk                 2


  k     km
kt   = g   kt
  ij         ijm
```

Note that only the mixed-index tensor is calculated and stored in the array `kt`.

The nonmetricity coefficients are calculated from the user-supplied nonmetricity vector `nm`. From this, the nonmetricity coefficients `nmc` are computed as follows:

```
              k    k        km
        -nm  D  - D  nm  + g   nm  g
   k       i  j    i   j         m  ij
nmc  = ------------------------------
```

```
ij                2
```

where D stands for the Kronecker-delta.

When `ctorsion_flag` is set to `true`, the values of `kt` are subtracted from the mixed-indexed connection coefficients computed by `christof` and stored in `mcs`. Similarly, if `cnonmet_flag` is set to `true`, the values of `nmc` are subtracted from the mixed-indexed connection coefficients.

If necessary, `christof` calls the functions `contortion` and `nonmetricity` in order to compute `kt` and `nm`.

`contortion` (*tr*) [Function]
Computes the (2,1) contortion coefficients from the torsion tensor *tr*.

`nonmetricity` (*nm*) [Function]
Computes the (2,1) nonmetricity coefficients from the nonmetricity vector *nm*.

51.2.7 Miscellaneous features

`ctransform` (*M*) [Function]
A function in the `ctensor` (component tensor) package which will perform a coordinate transformation upon an arbitrary square symmetric matrix *M*. The user must input the functions which define the transformation. (Formerly called `transform`.) These may also be supplied in the form of a list as an optional second argument.

`findde` (*A*, *n*) [Function]
returns a list of the unique differential equations (expressions) corresponding to the elements of the *n* dimensional square array *A*. Presently, *n* may be 2 or 3. `deindex` is a global list containing the indices of *A* corresponding to these unique differential equations. For the Einstein tensor (`ein`), which is a two dimensional array, if computed for the metric in the example below, `findde` gives the following independent differential equations:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2)                                true
(%i3) dim:4;
(%o3)                                 4
(%i4) lg:matrix([a, 0, 0, 0], [ 0, x^2, 0, 0],
                              [0, 0, x^2*sin(y)^2, 0], [0,0,0,-d]);
                          [ a  0       0        0  ]
                          [                        ]
                          [     2                  ]
                          [ 0  x       0        0  ]
(%o4)                     [                        ]
                          [         2    2         ]
                          [ 0  0   x  sin (y)   0  ]
                          [                        ]
```

```
                                [ 0  0       0       - d ]
(%i5) depends([a,d],x);
(%o5)                                [a(x), d(x)]
(%i6) ct_coords:[x,y,z,t];
(%o6)                                [x, y, z, t]
(%i7) cmetric();
(%o7)                                    done
(%i8) einstein(false);
(%o8)                                    done
(%i9) findde(ein,2);
                                                  2
(%o9) [d  x - a d + d, 2 a d d    x - a (d )  x - a  d d  x
        x                      x x         x        x    x

                                                  2          2
                                + 2 a d d   - 2 a  d , a  x + a  - a]
                                         x       x      x
(%i10) deindex;
(%o10)                          [[1, 1], [2, 2], [4, 4]]
```

cograd () [Function]

Computes the covariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example under `contragrad` illustrates.

contragrad () [Function]

Computes the contravariant gradient of a scalar function allowing the user to choose the corresponding vector name as the example below for the Schwarzschild metric illustrates:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2)                                    true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3)                                    done
(%i4) depends(f,r);
(%o4)                                   [f(r)]
(%i5) cograd(f,g1);
(%o5)                                    done
(%i6) listarray(g1);
(%o6)                                [0, f , 0, 0]
                                          r
(%i7) contragrad(f,g2);
(%o7)                                    done
(%i8) listarray(g2);
                                    f  r - 2 f  m
                                     r        r
(%o8)                           [0, -------------, 0, 0]
```

```
                                        r
```

`dscalar` () [Function]

computes the tensor d'Alembertian of the scalar function once dependencies have been declared upon the function. For example:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) derivabbrev:true;
(%o2)                                true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3)                                done
(%i4) depends(p,r);
(%o4)                              [p(r)]
(%i5) factor(dscalar(p));
                          2
                 p    r  - 2 m p    r + 2 p  r - 2 m p
                  r r           r r        r          r
(%o5)            ------------------------------------
                                  2
                                 r
```

`checkdiv` () [Function]

computes the covariant divergence of the mixed second rank tensor (whose first index must be covariant) by printing the corresponding n components of the vector field (the divergence) where n = `dim`. If the argument to the function is `g` then the divergence of the Einstein tensor will be formed and must be zero. In addition, the divergence (vector) is given the array name `div`.

`cgeodesic` (*dis*) [Function]

A function in the `ctensor` (component tensor) package. `cgeodesic` computes the geodesic equations of motion for a given metric. They are stored in the array `geod[i]`. If the argument *dis* is `true` then these equations are displayed.

`bdvac` (*f*) [Function]

generates the covariant components of the vacuum field equations of the Brans- Dicke gravitational theory. The scalar field is specified by the argument *f*, which should be a (quoted) function name with functional dependencies, e.g., `'p(x)`.

The components of the second rank covariant field tensor are represented by the array `bd`.

`invariant1` () [Function]

generates the mixed Euler- Lagrange tensor (field equations) for the invariant density of R^2. The field equations are the components of an array named `inv1`.

`invariant2` () [Function]

*** NOT YET IMPLEMENTED ***

generates the mixed Euler- Lagrange tensor (field equations) for the invariant density of `ric[i,j]*uriem[i,j]`. The field equations are the components of an array named `inv2`.

`bimetric` () [Function]

*** NOT YET IMPLEMENTED ***

generates the field equations of Rosen's bimetric theory. The field equations are the components of an array named `rosen`.

51.2.8 Utility functions

`diagmatrixp` (*M*,*n*) [Function]

Returns `true` if the first *n* rows and *n* columns of *M* form a diagonal matrix or (2D) array.

`symmetricp` (*M*, *n*) [Function]

Returns `true` if *M* is a *n* by *n* symmetric matrix or two-dimensional array, otherwise `false`.

If *n* is less than the size of *M*, `symmetricp` considers only the *n* by *n* submatrix (respectively, subarray) comprising rows 1 through *n* and columns 1 through *n*.

`ntermst` (*f*) [Function]

gives the user a quick picture of the "size" of the doubly subscripted tensor (array) *f*. It prints two element lists where the second element corresponds to NTERMS of the components specified by the first elements. In this way, it is possible to quickly find the non-zero expressions and attempt simplification.

`cdisplay` (*ten*) [Function]

displays all the elements of the tensor *ten*, as represented by a multidimensional array. Tensors of rank 0 and 1, as well as other types of variables, are displayed as with `ldisplay`. Tensors of rank 2 are displayed as 2-dimensional matrices, while tensors of higher rank are displayed as a list of 2-dimensional matrices. For instance, the Riemann-tensor of the Schwarzschild metric can be viewed as:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) ratfac:true;
(%o2)                                true
(%i3) ct_coordsys(exteriorschwarzschild,all);
(%o3)                                done
(%i4) riemann(false);
(%o4)                                done
(%i5) cdisplay(riem);
          [ 0              0                   0           0     ]
          [                                                      ]
          [                                2                     ]
          [      3 m (r - 2 m)   m    2 m                       ]
          [ 0  - ------------- + -- - ----      0           0     ]
          [            4          3     4                        ]
          [           r          r     r                         ]
          [                                                      ]
riem    = [                                m (r - 2 m)           ]
    1, 1  [ 0              0              -----------      0     ]
```

```
[                                  4                  ]
[                                 r                   ]
[                                                     ]
[                                        m (r - 2 m) ]
[ 0              0                  0    ----------- ]
[                                             4       ]
[                                            r        ]

                      [    2 m (r - 2 m)        ]
                      [ 0  -------------  0  0 ]
                      [          4              ]
                      [         r               ]
          riem     = [                         ]
              1, 2   [ 0        0        0  0 ]
                      [                         ]
                      [ 0        0        0  0 ]
                      [                         ]
                      [ 0        0        0  0 ]

                      [         m (r - 2 m)     ]
                      [ 0  0  - -----------  0 ]
                      [              4          ]
                      [             r           ]
          riem     = [                         ]
              1, 3   [ 0  0        0        0 ]
                      [                         ]
                      [ 0  0        0        0 ]
                      [                         ]
                      [ 0  0        0        0 ]

                      [            m (r - 2 m) ]
                      [ 0  0  0  - ----------- ]
                      [                 4       ]
                      [                r        ]
          riem     = [                         ]
              1, 4   [ 0  0  0        0        ]
                      [                         ]
                      [ 0  0  0        0        ]
                      [                         ]
                      [ 0  0  0        0        ]

                     [       0         0  0  0 ]
                     [                         ]
                     [       2 m               ]
                     [ - ------------  0  0  0 ]
         riem     = [    2                     ]
             2, 1   [   r  (r - 2 m)           ]
```

$$\begin{bmatrix} & & & \\ 0 & 0 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{riem}_{2,2} = \begin{bmatrix} \frac{2\,m}{r^2\,(r-2\,m)} & 0 & 0 & 0 \\ 0 & 0 & -\frac{m}{r^2\,(r-2\,m)} & 0 \\ 0 & 0 & 0 & \frac{m}{r^2\,(r-2\,m)} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{riem}_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{r^2\,(r-2\,m)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{riem}_{2,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{r^2\,(r-2\,m)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
                    [            ]
        riem     = [ m          ]
            3, 1   [ -  0  0  0 ]
                    [ r          ]
                    [            ]
                    [ 0  0  0  0 ]

                    [ 0  0  0  0 ]
                    [            ]
                    [ 0  0  0  0 ]
                    [            ]
        riem     = [    m       ]
            3, 2   [ 0  -  0  0 ]
                    [    r       ]
                    [            ]
                    [ 0  0  0  0 ]

                [   m                      ]
                [ - -   0   0       0      ]
                [   r                      ]
                [                          ]
                [        m                 ]
                [  0   - -  0       0      ]
riem     =      [        r                 ]
    3, 3        [                          ]
                [  0    0   0       0      ]
                [                          ]
                [              2 m - r     ]
                [  0    0   0  ------- + 1 ]
                [                 r        ]

                    [ 0  0  0    0   ]
                    [                ]
                    [ 0  0  0    0   ]
                    [                ]
     riem     =     [            2 m ]
         3, 4       [ 0  0  0  - --- ]
                    [             r  ]
                    [                ]
                    [ 0  0  0    0   ]

                [       0        0  0  0 ]
                [                        ]
                [       0        0  0  0 ]
                [                        ]
 riem     =     [       0        0  0  0 ]
     4, 1       [                        ]
```

```
                                  [      2                 ]
                                  [ m sin (theta)          ]
                                  [ -------------  0  0  0 ]
                                  [       r                ]

                                  [ 0        0        0  0 ]
                                  [                        ]
                                  [ 0        0        0  0 ]
                                  [                        ]
                     riem     = [ 0        0        0  0 ]
                         4, 2   [                        ]
                                  [         2              ]
                                  [    m sin (theta)       ]
                                  [ 0  -------------  0  0 ]
                                  [          r             ]

                                [ 0  0          0          0 ]
                                [                            ]
                                [ 0  0          0          0 ]
                                [                            ]
                   riem     = [ 0  0          0          0 ]
                       4, 3   [                            ]
                                [                2           ]
                                [         2 m sin (theta)    ]
                                [ 0  0  - ---------------  0 ]
                                [                r           ]

            [        2                                                ]
            [   m sin (theta)                                         ]
            [ - -------------         0                0         0 ]
            [         r                                               ]
            [                                                         ]
            [                         2                               ]
            [                   m sin (theta)                         ]
riem     = [        0         - -------------         0         0 ]
    4, 4   [                          r                               ]
            [                                                         ]
            [                                         2               ]
            [                                  2 m sin (theta)        ]
            [        0                0         ---------------  0 ]
            [                                         r               ]
            [                                                         ]
            [        0                0                0         0 ]

(%o5)                                done
```

`deleten` (*L*, *n*) [Function]
Returns a new list consisting of *L* with the *n*'th element deleted.

51.2.9 Variables used by `ctensor`

`dim` [Option variable]
Default value: 4

An option in the `ctensor` (component tensor) package. `dim` is the dimension of the manifold with the default 4. The command `dim: n` will reset the dimension to any other value `n`.

`diagmetric` [Option variable]
Default value: `false`

An option in the `ctensor` (component tensor) package. If `diagmetric` is `true` special routines compute all geometrical objects (which contain the metric tensor explicitly) by taking into consideration the diagonality of the metric. Reduced run times will, of course, result. Note: this option is set automatically by `csetup` if a diagonal metric is specified.

`ctrgsimp` [Option variable]
Causes trigonometric simplifications to be used when tensors are computed. Presently, `ctrgsimp` affects only computations involving a moving frame.

`cframe_flag` [Option variable]
Causes computations to be performed relative to a moving frame as opposed to a holonomic metric. The frame is defined by the inverse frame array `fri` and the frame metric `lfg`. For computations using a Cartesian frame, `lfg` should be the unit matrix of the appropriate dimension; for computations in a Lorentz frame, `lfg` should have the appropriate signature.

`ctorsion_flag` [Option variable]
Causes the contortion tensor to be included in the computation of the connection coefficients. The contortion tensor itself is computed by `contortion` from the user-supplied tensor `tr`.

`cnonmet_flag` [Option variable]
Causes the nonmetricity coefficients to be included in the computation of the connection coefficients. The nonmetricity coefficients are computed from the user-supplied nonmetricity vector `nm` by the function `nonmetricity`.

`ctayswitch` [Option variable]
If set to `true`, causes some `ctensor` computations to be carried out using Taylor-series expansions. Presently, `christof`, `ricci`, `uricci`, `einstein`, and `weyl` take into account this setting.

`ctayvar` [Option variable]
Variable used for Taylor-series expansion if `ctayswitch` is set to `true`.

`ctaypov` [Option variable]
Maximum power used in Taylor-series expansion when `ctayswitch` is set to `true`.

`ctaypt` [Option variable]

Point around which Taylor-series expansion is carried out when `ctayswitch` is set to `true`.

`gdet` [System variable]

The determinant of the metric tensor `lg`. Computed by `cmetric` when `cframe_flag` is set to `false`.

`ratchristof` [Option variable]

Causes rational simplification to be applied by `christof`.

`rateinstein` [Option variable]

Default value: `true`

If `true` rational simplification will be performed on the non-zero components of Einstein tensors; if `ratfac` is `true` then the components will also be factored.

`ratriemann` [Option variable]

Default value: `true`

One of the switches which controls simplification of Riemann tensors; if `true`, then rational simplification will be done; if `ratfac` is `true` then each of the components will also be factored.

`ratweyl` [Option variable]

Default value: `true`

If `true`, this switch causes the `weyl` function to apply rational simplification to the values of the Weyl tensor. If `ratfac` is `true`, then the components will also be factored.

`lfg` [Variable]

The covariant frame metric. By default, it is initialized to the 4-dimensional Lorentz frame with signature (+,+,+,-). Used when `cframe_flag` is `true`.

`ufg` [Variable]

The inverse frame metric. Computed from `lfg` when `cmetric` is called while `cframe_flag` is set to `true`.

`riem` [Variable]

The (3,1) Riemann tensor. Computed when the function `riemann` is invoked. For information about index ordering, see the description of `riemann`.

If `cframe_flag` is `true`, `riem` is computed from the covariant Riemann-tensor `lriem`.

`lriem` [Variable]

The covariant Riemann tensor. Computed by `lriemann`.

`uriem` [Variable]

The contravariant Riemann tensor. Computed by `uriemann`.

`ric` [Variable]

The covariant Ricci-tensor. Computed by `ricci`.

`uric` [Variable]
The mixed-index Ricci-tensor. Computed by `uricci`.

`lg` [Variable]
The metric tensor. This tensor must be specified (as a `dim` by `dim` matrix) before other computations can be performed.

`ug` [Variable]
The inverse of the metric tensor. Computed by `cmetric`.

`weyl` [Variable]
The Weyl tensor. Computed by `weyl`.

`fb` [Variable]
Frame bracket coefficients, as computed by `frame_bracket`.

`kinvariant` [Variable]
The Kretschmann invariant. Computed by `rinvariant`.

`np` [Variable]
A Newman-Penrose null tetrad. Computed by `nptetrad`.

`npi` [Variable]
The raised-index Newman-Penrose null tetrad. Computed by `nptetrad`. Defined as `ug.np`. The product `np.transpose(npi)` is constant:

```
(%i39) trigsimp(np.transpose(npi));
                                [  0   - 1  0  0 ]
                                [                ]
                                [ - 1   0   0  0 ]
(%o39)                          [                ]
                                [  0    0   0  1 ]
                                [                ]
                                [  0    0   1  0 ]
```

`tr` [Variable]
User-supplied rank-3 tensor representing torsion. Used by `contortion`.

`kt` [Variable]
The contortion tensor, computed from `tr` by `contortion`.

`nm` [Variable]
User-supplied nonmetricity vector. Used by `nonmetricity`.

`nmc` [Variable]
The nonmetricity coefficients, computed from `nm` by `nonmetricity`.

`tensorkill` [System variable]
Variable indicating if the tensor package has been initialized. Set and used by `csetup`, reset by `init_ctensor`.

`ct_coords` [Option variable]

Default value: `[]`

An option in the `ctensor` (component tensor) package. `ct_coords` contains a list of coordinates. While normally defined when the function `csetup` is called, one may redefine the coordinates with the assignment `ct_coords: [j1, j2, ..., jn]` where the j's are the new coordinate names. See also `csetup`.

51.2.10 Reserved names

The following names are used internally by the `ctensor` package and should not be redefined:

Name	Description
`_lg()`	Evaluates to lfg if frame metric used, lg otherwise
`_ug()`	Evaluates to ufg if frame metric used, ug otherwise
`cleanup()`	Removes items from the deindex list
`contract4()`	Used by psi()
`filemet()`	Used by csetup() when reading the metric from a file
`findde1()`	Used by findde()
`findde2()`	Used by findde()
`findde3()`	Used by findde()
`kdelt()`	Kronecker-delta (not generalized)
`newmet()`	Used by csetup() for setting up a metric interactively
`setflags()`	Used by init_ctensor()
`readvalue()`	
`resimp()`	
`sermet()`	Used by csetup() for entering a metric as Taylor-series
`txyzsum()`	
`tmetric()`	Frame metric, used by cmetric() when cframe_flag:true
`triemann()`	Riemann-tensor in frame base, used when cframe_flag:true
`tricci()`	Ricci-tensor in frame base, used when cframe_flag:true
`trrc()`	Ricci rotation coefficients, used by christof()
`yesp()`	

51.2.11 Changes

In November, 2004, the `ctensor` package was extensively rewritten. Many functions and variables have been renamed in order to make the package compatible with the commercial version of Macsyma.

New Name	Old Name	Description
`ctaylor()`	`DLGTAYLOR()`	Taylor-series expansion of an expression
`lgeod[]`	`EM`	Geodesic equations
`ein[]`	`G[]`	Mixed Einstein-tensor
`ric[]`	`LR[]`	Mixed Ricci-tensor
`ricci()`	`LRICCICOM()`	Compute the mixed Ricci-tensor
`ctaypov`	`MINP`	Maximum power in Taylor-series expansion
`cgeodesic()`	`MOTION`	Compute geodesic equations
`ct_coords`	`OMEGA`	Metric coordinates

ctayvar	PARAM	Taylor-series expansion variable
lriem[]	R[]	Covariant Riemann-tensor
uriemann()	RAISERIEMANN()	Compute the contravariant Riemann-tensor
ratriemann	RATRIEMAN	Rational simplif. of the Riemann-tensor
uric[]	RICCI[]	Contravariant Ricci-tensor
uricci()	RICCICOM()	Compute the contravariant Ricci-tensor
cmetric()	SETMETRIC()	Set up the metric
ctaypt	TAYPT	Point for Taylor-series expansion
ctayswitch	TAYSWITCH	Taylor-series setting switch
csetup()	TSETUP()	Start interactive setup session
ctransform()	TTRANSFORM()	Interactive coordinate transformation
uriem[]	UR[]	Contravariant Riemann-tensor
weyl[]	W[]	(3,1) Weyl-tensor

52 Package descriptive

52.1 Introduction to descriptive

Package `descriptive` contains a set of functions for making descriptive statistical computations and graphing. Together with the source code there are three data sets in your Maxima tree: `pidigits.data`, `wind.data` and `biomed.data`.

Any statistics manual can be used as a reference to the functions in package `descriptive`.

For comments, bugs or suggestions, please contact me at *'riotorto AT yahoo DOT com'*.

Here is a simple example on how the descriptive functions in `descriptive` do they work, depending on the nature of their arguments, lists or matrices,

```
(%i1) load ("descriptive")$
(%i2) /* univariate sample */   mean ([a, b, c]);
                               c + b + a
(%o2)                          ---------
                                   3
(%i3) matrix ([a, b], [c, d], [e, f]);
                               [ a  b ]
                               [      ]
(%o3)                          [ c  d ]
                               [      ]
                               [ e  f ]
(%i4) /* multivariate sample */ mean (%);
                         e + c + a  f + d + b
(%o4)                   [---------, ---------]
                             3          3
```

Note that in multivariate samples the mean is calculated for each column.

In case of several samples with possible different sizes, the Maxima function `map` can be used to get the desired results for each sample,

```
(%i1) load ("descriptive")$
(%i2) map (mean, [[a, b, c], [d, e]]);
                           c + b + a  e + d
(%o2)                     [---------, -----]
                               3        2
```

In this case, two samples of sizes 3 and 2 were stored into a list.

Univariate samples must be stored in lists like

```
(%i1) s1 : [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5];
(%o1)           [3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5]
```

and multivariate samples in matrices as in

```
(%i1) s2 : matrix ([13.17, 9.29], [14.71, 16.88], [18.50, 16.88],
             [10.58, 6.63], [13.33, 13.25], [13.21,  8.12]);
                          [ 13.17  9.29  ]
                          [              ]
                          [ 14.71  16.88 ]
                          [              ]
                          [ 18.5   16.88 ]
(%o1)                     [              ]
                          [ 10.58  6.63  ]
                          [              ]
                          [ 13.33  13.25 ]
                          [              ]
                          [ 13.21  8.12  ]
```

In this case, the number of columns equals the random variable dimension and the number of rows is the sample size.

Data can be introduced by hand, but big samples are usually stored in plain text files. For example, file `pidigits.data` contains the first 100 digits of number `%pi`:

```
3
1
4
1
5
9
2
6
5
3 ...
```

In order to load these digits in Maxima,

```
(%i1) s1 : read_list (file_search ("pidigits.data"))$
(%i2) length (s1);
(%o2)                          100
```

On the other hand, file `wind.data` contains daily average wind speeds at 5 meteorological stations in the Republic of Ireland (This is part of a data set taken at 12 meteorological stations. The original file is freely downloadable from the StatLib Data Repository and its analysis is discussed in Haslett, J., Raftery, A. E. (1989) *Space-time Modelling with Long-memory Dependence: Assessing Ireland's Wind Power Resource, with Discussion*. Applied Statistics 38, 1-50). This loads the data:

```
(%i1) s2 : read_matrix (file_search ("wind.data"))$
(%i2) length (s2);
(%o2)                          100
(%i3) s2 [%]; /* last record */
(%o3)            [3.58, 6.0, 4.58, 7.62, 11.25]
```

Some samples contain non numeric data. As an example, file `biomed.data` (which is part of another bigger one downloaded from the StatLib Data Repository) contains four blood measures taken from two groups of patients, `A` and `B`, of different ages,

```
(%i1) s3 : read_matrix (file_search ("biomed.data"))$
(%i2) length (s3);
(%o2)                          100
(%i3) s3 [1]; /* first record */
(%o3)            [A, 30, 167.0, 89.0, 25.6, 364]
```

The first individual belongs to group `A`, is 30 years old and his/her blood measures were 167.0, 89.0, 25.6 and 364.

One must take care when working with categorical data. In the next example, symbol `a` is assigned a value in some previous moment and then a sample with categorical value `a` is taken,

```
(%i1) a : 1$
(%i2) matrix ([a, 3], [b, 5]);
                             [ 1  3 ]
(%o2)                        [      ]
                             [ b  5 ]
```

52.2 Functions and Variables for data manipulation

`build_sample` [Function]

`build_sample` (*list*)
`build_sample` (*matrix*)

Builds a sample from a table of absolute frequencies. The input table can be a matrix or a list of lists, all of them of equal size. The number of columns or the length of the lists must be greater than 1. The last element of each row or list is interpreted as the absolute frequency. The output is always a sample in matrix form.

Examples:

Univariate frequency table.

```
(%i1) load ("descriptive")$
(%i2) sam1: build_sample([[6,1], [j,2], [2,1]]);
                               [ 6 ]
                               [   ]
                               [ j ]
(%o2)                          [   ]
                               [ j ]
                               [   ]
                               [ 2 ]
(%i3) mean(sam1);
                               j + 4
(%o3)                         [-----]
                                 2
(%i4) barsplot(sam1) $
```

Multivariate frequency table.

```
(%i1) load ("descriptive")$
```

```
(%i2) sam2: build_sample([[6,3,1], [5,6,2], [u,2,1],[6,8,2]]) ;
                              [ 6  3 ]
                              [      ]
                              [ 5  6 ]
                              [      ]
                              [ 5  6 ]
(%o2)                         [      ]
                              [ u  2 ]
                              [      ]
                              [ 6  8 ]
                              [      ]
                              [ 6  8 ]
(%i3) cov(sam2);
      [   2                  2                                  ]
      [  u  + 158   (u + 28)     2 u + 174   11 (u + 28)        ]
      [  -------- - ---------    --------- - -----------        ]
(%o3) [     6          36            6           12             ]
      [                                                         ]
      [ 2 u + 174   11 (u + 28)           21                    ]
      [ --------- - -----------           --                    ]
      [     6           12                4                     ]
(%i4) barsplot(sam2, grouping=stacked) $
```

`continuous_freq` [Function]

`continuous_freq` (*data*)
`continuous_freq` (*data*, *m*)

Divides the range of *data* into intervals, and counts how many values fall into each one.

A value x falls into an interval with left and right endpoints a and b if and only if `x > a` and `x <= b`, except for the first (least or leftmost) interval, for which `x >= a` and `x <= b`. That is, an interval excludes its left endpoint and includes its right endpoint, except for the first interval, which includes both the left and right endpoints.

data must be a list of numbers, or 1-dimensional array (as created by `make_array`).

m is optional, and equals either the number of classes (10 by default), or a list of two elements (the least and greatest values to be counted), or a list of three elements (the least and greatest values to be counted, and the number of classes), or a set containing the endpoints of the class intervals.

It is assumed that class intervals are contiguous. That is, the right endpoint of one interval is equal to the left endpoint of the next.

`continuous_freq` returns a list of two lists. The first list comprises all the endpoints of the class intervals, concatenated into a single list. The second list contains the class counts for the intervals corresponding to elements of the first list.

If sample values are all equal, this function returns exactly one class of width 2.

Examples:

Optional argument indicates the number of classes we want. The first list in the output contains the interval limits, and the second the corresponding counts: there are 16 digits inside the interval `[0, 1.8]`, 24 digits in `(1.8, 3.6]`, and so on.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, 5);
                 9  18  27  36
(%o3)     [[0, -, --, --, --, 9], [16, 24, 18, 17, 25]]
                 5  5   5   5
```

Optional argument indicates we want 7 classes with limits -2 and 12:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, [-2,12,7]);
(%o3) [[- 2, 0, 2, 4, 6, 8, 10, 12], [8, 20, 22, 17, 20, 13, 0]]
```

Optional argument indicates we want the default number of classes with limits -2 and 12:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) continuous_freq (s1, [-2,12]);
                 3  4  11  18     32  39  46  53
(%o3) [[- 2, - -, -, --, --, 5, --, --, --, --, 12],
                 5  5  5   5      5   5   5   5
                                [0, 8, 20, 12, 18, 9, 8, 25, 0, 0]]
```

The first argument may be an array.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) a1 : make_array (fixnum, length (s1)) $
(%i4) fillarray (a1, s1);
(%o4) {Lisp Array: #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2\
 6 4 3 3 8 3 2 7 9 5
                 0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7\
 4 9 4 4 5 9 2
                 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9 8 6 2 8 0 3 4 8\
 2 5 3 4 2 1 1
                 7 0 6 7)}
(%i5) continuous_freq (a1);
           9   9  27  18  9  27  63  36  81
(%o5) [[0, --, -, --, --, -, --, --, --, --, 9],
           10  5  10  5   2  5   10  5   10
                              [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]
```

`discrete_freq` (*data*) [Function]

Counts absolute frequencies in discrete samples, both numeric and categorical. Its sole argument is a list, or 1-dimensional array (as created by `make_array`).

Examples:

```
(%i1) load ("descriptive")$
```

```
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) discrete_freq (s1);
(%o3) [[0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
                              [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]
```

In the return value, the first list gives the sample values, and the second, their absolute frequencies.

The argument may be an array.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) a1 : make_array (fixnum, length (s1)) $
(%i4) fillarray (a1, s1);
(%o4) {Lisp Array: #(3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2\
 6 4 3 3 8 3 2 7 9 5
                0 2 8 8 4 1 9 7 1 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7\
 4 9 4 4 5 9 2
                3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9 8 6 2 8 0 3 4 8\
 2 5 3 4 2 1 1
                7 0 6 7)}
(%i5) discrete_freq (a1);
(%o5) [[0, 1, 2, 3, 4, 5, 6, 7, 8, 9],
                              [8, 8, 12, 12, 10, 8, 9, 8, 12, 13]]
```

`standardize` [Function]

`standardize` (*list*)
`standardize` (*matrix*)

Subtracts to each element of the list the sample mean and divides the result by the standard deviation. When the input is a matrix, `standardize` subtracts to each row the multivariate mean, and then divides each component by the corresponding standard deviation.

`subsample` [Function]

`subsample` (*data_matrix*, *predicate_function*)
`subsample` (*data_matrix*, *predicate_function*, *col_num1*, *col_num2*, ...)

This is a sort of variant of the Maxima `submatrix` function. The first argument is the data matrix, the second is a predicate function and optional additional arguments are the numbers of the columns to be taken.

Examples:

These are multivariate records in which the wind speed in the first meteorological station were greater than 18. See that in the lambda expression the *i*-th component is referred to as `v[i]`.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
```

```
(%i3) subsample (s2, lambda([v], v[1] > 18));
              [ 19.38  15.37  15.12  23.09  25.25 ]
              [                                    ]
              [ 18.29  18.66  19.08  26.08  27.63 ]
(%o3)         [                                    ]
              [ 20.25  21.46  19.95  27.71  23.38 ]
              [                                    ]
              [ 18.79  18.96  14.46  26.38  21.84 ]
```

In the following example, we request only the first, second and fifth components of those records with wind speeds greater or equal than 16 in station number 1 and less than 25 knots in station number 4. The sample contains only data from stations 1, 2 and 5. In this case, the predicate function is defined as an ordinary Maxima function.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) g(x):= x[1] >= 16 and x[4] < 25$
(%i4) subsample (s2, g, 1, 2, 5);
                    [ 19.38  15.37  25.25 ]
                    [                     ]
                    [ 17.33  14.67  19.58 ]
(%o4)               [                     ]
                    [ 16.92  13.21  21.21 ]
                    [                     ]
                    [ 17.25  18.46  23.87 ]
```

Here is an example with the categorical variables of `biomed.data`. We want the records corresponding to those patients in group `B` who are older than 38 years.

```
(%i1) load ("descriptive")$
(%i2) s3 : read_matrix (file_search ("biomed.data"))$
(%i3) h(u):= u[1] = B and u[2] > 38 $
(%i4) subsample (s3, h);
                [ B  39  28.0  102.3  17.1  146 ]
                [                               ]
                [ B  39  21.0  92.4   10.3  197 ]
                [                               ]
                [ B  39  23.0  111.5  10.0  133 ]
                [                               ]
                [ B  39  26.0  92.6   12.3  196 ]
(%o4)           [                               ]
                [ B  39  25.0  98.7   10.0  174 ]
                [                               ]
                [ B  39  21.0  93.2   5.9   181 ]
                [                               ]
                [ B  39  18.0  95.0   11.3  66  ]
                [                               ]
                [ B  39  39.0  88.5   7.6   168 ]
```

Probably, the statistical analysis will involve only the blood measures,

```
(%i1) load ("descriptive")$
```

```
(%i2) s3 : read_matrix (file_search ("biomed.data"))$
(%i3) subsample (s3, lambda([v], v[1] = B and v[2] > 38),
          3, 4, 5, 6);
                  [ 28.0  102.3  17.1  146 ]
                  [                        ]
                  [ 21.0  92.4   10.3  197 ]
                  [                        ]
                  [ 23.0  111.5  10.0  133 ]
                  [                        ]
                  [ 26.0  92.6   12.3  196 ]
(%o3)             [                        ]
                  [ 25.0  98.7   10.0  174 ]
                  [                        ]
                  [ 21.0  93.2   5.9   181 ]
                  [                        ]
                  [ 18.0  95.0   11.3  66  ]
                  [                        ]
                  [ 39.0  88.5   7.6   168 ]
```

This is the multivariate mean of `s3`,

```
(%i1) load ("descriptive")$
(%i2) s3 : read_matrix (file_search ("biomed.data"))$
(%i3) mean (s3);
       13 B + 7 A  317
(%o3) [----------, ---, 87.178, 0.06 NA + 81.44999999999999,
           20      10
                                                  3 NA + 19587
                                18.122999999999998, ------------]
                                                      100
```

Here, the first component is meaningless, since `A` and `B` are categorical, the second component is the mean age of individuals in rational form, and the fourth and last values exhibit some strange behaviour. This is because symbol `NA` is used here to indicate *non available* data, and the two means are nonsense. A possible solution would be to take out from the matrix those rows with `NA` symbols, although this deserves some loss of information.

```
(%i1) load ("descriptive")$
(%i2) s3 : read_matrix (file_search ("biomed.data"))$
(%i3) g(v):= v[4] # NA and v[6] # NA $
(%i4) mean (subsample (s3, g, 3, 4, 5, 6));
(%o4) [79.4923076923077, 86.2032967032967, 16.93186813186813,
                                                          2514
                                                          ----]
                                                           13
```

`transform_sample` (*matrix*, *varlist*, *exprlist*) [Function]

Transforms the sample *matrix*, where each column is called according to *varlist*, following expressions in *exprlist*.

Examples:

The second argument assigns names to the three columns. With these names, a list of expressions define the transformation of the sample.

```
(%i1) load ("descriptive")$
(%i2) data: matrix([3,2,7],[3,7,2],[8,2,4],[5,2,4]) $
(%i3) transform_sample(data, [a,b,c], [c, a*b, log(a)]);
                                [ 7  6   log(3) ]
                                [               ]
                                [ 2  21  log(3) ]
(%o3)                           [               ]
                                [ 4  16  log(8) ]
                                [               ]
                                [ 4  10  log(5) ]
```

Add a constant column and remove the third variable.

```
(%i1) load ("descriptive")$
(%i2) data: matrix([3,2,7],[3,7,2],[8,2,4],[5,2,4]) $
(%i3) transform_sample(data, [a,b,c], [makelist(1,k,length(data)),a,b]);
                                   [ 1  3  2 ]
                                   [         ]
                                   [ 1  3  7 ]
(%o3)                              [         ]
                                   [ 1  8  2 ]
                                   [         ]
                                   [ 1  5  2 ]
```

52.3 Functions and Variables for descriptive statistics

`mean` [Function]

`mean` (*x*)
`mean` (*x*, *w*)

Returns the sample mean. *x* must be a list or matrix.

When *x* is a list, `mean` returns the sample mean of *x*.

When *x* is a matrix, `mean` returns a list comprising the sample mean of each column.

w is an optional per-datum weight. *w* must either be 1, in which case every datum *x[i]* is given equal weight, or a list of the same length as *x*, in which case the weight for *x[i]* is given by *w[i]*. The elements of *w* must be nonnegative and not all zero; it is not required that they sum to 1.

The unweighted sample mean is defined as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The weighted sample mean is defined as

$$\bar{x} = \frac{1}{Z}\sum_{i=1}^{n} w_i x_i$$

where Z is the sum of the weights,

$$Z = \sum_{i=1}^{n} w_i$$

Examples:

Sample mean of a list.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) mean (s1);
                                471
(%o3)                           ---
                                100
```

Sample mean of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) mean (s2);
(%o3) [9.9485, 10.160700000000004, 10.868499999999997,
                          15.716600000000001, 14.844100000000001]
```

Weighted sample mean of a list.

```
(%i1) load ("descriptive")$
(%i2) mean ([a, b, c, d], [1, 2, 3, 4]);
                       4 d + 3 c + 2 b + a
(%o2)                  -------------------
                               10
```

Weighted sample mean of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) mm: matrix ([p, q, r], [s, t, u]);
                            [ p  q  r ]
(%o2)                       [         ]
                            [ s  t  u ]
(%i3) mean (mm, [vv, ww]);
                 s ww + p vv  t ww + q vv  u ww + r vv
(%o3)          [-----------, -----------, -----------]
                    ww + vv      ww + vv      ww + vv
```

`var` [Function]

`var` (*x*)
`var` (*x*, *w*)

Returns the sample variance. *x* must be a list or matrix.

When *x* is a list, `var` returns the sample variance of *x*.

When *x* is a matrix, `var` returns a list comprising the sample variance of each column.

w is an optional per-datum weight. *w* must either be 1, in which case every datum *x[i]* is given equal weight, or a list of the same length as *x*, in which case the weight

for *x[i]* is given by *w[i]*. The elements of *w* must be nonnegative and not all zero; it is not required that they sum to 1.

The unweighted sample variance is defined as

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

The weighted sample variance is defined as

$$\frac{1}{Z}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where Z is the sum of the weights,

$$Z = \sum_{i=1}^{n} w_i$$

Example:

Sample variance of a list.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var (s1), numer;
(%o3)                   8.425899999999999
```

Sample variance of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) var (s2);
(%o3) [17.22190675000001, 14.987736510000005,
       15.475728749999998, 32.17651044000001, 24.423076190000007]
```

Weighted sample variance of a list.

```
(%i1) load ("descriptive")$
(%i2) var ([a - b, a, a + b], [3, 5, 7]);
                                    2
                               134 b
(%o2)                          ------
                                225
```

Weighted sample variance of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) mm: matrix ([a - b, c - d], [a, c], [a + b, c + d]);
                          [ a - b  c - d ]
                          [              ]
(%o2)                     [   a      c   ]
                          [              ]
                          [ b + a  d + c ]
```

```
(%i3) var (mm, [3, 5, 7]);
                                  2       2
                             134 b   134 d
(%o3)                       [------, ------]
                              225     225
```

See also function var1.

var1 [Function]

var1 (*list*)
var1 (*matrix*)

This is the sample variance, defined as

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) var1 (s1), numer;
(%o3)                     8.5110101010101
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) var1 (s2);
(%o5) [17.395865404040414, 15.139127787878794,
       15.632049242424243, 32.50152569696971, 24.669773929292937]
```

See also function var.

std [Function]

std (*x*)
std (*x*, *w*)

Returns the sample standard deviation. *x* must be a list or matrix.

When *x* is a list, std returns the sample standard deviation of *x*, which is defined as the square root of the sample variance, as computed by var.

When *x* is a matrix, std returns a list comprising the sample standard deviation of each column.

w is an optional per-datum weight. *w* must either be 1, in which case every datum *x[i]* is given equal weight, or a list of the same length as *x*, in which case the weight for *x[i]* is given by *w[i]*. The elements of *w* must be nonnegative and not all zero; it is not required that they sum to 1.

Example:

Sample standard deviation of a list.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std (s1), numer;
(%o3)                  2.9027400848164135
```

Sample standard deviation of each column of a matrix.

```
(%i1) load ("descriptive")$
```

```
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) std (s2);
(%o3) [4.149928523480858, 3.8713998127292415,
       3.9339202775348663, 5.672434260526957, 4.941970881136392]
```

See also functions `var` and `std1`.

`std1` [Function]

`std1` (*list*)
`std1` (*matrix*)

This is the square root of the function `var1`, the variance with denominator $n - 1$.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) std1 (s1), numer;
(%o3)                   2.917363553109228
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) std1 (s2);
(%o5) [4.170835096721089, 3.8909032097803196,
       3.9537386411375555, 5.701010936401517, 4.966867617451963]
```

See also functions `var1` and `std`.

`noncentral_moment` [Function]

`noncentral_moment` (*x*, *k*)
`noncentral_moment` (*x*, *k*, *w*)

Returns the noncentral moment of order *k*. *x* must be a list or matrix.

When *x* is a list, `noncentral_moment` returns the noncentral moment of order *k* of *x*.

When *x* is a matrix, `noncentral_moment` returns a list comprising the noncentral moment of order *k* of each column.

w is an optional per-datum weight. *w* must either be 1, in which case every datum *x[i]* is given equal weight, or a list of the same length as *x*, in which case the weight for *x[i]* is given by *w[i]*. The elements of *w* must be nonnegative and not all zero; it is not required that they sum to 1.

The unweighted noncentral moment of order *k* is defined as

$$\frac{1}{n}\sum_{i=1}^{n} x_i^k$$

The weighted noncentral moment of order *k* is defined as

$$\frac{1}{Z}\sum_{i=1}^{n} w_i x_i^k$$

where Z is the sum of the weights,

$$Z = \sum_{i=1}^{n} w_i$$

Examples:

First noncentral moment of a list. The first noncentral moment is equal to the sample mean.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) noncentral_moment (s1, 1), numer;
(%o3)                          4.71
(%i4) mean (s1), numer;
(%o4)                          4.71
```

Fifth noncentral moment of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) noncentral_moment (s2, 5);
(%o3) [319793.87247615046, 320532.19238924625,
       391249.56213815557, 2502278.205988911, 1691881.7977422548]
```

See also function central_moment.

central_moment [Function]

central_moment (*x*, *k*)
central_moment (*x*, *k*, *w*)

Returns the central moment of order *k*. *x* must be a list or matrix.

When *x* is a list, central_moment returns the central moment of order *k* of *x*.

When *x* is a matrix, central_moment returns a list comprising the central moment of order *k* of each column.

w is an optional per-datum weight. *w* must either be 1, in which case every datum *x[i]* is given equal weight, or a list of the same length as *x*, in which case the weight for *x[i]* is given by *w[i]*. The elements of *w* must be nonnegative and not all zero; it is not required that they sum to 1.

The unweighted central moment of order k is defined as

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^k$$

The weighted central moment of order k is defined as

$$\frac{1}{Z}\sum_{i=1}^{n} w_i(x_i - \bar{x})^k$$

where Z is the sum of the weights,

$$Z = \sum_{i=1}^{n} w_i$$

Examples:

Second central moment of a list. The second central moment is equal to the sample variance.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) central_moment (s1, 2), numer;
(%o3)                    8.425899999999999
(%i4) var (s1), numer;
(%o4)                    8.425899999999999
```

Third central moment of each column of a matrix.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) central_moment (s1, 2), numer; /* the variance */
(%o3)                    8.425899999999999
(%i5) s2 : read_matrix (file_search ("wind.data"))$
(%i6) central_moment (s2, 3);
(%o6) [11.29584771375004, 16.97988248298583, 5.626661952750102,
                                37.5986572057918, 25.85981904394192]
```

See also functions `central_moment` and `mean`.

`cv` [Function]

`cv` (*list*)
`cv` (*matrix*)

Returns the variation coefficient, defined as the sample standard deviation `std` divided by the `mean`.

Examples:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) cv (s1), numer;
(%o3)                   0.6162930116383044
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) cv (s2);
(%o5) [0.4171411291632767, 0.38101703748061055,
       0.3619561372346568, 0.3609199356430116, 0.3329249251309538]
```

See also functions `std` and `mean`.

`smin` [Function]

`smin` (*list*)
`smin` (*matrix*)

This is the minimum value of the sample *list*. When the argument is a matrix, `smin` returns a list containing the minimum values of the columns, which are associated to statistical variables.

Examples:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) smin (s1);
(%o3)                            0
```

```
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) smin (s2);
(%o5)             [0.58, 0.5, 2.67, 5.25, 5.17]
```

See also function smax.

smax [Function]

smax (*list*)
smax (*matrix*)

This is the maximum value of the sample *list*. When the argument is a matrix, smax returns a list containing the maximum values of the columns, which are associated to statistical variables.

Examples:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) smax (s1);
(%o3)                           9
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) smax (s2);
(%o5)          [20.25, 21.46, 20.04, 29.63, 27.63]
```

See also function smin.

range [Function]

range (*list*)
range (*matrix*)

The range is the difference between the extreme values.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) range (s1);
(%o3)                           9
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) range (s2);
(%o5)   [19.67, 20.96, 17.369999999999997, 24.38, 22.46]
```

quantile [Function]

quantile (*list*, *p*)
quantile (*matrix*, *p*)

This is the p-quantile, with p a number in $[0, 1]$, of the sample *list*. Although there are several definitions for the sample quantile (Hyndman, R. J., Fan, Y. (1996) *Sample quantiles in statistical packages*. American Statistician, 50, 361-365), the one based on linear interpolation is implemented in package Chapter 52 [Package descriptive], page 751,

Examples:

Input is a list. First and third quartiles are computed.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
```

```
(%i3) [quantile (s1, 1/4), quantile (s1, 3/4)], numer;
(%o3)                      [2.0, 7.25]
```

Input is a matrix. First quartile is computed for each column.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) quantile (s2, 1/4);
(%o3)    [7.2575, 7.477500000000001, 7.82, 11.28, 11.48]
```

`median` [Function]

`median` (*list*)
`median` (*matrix*)

Once the sample is ordered, if the sample size is odd the median is the central value, otherwise it is the mean of the two central values.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) median (s1);
                                 9
(%o3)                            -
                                 2
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) median (s2);
(%o5)   [10.059999999999999, 9.855, 10.73, 15.48, 14.105]
```

The median is the 1/2-quantile.

See also function `quantile`.

`qrange` [Function]

`qrange` (*x*)

Returns the interquartile range, defined as the difference between the third and first quartiles: `quantile(`*x*`, 3/4) - quantile(`*x*`, 1/4)`

x must be a list or matrix. When *x* is a matrix, `qrange` returns the interquartile range for each column.

Examples:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) qrange (s1);
                                21
(%o3)                           --
                                4
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) qrange (s2);
(%o5) [5.385, 5.572499999999998, 6.022500000000001,
                            8.729999999999999, 6.649999999999999]
```

See also function `quantile`.

mean_deviation [Function]

mean_deviation (*list*)
mean_deviation (*matrix*)

The mean deviation, defined as

$$\frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$$

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) mean_deviation (s1);
                                 51
(%o3)                            --
                                 20
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) mean_deviation (s2);
(%o5) [3.2879599999999987, 3.075342, 3.2390700000000003,
                            4.715664000000001, 4.028546000000002]
```

See also function mean.

median_deviation [Function]

median_deviation (*list*)
median_deviation (*matrix*)

The median deviation, defined as

$$\frac{1}{n}\sum_{i=1}^{n}|x_i - med|$$

where med is the median of *list*.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) median_deviation (s1);
                                 5
(%o3)                            -
                                 2
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) median_deviation (s2);
(%o5) [2.75, 2.7550000000000003, 3.08, 4.315, 3.3099999999999996]
```

See also function mean.

harmonic_mean [Function]

harmonic_mean (*list*)
harmonic_mean (*matrix*)

The harmonic mean, defined as

$$\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$$

Example:

```
(%i1) load ("descriptive")$
(%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(%i3) harmonic_mean (y), numer;
(%o3)                  3.9018580276322052
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) harmonic_mean (s2);
(%o5) [6.948015590052786, 7.391967752360356, 9.055658197151745,
                           13.441990281936924, 13.01439145898509]
```

See also functions `mean` and `geometric_mean`.

`geometric_mean` [Function]

`geometric_mean` (*list*)
`geometric_mean` (*matrix*)

The geometric mean, defined as

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

Example:

```
(%i1) load ("descriptive")$
(%i2) y : [5, 7, 2, 5, 9, 5, 6, 4, 9, 2, 4, 2, 5]$
(%i3) geometric_mean (y), numer;
(%o3)                   4.454845412337012
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) geometric_mean (s2);
(%o5) [8.82476274347979, 9.22652604739361, 10.044267571488904,
                           14.612741263490207, 13.96184163444275]
```

See also functions `mean` and `harmonic_mean`.

`kurtosis` [Function]

`kurtosis` (*list*)
`kurtosis` (*matrix*)

The kurtosis coefficient, defined as

$$\frac{1}{ns^4}\sum_{i=1}^{n}(x_i - \bar{x})^4 - 3$$

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) kurtosis (s1), numer;
(%o3)                   - 1.273247946514421
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) kurtosis (s2);
(%o5) [- 0.2715445622195385, 0.119998784429451,
- 0.42752334904828615, - 0.6405361979019522,
- 0.4952382132352935]
```

See also functions `mean`, `var` and `skewness`.

skewness [Function]

skewness (*list*)
skewness (*matrix*)

The skewness coefficient, defined as

$$\frac{1}{ns^3}\sum_{i=1}^{n}(x_i-\bar{x})^3$$

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) skewness (s1), numer;
(%o3)                 0.009196180476450424
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) skewness (s2);
(%o5) [0.1580509020000978, 0.2926379232061854,
   0.09242174416107717, 0.20599843481486865, 0.21425202488908313]
```

See also functions mean,, var and kurtosis.

pearson_skewness [Function]

pearson_skewness (*list*)
pearson_skewness (*matrix*)

Pearson's skewness coefficient, defined as

$$\frac{3\ (\bar{x}-med)}{s}$$

where *med* is the median of *list*.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) pearson_skewness (s1), numer;
(%o3)                  0.21594840290938955
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) pearson_skewness (s2);
(%o5) [- 0.08019976629211892, 0.2357036272952649,
   0.10509040624912039, 0.12450423405923679, 0.44641817958045193]
```

See also functions mean, var and median.

quartile_skewness [Function]

quartile_skewness (*list*)
quartile_skewness (*matrix*)

The quartile skewness coefficient, defined as

$$\frac{c_{\frac{3}{4}}-2\,c_{\frac{1}{2}}+c_{\frac{1}{4}}}{c_{\frac{3}{4}}-c_{\frac{1}{4}}}$$

where c_p is the p-quantile of sample *list*.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) quartile_skewness (s1), numer;
(%o3)                 0.047619047619047616
(%i4) s2 : read_matrix (file_search ("wind.data"))$
(%i5) quartile_skewness (s2);
(%o5) [- 0.040854224698235304, 0.14670255720053824,
   0.033623910336239196, 0.03780068728522298, 0.2105263157894735]
```

See also function `quantile`.

`km` [Function]

`km` (*list*, *option* ...)
`km` (*matrix*, *option* ...)

Kaplan Meier estimator of the survival, or reliability, function $S(x) = 1 - F(x)$.

Data can be introduced as a list of pairs, or as a two column matrix. The first component is the observed time, and the second component a censoring index (1 = non censored, 0 = right censored).

The optional argument is the name of the variable in the returned expression, which is x by default.

Examples:

Sample as a list of pairs.

```
(%i1) load ("descriptive")$
(%i2) S: km([[2,1], [3,1], [5,0], [8,1]]);
                         charfun((3 <= x) and (x < 8))
(%o2) charfun(x < 0) + ------------------------------
                                     2
   3 charfun((2 <= x) and (x < 3))
 + -------------------------------
                   4
 + charfun((0 <= x) and (x < 2))
(%i3) load ("draw")$
(%i4) draw2d(
  line_width = 3, grid = true,
  explicit(S, x, -0.1, 10))$
```

Estimate survival probabilities.

```
(%i1) load ("descriptive")$
(%i2) S(t):= ''(km([[2,1], [3,1], [5,0], [8,1]], t)) $
(%i3) S(6);
                                  1
(%o3)                             -
                                  2
```

cdf_empirical [Function]

cdf_empirical (*list*, *option* ...)
cdf_empirical (*matrix*, *option* ...)

Empirical distribution function $F(x)$.

Data can be introduced as a list of numbers, or as an one column matrix.

The optional argument is the name of the variable in the returned expression, which is x by default.

Example:

Empirical distribution function.

```
(%i1) load ("descriptive")$
(%i2) F(x):= ''(cdf_empirical([1,3,3,5,7,7,7,8,9]));
(%o2) F(x) := (charfun(x >= 9) + charfun(x >= 8)
 + 3 charfun(x >= 7) + charfun(x >= 5) + 2 charfun(x >= 3)
 + charfun(x >= 1))/9
(%i3) F(6);
                                    4
(%o3)                               -
                                    9
(%i4) load("draw")$
(%i5) draw2d(
   line_width = 3,
   grid       = true,
   explicit(F(z), z, -2, 12)) $
```

cov [Function]

cov (*X*)
cov (*X*, *w*)

Returns the sample covariance matrix. X must be a matrix.

The sample covariance matrix has the same number of rows and columns, both equal to the number of columns of X; each diagonal element *X[i, i]* is equal to the sample variance of the i'th column, and each off-diagonal element *X[i, j]* is equal to the sample covariance of the i'th and j'th columns.

w is an optional per-datum weight. w must either be 1, in which case every datum *X[i]* is given equal weight, or a list of the same length as X, in which case the weight for *X[i]* is given by *w[i]*. The elements of w must be nonnegative and not all zero; it is not required that they sum to 1.

The unweighted sample covariance is defined as

$$S = \frac{1}{n}\sum_{j=1}^{n} \left(X_j - \bar{X}\right)\ \left(X_j - \bar{X}\right)'$$

where *X[j]* is the j'th row of the sample matrix.

The weighted sample covariance is defined as

$$S = \frac{1}{Z}\sum_{j=1}^{n} w_j \left(X_j - \bar{X}\right)\ \left(X_j - \bar{X}\right)'$$

where Z is the sum of the weights,

$$Z = \sum_{i=1}^{n} w_i$$

Example:

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) fpprintprec : 7$
(%i4) cov (s2);
      [ 17.22191  13.61811  14.37217  19.39624  15.42162 ]
      [                                                  ]
      [ 13.61811  14.98774  13.30448  15.15834  14.9711  ]
      [                                                  ]
(%o4) [ 14.37217  13.30448  15.47573  17.32544  16.18171 ]
      [                                                  ]
      [ 19.39624  15.15834  17.32544  32.17651  20.44685 ]
      [                                                  ]
      [ 15.42162  14.9711   16.18171  20.44685  24.42308 ]
```

See also function `cov1`.

`cov1` (*matrix*) [Function]

The covariance matrix of the multivariate sample, defined as

$$\frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}\right)\left(X_j - \bar{X}\right)'$$

where X_j is the j-th row of the sample matrix.

Example:

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) fpprintprec : 7$
(%i4) cov1 (s2);
      [ 17.39587  13.75567  14.51734  19.59216  15.5774  ]
      [                                                  ]
      [ 13.75567  15.13913  13.43887  15.31145  15.12232 ]
      [                                                  ]
(%o4) [ 14.51734  13.43887  15.63205  17.50044  16.34516 ]
      [                                                  ]
      [ 19.59216  15.31145  17.50044  32.50153  20.65338 ]
      [                                                  ]
      [ 15.5774   15.12232  16.34516  20.65338  24.66977 ]
```

See also function `cov`.

`global_variances` [Function]

`global_variances` (*matrix*)
`global_variances` (*matrix*, *options* ...)

Function `global_variances` returns a list of global variance measures:

- *total variance*: `trace(S_1)`,

- *mean variance*: `trace(S_1)/p`,
- *generalized variance*: `determinant(S_1)`,
- *generalized standard deviation*: `sqrt(determinant(S_1))`,
- *effective variance* `determinant(S_1)^(1/p)`, (defined in: Peña, D. (2002) *Análisis de datos multivariantes*; McGraw-Hill, Madrid.)
- *effective standard deviation*: `determinant(S_1)^(1/(2*p))`.

where p is the dimension of the multivariate random variable and S_1 the covariance matrix returned by `cov1`.

Option:

- `'data`, default `'true`, indicates whether the input matrix contains the sample data, in which case the covariance matrix `cov1` must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Examples:

Calculate the `global_variances` from sample data.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) global_variances (s2);
(%o3) [105.33834206060595, 21.06766841212119, 12874.34690469686,
       113.46517926085015, 6.636590811800794, 2.5761581496097623]
```

Calculate the `global_variances` from the covariance matrix.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) s : cov1 (s2)$
(%i4) global_variances (s, data=false);
(%o4) [105.33834206060595, 21.06766841212119, 12874.34690469686,
       113.46517926085015, 6.636590811800794, 2.5761581496097623]
```

See also `cov` and `cov1`.

`cor` [Function]

`cor` (*matrix*)
`cor` (*matrix*, *logical_value*)

The correlation matrix of the multivariate sample.

Option:

- `'data`, default `'true`, indicates whether the input matrix contains the sample data, in which case the covariance matrix `cov1` must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Examples:

```
(%i1) load ("descriptive")$
(%i2) fpprintprec : 7 $
(%i3) s2 : read_matrix (file_search ("wind.data"))$
```

```
(%i4) cor (s2);
      [    1.0     0.8476339  0.8803515  0.8239624  0.7519506 ]
      [                                                       ]
      [ 0.8476339     1.0     0.8735834  0.6902622  0.782502  ]
      [                                                       ]
(%o4) [ 0.8803515  0.8735834     1.0     0.7764065  0.8323358 ]
      [                                                       ]
      [ 0.8239624  0.6902622  0.7764065     1.0     0.7293848 ]
      [                                                       ]
      [ 0.7519506  0.782502   0.8323358  0.7293848     1.0    ]
```

Calculate the correlation matrix from the covariance matrix.

```
(%i1) load ("descriptive")$
(%i2) fpprintprec : 7 $
(%i3) s2 : read_matrix (file_search ("wind.data"))$
(%i4) s : cov1 (s2)$
(%i5) cor (s, data=false); /* this is faster */
      [    1.0     0.8476339  0.8803515  0.8239624  0.7519506 ]
      [                                                       ]
      [ 0.8476339     1.0     0.8735834  0.6902622  0.782502  ]
      [                                                       ]
(%o5) [ 0.8803515  0.8735834     1.0     0.7764065  0.8323358 ]
      [                                                       ]
      [ 0.8239624  0.6902622  0.7764065     1.0     0.7293848 ]
      [                                                       ]
      [ 0.7519506  0.782502   0.8323358  0.7293848     1.0    ]
```

See also `cov` and `cov1`.

`list_correlations` [Function]

`list_correlations` (*matrix*)
`list_correlations` (*matrix*, *options* ...)

Function `list_correlations` returns a list of correlation measures:

- *precision matrix*: the inverse of the covariance matrix S_1,

$$S_1^{-1} = \left(s^{ij}\right)_{i,j=1,2,...,p}$$

- *multiple correlation vector*: $(R_1^2, R_2^2, ..., R_p^2)$, with

$$R_i^2 = 1 - \frac{1}{s^{ii} s_{ii}}$$

  being an indicator of the goodness of fit of the linear multivariate regression model on X_i when the rest of variables are used as regressors.

- *partial correlation matrix*: with element (i, j) being

$$r_{ij.rest} = -\frac{s^{ij}}{\sqrt{s^{ii} s^{jj}}}$$

Option:

- 'data, default 'true, indicates whether the input matrix contains the sample data, in which case the covariance matrix cov1 must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

Example:

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) z : list_correlations (s2)$
(%i4) fpprintprec : 5$
(%i5) precision_matrix: z[1];
(%o5)
    [  0.38486   - 0.13856   - 0.15626   - 0.10239    0.031179  ]
    [                                                           ]
    [ - 0.13856   0.34107    - 0.15233    0.038447   - 0.052842 ]
    [                                                           ]
    [ - 0.15626  - 0.15233    0.47296    - 0.024816  - 0.10054  ]
    [                                                           ]
    [ - 0.10239   0.038447   - 0.024816   0.10937    - 0.034033 ]
    [                                                           ]
    [ 0.031179   - 0.052842  - 0.10054   - 0.034033   0.14834   ]
(%i6) multiple_correlation_vector: z[2];
(%o6)     [0.85063, 0.80634, 0.86474, 0.71867, 0.72675]
(%i7) partial_correlation_matrix: z[3];
      [   - 1.0     0.38244   0.36627   0.49908   - 0.13049 ]
      [                                                     ]
      [  0.38244     - 1.0    0.37927  - 0.19907   0.23492  ]
      [                                                     ]
(%o7) [  0.36627    0.37927    - 1.0    0.10911    0.37956  ]
      [                                                     ]
      [  0.49908   - 0.19907  0.10911    - 1.0     0.26719  ]
      [                                                     ]
      [ - 0.13049   0.23492   0.37956   0.26719     - 1.0   ]
```

See also cov and cov1.

principal_components [Function]

principal_components (*matrix*)
principal_components (*matrix*, *options* ...)

Calculates the principal components of a multivariate sample. Principal components are used in multivariate statistical analysis to reduce the dimensionality of the sample.

Option:

- 'data, default 'true, indicates whether the input matrix contains the sample data, in which case the covariance matrix cov1 must be calculated, or not, and then the covariance matrix (symmetric) must be given, instead of the data.

The output of function principal_components is a list with the following results:

- variances of the principal components,

- percentage of total variance explained by each principal component,
- rotation matrix.

Examples:

In this sample, the first component explains 83.13 per cent of total variance.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) fpprintprec:4 $
(%i4) res: principal_components(s2);
0 errors, 0 warnings
(%o4) [[87.57, 8.753, 5.515, 1.889, 1.613],
[83.13, 8.31, 5.235, 1.793, 1.531],
[ .4149  .03379   - .4757  - 0.581   - .5126 ]
[                                            ]
[ 0.369  - .3657  - .4298   .7237    - .1469 ]
[                                            ]
[ .3959  - .2178  - .2181  - .2749    .8201  ]]
[                                            ]
[ .5548   .7744    .1857    .2319    .06498  ]
[                                            ]
[ .4765  - .4669   0.712   - .09605  - .1969 ]
(%i5) /* accumulated percentages  */
    block([ap: copy(res[2])],
      for k:2 thru length(ap) do ap[k]: ap[k]+ap[k-1],
      ap);
(%o5)                [83.13, 91.44, 96.68, 98.47, 100.0]
(%i6) /* sample dimension */
      p: length(first(res));
(%o6)                                5
(%i7) /* plot percentages to select number of
         principal components for further work */
     draw2d(
        fill_density = 0.2,
        apply(bars, makelist([k, res[2][k], 1/2], k, p)),
        points_joined = true,
        point_type    = filled_circle,
        point_size    = 3,
        points(makelist([k, res[2][k]], k, p)),
        xlabel = "Variances",
        ylabel = "Percentages",
        xtics  = setify(makelist([concat("PC",k),k], k, p))) $
```

In case the covariance matrix is known, it can be passed to the function, but option `data=false` must be used.

```
(%i1) load ("descriptive")$
(%i2) S: matrix([1,-2,0],[-2,5,0],[0,0,2]);
                                [  1   - 2  0 ]
```

```
                                  [            ]
(%o2)                             [ - 2   5   0 ]
                                  [            ]
                                  [  0    0   2 ]
(%i3) fpprintprec:4 $
(%i4) /* the argument is a covariance matrix */
      res: principal_components(S, data=false);
0 errors, 0 warnings
                                                    [ - .3827  0.0  .9239 ]
                                                    [                     ]
(%o4) [[5.828, 2.0, .1716], [72.86, 25.0, 2.145], [  .9239   0.0  .3827 ]]
                                                    [                     ]
                                                    [   0.0    1.0   0.0  ]
(%i5) /* transformation to get the principal components
         from original records */
      matrix([a1,b2,c3],[a2,b2,c2]).last(res);
             [ .9239 b2 - .3827 a1  1.0 c3  .3827 b2 + .9239 a1 ]
(%o5)        [                                                  ]
             [ .9239 b2 - .3827 a2  1.0 c2  .3827 b2 + .9239 a2 ]
```

52.4 Functions and Variables for statistical graphs

`barsplot` (*data1*, *data2*, ..., *option_1*, *option_2*, ...) [Function]

Plots bars diagrams for discrete statistical variables, both for one or multiple samples.

data can be a list of outcomes representing one sample, or a matrix of m rows and n columns, representing n samples of size m each.

Available options are:

- *box_width* (default, `3/4`): relative width of rectangles. This value must be in the range `[0,1]`.
- *grouping* (default, `clustered`): indicates how multiple samples are shown. Valid values are: `clustered` and `stacked`.
- *groups_gap* (default, `1`): a positive integer number representing the gap between two consecutive groups of bars.
- *bars_colors* (default, `[]`): a list of colors for multiple samples. When there are more samples than specified colors, the extra necessary colors are chosen at random. See `color` to learn more about them.
- *frequency* (default, `absolute`): indicates the scale of the ordinates. Possible values are: `absolute`, `relative`, and `percent`.
- *ordering* (default, `orderlessp`): possible values are `orderlessp` or `ordergreatp`, indicating how statistical outcomes should be ordered on the x-axis.
- *sample_keys* (default, `[]`): a list with the strings to be used in the legend. When the list length is other than 0 or the number of samples, an error message is returned.
- *start_at* (default, `0`): indicates where the plot begins to be plotted on the x axis.

- All global `draw` options, except `xtics`, which is internally assigned by `barsplot`. If you want to set your own values for this option or want to build complex scenes, make use of `barsplot_description`. See example below.
- The following local Chapter 55 [Package draw], page 847, options: `key`, `color_draw`, `fill_color`, `fill_density` and `line_width`. See also `barsplot`.

There is also a function `wxbarsplot` for creating embedded histograms in interfaces wxMaxima and iMaxima. `barsplot` in a multiplot context.

Examples:

Univariate sample in matrix form. Absolute frequencies.

```
(%i1) load ("descriptive")$
(%i2) m : read_matrix (file_search ("biomed.data"))$
(%i3) barsplot(
  col(m,2),
  title        = "Ages",
  xlabel       = "years",
  box_width    = 1/2,
  fill_density = 3/4)$
```

Two samples of different sizes, with relative frequencies and user declared colors.

```
(%i1) load ("descriptive")$
(%i2) l1:makelist(random(10),k,1,50)$
(%i3) l2:makelist(random(10),k,1,100)$
(%i4) barsplot(
   l1,l2,
   box_width = 1,
   fill_density = 1,
   bars_colors = [black, grey],
   frequency = relative,
   sample_keys = ["A", "B"])$
```

Four non numeric samples of equal size.

```
(%i1) load ("descriptive")$
(%i2) barsplot(
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  title      = "Asking for something to four groups",
  ylabel     = "# of individuals",
  groups_gap = 3,
  fill_density = 0.5,
  ordering = ordergreatp)$
```

Stacked bars.

```
(%i1) load ("descriptive")$
```

```
(%i2) barsplot(
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  makelist([Yes, No, Maybe][random(3)+1],k,1,50),
  title      = "Asking for something to four groups",
  ylabel     = "# of individuals",
  grouping   = stacked,
  fill_density = 0.5,
  ordering = ordergreatp)$
```

For bars diagrams related options, see `barsplot` of package Chapter 55 [Package draw], page 847, See also functions `histogram` and `piechart`.

`barsplot_description` (`...`) [Function]

Function `barsplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

Example: `barsplot` in a multiplot context.

```
(%i1) load ("descriptive")$
(%i2) l1:makelist(random(10),k,1,50)$
(%i3) l2:makelist(random(10),k,1,100)$
(%i4) bp1 :
        barsplot_description(
         l1,
         box_width = 1,
         fill_density = 0.5,
         bars_colors = [blue],
         frequency = relative)$
(%i5) bp2 :
        barsplot_description(
         l2,
         box_width = 1,
         fill_density = 0.5,
         bars_colors = [red],
         frequency = relative)$
(%i6) draw(gr2d(bp1), gr2d(bp2))$
```

`boxplot` (*data*) [Function]

`boxplot` (*data*, *option_1*, *option_2*, ...)

This function plots box-and-whisker diagrams. Argument *data* can be a list, which is not of great interest, since these diagrams are mainly used for comparing different samples, or a matrix, so it is possible to compare two or more components of a multivariate statistical variable. But it is also allowed *data* to be a list of samples with possible different sample sizes, in fact this is the only function in package `descriptive` that admits this type of data structure.

The box is plotted from the first quartile to the third, with an horizontal segment situated at the second quartile or median. By default, lower and

upper whiskers are plotted at the minimum and maximum values, respectively. Option *range* can be used to indicate that values greater than `quantile(x,3/4)+range*(quantile(x,3/4)-quantile(x,1/4))` or less than `quantile(x,1/4)-range*(quantile(x,3/4)-quantile(x,1/4))` must be considered as outliers, in which case they are plotted as isolated points, and the whiskers are located at the extremes of the rest of the sample.

Available options are:

- *box_width* (default, `3/4`): relative width of boxes. This value must be in the range `[0,1]`.
- *box_orientation* (default, `vertical`): possible values: `vertical` and `horizontal`.
- *range* (default, `inf`): positive coefficient of the interquartilic range to set outliers boundaries.
- *outliers_size* (default, `1`): circle size for isolated outliers.
- All `draw` options, except `points_joined`, `point_size`, `point_type`, `xtics`, `ytics`, `xrange`, and `yrange`, which are internally assigned by `boxplot`. If you want to set your own values for this options or want to build complex scenes, make use of `boxplot_description`.
- The following local `draw` options: `key`, `color`, and `line_width`.

There is also a function `wxboxplot` for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

Box-and-whisker diagram from a multivariate sample.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix(file_search("wind.data"))$
(%i3) boxplot(s2,
  box_width  = 0.2,
  title      = "Windspeed in knots",
  xlabel     = "Stations",
  color      = red,
  line_width = 2)$
```

Box-and-whisker diagram from three samples of different sizes.

```
(%i1) load ("descriptive")$
(%i2) A :
 [[6, 4, 6, 2, 4, 8, 6, 4, 6, 4, 3, 2],
  [8, 10, 7, 9, 12, 8, 10],
  [16, 13, 17, 12, 11, 18, 13, 18, 14, 12]]$
(%i3) boxplot (A, box_orientation = horizontal)$
```

Option *range* can be used to handle outliers.

```
(%i1)  load ("descriptive")$
 B: [[7, 15, 5, 8, 6, 5, 7, 3, 1],
     [10, 8, 12, 8, 11, 9, 20],
     [23, 17, 19, 7, 22, 19]] $
 boxplot (B, range=1)$
 boxplot (B, range=1.5, box_orientation = horizontal)$
 draw2d(
    boxplot_description(
       B,
       range            = 1.5,
       line_width       = 3,
       outliers_size    = 2,
       color            = red,
       background_color = light_gray),
    xtics = {["Low",1],["Medium",2],["High",3]}) $
```

`boxplot_description` (*...*) [Function]

Function `boxplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

`histogram` [Function]

`histogram` (*list*)
`histogram` (*list*, *option_1*, *option_2*, *...*)
`histogram` (*one_column_matrix*)
`histogram` (*one_column_matrix*, *option_1*, *option_2*, *...*)
`histogram` (*one_row_matrix*)
`histogram` (*one_row_matrix*, *option_1*, *option_2*, *...*)

Constructs and displays a histogram from a data sample. Data must be stored as a list of numbers, or a matrix of one row or one column.

Optional arguments:

- `nclasses` (default, 10): the number of classes (also called bins) in the histogram, or a list of two numbers (the least and greatest values included in the histogram), or a list of three numbers (the least and greatest values included in the histogram, and the number of classes), or a set containing the endpoints of the class intervals, or a symbol specifying the name of one of three algorithms to automatically determine the number of classes: `fd` (Ref. [1]), `scott` (Ref. [2]), or `sturges` (Ref. [3]).

  A class interval excludes its left endpoint and includes its right endpoint, except for the first interval, which includes both the left and right endpoints. It is assumed that class intervals are contiguous. That is, the right endpoint of one interval is equal to the left endpoint of the next.
- `frequency` (default, `absolute`): indicates the scale of the vertical axis. Possible values are: `absolute` (heights of bars add up to number of data), `relative` (heights of bars add up to 1), `percent` (heights of bars add up to 100), and `density` (total area of histogram is 1).
- `htics` (default, `auto`): format of tic marks on the horizontal axis. Possible values are: `auto` (tics are placed automatically), `endpoints` (tics are placed at the

divisions between classes), `intervals` (classes are labeled with the corresponding intervals), or a list of labels, one for each class.

- All global `draw` options, except `xrange`, `yrange`, and `xtics`, which are internally assigned by `histogram`. If you want to set your own values for these options, make use of `histogram_description`.
- The following local Chapter 55 [Package draw], page 847, options: `key`, `fill_color`, `fill_density`, and `line_width`. Note that the outlines of bars, as well as the interior of bars when `fill_density` is nonzero, are drawn with `fill_color`, not `color`.

`histogram` honors the global option `histogram_skyline`. When `histogram_skyline` is `true`, `histogram` and `histogram_description` construct "skyline" plots, which shows the outline of the histogram bars, instead of drawing all the vertical segments. Otherwise (the default), histograms are displayed with bars showing vertical segments.

There is also a function `wxhistogram` for creating embedded histograms in interfaces wxMaxima and iMaxima.

See also `continuous_freq`, which, like `histogram`, counts data in intervals, but returns the counts instead of displaying a graphic representation.

See also `barsplot`.

Examples:

A simple histogram with eight classes:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (
     s1,
     nclasses     = 8,
     title        = "pi digits",
     xlabel       = "digits",
     ylabel       = "Absolute frequency",
     fill_color   = grey,
     fill_density = 0.6)$
```

Setting the limits of the histogram to -2 and 12, with 3 classes. Also, we introduce predefined tics:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram (
     s1,
     nclasses     = [-2,12,3],
     htics        = ["A", "B", "C"],
     terminal     = png,
     fill_color   = "#23afa0",
     fill_density = 0.6)$
```

Bounds for varying class widths.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
```

```
(%i3) histogram (s1, nclasses = {0,3,6,7,11})$
```

Freedman-Diaconis formula for the number of classes.

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
(%i3) histogram(s1, nclasses=fd) $
```

References:

[1] Freedman, D., and Diaconis, P. (1981) On the histogram as a density estimator: L_2 theory. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 57, 453-476.

[2] Scott, D. W. (1979) On optimal and data-based histograms. Biometrika 66, 605-610.

[3] Sturges, H. A. (1926) The choice of a class interval. Journal of the American Statistical Association 21, 65-66.

`histogram_description (...)` [Function]

Creates a graphic object which represents a histogram. Such an object is suitable for creating complex scenes together with other graphic objects, to be displayed by `draw2d`.

`histogram_description` takes the same arguments as the stand-alone function `histogram`. See `histogram` for more information.

Example:

We make use of `histogram_description` for setting `xrange` and adding an explicit curve into the scene:

```
(%i1) load ("descriptive")$
(%i2) ( load("distrib"),
        m: 14, s: 2,
        s2: random_normal(m, s, 1000) ) $
(%i3) draw2d(
        grid   = true,
        xrange = [5, 25],
        histogram_description(
          s2,
          nclasses     = 9,
          frequency    = density,
          fill_density = 0.5),
        explicit(pdf_normal(x,m,s), x, m - 3*s, m + 3* s))$
```

`histogram_skyline` [Option variable]

Default value: `false`

When `histogram_skyline` is `true`, `histogram` and `histogram_description` construct "skyline" plots, which shows the outline of the histogram bars, instead of drawing all the vertical segments.

The outline is drawn with the current `fill_color` (not the current `color`). The interior of the histogram is filled with `fill_color`, but only if `fill_density` is nonzero.

Otherwise, histograms are displayed with bars showing vertical segments.

Examples:

Construct a skyline histogram, and an ordinary histogram for comparison, on the same plot.

```
(%i1) load ("descriptive") $
(%i2) L: read_list (file_search ("pidigits.data")) $
(%i3) histogram_skyline: true $
(%i4) skyline_hist: histogram_description (L) $
(%i5) histogram_skyline: false $
(%i6) ordinary_hist: histogram_description (L) $
(%i7) draw (gr2d (skyline_hist), gr2d (ordinary_hist)) $
```

Continuing the preceding example. Set display options for `fill_color` and `fill_density`.

```
(%i8) histogram_skyline: true $
(%i9) skyline_hist: histogram_description (L, fill_color = blue, fill_density = 0
(%i10) histogram_skyline: false $
(%i11) ordinary_hist: histogram_description (L, fill_color = blue, fill_density =
(%i12) draw (gr2d (skyline_hist), gr2d (ordinary_hist)) $
```

`piechart` [Function]

`piechart` (*list*)
`piechart` (*list*, *option_1*, *option_2*, ...)
`piechart` (*one_column_matrix*)
`piechart` (*one_column_matrix*, *option_1*, *option_2*, ...)
`piechart` (*one_row_matrix*)
`piechart` (*one_row_matrix*, *option_1*, *option_2*, ...)

Similar to `barsplot`, but plots sectors instead of rectangles.

Available options are:

- *sector_colors* (default, `[]`): a list of colors for sectors. When there are more sectors than specified colors, the extra necessary colors are chosen at random. See `color` to learn more about them.
- *pie_center* (default, `[0,0]`): diagram's center.
- *pie_radius* (default, `1`): diagram's radius.
- All global `draw` options, except `key`, which is internally assigned by `piechart`. If you want to set your own values for this option or want to build complex scenes, make use of `piechart_description`.
- The following local `draw` options: `key`, `color`, `fill_density` and `line_width`. See also `ellipse`

There is also a function `wxpiechart` for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

```
(%i1) load ("descriptive")$
(%i2) s1 : read_list (file_search ("pidigits.data"))$
```

```
(%i3) piechart(
  s1,
  xrange = [-1.1, 1.3],
  yrange = [-1.1, 1.1],
  title  = "Digit frequencies in pi")$
```

See also function barsplot.

piechart_description (...) [Function]

Function piechart_description creates a graphic object suitable for creating complex scenes, together with other graphic objects.

scatterplot [Function]

scatterplot (*list*)
scatterplot (*list*, *option_1*, *option_2*, ...)
scatterplot (*matrix*)
scatterplot (*matrix*, *option_1*, *option_2*, ...)

Plots scatter diagrams both for univariate (*list*) and multivariate (*matrix*) samples.

Available options are the same admitted by histogram.

There is also a function wxscatterplot for creating embedded histograms in interfaces wxMaxima and iMaxima.

Examples:

Univariate scatter diagram from a simulated Gaussian sample.

```
(%i1) load ("descriptive")$
(%i2) load ("distrib")$
(%i3) scatterplot(
  random_normal(0,1,200),
  xaxis      = true,
  point_size = 2,
  dimensions = [600,150])$
```

Two dimensional scatter plot.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(
 submatrix(s2, 1,2,3),
 title      = "Data from stations #4 and #5",
 point_type = diamant,
 point_size = 2,
 color      = blue)$
```

Three dimensional scatter plot.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) scatterplot(submatrix (s2, 1,2), nclasses=4)$
```

Five dimensional scatter plot, with five classes histograms.

```
(%i1) load ("descriptive")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
```

```
(%i3) scatterplot(
  s2,
  nclasses     = 5,
  frequency    = relative,
  fill_color   = blue,
  fill_density = 0.3,
  xtics        = 5)$
```

For plotting isolated or line-joined points in two and three dimensions, see `points`. See also `histogram`.

`scatterplot_description` (`...`) [Function]

Function `scatterplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

`starplot` (*data1*, *data2*, ..., *option_1*, *option_2*, ...) [Function]

Plots star diagrams for discrete statistical variables, both for one or multiple samples.

data can be a list of outcomes representing one sample, or a matrix of m rows and n columns, representing n samples of size m each.

Available options are:

- *stars_colors* (default, `[]`): a list of colors for multiple samples. When there are more samples than specified colors, the extra necessary colors are chosen at random. See `color` to learn more about them.
- *frequency* (default, `absolute`): indicates the scale of the radii. Possible values are: `absolute` and `relative`.
- *ordering* (default, `orderlessp`): possible values are `orderlessp` or `ordergreatp`, indicating how statistical outcomes should be ordered.
- *sample_keys* (default, `[]`): a list with the strings to be used in the legend. When the list length is other than 0 or the number of samples, an error message is returned.
- *star_center* (default, `[0,0]`): diagram's center.
- *star_radius* (default, `1`): diagram's radius.
- All global `draw` options, except `points_joined`, `point_type`, and `key`, which are internally assigned by `starplot`. If you want to set your own values for this options or want to build complex scenes, make use of `starplot_description`.
- The following local `draw` option: `line_width`.

There is also a function `wxstarplot` for creating embedded histograms in interfaces wxMaxima and iMaxima.

Example:

Plot based on absolute frequencies. Location and radius defined by the user.

```
(%i1) load ("descriptive")$
(%i2) l1: makelist(random(10),k,1,50)$
(%i3) l2: makelist(random(10),k,1,200)$
```

```
(%i4) starplot(
        l1, l2,
        stars_colors = [blue,red],
        sample_keys = ["1st sample", "2nd sample"],
        star_center = [1,2],
        star_radius = 4,
        proportional_axes = xy,
        line_width = 2 ) $
```

`starplot_description` (...) [Function]

Function `starplot_description` creates a graphic object suitable for creating complex scenes, together with other graphic objects.

`stemplot` [Function]

`stemplot` (*data*)
`stemplot` (*data*, *option*)

Plots stem and leaf diagrams.

The only available option is:

- *leaf_unit* (default, `1`): indicates the unit of the leaves; must be a power of 10.

Example:

```
(%i1) load ("descriptive")$
(%i2) load("distrib")$
```

```
(%i3) stemplot(
        random_normal(15, 6, 100),
        leaf_unit = 0.1);
-5|4
 0|37
 1|7
 3|6
 4|4
 5|4
 6|57
 7|0149
 8|3
 9|1334588
10|07888
11|01144467789
12|12566889
13|24778
14|047
15|223458
16|4
17|11557
18|000247
19|4467799
20|00
21|1
22|2335
23|01457
24|12356
25|455
27|79
key: 6|3 =  6.3
(%o3)                  done
```

53 Package diag

53.1 Functions and Variables for diag

`diag` (*lm*) [Function]

Constructs a matrix that is the block sum of the elements of *lm*. The elements of *lm* are assumed to be matrices; if an element is scalar, it treated as a 1 by 1 matrix.

The resulting matrix will be square if each of the elements of *lm* is square.

Example:

```
(%i1) load("diag")$

(%i2) a1:matrix([1,2,3],[0,4,5],[0,0,6])$

(%i3) a2:matrix([1,1],[1,0])$

(%i4) diag([a1,x,a2]);
                  [ 1  2  3  0  0  0 ]
                  [                  ]
                  [ 0  4  5  0  0  0 ]
                  [                  ]
                  [ 0  0  6  0  0  0 ]
(%o4)             [                  ]
                  [ 0  0  0  x  0  0 ]
                  [                  ]
                  [ 0  0  0  0  1  1 ]
                  [                  ]
                  [ 0  0  0  0  1  0 ]
(%i5) diag ([matrix([1,2]), 3]);
                       [ 1  2  0 ]
(%o5)                  [         ]
                       [ 0  0  3 ]
```

To use this function write first `load("diag")`.

`JF` (*lambda*,*n*) [Function]

Returns the Jordan cell of order *n* with eigenvalue *lambda*.

Example:

```
(%i1) load("diag")$

(%i2) JF(2,5);
                   [ 2  1  0  0  0 ]
                   [               ]
                   [ 0  2  1  0  0 ]
                   [               ]
(%o2)              [ 0  0  2  1  0 ]
                   [               ]
```

```
                      [ 0  0  0  2  1 ]
                      [               ]
                      [ 0  0  0  0  2 ]
(%i3) JF(3,2);
                           [ 3  1 ]
(%o3)                      [      ]
                           [ 0  3 ]
```

To use this function write first `load("diag")`.

`jordan` (*mat*) [Function]

Returns the Jordan form of matrix *mat*, encoded as a list in a particular format. To get the corresponding matrix, call the function `dispJordan` using the output of `jordan` as the argument.

The elements of the returned list are themselves lists. The first element of each is an eigenvalue of *mat*. The remaining elements are positive integers which are the lengths of the Jordan blocks for this eigenvalue. These integers are listed in decreasing order. Eigenvalues are not repeated.

The functions `dispJordan`, `minimalPoly` and `ModeMatrix` expect the output of a call to `jordan` as an argument. If you construct this argument by hand, rather than by calling `jordan`, you must ensure that each eigenvalue only appears once and that the block sizes are listed in decreasing order, otherwise the functions might give incorrect answers.

Example:

```
(%i1) load("diag")$
(%i2) A: matrix([2,0,0,0,0,0,0,0],
                [1,2,0,0,0,0,0,0],
                [-4,1,2,0,0,0,0,0],
                [2,0,0,2,0,0,0,0],
                [-7,2,0,0,2,0,0,0],
                [9,0,-2,0,1,2,0,0],
                [-34,7,1,-2,-1,1,2,0],
                [145,-17,-16,3,9,-2,0,3])$
(%i3) jordan (A);
(%o3)               [[2, 3, 3, 1], [3, 1]]
(%i4) dispJordan (%);
                   [ 2  1  0  0  0  0  0  0 ]
                   [                        ]
                   [ 0  2  1  0  0  0  0  0 ]
                   [                        ]
                   [ 0  0  2  0  0  0  0  0 ]
                   [                        ]
                   [ 0  0  0  2  1  0  0  0 ]
(%o4)              [                        ]
                   [ 0  0  0  0  2  1  0  0 ]
                   [                        ]
                   [ 0  0  0  0  0  2  0  0 ]
```

```
[                       ]
[ 0  0  0  0  0  0  2  0 ]
[                       ]
[ 0  0  0  0  0  0  0  3 ]
```

To use this function write first `load("diag")`. See also `dispJordan` and `minimalPoly`.

`dispJordan` (*l*) [Function]

Returns a matrix in Jordan canonical form (JCF) corresponding to the list of eigenvalues and multiplicities given by *l*. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.

Example:

```
(%i1) load("diag")$

(%i2) b1:matrix([0,0,1,1,1],
                [0,0,0,1,1],
                [0,0,0,0,1],
                [0,0,0,0,0],
                [0,0,0,0,0])$

(%i3) jordan(b1);
(%o3)                  [[0, 3, 2]]
(%i4) dispJordan(%);
                     [ 0  1  0  0  0 ]
                     [               ]
                     [ 0  0  1  0  0 ]
                     [               ]
(%o4)                [ 0  0  0  0  0 ]
                     [               ]
                     [ 0  0  0  0  1 ]
                     [               ]
                     [ 0  0  0  0  0 ]
```

To use this function write first `load("diag")`. See also `jordan` and `minimalPoly`.

`minimalPoly` (*l*) [Function]

Returns the minimal polynomial of the matrix whose Jordan form is described by the list *l*. This list should be in the format given by the `jordan` function. See `jordan` for details of this format.

Example:

```
(%i1) load("diag")$

(%i2) a:matrix([2,1,2,0],
               [-2,2,1,2],
               [-2,-1,-1,1],
               [3,1,2,-1])$

(%i3) jordan(a);
```

```
(%o3)                [[- 1, 1], [1, 3]]
(%i4) minimalPoly(%);
                                3
(%o4)                (x - 1)  (x + 1)
```

To use this function write first `load("diag")`. See also `jordan` and `dispJordan`.

`ModeMatrix` (*A*, [*jordan_info*]) [Function]

Returns an invertible matrix M such that $(Mm1).A.M$ is the Jordan form of A.

To calculate this, Maxima must find the Jordan form of A, which might be quite computationally expensive. If that has already been calculated by a previous call to `jordan`, pass it as a second argument, *jordan_info*. See `jordan` for details of the required format.

Example:

```
(%i1) load("diag")$
(%i2) A: matrix([2,1,2,0], [-2,2,1,2], [-2,-1,-1,1], [3,1,2,-1])$
(%i3) M: ModeMatrix (A);
                       [  1    - 1   1   1 ]
                       [                   ]
                       [   1               ]
                       [ - -   - 1   0   0 ]
                       [   9               ]
                       [                   ]
(%o3)                  [   13              ]
                       [ - --   1   - 1  0 ]
                       [   9               ]
                       [                   ]
                       [  17               ]
                       [  --   - 1   1   1 ]
                       [  9                ]
(%i4) is ((M^^-1) . A . M = dispJordan (jordan (A)));
(%o4)                          true
```

Note that, in this example, the Jordan form of `A` is computed twice. To avoid this, we could have stored the output of `jordan(A)` in a variable and passed that to both `ModeMatrix` and `dispJordan`.

To use this function write first `load("diag")`. See also `jordan` and `dispJordan`.

`mat_function` (*f*,*A*) [Function]

Returns $f(A)$, where f is an analytic function and A a matrix. This computation is based on the Taylor expansion of *f*. It is not efficient for numerical evaluation, but can give symbolic answers for small matrices.

Example 1:

The exponential of a matrix. We only give the first row of the answer, since the output is rather large.

```
(%i1) load("diag")$
(%i2) A: matrix ([0,1,0], [0,0,1], [-1,-3,-3])$
```

```
(%i3) ratsimp (mat_function (exp, t*A)[1]);
          2             - t                   2   - t
        (t  + 2 t + 2) %e       2        - t  t  %e
(%o3)   [--------------------, (t  + t) %e   , --------]
                 2                               2
```

Example 2:

Comparison with the Taylor series for the exponential and also comparing `exp(%i*A)` with sine and cosine.

```
(%i1) load("diag")$
(%i2) A: matrix ([0,1,1,1],
                 [0,0,0,1],
                 [0,0,0,1],
                 [0,0,0,0])$
(%i3) ratsimp (mat_function (exp, t*A));
                      [           2     ]
                      [ 1  t  t  t  + t ]
                      [                 ]
(%o3)                 [ 0  1  0    t    ]
                      [                 ]
                      [ 0  0  1    t    ]
                      [                 ]
                      [ 0  0  0    1    ]
(%i4) minimalPoly (jordan (A));
                             3
(%o4)                       x
(%i5) ratsimp (ident(4) + t*A + 1/2*(t^2)*A^^2);
                      [           2     ]
                      [ 1  t  t  t  + t ]
                      [                 ]
(%o5)                 [ 0  1  0    t    ]
                      [                 ]
                      [ 0  0  1    t    ]
                      [                 ]
                      [ 0  0  0    1    ]
(%i6) ratsimp (mat_function (exp, %i*t*A));
                  [                        2 ]
                  [ 1  %i t  %i t  %i t - t  ]
                  [                          ]
(%o6)             [ 0   1     0      %i t    ]
                  [                          ]
                  [ 0   0     1      %i t    ]
                  [                          ]
                  [ 0   0     0        1     ]
```

```
(%i7) ratsimp (mat_function (cos, t*A) + %i*mat_function (sin, t*A));
                  [                          2 ]
                  [ 1  %i t  %i t  %i t - t  ]
                  [                          ]
(%o7)             [ 0   1     0      %i t    ]
                  [                          ]
                  [ 0   0     1      %i t    ]
                  [                          ]
                  [ 0   0     0        1     ]
```

Example 3:

Power operations.

```
(%i1) load("diag")$
(%i2) A: matrix([1,2,0], [0,1,0], [1,0,1])$
(%i3) integer_pow(x) := block ([k], declare (k, integer), x^k)$
(%i4) mat_function (integer_pow, A);
                        [ 1     2 k     0 ]
                        [                 ]
(%o4)                   [ 0      1      0 ]
                        [                 ]
                        [ k  (k - 1) k  1 ]
(%i5) A^^20;
                          [ 1   40   0 ]
                          [            ]
(%o5)                     [ 0    1   0 ]
                          [            ]
                          [ 20  380  1 ]
```

To use this function write first `load("diag")`.

54 Package distrib

54.1 Introduction to distrib

Package `distrib` contains a set of functions for making probability computations on both discrete and continuous univariate models.

What follows is a short reminder of basic probabilistic related definitions.

Let $f(x)$ be the *density function* of an absolute continuous random variable X. The *cumulative distribution function* is defined as

$$F(x) = \int_{-\infty}^{x} f(u)\ du$$

which equals the probability $\Pr(X \leq x)$.

The *mean* value is a localization parameter and is defined as

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\ dx$$

The *variance* is a measure of variation,

$$V[X] = \int_{-\infty}^{\infty} f(x)\ (x - E[X])^2\ dx$$

which is a positive real number. The square root of the variance is the *standard deviation*, $D[x] = \sqrt{V[X]}$, and it is another measure of variation.

The *skewness coefficient* is a measure of non-symmetry,

$$SK[X] = \frac{\int_{-\infty}^{\infty} f(x)\ (x - E[X])^3\ dx}{D[X]^3}$$

And the *kurtosis coefficient* measures the peakedness of the distribution,

$$KU[X] = \frac{\int_{-\infty}^{\infty} f(x)\ (x - E[X])^4\ dx}{D[X]^4} - 3$$

If X is gaussian, $KU[X] = 0$. In fact, both skewness and kurtosis are shape parameters used to measure the non–gaussianity of a distribution.

If the random variable X is discrete, the density, or *probability*, function $f(x)$ takes positive values within certain countable set of numbers x_i, and zero elsewhere. In this case, the cumulative distribution function is

$$F(x) = \sum_{x_i \leq x} f(x_i)$$

The mean, variance, standard deviation, skewness coefficient and kurtosis coefficient take the form

$$E[X] = \sum_{x_i} x_i f(x_i),$$
$$V[X] = \sum_{x_i} f(x_i) (x_i - E[X])^2,$$
$$D[X] = \sqrt{V[X]},$$
$$SK[X] = \frac{\sum_{x_i} f(x) \ (x - E[X])^3 \ dx}{D[X]^3},$$
$$KU[X] = \frac{\sum_{x_i} f(x) \ (x - E[X])^4 \ dx}{D[X]^4} - 3,$$

respectively.

There is a naming convention in package `distrib`. Every function name has two parts, the first one makes reference to the function or parameter we want to calculate,

```
Functions:
   Density function            (pdf_*)
   Distribution function       (cdf_*)
   Quantile                    (quantile_*)
   Mean                        (mean_*)
   Variance                    (var_*)
   Standard deviation          (std_*)
   Skewness coefficient        (skewness_*)
   Kurtosis coefficient        (kurtosis_*)
   Random variate              (random_*)
```

The second part is an explicit reference to the probabilistic model,

```
Continuous distributions:
   Normal              (*normal)
   Student             (*student_t)
   Chi^2               (*chi2)
   Noncentral Chi^2    (*noncentral_chi2)
   F                   (*f)
   Exponential         (*exp)
   Lognormal           (*lognormal)
   Gamma               (*gamma)
   Beta                (*beta)
   Continuous uniform  (*continuous_uniform)
   Logistic            (*logistic)
   Pareto              (*pareto)
   Weibull             (*weibull)
   Rayleigh            (*rayleigh)
   Laplace             (*laplace)
   Cauchy              (*cauchy)
   Gumbel              (*gumbel)
```

```
Discrete distributions:
   Binomial             (*binomial)
   Poisson              (*poisson)
   Bernoulli            (*bernoulli)
   Geometric            (*geometric)
   Discrete uniform     (*discrete_uniform)
   hypergeometric       (*hypergeometric)
   Negative binomial    (*negative_binomial)
   Finite discrete      (*general_finite_discrete)
```

For example, `pdf_student_t(x,n)` is the density function of the Student distribution with n degrees of freedom, `std_pareto(a,b)` is the standard deviation of the Pareto distribution with parameters a and b and `kurtosis_poisson(m)` is the kurtosis coefficient of the Poisson distribution with mean m.

In order to make use of package `distrib` you need first to load it by typing

```
(%i1) load("distrib")$
```

For comments, bugs or suggestions, please contact the author at *'riotorto AT yahoo DOT com'*.

54.2 Functions and Variables for continuous distributions

Maxima knows the following kinds of continuous distributions.

54.2.1 Normal Random Variable

Normal random variables (also called Gaussian) is denoted by $Normal(m, s)$ where m is the mean and $s > 0$ is the standard deviation.

`pdf_normal (x,m,s)` [Function]

Returns the value at x of the density function of a $Normal(m, s)$ random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; m, s) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}}$$

`cdf_normal (x,m,s)` [Function]

Returns the value at x of the cumulative distribution function of a $Normal(m, s)$ random variable, with $s > 0$. This function is defined in terms of Maxima's built-in error function `erf`.

The cdf can be written analytically:

$$F(x; m, s) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x-m}{s\sqrt{2}}\right)$$

```
(%i1) load ("distrib")$
(%i2) cdf_normal(x,m,s);
                                 x - m
                            erf(---------)
                                sqrt(2) s    1
(%o2)                       -------------- + -
                                   2         2
```

See also erf.

quantile_normal (q,m,s) [Function]

Returns the q-quantile of a $Normal(m, s)$ random variable, with $s > 0$; in other words, this is the inverse of cdf_normal. Argument q must be an element of $[0, 1]$. To make use of this function, write first load("distrib").

```
(%i1) load ("distrib")$
(%i2) quantile_normal(95/100,0,1);
                                          9
(%o2)                sqrt(2) inverse_erf(--)
                                         10
(%i3) float(%);
(%o3)                  1.6448536269514724
```

mean_normal (m,s) [Function]

Returns the mean of a $Normal(m, s)$ random variable, with $s > 0$. To make use of this function, write first load("distrib").

The mean is

$$E[X] = m$$

var_normal (m,s) [Function]

Returns the variance of a $Normal(m, s)$ random variable, with $s > 0$. To make use of this function, write first load("distrib").

The variance is

$$V[X] = s^2$$

std_normal (m,s) [Function]

Returns the standard deviation of a $Normal(m, s)$ random variable, with $s > 0$, namely s. To make use of this function, write first load("distrib").

The standard deviation is

$$D[X] = s$$

skewness_normal (m,s) [Function]

Returns the skewness coefficient of a $Normal(m, s)$ random variable, with $s > 0$. To make use of this function, write first load("distrib").

The skewness coefficient is

$$SK[X] = 0$$

kurtosis_normal (m,s) [Function]

Returns the kurtosis coefficient of a $Normal(m, s)$ random variable, with $s > 0$, which is always equal to 0. To make use of this function, write first load("distrib").

The kurtosis coefficient is

$$KU[X] = 0$$

random_normal (m,s) [Function]

random_normal (m,s,n)

Returns a $Normal(m, s)$ random variate, with $s > 0$. Calling random_normal with a third argument n, a random sample of size n will be simulated.

This is an implementation of the Box-Mueller algorithm, as described in Knuth, D.E. (1981) *Seminumerical Algorithms. The Art of Computer Programming.* Addison-Wesley.

To make use of this function, write first `load("distrib")`.

54.2.2 Student's t Random Variable

Student's t random variable is denoted by $t(n)$ where n is the degrees of freedom with $n > 0$. If Z is a $Normal(0, 1)$ variable and V is an independent χ^2 random variable with n degrees of freedom, then

$$\frac{Z}{\sqrt{V/n}}$$

has a Student's t-distribution with n degrees of freedom.

`pdf_student_t (x,n)` [Function]

Returns the value at x of the density function of a Student random variable $t(n)$, with $n > 0$ degrees of freedom. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n) = \left[\sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^{-1} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

`cdf_student_t (x,n)` [Function]

Returns the value at x of the cumulative distribution function of a Student random variable $t(n)$, with $n > 0$ degrees of freedom.

The cdf is

$$F(x; n) = \begin{cases} 1 - \frac{1}{2} I_t\left(\frac{n}{2}, \frac{1}{2}\right) & x \geq 0 \\ \frac{1}{2} I_t\left(\frac{n}{2}, \frac{1}{2}\right) & x < 0 \end{cases}$$

where $t = n/(n + x^2)$ and $I_t(a, b)$ is the [beta_incomplete_regularized], page 341, function.

```
(%i1) load ("distrib")$
(%i2) cdf_student_t(1/2, 7/3);
                                              7  1  28
                   beta_incomplete_regularized(-, -, --)
                                              6  2  31
(%o2)         1 - ------------------------------------
                                   2
(%i3) float(%);
(%o3)                    0.6698450596140415
```

`quantile_student_t (q,n)` [Function]

Returns the q-quantile of a Student random variable $t(n)$, with $n > 0$; in other words, this is the inverse of `cdf_student_t`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_student_t (n)` [Function]

Returns the mean of a Student random variable $t(n)$, with $n > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = 0$$

`var_student_t (n)` [Function]

Returns the variance of a Student random variable $t(n)$, with $n > 2$.

The variance is

$$V[X] = \frac{n}{n-2}$$

```
(%i1) load ("distrib")$
(%i2) var_student_t(n);
                                    n
(%o2)                             -----
                                  n - 2
```

`std_student_t (n)` [Function]

Returns the standard deviation of a Student random variable $t(n)$, with $n > 2$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \sqrt{\frac{n}{n-2}}$$

`skewness_student_t (n)` [Function]

Returns the skewness coefficient of a Student random variable $t(n)$, with $n > 3$, which is always equal to 0. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = 0$$

`kurtosis_student_t (n)` [Function]

Returns the kurtosis coefficient of a Student random variable $t(n)$, with $n > 4$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{6}{n-4}$$

`random_student_t (n)` [Function]
`random_student_t (n,m)`

Returns a Student random variate $t(n)$, with $n > 0$. Calling `random_student_t` with a second argument m, a random sample of size m will be simulated.

The implemented algorithm is based on the fact that if Z is a normal random variable $Normal(0, 1)$ and S^2 is a χ^2 random variable with n degrees of freedom, $\chi^2(n)$, then

$$X = \frac{Z}{\sqrt{\frac{S^2}{n}}}$$

is a Student random variable with n degrees of freedom, $t(n)$.

To make use of this function, write first `load("distrib")`.

54.2.3 Noncentral Student's t Random Variable

Let *ncp* be the non-centrality parameter, n be the degrees of freedom for the non-central Student's t random variable.

Then let X be a $Normal(n, ncp)$ and S^2 be an independent χ^2 random variable with n degrees of freedom, the random variable

$$U = \frac{X}{\sqrt{\frac{S^2}{n}}}$$

has a non-central Student's t distribution with non-centrality parameter *ncp*.

`pdf_noncentral_student_t (x,n,ncp)` [Function]

Returns the value at x of the density function of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 0$ degrees of freedom and noncentrality parameter *ncp*. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n, \mu) = \left[\sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^{-1} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} e^{-\mu^2/2} \left[A_n(x; \mu) + B_n(x; \mu)\right]$$

where

$$A_n(x; \mu) = {}_1F_1\left(\frac{n+1}{2}; \frac{1}{2}; \frac{\mu^2 x^2}{2\left(x^2 + n\right)}\right)$$

$$B_n(x; \mu) = \frac{\sqrt{2}\mu x}{\sqrt{x^2 + n}} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n+1}{2}\right)} {}_1F_1\left(\frac{n}{2} + 1; \frac{3}{2}; \frac{\mu^2 x^2}{2\left(x^2 + n\right)}\right)$$

and μ is the non-centrality parameter *ncp*.

Sometimes an extra work is necessary to get the final result.

```
(%i1) load ("distrib")$
```

```
(%i2) expand(pdf_noncentral_student_t(3,5,0.1));
rat: replaced 0.018898223650461364 by 15934951/843198350 = 0.018898223650461364

rat: replaced -8.734356480209641 by -294697965/33740089 = -8.734356480209641

rat: replaced 4.136255165816327 by 51033443/12338079 = 4.136255165816332

rat: replaced 1.0806143216420299 by 49366521/45683756 = 1.0806143216420296

rat: replaced 0.0565127306411839 by 5608717/99246965 = 0.05651273064118384

rat: replaced -300.8069396896258 by -79782423/265228 = -300.80693968962555

rat: replaced 160.62691761849732 by 178374907/1110492 = 160.62691761849703
                            7/2                          7/2
      0.042964144174009046 5      1.3236503072892878e-6 5
(%o2) ------------------------ + -------------------------
         3/2   5/2                        sqrt(%pi)
        2    14    sqrt(%pi)
                                                                7/2
                                          1.94793720435093e-4 5
                                        + ------------------------
                                                   %pi
(%i3) float(%);
(%o3)                 0.020805931594056706
```

cdf_noncentral_student_t (*x*,*n*,*ncp*) [Function]

Returns the value at x of the cumulative distribution function of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 0$ degrees of freedom and noncentrality parameter ncp. This function has no closed form and it is numerically computed.

```
(%i1) load ("distrib")$
(%i2) cdf_noncentral_student_t(-2,5,-5);
(%o2)                   0.995203009331975
```

quantile_noncentral_student_t (*q*,*n*,*ncp*) [Function]

Returns the q-quantile of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 0$ degrees of freedom and noncentrality parameter ncp; in other words, this is the inverse of `cdf_noncentral_student_t`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

mean_noncentral_student_t (*n*,*ncp*) [Function]

Returns the mean of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 1$ degrees of freedom and noncentrality parameter ncp. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{\mu\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}$$

where μ is the noncentrality parameter *ncp*.

```
(%i1) load ("distrib")$
(%i2) mean_noncentral_student_t(df,k);
                              df - 1
                          gamma(------) sqrt(df) k
                                2
(%o2)                     -----------------------
                                              df
                             sqrt(2) gamma(--)
                                              2
```

`var_noncentral_student_t` (*n*,*ncp*) [Function]

Returns the variance of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 2$ degrees of freedom and noncentrality parameter *ncp*. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{n(\mu^2+1)}{n-2} - \frac{n\mu^2\,\Gamma\left(\frac{n-1}{2}\right)^2}{2\Gamma\left(\frac{n}{2}\right)^2}$$

where μ is the noncentrality parameter *ncp*.

`std_noncentral_student_t` (*n*,*ncp*) [Function]

Returns the standard deviation of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 2$ degrees of freedom and noncentrality parameter *ncp*. To make use of this function, write first `load("distrib")`.

The standard deviation is

`skewness_noncentral_student_t` (*n*,*ncp*) [Function]

Returns the skewness coefficient of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 3$ degrees of freedom and noncentrality parameter *ncp*. To make use of this function, write first `load("distrib")`.

If U is a non-central Student's t random variable with n degrees of freedom and a noncentrality parameter μ, the skewness is

$$SK[U] = \frac{\mu\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n}{2}\right)\sigma^3}\left(\frac{n\,(2n+\mu^2-3)}{(n-3)\,(n-2)} - 2\sigma^2\right)$$

$$\sigma^2 = \frac{n\,(\mu^2+1)}{n-2} - \frac{n\mu^2\,\Gamma\left(\frac{n-1}{2}\right)^2}{2\Gamma\left(\frac{n}{2}\right)^2}$$

`kurtosis_noncentral_student_t (n,ncp)` [Function]

Returns the kurtosis coefficient of a noncentral Student random variable $nc\_t(n, ncp)$, with $n > 4$ degrees of freedom and noncentrality parameter ncp. To make use of this function, write first `load("distrib")`.

If U is a non-central Student's t random variable with n degrees of freedom and a noncentrality parameter μ, the kurtosis is

$$\begin{aligned} KU[U] &= \frac{\mu_4}{\sigma^4} - 3 \\ \mu_4 &= \frac{\left(\mu^4 + 6\mu^2 + 3\right) n^2}{(n-4)(n-2)} - \left(\frac{n\left(3(3n-5) + \mu^2(n+1)\right)}{(n-3)(n-2)} - 3\sigma^2\right) F \\ \sigma^2 &= \frac{n\left(\mu^2+1\right)}{n-2} - \frac{n\mu^2\Gamma\left(\frac{n-1}{2}\right)^2}{2\Gamma\left(\frac{n}{2}\right)^2} \\ F &= \frac{n\mu^2\Gamma\left(\frac{n-1}{2}\right)^2}{2\sigma^4\Gamma\left(\frac{n}{2}\right)^2} \end{aligned}$$

`random_noncentral_student_t (n,ncp)` [Function]
`random_noncentral_student_t (n,ncp,m)`

Returns a noncentral Student random variate $nc\_t(n, ncp)$, with $n > 0$. Calling `random_noncentral_student_t` with a third argument m, a random sample of size m will be simulated.

The implemented algorithm is based on the fact that if X is a normal random variable $Normal(ncp, 1)$ and S^2 is a χ^2 random variable with n degrees of freedom, $\chi^2(n)$, then

$$U = \frac{X}{\sqrt{\frac{S^2}{n}}}$$

is a noncentral Student random variable with n degrees of freedom and noncentrality parameter ncp, $nc\_t(n, ncp)$.

To make use of this function, write first `load("distrib")`.

54.2.4 Chi-squared Random Variable

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $Normal(0, 1)$ variables. Then

$$X^2 = \sum_{i=1}^{n} X_i^2$$

is said to follow a chi-square distribution with n degrees of freedom.

`pdf_chi2 (x,n)` [Function]

Returns the value at x of the density function of a Chi-square random variable $\chi^2(n)$, with $n > 0$. The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The pdf is

$$f(x;n) = \begin{cases} \dfrac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma\left(\dfrac{n}{2}\right)} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) pdf_chi2(x,n);
                        - x/2  n/2 - 1
                      %e      x         unit_step(x)
(%o2)                 ------------------------------
                                n/2       n
                               2    gamma(-)
                                         2
```

`cdf_chi2 (x,n)` [Function]

Returns the value at x of the cumulative distribution function of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The cdf is

$$F(x;n) = \begin{cases} 1 - Q\left(\dfrac{n}{2}, \dfrac{x}{2}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $Q(a,z)$ is the [gamma_incomplete_regularized], page 335, function.

```
(%i1) load ("distrib")$
(%i2) cdf_chi2(3,4);
                                                        3
(%o2)        1 - gamma_incomplete_regularized(2, -)
                                                        2
(%i3) float(%);
(%o3)                  0.44217459962892525
```

`quantile_chi2 (q,n)` [Function]

Returns the q-quantile of a Chi-square random variable $\chi^2(n)$, with $n > 0$; in other words, this is the inverse of `cdf_chi2`. Argument q must be an element of $[0,1]$.

This function has no closed form and it is numerically computed.

```
(%i1) load ("distrib")$
(%i2) quantile_chi2(0.99,9);
(%o2)                   21.66599433346194
```

`mean_chi2 (n)` [Function]

Returns the mean of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The mean is

$$E[X] = n$$

```
(%i1) load ("distrib")$
(%i2) mean_chi2(n);
(%o2)                           n
```

`var_chi2 (n)` [Function]

Returns the variance of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The variance is

$$V[X] = 2n$$

```
(%i1) load ("distrib")$
(%i2) var_chi2(n);
(%o2)                          2 n
```

`std_chi2 (n)` [Function]

Returns the standard deviation of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The standard deviation is

$$D[X] = \sqrt{2n}$$

```
(%i1) load ("distrib")$
(%i2) std_chi2(n);
(%o2)                       sqrt(2) sqrt(n)
```

`skewness_chi2 (n)` [Function]

Returns the skewness coefficient of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The skewness coefficient is

$$SK[X] = \sqrt{\frac{8}{n}}$$

```
(%i1) load ("distrib")$
(%i2) skewness_chi2(n);
                                  3/2
                                 2
(%o2)                           -------
                                sqrt(n)
```

`kurtosis_chi2 (n)` [Function]

Returns the kurtosis coefficient of a Chi-square random variable $\chi^2(n)$, with $n > 0$.

The $\chi^2(n)$ random variable is equivalent to the $\Gamma(n/2, 2)$.

The kurtosis coefficient is

$$KU[X] = \frac{12}{n}$$

```
(%i1) load ("distrib")$
(%i2) kurtosis_chi2(n);
                                  12
(%o2)                             --
                                  n
```

`random_chi2 (n)` [Function]
`random_chi2 (n,m)`

Returns a Chi-square random variate $\chi^2(n)$, with $n > 0$. Calling `random_chi2` with a second argument m, a random sample of size m will be simulated.

The simulation is based on the Ahrens-Cheng algorithm. See `random_gamma` for details.

To make use of this function, write first `load("distrib")`.

54.2.5 Noncentral Chi-squared Random Variable

Let $X_1, X_2, ..., X_n$ be n independent normally distributed random variables with means μ_k and unit variances. Then the random variable

$$\sum_{k=1}^{n} X_k^2$$

has a noncentral χ^2 distribution. The number of degrees of freedom is n, and the noncentrality parameter is defined by

$$\sum_{k=1}^{n} \mu_k^2$$

`pdf_noncentral_chi2 (x,n,ncp)` [Function]

Returns the value at x of the density function of a noncentral χ^2 random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first `load("distrib")`.

For $x < 0$, the pdf is 0, and for $x \geq 0$ the pdf is

$$f(x; n, \lambda) = \frac{1}{2} e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{n/4-1/2} I_{\frac{n}{2}-1}\left(\sqrt{n\lambda}\right)$$

`cdf_noncentral_chi2 (x,n,ncp)` [Function]

Returns the value at x of the cumulative distribution function of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$. To make use of this function, write first `load("distrib")`.

`quantile_noncentral_chi2 (q,n,ncp)` [Function]

Returns the q-quantile of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$; in other words, this is the inverse of `cdf_noncentral_chi2`. Argument q must be an element of $[0, 1]$.

This function has no closed form and it is numerically computed.

`mean_noncentral_chi2 (n,ncp)` [Function]

Returns the mean of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The mean is

$$E[X] = n + \mu$$

where μ is the noncentrality parameter *ncp*.

`var_noncentral_chi2 (n,ncp)` [Function]

Returns the variance of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The variance is

$$V[X] = 2(n + 2\mu)$$

where μ is the noncentrality parameter *ncp*.

`std_noncentral_chi2 (n,ncp)` [Function]

Returns the standard deviation of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The standard deviation is

$$D[X] = \sqrt{2(n + 2\mu)}$$

where μ is the noncentrality parameter *ncp*.

`skewness_noncentral_chi2 (n,ncp)` [Function]

Returns the skewness coefficient of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The skewness coefficient is

$$SK[X] = \frac{2^{3/2}(n + 3\mu)}{(n + 2\mu)^{3/2}}$$

where μ is the noncentrality parameter *ncp*.

`kurtosis_noncentral_chi2 (n,ncp)` [Function]

Returns the kurtosis coefficient of a noncentral Chi-square random variable m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$.

The kurtosis coefficient is

$$KU[X] = \frac{12(n + 4\mu)}{(2 + 2\mu)^2}$$

where μ is the noncentrality parameter *ncp*.

`random_noncentral_chi2 (n,ncp)` [Function]
`random_noncentral_chi2 (n,ncp,m)`

Returns a noncentral Chi-square random variate m4_noncentral_chi2(n,ncp) , with $n > 0$ and noncentrality parameter $ncp \geq 0$. Calling `random_noncentral_chi2` with a third argument m, a random sample of size m will be simulated.

To make use of this function, write first `load("distrib")`.

54.2.6 F Random Variable

Let S_1 and S_2 be independent random variables with a χ^2 distribution with degrees of freedom n and m, respectively. Then

$$F = \frac{S_1/n}{S_2/m}$$

has an F distribution with n and m degrees of freedom.

`pdf_f (x,m,n)` [Function]

Returns the value at x of the density function of a F random variable $F(m, n)$, with $m, n > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; m, n) = \begin{cases} B\left(\frac{m}{2}, \frac{n}{2}\right)^{-1} \left(\frac{m}{n}\right)^{m/2} x^{m/2-1} \left(1 + \frac{m}{n}x\right)^{-(n+m)/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

`cdf_f (x,m,n)` [Function]

Returns the value at x of the cumulative distribution function of a F random variable $F(m, n)$, with $m, n > 0$.

The cdf is

$$F(x; m, n) = \begin{cases} 1 - I_z\left(\frac{m}{2}, \frac{n}{2}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$z = \frac{n}{mx + n}$$

and $I_z(a, b)$ is the [beta_incomplete_regularized], page 341, function.

```
(%i1) load ("distrib")$
(%i2) cdf_f(2,3,9/4);
                                                 9  3  3
(%o2)       1 - beta_incomplete_regularized(-, -, --)
                                                 8  2  11
(%i3) float(%);
(%o3)                  0.6675672817900802
```

`quantile_f (q,m,n)` [Function]

Returns the q-quantile of a F random variable $F(m, n)$, with $m, n > 0$; in other words, this is the inverse of `cdf_f`. Argument q must be an element of $[0, 1]$.

```
(%i1) load ("distrib")$
(%i2) quantile_f(2/5,sqrt(3),5);
(%o2)                  0.5189478385736904
```

`mean_f (m,n)` [Function]

Returns the mean of a F random variable $F(m, n)$, with $m > 0, n > 2$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{n}{n-2}$$

`var_f` (*m*,*n*) [Function]

Returns the variance of a F random variable $F(m,n)$, with $m > 0, n > 4$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{2n^2(n+m-2)}{m(n-4)(n-2)^2}$$

`std_f` (*m*,*n*) [Function]

Returns the standard deviation of a F random variable $F(m,n)$, with $m > 0, n > 4$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{\sqrt{2}\,n}{n-2}\sqrt{\frac{n+m-2}{m(n-4)}}$$

`skewness_f` (*m*,*n*) [Function]

Returns the skewness coefficient of a F random variable $F(m,n)$, with $m > 0, n > 6$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{(n+2m-2)\sqrt{8(n-4)}}{(n-6)\sqrt{m(n+m-2)}}$$

`kurtosis_f` (*m*,*n*) [Function]

Returns the kurtosis coefficient of a F random variable $F(m,n)$, with $m > 0, n > 8$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = 12\frac{m(n+m-2)(5n-22)+(n-4)(n-2)^2}{m(n-8)(n-6)(n+m-2)}$$

`random_f` (*m*,*n*) [Function]
`random_f` (*m*,*n*,*k*)

Returns a F random variate $F(m,n)$, with $m, n > 0$. Calling `random_f` with a third argument k, a random sample of size k will be simulated.

The simulation algorithm is based on the fact that if X is a $Chi^2(m)$ random variable and Y is a $\chi^2(n)$ random variable, then

$$F = \frac{nX}{mY}$$

is a F random variable with m and n degrees of freedom, $F(m,n)$.

To make use of this function, write first `load("distrib")`.

54.2.7 Exponential Random Variable

The *exponential distribution* is the probability distribution of the time between events in a process where the events occur continuously and independently at a constant average rate.

`pdf_exp (x,m)` [Function]

Returns the value at x of the density function of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The pdf is

$$f(x; m) = \begin{cases} me^{-mx} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) pdf_exp(x,m);
                            - m x
(%o2)                  %e      m unit_step(x)
```

`cdf_exp (x,m)` [Function]

Returns the value at x of the cumulative distribution function of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The cdf is

$$F(x; m) = \begin{cases} 1 - e^{-mx} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) cdf_exp(x,m);
                               - m x
(%o2)                (1 - %e     ) unit_step(x)
```

`quantile_exp (q,m)` [Function]

Returns the q-quantile of an *Exponential*(m) random variable, with $m > 0$; in other words, this is the inverse of `cdf_exp`. Argument q must be an element of $[0, 1]$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

```
(%i1) load ("distrib")$
(%i2) quantile_exp(0.56,5);
(%o2)                  0.1641961104139661
(%i3) quantile_exp(0.56,m);
                       0.8209805520698303
(%o3)                  ------------------
                              m
```

`mean_exp (m)` [Function]

Returns the mean of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The mean is

$$E[X] = \frac{1}{m}$$

```
(%i1) load ("distrib")$
(%i2) mean_exp(m);
                                  1
(%o2)                             -
                                  m
```

`var_exp` (*m*) [Function]

Returns the variance of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The variance is

$$V[X] = \frac{1}{m^2}$$

```
(%i1) load ("distrib")$
(%i2) var_exp(m);
                                  1
(%o2)                             --
                                   2
                                  m
```

`std_exp` (*m*) [Function]

Returns the standard deviation of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The standard deviation is

$$D[X] = \frac{1}{m}$$

```
(%i1) load ("distrib")$
(%i2) std_exp(m);
                                  1
(%o2)                             -
                                  m
```

`skewness_exp` (*m*) [Function]

Returns the skewness coefficient of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The skewness coefficient is

$$SK[X] = 2$$

```
(%i1) load ("distrib")$
(%i2) skewness_exp(m);
(%o2)                             2
```

`kurtosis_exp` (*m*) [Function]

Returns the kurtosis coefficient of an *Exponential*(m) random variable, with $m > 0$.

The *Exponential*(m) random variable is equivalent to the *Weibull*$(1, 1/m)$.

The kurtosis coefficient is

$$KU[X] = 6$$

```
(%i1) load ("distrib")$
(%i2) kurtosis_exp(m);
(%o2)                             6
```

`random_exp (m)` [Function]
`random_exp (m,k)`

Returns an *Exponential*(m) random variate, with $m > 0$. Calling `random_exp` with a second argument k, a random sample of size k will be simulated.

The simulation algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.8 Lognormal Random Variable

The *lognormal* distribution is distribution for a random variable whose logarithm is normally distributed.

`pdf_lognormal (x,m,s)` [Function]

Returns the value at x of the density function of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; m, s) = \begin{cases} \dfrac{1}{xs\sqrt{2\pi}} \exp\left(-\dfrac{(\log x - m)^2}{2s^2}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

`cdf_lognormal (x,m,s)` [Function]

Returns the value at x of the cumulative distribution function of a *Lognormal*(m, s) random variable, with $s > 0$. This function is defined in terms of Maxima's built-in error function `erf`.

The cdf is

$$F(x; m, s) = \begin{cases} \dfrac{1}{2}\left[1 + \operatorname{erf}\left(\dfrac{\log x - m}{s\sqrt{2}}\right)\right] & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) cdf_lognormal(x,m,s);
                                    log(x) - m
                                 erf(----------)
                                    sqrt(2) s     1
(%o2)          unit_step(x) (--------------- + -)
                                       2          2
```

See also `erf`.

`quantile_lognormal (q,m,s)` [Function]

Returns the q-quantile of a *Lognormal*(m, s) random variable, with $s > 0$; in other words, this is the inverse of `cdf_lognormal`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

```
(%i1) load ("distrib")$
(%i2) quantile_lognormal(95/100,0,1);
                          sqrt(2) inverse_erf(9/10)
(%o2)                    %e
(%i3) float(%);
(%o3)                      5.180251602233015
```

`mean_lognormal (m,s)` [Function]

Returns the mean of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \exp\left(m + \frac{s^2}{2}\right)$$

`var_lognormal (m,s)` [Function]

Returns the variance of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \left(\exp\left(s^2\right) - 1\right) \exp\left(2m + s^2\right)$$

`std_lognormal (m,s)` [Function]

Returns the standard deviation of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \sqrt{\left(\exp\left(s^2\right) - 1\right)} \exp\left(m + \frac{s^2}{2}\right)$$

`skewness_lognormal (m,s)` [Function]

Returns the skewness coefficient of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \left(\exp\left(s^2\right) + 2\right) \sqrt{\exp\left(s^2\right) - 1}$$

`kurtosis_lognormal (m,s)` [Function]

Returns the kurtosis coefficient of a *Lognormal*(m, s) random variable, with $s > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \exp\left(4s^2\right) + 2\exp\left(3s^2\right) + 3\exp\left(2s^2\right) - 3$$

`random_lognormal (m,s)` [Function]
`random_lognormal (m,s,n)`

Returns a *Lognormal*(m, s) random variate, with $s > 0$. Calling `random_lognormal` with a third argument n, a random sample of size n will be simulated.

Log-normal variates are simulated by means of random normal variates. See `random_normal` for details.

To make use of this function, write first `load("distrib")`.

54.2.9 Gamma Random Variable

The *gamma distribution* is a two-parameter family of probability distributions. Maxima uses the parameterization using the shape and scale for the first and second parameters of the distribution.

`pdf_gamma (x,a,b)` [Function]

Returns the value at x of the density function of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The shape parameter is a, and the scale parameter is b.

The pdf is

$$f(x;a,b) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}$$

`cdf_gamma (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $\Gamma(a,b)$ random variable, with $a, b > 0$.

The cdf is

$$F(x;a,b) = \begin{cases} 1 - Q(a, \frac{x}{b}) & \text{for } x \geq 0 \\ \\ 0 & \text{for } x < 0 \end{cases}$$

where $Q(a,z)$ is the [gamma_incomplete_regularized], page 335, function.

```
(%i1) load ("distrib")$
(%i2) cdf_gamma(3,5,21);
                                                      1
(%o2)        1 - gamma_incomplete_regularized(5, -)
                                                      7
(%i3) float(%);
(%o3)                 4.402663157376807e-7
```

`quantile_gamma (q,a,b)` [Function]

Returns the q-quantile of a $\Gamma(a,b)$ random variable, with $a, b > 0$; in other words, this is the inverse of `cdf_gamma`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_gamma (a,b)` [Function]

Returns the mean of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = ab$$

`var_gamma (a,b)` [Function]

Returns the variance of a $\Gamma(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = ab^2$$

`std_gamma (a,b)` [Function]

Returns the standard deviation of a $\Gamma(a,b)$ random variable, with $a,b>0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = b\sqrt{a}$$

`skewness_gamma (a,b)` [Function]

Returns the skewness coefficient of a $\Gamma(a,b)$ random variable, with $a,b>0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{2}{\sqrt{a}}$$

`kurtosis_gamma (a,b)` [Function]

Returns the kurtosis coefficient of a $\Gamma(a,b)$ random variable, with $a,b>0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{6}{a}$$

`random_gamma (a,b)` [Function]
`random_gamma (a,b,n)`

Returns a $\Gamma(a,b)$ random variate, with $a,b>0$. Calling `random_gamma` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is a combination of two procedures, depending on the value of parameter a:

For $a \geq 1$, Cheng, R.C.H. and Feast, G.M. (1979). *Some simple gamma variate generators*. Appl. Stat., 28, 3, 290-295.

For $0 < a < 1$, Ahrens, J.H. and Dieter, U. (1974). *Computer methods for sampling from gamma, , poisson and binomial distributions*. Computing, 12, 223-246.

To make use of this function, write first `load("distrib")`.

54.2.10 Beta Random Variable

The *beta* distribution is a family of distributions defined over $[0,1]$ parameterized by two positive shape parameters a, and b.

`pdf_beta (x,a,b)` [Function]

Returns the value at x of the density function of a $Beta(a,b)$ random variable, with $a,b>0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x;a,b) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

`cdf_beta (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $Beta(a,b)$ random variable, with $a,b>0$.

The cdf is

$$F(x;a,b) = \begin{cases} 0 & x < 0 \\ I_x(a,b) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) cdf_beta(1/3,15,2);
                                  11
(%o2)                           --------
                                14348907
(%i3) float(%);
(%o3)                  7.666089131388195e-7
```

`quantile_beta` (*q*,*a*,*b*) [Function]

Returns the q-quantile of a $Beta(a,b)$ random variable, with $a, b > 0$; in other words, this is the inverse of `cdf_beta`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_beta` (*a*,*b*) [Function]

Returns the mean of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{a}{a+b}$$

`var_beta` (*a*,*b*) [Function]

Returns the variance of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{ab}{(a+b)^2(a+b+1)}$$

`std_beta` (*a*,*b*) [Function]

Returns the standard deviation of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{1}{a+b}\sqrt{\frac{ab}{a+b+1}}$$

`skewness_beta` (*a*,*b*) [Function]

Returns the skewness coefficient of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{2(b-a)\sqrt{a+b+1}}{(a+b+2)\sqrt{ab}}$$

`kurtosis_beta` (*a*,*b*) [Function]

Returns the kurtosis coefficient of a $Beta(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{3(a+b+1)\left(2(a+b)^2+ab(a+b-6)\right)}{ab(a+b+2)(a+b+3)} - 3$$

`random_beta (a,b)` [Function]
`random_beta (a,b,n)`

Returns a $Beta(a,b)$ random variate, with $a, b > 0$. Calling `random_beta` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is defined in Cheng, R.C.H. (1978). *Generating Beta Variates with Nonintegral Shape Parameters*. Communications of the ACM, 21:317-322

To make use of this function, write first `load("distrib")`.

54.2.11 Continuous Uniform Random Variable

The *continuous uniform* distribution is constant over the interval $[a,b]$ and is zero elsewhere.

`pdf_continuous_uniform (x,a,b)` [Function]

Returns the value at x of the density function of a $ContinuousUniform(a,b)$ random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The pdf

$$f(x;a,b) = \begin{cases} \dfrac{1}{b-a} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and is 0 otherwise.

`cdf_continuous_uniform (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $ContinuousUniform(a,b)$ random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x;a,b) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

`quantile_continuous_uniform (q,a,b)` [Function]

Returns the q-quantile of a $ContinuousUniform(a,b)$ random variable, with $a < b$; in other words, this is the inverse of `cdf_continuous_uniform`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_continuous_uniform (a,b)` [Function]

Returns the mean of a $ContinuousUniform(a,b)$ random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{a+b}{2}$$

`var_continuous_uniform (a,b)` [Function]

Returns the variance of a *ContinuousUniform*(a, b) random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{(b-a)^2}{12}$$

`std_continuous_uniform (a,b)` [Function]

Returns the standard deviation of a *ContinuousUniform*(a, b) random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{b-a}{2\sqrt{3}}$$

`skewness_continuous_uniform (a,b)` [Function]

Returns the skewness coefficient of a *ContinuousUniform*(a, b) random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = 0$$

`kurtosis_continuous_uniform (a,b)` [Function]

Returns the kurtosis coefficient of a *ContinuousUniform*(a, b) random variable, with $a < b$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = -\frac{6}{5}$$

`random_continuous_uniform (a,b)` [Function]
`random_continuous_uniform (a,b,n)`

Returns a *ContinuousUniform*(a, b) random variate, with $a < b$. Calling `random_continuous_uniform` with a third argument n, a random sample of size n will be simulated.

This is a direct application of the `random` built-in Maxima function.

See also `random`. To make use of this function, write first `load("distrib")`.

54.2.12 Logistic Random Variable

The *logistic* distribution is a continuous distribution where its cumulative distribution function is the logistic function.

`pdf_logistic (x,a,b)` [Function]

Returns the value at x of the density function of a *Logistic*(a, b) random variable , with $b > 0$. To make use of this function, write first `load("distrib")`.

a is the location parameter and b is the scale parameter.

The pdf is

$$f(x; a, b) = \frac{e^{-(x-a)/b}}{b\left(1 + e^{-(x-a)/b}\right)^2}$$

`cdf_logistic (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a *Logistic*(a, b) random variable , with $b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; a, b) = \frac{1}{1 + e^{-(x-a)/b}}$$

`quantile_logistic (q,a,b)` [Function]

Returns the q-quantile of a *Logistic*(a, b) random variable , with $b > 0$; in other words, this is the inverse of `cdf_logistic`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_logistic (a,b)` [Function]

Returns the mean of a *Logistic*(a, b) random variable , with $b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = a$$

`var_logistic (a,b)` [Function]

Returns the variance of a *Logistic*(a, b) random variable , with $b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{\pi^2 b^2}{3}$$

`std_logistic (a,b)` [Function]

Returns the standard deviation of a *Logistic*(a, b) random variable , with $b > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{\pi b}{\sqrt{3}}$$

`skewness_logistic (a,b)` [Function]

Returns the skewness coefficient of a *Logistic*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = 0$$

`kurtosis_logistic (a,b)` [Function]

Returns the kurtosis coefficient of a *Logistic*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{6}{5}$$

`random_logistic (a,b)` [Function]
`random_logistic (a,b,n)`

Returns a *Logistic*(a, b) random variate, with $b > 0$. Calling `random_logistic` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.13 Pareto Random Variable

`pdf_pareto (x,a,b)` [Function]

Returns the value at x of the density function of a $Pareto(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; a, b) = \begin{cases} \dfrac{ab^a}{x^{a+1}} & \text{for } x \geq b \\ 0 & \text{for } x < b \end{cases}$$

`cdf_pareto (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $Pareto(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; a, b) = \begin{cases} 1 - \left(\dfrac{b}{x}\right)^a & \text{for } x \geq b \\ 0 & \text{for } x < b \end{cases}$$

`quantile_pareto (q,a,b)` [Function]

Returns the q-quantile of a $Pareto(a, b)$ random variable, with $a, b > 0$; in other words, this is the inverse of `cdf_pareto`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_pareto (a,b)` [Function]

Returns the mean of a $Pareto(a, b)$ random variable, with $a > 1, b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{ab}{a-1}$$

`var_pareto (a,b)` [Function]

Returns the variance of a $Pareto(a, b)$ random variable, with $a > 2, b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{ab^2}{(a-2)(a-1)^2}$$

`std_pareto (a,b)` [Function]

Returns the standard deviation of a $Pareto(a, b)$ random variable, with $a > 2, b > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{b}{a-1}\sqrt{\frac{a}{a-2}}$$

`skewness_pareto (a,b)` [Function]

Returns the skewness coefficient of a $Pareto(a, b)$ random variable, with $a > 3, b > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{2(a+1)}{a-3}\sqrt{\frac{a-2}{a}}$$

`kurtosis_pareto (a,b)` [Function]

Returns the kurtosis coefficient of a $Pareto(a,b)$ random variable, with $a > 4, b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{6\left(a^3 + a^2 - 6 * a - 2\right)}{a(a-3)(a-4)} - 3$$

`random_pareto (a,b)` [Function]

`random_pareto (a,b,n)`

Returns a $Pareto(a,b)$ random variate, with $a > 0, b > 0$. Calling `random_pareto` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.14 Weibull Random Variable

`pdf_weibull (x,a,b)` [Function]

Returns the value at x of the density function of a $Weibull(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x;a,b) = \begin{cases} \frac{1}{b}\left(\frac{x}{b}\right)^{a-1} e^{-(x/b)^a} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

`cdf_weibull (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $Weibull(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x;a,b) = \begin{cases} 1 - e^{-(x/b)^a} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

`quantile_weibull (q,a,b)` [Function]

Returns the q-quantile of a $Weibull(a,b)$ random variable, with $a, b > 0$; in other words, this is the inverse of `cdf_weibull`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_weibull (a,b)` [Function]

Returns the mean of a $Weibull(a,b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = b\Gamma\left(1 + \frac{1}{a}\right)$$

`var_weibull (a,b)` [Function]

Returns the variance of a $Weibull(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = b^2 \left[\Gamma\left(1 + \frac{2}{a}\right) - \Gamma\left(1 + \frac{1}{a}\right)^2 \right]$$

`std_weibull (a,b)` [Function]

Returns the standard deviation of a $Weibull(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$D[X] = b\sqrt{\Gamma\left(1 + \frac{2}{a}\right) - \Gamma\left(1 + \frac{1}{a}\right)^2}$$

`skewness_weibull (a,b)` [Function]

Returns the skewness coefficient of a $Weibull(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{\Gamma\left(1 + \frac{3}{a}\right) - 3\Gamma\left(1 + \frac{1}{a}\right)\Gamma\left(1 + \frac{2}{a}\right) + 2\Gamma\left(1 + \frac{1}{a}\right)^3}{\left[\Gamma\left(1 + \frac{2}{a}\right) - \Gamma\left(1 + \frac{1}{a}\right)^2\right]^{3/2}}$$

`kurtosis_weibull (a,b)` [Function]

Returns the kurtosis coefficient of a $Weibull(a, b)$ random variable, with $a, b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{\Gamma_4 - 4\Gamma_1\Gamma_3 + 6\Gamma_1^2\Gamma_2 - 3\Gamma_1^4}{\left[\Gamma_2 - \Gamma_1^2\right]^2} - 3$$

where $\Gamma_k = \Gamma\left(1 + k/a\right)$.

`random_weibull (a,b)` [Function]
`random_weibull (a,b,n)`

Returns a $Weibull(a, b)$ random variate, with $a, b > 0$. Calling `random_weibull` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.15 Rayleigh Random Variable

The *Rayleigh* distribution coincides with the χ^2 distribution with two degrees of freedom.

`pdf_rayleigh (x,b)` [Function]

Returns the value at x of the density function of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The pdf is

$$f(x;b) = \begin{cases} 2b^2 x e^{-b^2 x^2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) pdf_rayleigh(x,b);
                              2  2
                          - b  x   2
(%o2)                2 %e         b  x unit_step(x)
```

`cdf_rayleigh (x,b)` [Function]

Returns the value at x of the cumulative distribution function of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The cdf is

$$F(x;b) = \begin{cases} 1 - e^{-b^2 x^2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

```
(%i1) load ("distrib")$
(%i2) cdf_rayleigh(x,b);
                                 2  2
                             - b  x
(%o2)                (1 - %e         ) unit_step(x)
```

`quantile_rayleigh (q,b)` [Function]

Returns the q-quantile of a *Rayleigh*(b) random variable, with $b > 0$; in other words, this is the inverse of `cdf_rayleigh`. Argument q must be an element of $[0, 1]$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

```
(%i1) load ("distrib")$
(%i2) quantile_rayleigh(0.99,b);
                          2.1459660262893467
(%o2)                     ------------------
                                  b
```

`mean_rayleigh (b)` [Function]

Returns the mean of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The mean is

$$E[X] = \frac{\sqrt{\pi}}{2b}$$

```
(%i1) load ("distrib")$
```

```
(%i2) mean_rayleigh(b);
                                 sqrt(%pi)
(%o2)                            ---------
                                    2 b
```

`var_rayleigh (b)` [Function]

Returns the variance of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The variance is

$$V[X] = \frac{1}{b^2}\left(1 - \frac{\pi}{4}\right)$$

```
(%i1) load ("distrib")$
(%i2) var_rayleigh(b);
                                     %pi
                                  1 - ---
                                      4
(%o2)                             -------
                                     2
                                    b
```

`std_rayleigh (b)` [Function]

Returns the standard deviation of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The standard deviation is

$$D[X] = \frac{1}{b}\sqrt{1 - \frac{\pi}{4}}$$

```
(%i1) load ("distrib")$
(%i2) std_rayleigh(b);
                                       %pi
                                sqrt(1 - ---)
                                        4
(%o2)                           -------------
                                       b
```

`skewness_rayleigh (b)` [Function]

Returns the skewness coefficient of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The skewness coefficient is

$$SK[X] = \frac{2\sqrt{\pi}(\pi - 3)}{(4 - \pi)^{3/2}}$$

```
(%i1) load ("distrib")$
```

```
(%i2) skewness_rayleigh(b);
                              3/2
                          %pi      3 sqrt(%pi)
                          ------ - -----------
                            4           4
(%o2)                     --------------------
                                  %pi 3/2
                             (1 - ---)
                                   4
```

`kurtosis_rayleigh` (*b*) [Function]

Returns the kurtosis coefficient of a *Rayleigh*(b) random variable, with $b > 0$.

The *Rayleigh*(b) random variable is equivalent to the *Weibull*$(2, 1/b)$.

The kurtosis coefficient is

$$KU[X] = \frac{32 - 3\pi}{(4 - \pi)^2} - 3$$

```
(%i1) load ("distrib")$
(%i2) kurtosis_rayleigh(b);
                                      2
                                 3 %pi
                            2 - ------
                                  16
(%o2)                       ---------- - 3
                                 %pi 2
                            (1 - ---)
                                  4
```

`random_rayleigh` (*b*) [Function]
`random_rayleigh` (*b*,*n*)

Returns a *Rayleigh*(b) random variate, with $b > 0$. Calling `random_rayleigh` with a second argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.16 Laplace Random Variable

The *Laplace* distribution is a continuous probability distribution that is sometimes called the double exponential distribution because it can be thought of as two exponential distributions spliced back to back.

`pdf_laplace` (*x*,*a*,*b*) [Function]

Returns the value at x of the density function of a *Laplace*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

Here, a is the location parameter (or mean), and b is the scale parameter, related to the variance.

The pdf is

$$f(x;a,b) = \frac{1}{2b}\exp\left(-\frac{|x-a|}{b}\right)$$

`cdf_laplace (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x;a,b) = \begin{cases} \frac{1}{2}\exp\left(\frac{x-a}{b}\right) & \text{for } x < a \\ 1 - \frac{1}{2}\exp\left(\frac{x-a}{b}\right) & \text{for } x \geq a \end{cases}$$

`quantile_laplace (q,a,b)` [Function]

Returns the q-quantile of a *Laplace*(a,b) random variable, with $b > 0$; in other words, this is the inverse of `cdf_laplace`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_laplace (a,b)` [Function]

Returns the mean of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = a$$

`var_laplace (a,b)` [Function]

Returns the variance of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = 2b^2$$

`std_laplace (a,b)` [Function]

Returns the standard deviation of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \sqrt{2}b$$

`skewness_laplace (a,b)` [Function]

Returns the skewness coefficient of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = 0$$

`kurtosis_laplace (a,b)` [Function]

Returns the kurtosis coefficient of a *Laplace*(a,b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = 3$$

`random_laplace (a,b)` [Function]
`random_laplace (a,b,n)`

Returns a *Laplace*(a, b) random variate, with $b > 0$. Calling `random_laplace` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.17 Cauchy Random Variable

The *Cauchy* distribution (also known as the Lorentz distribution) is the distribution of of the ratio of two independent normally distributed random variables with mean zero.

Note that the mean, variance, standard deviation, skewness coefficient, and kurtosis coefficient are all undefined for the Cauchy distribution. The integrals do not converge in this case.

`pdf_cauchy (x,a,b)` [Function]

Returns the value at x of the density function of a *Cauchy*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; a, b) = \frac{b}{\pi\left((x-a)^2 + b^2\right)}$$

`cdf_cauchy (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a *Cauchy*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; a, b) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{x-a}{b}$$

`quantile_cauchy (q,a,b)` [Function]

Returns the q-quantile of a *Cauchy*(a, b) random variable, with $b > 0$; in other words, this is the inverse of `cdf_cauchy`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`random_cauchy (a,b)` [Function]
`random_cauchy (a,b,n)`

Returns a *Cauchy*(a, b) random variate, with $b > 0$. Calling `random_cauchy` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.2.18 Gumbel Random Variable

`pdf_gumbel (x,a,b)` [Function]

Returns the value at x of the density function of a *Gumbel*(a, b) random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; a, b) = \frac{1}{b}\exp\left[\frac{a-x}{b} - \exp\left(\frac{a-x}{b}\right)\right]$$

`cdf_gumbel (x,a,b)` [Function]

Returns the value at x of the cumulative distribution function of a $Gumbel(a, b)$ random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; a, b) = \exp\left[-\exp\left(\frac{a-x}{b}\right)\right]$$

`quantile_gumbel (q,a,b)` [Function]

Returns the q-quantile of a $Gumbel(a, b)$ random variable, with $b > 0$; in other words, this is the inverse of `cdf_gumbel`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_gumbel (a,b)` [Function]

Returns the mean of a $Gumbel(a, b)$ random variable, with $b > 0$.

The mean is

$$E[X] = a + b\gamma$$

```
(%i1) load ("distrib")$
(%i2) mean_gumbel(a,b);
(%o2)                        %gamma b + a
```

where symbol `%gamma` stands for the Euler-Mascheroni constant. See also `%gamma`.

`var_gumbel (a,b)` [Function]

Returns the variance of a $Gumbel(a, b)$ random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{\pi^2}{6}b^2$$

`std_gumbel (a,b)` [Function]

Returns the standard deviation of a $Gumbel(a, b)$ random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{\pi}{\sqrt{6}}b$$

`skewness_gumbel (a,b)` [Function]

Returns the skewness coefficient of a $Gumbel(a, b)$ random variable, with $b > 0$.

The skewness coefficient is

$$SK[X] = \frac{12\sqrt{6}}{\pi^3}\zeta(3)$$

```
(%i1) load ("distrib")$
(%i2) skewness_gumbel(a,b);
                                3/2
                           2 6    zeta(3)
(%o2)                      --------------
                                 3
                              %pi
```

where `zeta` stands for the Riemann's zeta function.

`kurtosis_gumbel (a,b)` [Function]

Returns the kurtosis coefficient of a $Gumbel(a, b)$ random variable, with $b > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{12}{5}$$

`random_gumbel (a,b)` [Function]
`random_gumbel (a,b,n)`

Returns a $Gumbel(a, b)$ random variate, with $b > 0$. Calling `random_gumbel` with a third argument n, a random sample of size n will be simulated.

The implemented algorithm is based on the general inverse method.

To make use of this function, write first `load("distrib")`.

54.3 Functions and Variables for discrete distributions

Maxima knows the following kinds of discrete distributions

54.3.1 General Finite Discrete Random Variable

`pdf_general_finite_discrete (x,v)` [Function]

Returns the value at x of the probability function of a general finite discrete random variable, with vector probabilities v, such that $Pr(X = i) = v_i$. Vector v can be a list of nonnegative expressions whose components will be normalized to get a vector of probabilities. To make use of this function, write first `load("distrib")`.

Note that $i = 1$ corresponds to the first element of v.

```
(%i1) load ("distrib")$
(%i2) pdf_general_finite_discrete(2, [1/7, 4/7, 2/7]);
                                4
(%o2)                           -
                                7
(%i3) pdf_general_finite_discrete(2, [1, 4, 2]);
                                4
(%o3)                           -
                                7
```

`cdf_general_finite_discrete (x,v)` [Function]

Returns the value at x of the cumulative distribution function of a general finite discrete random variable, with vector probabilities v.

See [pdf_general_finite_discrete], page 832, for more details.

```
(%i1) load ("distrib")$
(%i2) cdf_general_finite_discrete(2, [1/7, 4/7, 2/7]);
                                5
(%o2)                           -
                                7
(%i3) cdf_general_finite_discrete(2, [1, 4, 2]);
                                5
(%o3)                           -
                                7
```

```
(%i4) cdf_general_finite_discrete(2+1/2, [1, 4, 2]);
                                  5
(%o4)                             -
                                  7
```

`quantile_general_finite_discrete` (*q*,*v*) [Function]
Returns the *q*-quantile of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`mean_general_finite_discrete` (*v*) [Function]
Returns the mean of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`var_general_finite_discrete` (*v*) [Function]
Returns the variance of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`std_general_finite_discrete` (*v*) [Function]
Returns the standard deviation of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`skewness_general_finite_discrete` (*v*) [Function]
Returns the skewness coefficient of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`kurtosis_general_finite_discrete` (*v*) [Function]
Returns the kurtosis coefficient of a general finite discrete random variable, with vector probabilities *v*.

See [pdf_general_finite_discrete], page 832, for more details.

`random_general_finite_discrete` (*v*) [Function]
`random_general_finite_discrete` (*v*,*m*)
Returns a general finite discrete random variate, with vector probabilities *v*. Calling `random_general_finite_discrete` with a second argument *m*, a random sample of size *m* will be simulated.

See [pdf_general_finite_discrete], page 832, for more details.

```
(%i1) load ("distrib")$
(%i2) random_general_finite_discrete([1,3,1,5]);
(%o2)                            4
(%i3) random_general_finite_discrete([1,3,1,5], 10);
(%o3)           [4, 4, 2, 4, 2, 2, 4, 2, 2, 4]
```

54.3.2 Binomial Random Variable

The *binomial distribution* with parameters n and p is a discrete probability distribution. It consists of n independent experiments where each experiment consists of a Boolean-valued outcome where a success occurs with a probability p.

For example, a biased coin that comes up heads with probability p is tossed n times. Then the probability of exactly k heads in n tosses is given by the binomial distribution.

`pdf_binomial (x,n,p)` [Function]

Returns the value at x of the probability function of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n, p) = \binom{n}{x}(1-p)^{n-x}p^x$$

`cdf_binomial (x,n,p)` [Function]

Returns the value at x of the cumulative distribution function of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and n a positive integer.

The cdf is

$$F(x; n, p) = I_{1-p}(n - \lfloor x \rfloor, \lfloor x \rfloor + 1)$$

where $I_z(a, b)$ is the [beta_incomplete_regularized], page 341, function.

```
(%i1) load ("distrib")$
(%i2) cdf_binomial(5,7,1/6);
                                7775
(%o2)                           ----
                                7776
(%i3) float(%);
(%o3)                0.9998713991769548
```

`quantile_binomial (q,n,p)` [Function]

Returns the q-quantile of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and n a positive integer; in other words, this is the inverse of `cdf_binomial`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_binomial (n,p)` [Function]

Returns the mean of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = np$$

`var_binomial (n,p)` [Function]

Returns the variance of a $Binomial(n, p)$ random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = np(1-p)$$

`std_binomial (n,p)` [Function]

Returns the standard deviation of a *Binomial*(n, p) random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \sqrt{np(1-p)}$$

`skewness_binomial (n,p)` [Function]

Returns the skewness coefficient of a *Binomial*(n, p) random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{1-2p}{\sqrt{np(1-p)}}$$

`kurtosis_binomial (n,p)` [Function]

Returns the kurtosis coefficient of a *Binomial*(n, p) random variable, with $0 \leq p \leq 1$ and n a positive integer. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{1-6p(1-p)}{np(1-p)}$$

`random_binomial (n,p)` [Function]
`random_binomial (n,p,m)`

Returns a *Binomial*(n, p) random variate, with $0 \leq p \leq 1$ and n a positive integer. Calling `random_binomial` with a third argument m, a random sample of size m will be simulated.

The implemented algorithm is based on the one described in Kachitvichyanukul, V. and Schmeiser, B.W. (1988) *Binomial Random Variate Generation*. Communications of the ACM, 31, Feb., 216.

To make use of this function, write first `load("distrib")`.

54.3.3 Poisson Random Variable

The *Poisson distribution* is a discrete probability distribution. It is the probability that a given number of events occur in a fixed interval when the events occur independently of the time of the last event, and the events occur with a known constant rate.

`pdf_poisson (x,m)` [Function]

Returns the value at x of the probability function of a *Poisson*(m) random variable, with $m > 0$. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; m) = \frac{m^x e^{-m}}{x!}$$

`cdf_poisson (x,m)` [Function]

Returns the value at x of the cumulative distribution function of a *Poisson*(m) random variable, with $m > 0$.

The cdf is

$$F(x;m) = Q(\lfloor x \rfloor + 1, m)$$

where $Q(x,m)$ is the [gamma_incomplete_regularized], page 335, function.

```
(%i1) load ("distrib")$
(%i2) cdf_poisson(3,5);
(%o2)          gamma_incomplete_regularized(4, 5)
(%i3) float(%);
(%o3)                  0.26502591529736186
```

`quantile_poisson (q,m)` [Function]

Returns the q-quantile of a $Poisson(m)$ random variable, with $m > 0$; in other words, this is the inverse of `cdf_poisson`. Argument q must be an element of $[0,1]$. To make use of this function, write first `load("distrib")`.

`mean_poisson (m)` [Function]

Returns the mean of a $Poisson(m)$ random variable, with $m > 0$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = m$$

`var_poisson (m)` [Function]

Returns the variance of a $Poisson(m)$ random variable, with $m > 0$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = m$$

`std_poisson (m)` [Function]

Returns the standard deviation of a $Poisson(m)$ random variable, with $m > 0$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$V[X] = \sqrt{m}$$

`skewness_poisson (m)` [Function]

Returns the skewness coefficient of a $Poisson(m)$ random variable, with $m > 0$. To make use of this function, write first `load("distrib")`.

The skewness is

$$SK[X] = \frac{1}{\sqrt{m}}$$

`kurtosis_poisson (m)` [Function]

Returns the kurtosis coefficient of a Poisson random variable $Poi(m)$, with $m > 0$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{1}{m}$$

`random_poisson (m)` [Function]
`random_poisson (m,n)`

Returns a $Poisson(m)$ random variate, with $m > 0$. Calling `random_poisson` with a second argument n, a random sample of size n will be simulated.

The implemented algorithm is the one described in Ahrens, J.H. and Dieter, U. (1982) *Computer Generation of Poisson Deviates From Modified Normal Distributions*. ACM Trans. Math. Software, 8, 2, June,163-179.

To make use of this function, write first `load("distrib")`.

54.3.4 Bernoulli Random Variable

The *Bernoulli distribution* is a discrete probability distribution which takes on two values, 0 and 1. The value 1 occurs with probability p, and 0 occurs with probability $1 - p$.

It is equivalent to the $Binomial(1, p)$ distribution (see Section 54.3.2 [Binomial Random Variable], page 834)

`pdf_bernoulli (x,p)` [Function]

Returns the value at x of the probability function of a $Bernoulli(p)$ random variable, with $0 \leq p \leq 1$.

The $Bernoulli(p)$ random variable is equivalent to the $Binomial(1, p)$.

The mean is

$$f(x; p) = p^x (1 - p)^{1-x}$$

```
(%i1) load ("distrib")$
(%i2) pdf_bernoulli(1,p);
(%o2) if equal(p, 0) then 0 elseif equal(p, 1) then 1 else p
```

`cdf_bernoulli (x,p)` [Function]

Returns the value at x of the cumulative distribution function of a $Bernoulli(p)$ random variable, with $0 \leq p \leq 1$. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; p) = I_{1-p}(1 - \lfloor x \rfloor, \lfloor x \rfloor + 1)$$

`quantile_bernoulli (q,p)` [Function]

Returns the q-quantile of a $Bernoulli(p)$ random variable, with $0 \leq p \leq 1$; in other words, this is the inverse of `cdf_bernoulli`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_bernoulli (p)` [Function]

Returns the mean of a $Bernoulli(p)$ random variable, with $0 \leq p \leq 1$.

The $Bernoulli(p)$ random variable is equivalent to the $Binomial(1, p)$.

The mean is

$$E[X] = p$$

```
(%i1) load ("distrib")$
(%i2) mean_bernoulli(p);
(%o2)                          p
```

`var_bernoulli (p)` [Function]

Returns the variance of a *Bernoulli*(p) random variable, with $0 \leq p \leq 1$.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*$(1, p)$.

The variance is

$$V[X] = p(1-p)$$

```
(%i1) load ("distrib")$
(%i2) var_bernoulli(p);
(%o2)                          (1 - p) p
```

`std_bernoulli (p)` [Function]

Returns the standard deviation of a *Bernoulli*(p) random variable, with $0 \leq p \leq 1$.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*$(1, p)$.

The standard deviation is

$$D[X] = \sqrt{p(1-p)}$$

```
(%i1) load ("distrib")$
(%i2) std_bernoulli(p);
(%o2)                       sqrt((1 - p) p)
```

`skewness_bernoulli (p)` [Function]

Returns the skewness coefficient of a *Bernoulli*(p) random variable, with $0 \leq p \leq 1$.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*$(1, p)$.

The skewness coefficient is

$$SK[X] = \frac{1-2p}{\sqrt{p(1-p)}}$$

```
(%i1) load ("distrib")$
(%i2) skewness_bernoulli(p);
                                 1 - 2 p
(%o2)                       ---------------
                            sqrt((1 - p) p)
```

`kurtosis_bernoulli (p)` [Function]

Returns the kurtosis coefficient of a *Bernoulli*(p) random variable, with $0 \leq p \leq 1$.

The *Bernoulli*(p) random variable is equivalent to the *Binomial*$(1, p)$.

The kurtosis coefficient is

$$KU[X] = \frac{1-6p(1-p)}{p(1-p)}$$

```
(%i1) load ("distrib")$
(%i2) kurtosis_bernoulli(p);
                            1 - 6 (1 - p) p
(%o2)                       ---------------
                               (1 - p) p
```

`random_bernoulli (p)` [Function]
`random_bernoulli (p,n)`

Returns a *Bernoulli*(p) random variate, with $0 \leq p \leq 1$. Calling `random_bernoulli` with a second argument n, a random sample of size n will be simulated.

This is a direct application of the `random` built-in Maxima function.

See also `random`. To make use of this function, write first `load("distrib")`.

54.3.5 Geometric Random Variable

The *Geometric distribution* is a discrete probability distribution. It is the distribution of the number Bernoulli trials that fail before the first success.

Consider flipping a biased coin where heads occurs with probability p. Then the probability of $k-1$ tails in a row followed by heads is given by the *Geometric*(p) distribution.

`pdf_geometric (x,p)` [Function]

Returns the value at x of the probability function of a *Geometric*(p) random variable, with $0 < p \leq 1$

The pdf is

$$f(x;p) = p(1-p)^x$$

This is interpreted as the probability of x failures before the first success.

`load("distrib")` loads this function.

`cdf_geometric (x,p)` [Function]

Returns the value at x of the cumulative distribution function of a *Geometric*(p) random variable, with $0 < p \leq 1$

The cdf is

$$1-(1-p)^{1+\lfloor x \rfloor}$$

`load("distrib")` loads this function.

`quantile_geometric (q,p)` [Function]

Returns the q-quantile of a *Geometric*(p) random variable, with $0 < p <= 1$; in other words, this is the inverse of `cdf_geometric`. Argument q must be an element of $[0,1]$.

The probability from which the quantile is derived is defined as $p(1-p)^x$. This is interpreted as the probability of x failures before the first success.

`load("distrib")` loads this function.

`mean_geometric (p)` [Function]

Returns the mean of a *Geometric*(p) random variable, with $0 < p \leq 1$.

The mean is

$$E[X] = \frac{1}{p} - 1$$

The probability from which the mean is derived is defined as $p(1-p)^x$. This is interpreted as the probability of x failures before the first success.

`load("distrib")` loads this function.

`var_geometric (p)` [Function]

Returns the variance of a *Geometric*(p) random variable, with $0 < p \leq 1$.

The variance is

$$V[X] = \frac{1-p}{p^2}$$

`load("distrib")` loads this function.

`std_geometric (p)` [Function]

Returns the standard deviation of a *Geometric*(p) random variable, with $0 < p \leq 1$.

$$D[X] = \frac{\sqrt{1-p}}{p}$$

`load("distrib")` loads this function.

`skewness_geometric (p)` [Function]

Returns the skewness coefficient of a *Geometric*(p) random variable, with $0 < p \leq 1$.

The skewness coefficient is

$$SK[X] = \frac{2-p}{\sqrt{1-p}}$$

`load("distrib")` loads this function.

`kurtosis_geometric (p)` [Function]

Returns the kurtosis coefficient of a geometric random variable *Geometric*(p) , with $0 < p \leq 1$.

The kurtosis coefficient is

$$KU[X] = \frac{p^2 - 6p + 6}{1-p}$$

`load("distrib")` loads this function.

`random_geometric (p)` [Function]
`random_geometric (p,n)`

`random_geometric(p)` returns one random sample from a *Geometric*(p) distribution, with $0 < p \leq 1$.

`random_geometric(p, n)` returns a list of n random samples.

The algorithm is based on simulation of Bernoulli trials.

The probability from which the random sample is derived is defined as $p(1-p)^x$. This is interpreted as the probability of x failures before the first success.

`load("distrib")` loads this function.

54.3.6 Discrete Uniform Random Variable

The *Discrete uniform distribution* is a discrete probability distribution where a finite number of values are equally likely to occur. The values are $1, 2, 3, ..., n$.

For example throwing a fair die of 6 sides numbered 1 through 6 follows a *DiscreteUniform*(1/6) distribution.

`pdf_discrete_uniform (x,n)` [Function]

Returns the value at x of the probability function of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x, n) = \frac{1}{n}$$

`cdf_discrete_uniform (x,n)` [Function]

Returns the value at x of the cumulative distribution function of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; n) = \frac{\lfloor x \rfloor}{n}$$

`quantile_discrete_uniform (q,n)` [Function]

Returns the q-quantile of a *DiscreteUniform*(n) random variable, with n a strictly positive integer; in other words, this is the inverse of `cdf_discrete_uniform`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_discrete_uniform (n)` [Function]

Returns the mean of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{n + 1}{2}$$

`var_discrete_uniform (n)` [Function]

Returns the variance of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{n^2 - 1}{12}$$

`std_discrete_uniform (n)` [Function]

Returns the standard deviation of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{\sqrt{n^2 - 1}}{2\sqrt{3}}$$

`skewness_discrete_uniform (n)` [Function]

Returns the skewness coefficient of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = 0$$

`kurtosis_discrete_uniform (n)` [Function]

Returns the kurtosis coefficient of a *DiscreteUniform*(n) random variable, with n a strictly positive integer. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = -\frac{6(n^2+1)}{5(n^2-1)}$$

`random_discrete_uniform (n)` [Function]
`random_discrete_uniform (n,m)`

Returns a *DiscreteUniform*(n) random variate, with n a strictly positive integer. Calling `random_discrete_uniform` with a second argument m, a random sample of size m will be simulated.

This is a direct application of the `random` built-in Maxima function.

See also `random`. To make use of this function, write first `load("distrib")`.

54.3.7 Hypergeometric Random Variable

The *hypergeometric distribution* is a discrete probability distribution.

Let n_1 be the number of objects of a class A and n_2 be the number of objects of class B. We take out n objects, *without* replacement. Then the hypergeometric distribution is the probability that exactly k objects are from class A. Of course $n \leq n_1 + n_2$.

`pdf_hypergeometric (x,n_1,n_2,n)` [Function]

Returns the value at x of the probability function of a *Hypergeometric*$(n1, n2, n)$ random variable, with n_1, n_2 and n non negative integers and $n \leq n_1 + n_2$. Being n_1 the number of objects of class A, n_2 the number of objects of class B, and n the size of the sample without replacement, this function returns the probability of event "exactly x objects are of class A".

To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n_1, n_2, n) = \frac{\binom{n_1}{x}\binom{n_2}{n-x}}{\binom{n_2+n_1}{n}}$$

`cdf_hypergeometric (x,n_1,n_2,n)` [Function]

Returns the value at x of the cumulative distribution function of a *Hypergeometric*$(n1, n2, n)$ random variable, with n_1, n_2 and n non negative integers and $n \leq n_1 + n_2$. See `pdf_hypergeometric` for a more complete description.

To make use of this function, write first `load("distrib")`.

The cdf is

$$F(x; n_1, n_2, n) = \binom{n_2 + n_1}{n}^{-1} \sum_{k=0}^{\lfloor x \rfloor} \binom{n_1}{k} \binom{n_2}{n-k}$$

`quantile_hypergeometric (q,n1,n2,n)` [Function]

Returns the q-quantile of a *Hypergeometric*$(n1, n2, n)$ random variable, with *n1*, *n2* and n non negative integers and $n \leq n1 + n2$; in other words, this is the inverse of `cdf_hypergeometric`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_hypergeometric (n_1,n_2,n)` [Function]

Returns the mean of a discrete uniform random variable *Hypergeometric*(n_1, n_2, n) , with n_1, n_2 and n non negative integers and $n \leq n_1 + n_2$. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{nn_1}{n_2 + n_1}$$

`var_hypergeometric (n1,n2,n)` [Function]

Returns the variance of a hypergeometric random variable *Hypergeometric*(n_1, n_2, n) , with n_1, n_2 and n non negative integers and $n \leq n_1 + n_2$. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{nn_1n_2(n_1 + n_2 - n)}{(n_1 + n_2 - 1)(n_1 + n_2)^2}$$

`std_hypergeometric (n_1,n_2,n)` [Function]

Returns the standard deviation of a *Hypergeometric*(n_1, n_2, n) random variable, with n_1, n_2 and n non negative integers and $n \leq n_1 + n_2$. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{1}{n_1 + n_2} \sqrt{\frac{nn_1n_2(n_1 + n_2 - n)}{n_1 + n_2 - 1}}$$

`skewness_hypergeometric (n_1,n_2,n)` [Function]

Returns the skewness coefficient of a *Hypergeometric*$(n1, n2, n)$ random variable, with n_1, n_2 and n non negative integers and $n \leq n1 + n2$. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{(n_2 - n_2)(n_1 + n_2 - 2n)}{n_1 + n_2 - 2} \sqrt{\frac{n_1 + n_2 - 1}{nn_1n_2(n_1 + n_2 - n)}}$$

`kurtosis_hypergeometric (n_1,n_2,n)` [Function]

Returns the kurtosis coefficient of a *Hypergeometric*(n_1, n_2, n) random variable, with n_1, n_2 and n non negative integers and $n \leq n1 + n2$. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \left[\frac{C(1)C(0)^2}{nn_1n_2C(3)C(2)C(n)} \times \left(\frac{3n_1n_2\left((n-2)C(0)^2+6nC(n)-n^2C(0)\right)}{C(0)^2} - 6nC(n) + C(0)C(-1)\right)\right] - 3$$

where $C(k) = n_1 + n_2 - k$.

`random_hypergeometric (n1,n2,n)` [Function]
`random_hypergeometric (n1,n2,n,m)`

Returns a $Hypergeometric(n1, n2, n)$ random variate, with *n1*, *n2* and *n* non negative integers and $n \leq n_1 + n_2$. Calling `random_hypergeometric` with a fourth argument m, a random sample of size m will be simulated.

Algorithm described in Kachitvichyanukul, V., Schmeiser, B.W. (1985) *Computer generation of hypergeometric random variates.* Journal of Statistical Computation and Simulation 22, 127-145.

To make use of this function, write first `load("distrib")`.

54.3.8 Negative Binomial Random Variable

The *negative binomial distribution* is a discrete probability distribution. Suppose we have a sequence of Bernoulli trials where each trial has two outcomes called "success" and "failure" where "success" occurs with probability p and "failure" with probability $1 - p$. We observe the sequence until a predefined number r of successes have occurred. Then the number of failures seen will have a $NegativeBinomial(r, p)$ distribution.

`pdf_negative_binomial (x,n,p)` [Function]

Returns the value at x of the probability function of a $NegativeBinomial(n, p)$ random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The pdf is

$$f(x; n, p) = \binom{x+n-1}{n-1}(1-p)^x p^n$$

`cdf_negative_binomial (x,n,p)` [Function]

Returns the value at x of the cumulative distribution function of a $NegativeBinomial(n, p)$ random variable, with $0 < p \leq 1$ and n a positive number.

The cdf is

$$F(x; n, p) = I_p(n, \lfloor x \rfloor + 1)$$

where $I_p(a, b)$ is the [beta_incomplete_regularized], page 341, function.

```
(%i1) load ("distrib")$
(%i2) cdf_negative_binomial(3,4,1/8);
                                3271
(%o2)                           ------
                                524288
```

`quantile_negative_binomial (q,n,p)` [Function]
Returns the q-quantile of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number; in other words, this is the inverse of `cdf_negative_binomial`. Argument q must be an element of $[0, 1]$. To make use of this function, write first `load("distrib")`.

`mean_negative_binomial (n,p)` [Function]
Returns the mean of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The mean is

$$E[X] = \frac{n(1-p)}{p}$$

`var_negative_binomial (n,p)` [Function]
Returns the variance of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The variance is

$$V[X] = \frac{n(1-p)}{p^2}$$

`std_negative_binomial (n,p)` [Function]
Returns the standard deviation of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The standard deviation is

$$D[X] = \frac{\sqrt{n(1-p)}}{p}$$

`skewness_negative_binomial (n,p)` [Function]
Returns the skewness coefficient of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The skewness coefficient is

$$SK[X] = \frac{2-p}{\sqrt{n(1-p)}}$$

`kurtosis_negative_binomial (n,p)` [Function]
Returns the kurtosis coefficient of a *NegativeBinomial*(n, p) random variable, with $0 < p \leq 1$ and n a positive number. To make use of this function, write first `load("distrib")`.

The kurtosis coefficient is

$$KU[X] = \frac{p^2 - 6p + 6}{n(1-p)}$$

`random_negative_binomial` (*n*,*p*) [Function]
`random_negative_binomial` (*n*,*p*,*m*)

Returns a *NegativeBinomial*(n, p) random variate, with $0 < p \leq 1$ and n a positive number. Calling `random_negative_binomial` with a third argument m, a random sample of size m will be simulated.

Algorithm described in Devroye, L. (1986) *Non-Uniform Random Variate Generation*. Springer Verlag, p. 480.

To make use of this function, write first `load("distrib")`.

55 Package draw

55.1 Introduction to draw

`draw` is a Maxima-Gnuplot and a Maxima-VTK interface.

There are three main functions to be used at Maxima level:

- `draw2d`, draws a single 2D scene.
- `draw3d`, draws a single 3D scene.
- `draw`, can be filled with multiple `gr2d` and `gr3d` commands that each creates a draw scene all sharing the same window.

Each scene can contain any number of objects and `key=value` pairs with options for the scene or the following objects.

A selection of useful objects a scene can be made up from are:

- `explicit` plots a function.
- `implicit` plots all points an equation is true at.
- `points` plots points that are connected by lines if the option `points_joined` was set to `true` in a previous line of the current scene.
- `parametric` allows to specify separate expressions that calculate the x, y (and in 3d plots also for the z) variable.

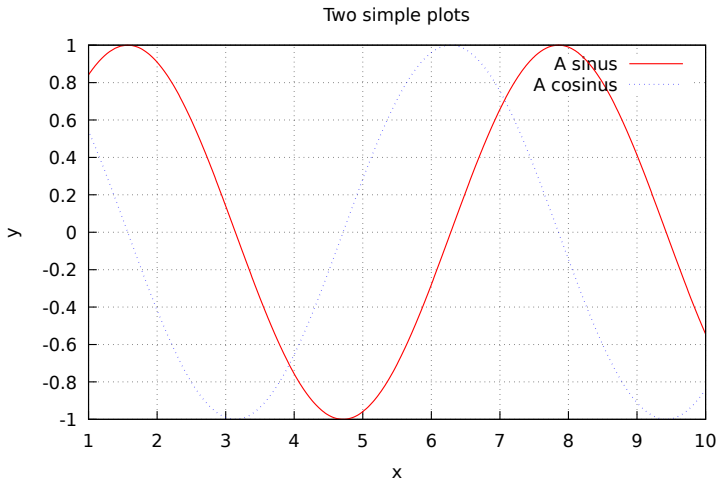
A short description of all draw commands and options including example plots (in the html and pdf version of this manual) can be found in the section See Section 55.2 [Functions and Variables for draw], page 848. An online version of the html manual can be found at https://maxima.sourceforge.io/docs/manual/maxima_singlepage.html#draw. More elaborated examples of this package can be found at the following locations:

http://riotorto.users.sourceforge.net/Maxima/gnuplot/
http://riotorto.users.sourceforge.net/Maxima/vtk/

Example:

```
(%i1) draw2d(
          title="Two simple plots",
          xlabel="x",ylabel="y",grid=true,

          color=red,key="A sinus",
          explicit(sin(x),x,1,10),
          color=blue,line_type=dots,key="A cosinus",
          explicit(cos(x),x,1,10)
)$
```



You need Gnuplot 4.2 or newer to run draw; If you are using wxMaxima as a front end `wxdraw`, `wxdraw2d` and `wxdraw3d` are drop-in replacements for draw that do the same as `draw`, `draw2d` and `draw3d` but embed the resulting plot in the worksheet.

If you want to use VTK with draw, you need VTK with the Python interface installed (the Chapter 57 [Package dynamics], page 969, uses VTK with the TCL interface!) and set the variable:

```
draw_renderer: 'vtk $
```

55.2 Functions and Variables for draw

55.2.1 Scenes

`gr2d` (*argument_1*, ...) [Scene constructor]

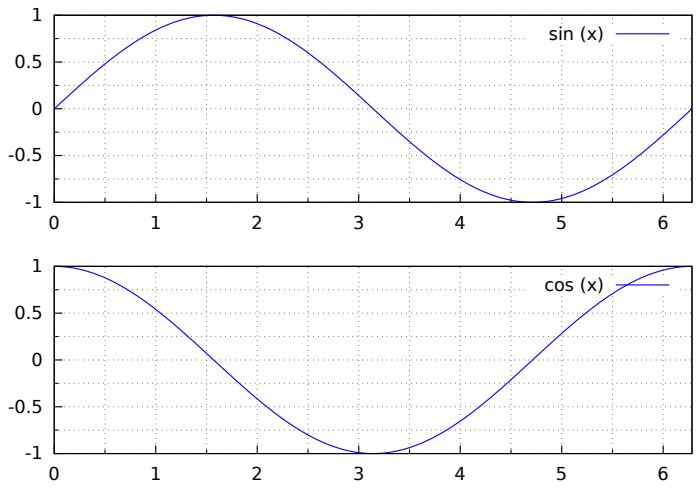
Function `gr2d` builds an object describing a 2D scene. Arguments are *graphic options*, *graphic objects*, or lists containing both graphic options and objects. This scene is interpreted sequentially: *graphic options* affect those *graphic objects* placed on its right. Some *graphic options* affect the global appearance of the scene.

This is the list of *graphic objects* available for scenes in two dimensions: `bars`, `ellipse`, `explicit`, `image`, `implicit`, `label`, `parametric`, `points`, `polar`, `polygon`, `quadrilateral`, `rectangle`, `triangle`, `vector` and `geomap` (this one defined in package `worldmap`).

See also `draw` and `draw2d`.

```
(%i1) draw(
    gr2d(
        key="sin (x)",grid=[2,2],
        explicit(
            sin(x),
            x,0,2*%pi
        )
    ),
    gr2d(
        key="cos (x)",grid=[2,2],
```

```
        explicit(
            cos(x),
            x,0,2*%pi
        )
    )
);
(%o1)           [gr2d(explicit), gr2d(explicit)]
```



`gr3d` (*argument_1*, ...) [Scene constructor]

Function `gr3d` builds an object describing a 3d scene. Arguments are *graphic options*, *graphic objects*, or lists containing both graphic options and objects. This scene is interpreted sequentially: *graphic options* affect those *graphic objects* placed on its right. Some *graphic options* affect the global appearance of the scene.

This is the list of *graphic objects* available for scenes in three dimensions: `cylindrical`, `elevation_grid`, `explicit`, `implicit`, `label`, `mesh`, `parametric`, `parametric_surface`, `points`, `quadrilateral`, `spherical`, `triangle`, `tube`, `vector`, and `geomap` (this one defined in package `worldmap`).

See also `draw` and `draw3d`.

55.2.2 Functions

`draw` (`<arg_1>, ...`) [Function]

Plots a series of scenes; its arguments are `gr2d` and/or `gr3d` objects, together with some options, or lists of scenes and options. By default, the scenes are put together in one column.

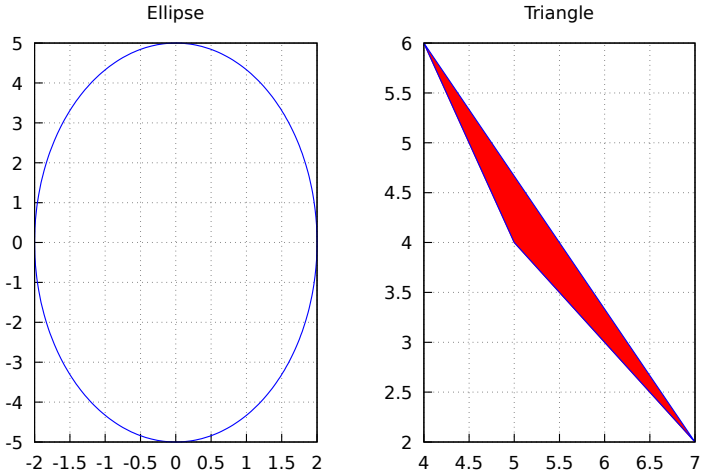
Besides scenes the function `draw` accepts the following global options: `terminal`, `columns`, `dimensions`, `file_name` and `delay`.

Functions `draw2d` and `draw3d` short cuts that can be used when only one scene is required, in two or three dimensions, respectively.

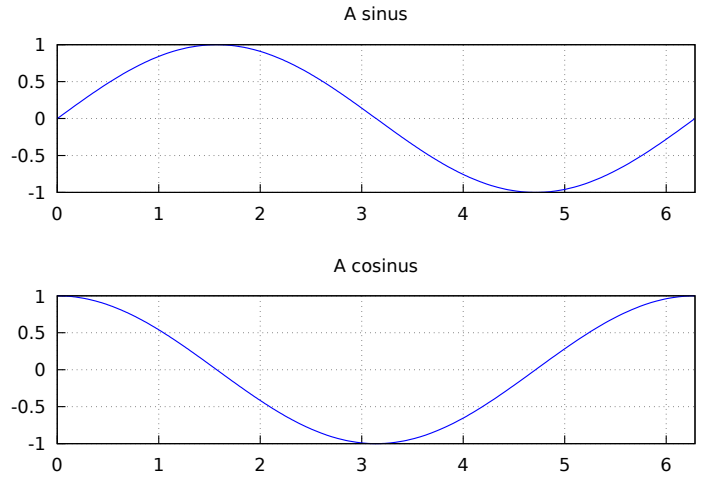
See also `gr2d` and `gr3d`.

Examples:

```
(%i1) scene1: gr2d(title="Ellipse",
                   nticks=300,
                   parametric(2*cos(t),5*sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="Triangle",
                   polygon([4,5,7],[6,4,2]))$
(%i3) draw(scene1, scene2, columns = 2)$
```




```
(%i1) scene1: gr2d(title="A sinus",
        grid=true,
        explicit(sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="A cosinus",
        grid=true,
        explicit(cos(t),t,0,2*%pi))$
(%i3) draw(scene1, scene2)$
```

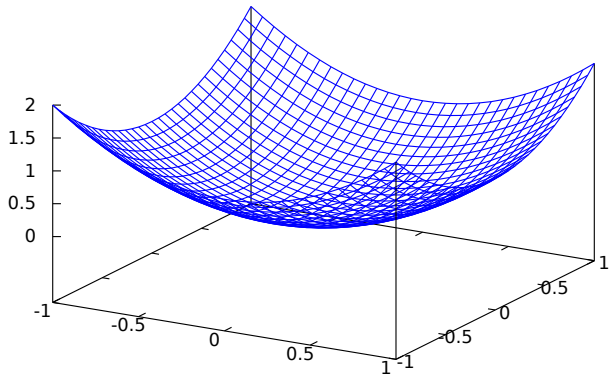


The following two draw sentences are equivalent:

```
(%i1) draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)));
(%o1)                          [gr3d(explicit)]
(%i2) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1));
(%o2)                          [gr3d(explicit)]
```

Creating an animated gif file:

```
(%i1) draw(
        delay     = 100,
        file_name = "zzz",
        terminal  = 'animated_gif,
        gr2d(explicit(x^2,x,-1,1)),
        gr2d(explicit(x^3,x,-1,1)),
        gr2d(explicit(x^4,x,-1,1)));
End of animation sequence
(%o1)          [gr2d(explicit), gr2d(explicit), gr2d(explicit)]
```



See also gr2d, gr3d, draw2d and draw3d.

draw2d (*argument_1*, ...) [Function]

This function is a shortcut for draw(gr2d(*options*, ..., *graphic_object*, ...)).

It can be used to plot a unique scene in 2d, as can be seen in most examples below.

See also draw and gr2d.

draw3d (*argument_1*, ...) [Function]

This function is a shortcut for draw(gr3d(*options*, ..., *graphic_object*, ...)).

It can be used to plot a unique scene in 3d, as can be seen in many examples below.

See also draw and gr3d.

draw_file (*graphic option*, ..., *graphic object*, ...) [Function]

Saves the current plot into a file. Accepted graphics options are: terminal, dimensions and file_name.

Example:

```
(%i1) /* screen plot */
```

```
      draw(gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1)))$
(%i2) /* same plot in eps format */
      draw_file(terminal  = eps,
                dimensions = [5,5]) $
```

`multiplot_mode` (*term*) [Function]

This function enables Maxima to work in one-window multiplot mode with terminal *term*; accepted arguments for this function are `screen`, `wxt`, `aquaterm`, `windows` and `none`.

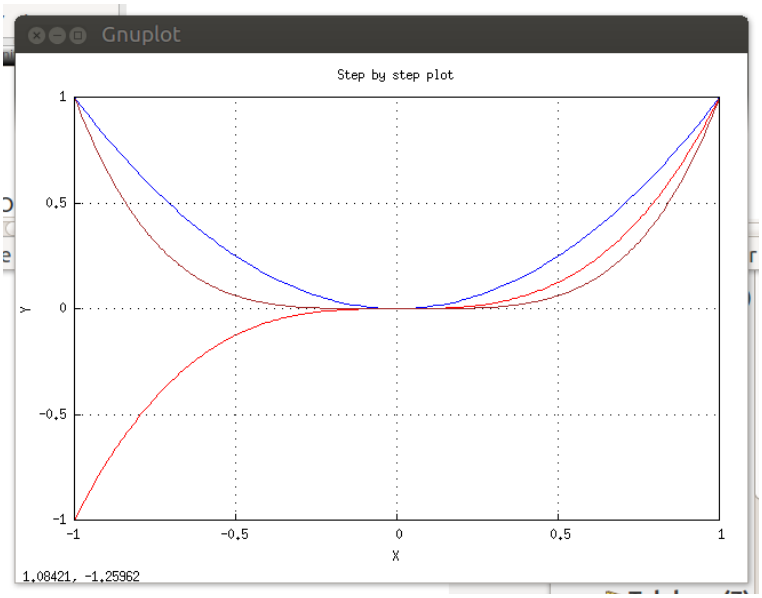
When multiplot mode is enabled, each call to `draw` sends a new plot to the same window, without erasing the previous ones. To disable the multiplot mode, write `multiplot_mode(none)`.

When multiplot mode is enabled, global option `terminal` is blocked and you have to disable this working mode before changing to another terminal.

On Windows this feature requires Gnuplot 5.0 or newer. Note, that just plotting multiple expressions into the same plot doesn't require multiplot: It can be done by just issuing multiple `explicit` or similar commands in a row.

Example:

```
(%i1) set_draw_defaults(
         xrange = [-1,1],
         yrange = [-1,1],
         grid   = true,
         title  = "Step by step plot" )$
(%i2) multiplot_mode(screen)$
(%i3) draw2d(color=blue,  explicit(x^2,x,-1,1))$
(%i4) draw2d(color=red,   explicit(x^3,x,-1,1))$
(%i5) draw2d(color=brown, explicit(x^4,x,-1,1))$
(%i6) multiplot_mode(none)$
```



`set_draw_defaults` (*graphic option*, ..., *graphic object*, ...) [Function]

Sets user graphics options. This function is useful for plotting a sequence of graphics with common graphics options. Calling this function without arguments removes user defaults.

Example:

```
(%i1) set_draw_defaults(
         xrange = [-10,10],
         yrange = [-2, 2],
         color  = blue,
         grid   = true)$
(%i2) /* plot with user defaults */
      draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$
(%i3) set_draw_defaults()$
(%i4) /* plot with standard defaults */
      draw2d(explicit(((1+x)**2/(1+x*x))-1,x,-10,10))$
```

55.2.3 Plot options for draw programs

`adapt_depth` [Graphic option]

Default value: 10

`adapt_depth` is the maximum number of splittings used by the adaptive plotting routine.

This option is relevant only for 2d `explicit` functions.

See also `nticks`

`allocation` [Graphic option]

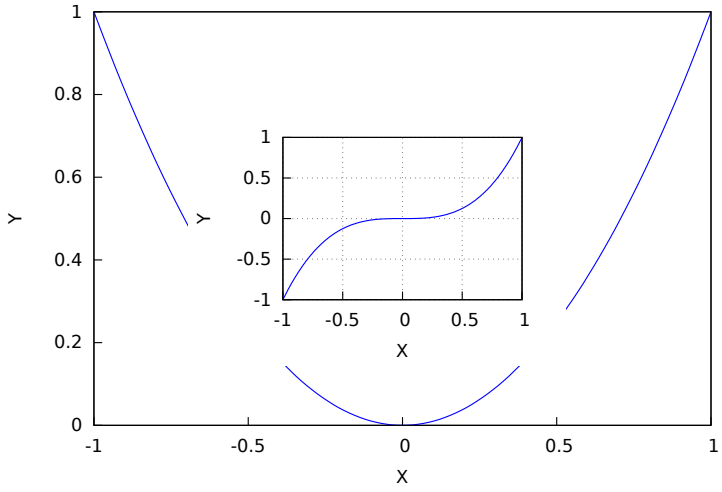
Default value: `false`

With option `allocation` it is possible to place a scene in the output window at will; this is of interest in multiplots. When `false`, the scene is placed automatically, depending on the value assigned to option `columns`. In any other case, `allocation` must be set to a list of two pairs of numbers; the first corresponds to the position of the lower left corner of the scene, and the second pair gives the width and height of the plot. All quantities must be given in relative coordinates, between 0 and 1.

Examples:

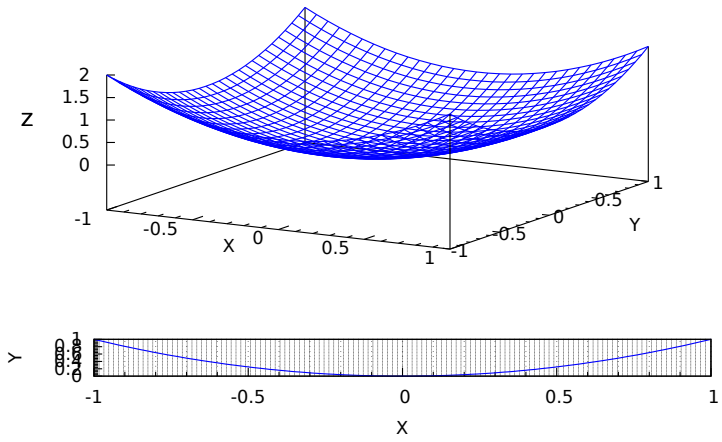
In site graphics.

```
(%i1) draw(
        gr2d(
          explicit(x^2,x,-1,1)),
        gr2d(
          allocation = [[1/4, 1/4],[1/2, 1/2]],
          explicit(x^3,x,-1,1),
          grid = true) ) $
```



Multiplot with selected dimensions.

```
(%i1) draw(
        terminal = wxt,
        gr2d(
          grid=[5,5],
          allocation = [[0, 0],[1, 1/4]],
          explicit(x^2,x,-1,1)),
        gr3d(
          allocation = [[0, 1/4],[1, 3/4]],
          explicit(x^2+y^2,x,-1,1,y,-1,1) ))$
```



See also option `columns`.

`axis_3d` [Graphic option]

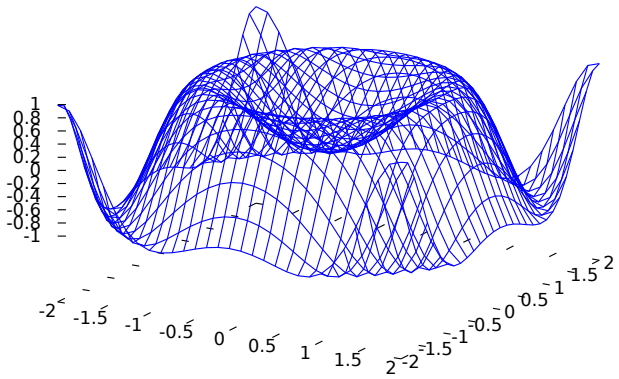
Default value: `true`

If `axis_3d` is `true`, the x, y and z axis are shown in 3d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(axis_3d = false,
             explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$
```



See also `axis_bottom`, `axis_left`, `axis_top`, and `axis_right` for axis in 2d.

`axis_bottom` [Graphic option]

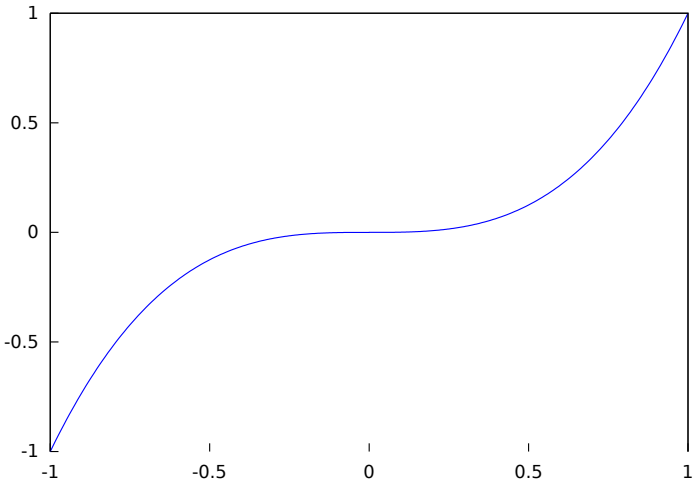
Default value: `true`

If `axis_bottom` is `true`, the bottom axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(axis_bottom = false,
             explicit(x^3,x,-1,1))$
```



See also `axis_left`, `axis_top`, `axis_right` and `axis_3d`.

`axis_left` [Graphic option]

Default value: `true`

If `axis_left` is `true`, the left axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(axis_left = false,
             explicit(x^3,x,-1,1))$
```

See also `axis_bottom`, `axis_top`, `axis_right` and `axis_3d`.

`axis_right` [Graphic option]

Default value: `true`

If `axis_right` is `true`, the right axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(axis_right = false,
             explicit(x^3,x,-1,1))$
```

See also `axis_bottom`, `axis_left`, `axis_top` and `axis_3d`.

`axis_top` [Graphic option]

Default value: `true`

If `axis_top` is `true`, the top axis is shown in 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(axis_top = false,
             explicit(x^3,x,-1,1))$
```

See also `axis_bottom`, `axis_left`, `axis_right`, and `axis_3d`.

`background_color` [Graphic option]

Default value: `white`

Sets the background color for terminals. Default background color is white.

Since this is a global graphics option, its position in the scene description does not matter.

This option does not work with terminals `epslatex` and `epslatex_standalone`.

See also `color`

`border` [Graphic option]

Default value: `true`

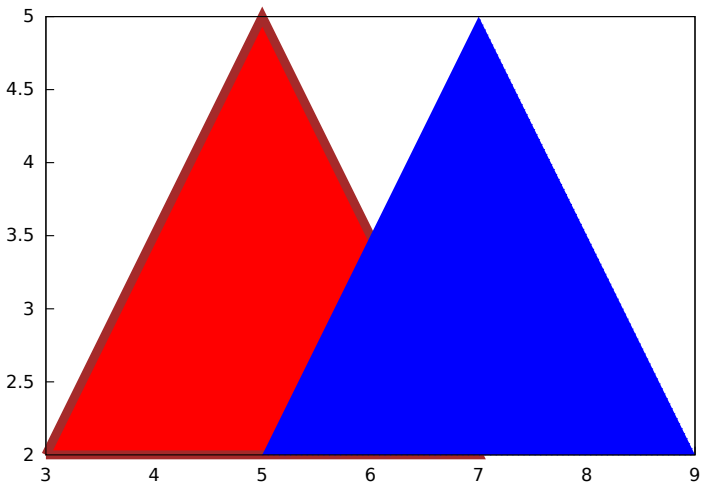
If `border` is `true`, borders of polygons are painted according to `line_type` and `line_width`.

This option affects the following graphic objects:

- `gr2d`: `polygon`, `rectangle` and `ellipse`.

Example:

```
(%i1) draw2d(color       = brown,
             line_width  = 8,
             polygon([[3,2],[7,2],[5,5]]),
             border      = false,
             fill_color  = blue,
             polygon([[5,2],[9,2],[7,5]]) )$
```



`capping` [Graphic option]

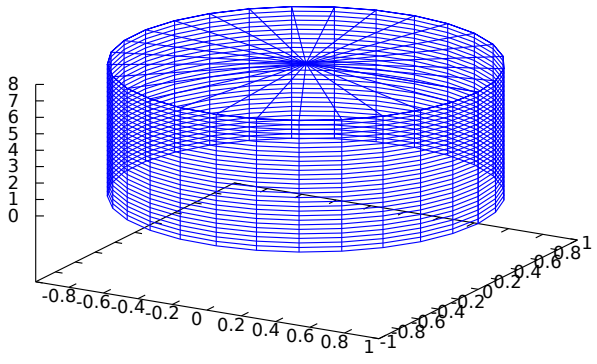
Default value: `[false, false]`

A list with two possible elements, `true` and `false`, indicating whether the extremes of a graphic object `tube` remain closed or open. By default, both extremes are left open.

Setting `capping = false` is equivalent to `capping = [false, false]`, and `capping = true` is equivalent to `capping = [true, true]`.

Example:

```
(%i1) draw3d(
        capping = [false, true],
        tube(0, 0, a, 1,
             a, 0, 8) )$
```



`cbrange` [Graphic option]

Default value: `auto`

If `cbrange` is `auto`, the range for the values which are colored when `enhanced3d` is not `false` is computed automatically. Values outside of the color range use color of the nearest extreme.

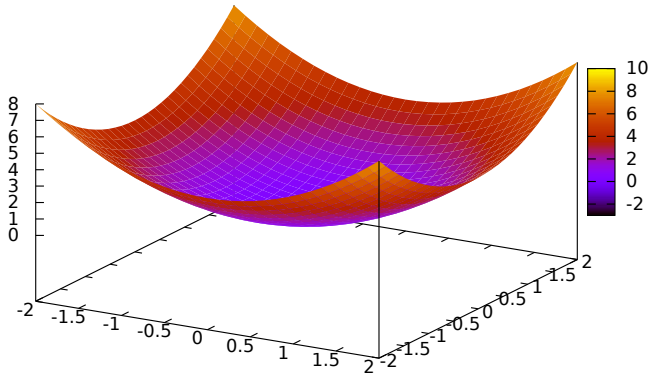
When `enhanced3d` or `colorbox` is `false`, option `cbrange` has no effect.

If the user wants a specific interval for the colored values, it must be given as a Maxima list, as in `cbrange=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d (
        enhanced3d     = true,
        color          = green,
        cbrange = [-3,10],
        explicit(x^2+y^2, x,-2,2,y,-2,2)) $
```



See also enhanced3d, colorbox and cbtics.

cbtics [Graphic option]

Default value: auto

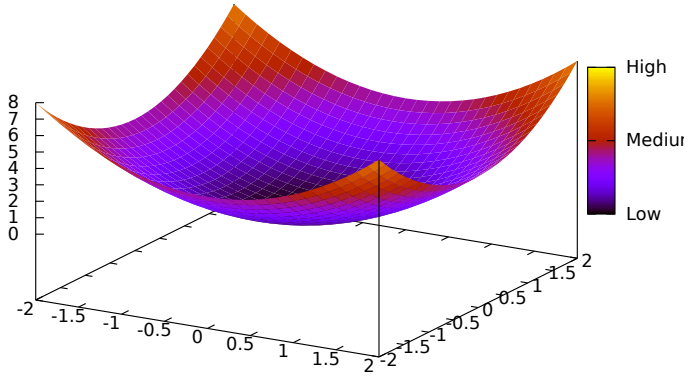
This graphic option controls the way tic marks are drawn on the colorbox when option enhanced3d is not false.

When enhanced3d or colorbox is false, option cbtics has no effect.

See xtics for a complete description.

Example :

```
(%i1) draw3d (
        enhanced3d = true,
        color      = green,
        cbtics  = {["High",10],["Medium",05],["Low",0]},
        cbrange = [0, 10],
        explicit(x^2+y^2, x,-2,2,y,-2,2)) $
```



See also enhanced3d, colorbox and cbrange.

color [Graphic option]

Default value: blue

color specifies the color for plotting lines, points, borders of polygons and labels.

Colors can be given as names or in hexadecimal *rgb* code. If a gnuplot version >= 5.0 is used and the terminal that is in use supports this *rgba* colors with transparency information are also supported.

Available color names are:

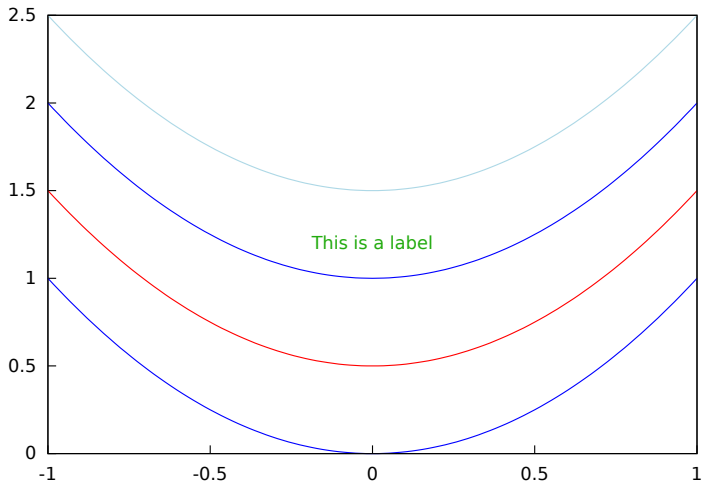
white	black	gray0	grey0
gray10	grey10	gray20	grey20
gray30	grey30	gray40	grey40
gray50	grey50	gray60	grey60
gray70	grey70	gray80	grey80
gray90	grey90	gray100	grey100

gray	grey	light_gray	light_grey
dark_gray	dark_grey	red	light_red
dark_red	yellow	light_yellow	dark_yellow
green	light_green	dark_green	spring_green
forest_green	sea_green	blue	light_blue
dark_blue	midnight_blue	navy	medium_blue
royalblue	skyblue	cyan	light_cyan
dark_cyan	magenta	light_magenta	dark_magenta
turquoise	light_turquoise	dark_turquoise	pink
light_pink	dark_pink	coral	light_coral
orange_red	salmon	light_salmon	dark_salmon
aquamarine	khaki	dark_khaki	goldenrod
light_goldenrod	dark_goldenrod	gold	beige
brown	orange	dark_orange	violet
dark_violet	plum	purple	

Cromatic components in hexadecimal code are introduced in the form `"#rrggbb"`.

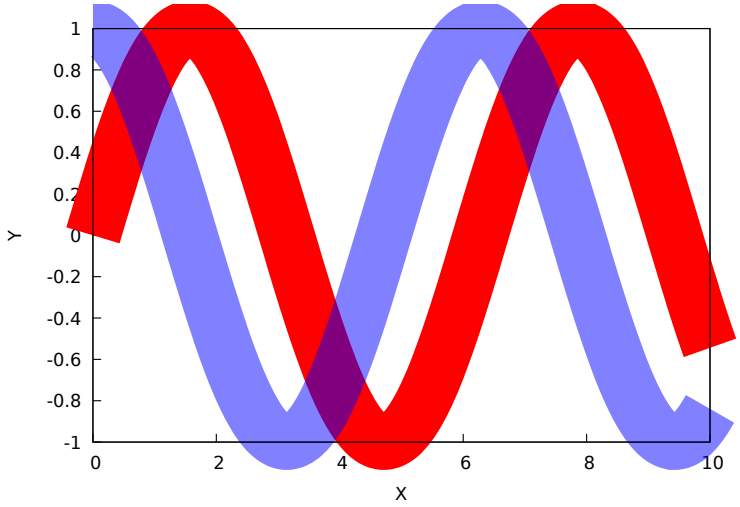
Example:

```
(%i1) draw2d(explicit(x^2,x,-1,1), /* default is black */
             color = red,
             explicit(0.5 + x^2,x,-1,1),
             color = blue,
             explicit(1 + x^2,x,-1,1),
             color = light_blue,
             explicit(1.5 + x^2,x,-1,1),
             color = "#23ab0f",
             label(["This is a label",0,1.2])  )$
```

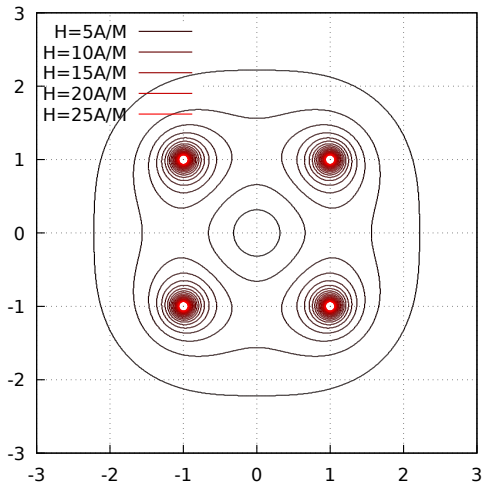



```
(%i1) draw2d(
             line_width=50,
             color="#FF0000",
             explicit(sin(x),x,0,10),
```

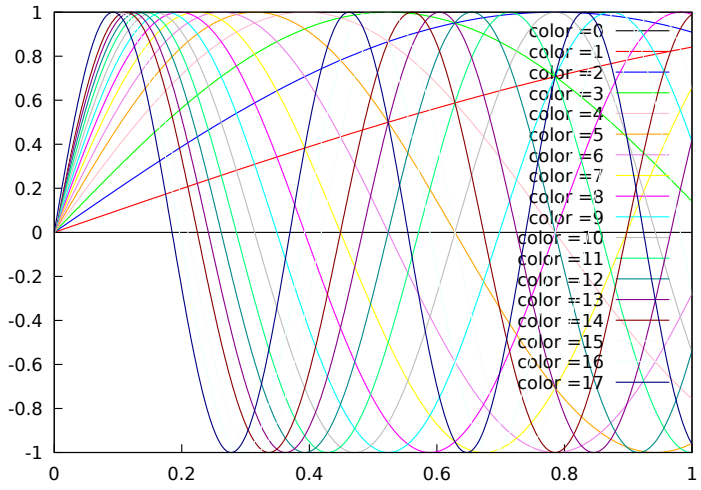
```
        color="#0000FF80",
        explicit(cos(x),x,0,10)
);
```




```
(%i1) H(p,p_0) := %i/(2*%pi*(p-p_0))$
(%i2) draw2d(
       proportional_axes=xy,
       ip_grid=[150,150],
       grid=true,
       makelist(
         [
           color=printf(false,"#~2,'0x~2,'0x~2,'0x",i*10,0,0),
           key_pos=top_left,
           key = if mod(i,5)=0 then sconcat("H=",i,"A/M") else "",
           implicit(
             cabs(H(x+%i*y,-1-%i)+H(x+%i*y,1+%i)-H(x+%i*y,1-%i)
                  -H(x+%i*y,-1+%i))=i/10,
             x,-3,3,
             y,-3,3
           )
         ],
         i,1,25
       )
      )$
```




```
(%i1) draw2d(
        "figures/draw_color4",
        makelist(
            [
                color=i,
                key=sconcat("color =",i),
                explicit(sin(i*x),x,0,1)
            ],
            i,0,17
        )
    )$
```



See also `fill_color`.

`colorbox` [Graphic option]

Default value: `true`

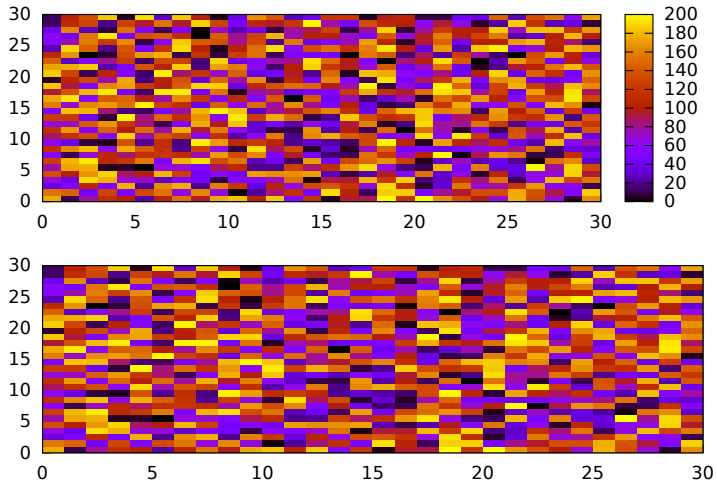
If `colorbox` is `true`, a color scale without label is drawn together with `image` 2D objects, or coloured 3d objects. If `colorbox` is `false`, no color scale is shown. If `colorbox` is a string, a color scale with label is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

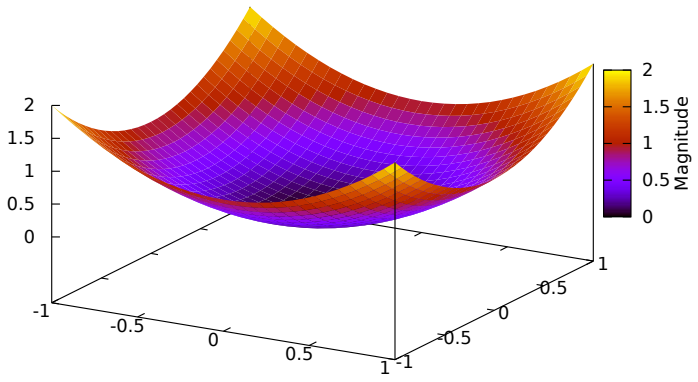
Color scale and images.

```
(%i1) im: apply('matrix,
                makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) draw(
          gr2d(image(im,0,0,30,30)),
          gr2d(colorbox = false, image(im,0,0,30,30))
      )$
```



Color scale and 3D coloured object.

```
(%i1) draw3d(
        colorbox   = "Magnitude",
        enhanced3d = true,
        explicit(x^2+y^2,x,-1,1,y,-1,1))$
```



See also `palette_draw`.

`columns` [Graphic option]

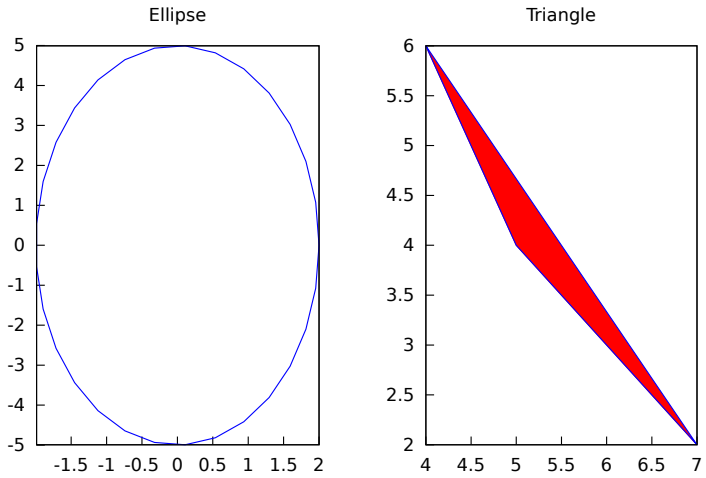
Default value: 1

`columns` is the number of columns in multiple plots.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.

Example:

```
(%i1) scene1: gr2d(title="Ellipse",
                   nticks=30,
                   parametric(2*cos(t),5*sin(t),t,0,2*%pi))$
(%i2) scene2: gr2d(title="Triangle",
                   polygon([4,5,7],[6,4,2]))$
(%i3) draw(scene1, scene2, columns = 2)$
```



`contour` [Graphic option]

Default value: `none`

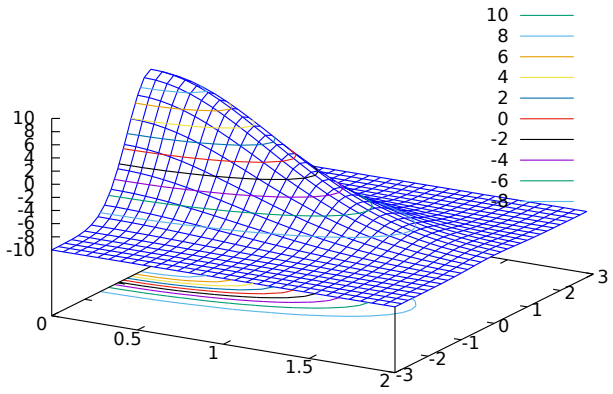
Option `contour` enables the user to select where to plot contour lines. Possible values are:

- `none`: no contour lines are plotted.
- `base`: contour lines are projected on the xy plane.
- `surface`: contour lines are plotted on the surface.
- `both`: two contour lines are plotted: on the xy plane and on the surface.
- `map`: contour lines are projected on the xy plane, and the view point is set just in the vertical.

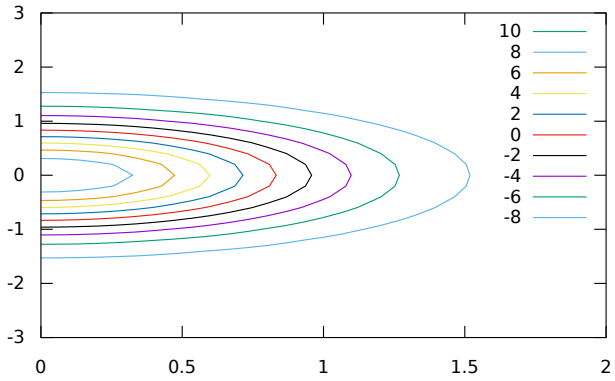
Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
             contour_levels = 15,
             contour        = both,
             surface_hide   = true) $
```




```
(%i1) draw3d(explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
             contour_levels = 15,
             contour        = map
      ) $
```



`contour_levels` [Graphic option]

Default value: 5

This graphic option controls the way contours are drawn. `contour_levels` can be set to a positive integer number, a list of three numbers or an arbitrary set of numbers:

- When option `contour_levels` is bounded to positive integer *n*, *n* contour lines will be drawn at equal intervals. By default, five equally spaced contours are plotted.
- When option `contour_levels` is bounded to a list of length three of the form `[lowest,s,highest]`, contour lines are plotted from `lowest` to `highest` in steps of `s`.
- When option `contour_levels` is bounded to a set of numbers of the form `{n1, n2, ...}`, contour lines are plotted at values `n1`, `n2`, ...

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

Ten equally spaced contour lines. The actual number of levels can be adjusted to give simple labels.

```
(%i1) draw3d(color = green,
             explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
             contour_levels = 10,
             contour        = both,
             surface_hide   = true) $
```

From -8 to 8 in steps of 4.

```
(%i1) draw3d(color = green,
             explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
             contour_levels = [-8,4,8],
             contour        = both,
             surface_hide   = true) $
```

Isolines at levels -7, -6, 0.8 and 5.

```
(%i1) draw3d(color = green,
```

```
                    explicit(20*exp(-x^2-y^2)-10,x,0,2,y,-3,3),
                    contour_levels = {-7, -6, 0.8, 5},
                    contour        = both,
                    surface_hide   = true) $
```

See also contour.

data_file_name [Graphic option]

Default value: "data.gnuplot"

This is the name of the file with the numeric data needed by Gnuplot to build the requested plot.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

See example in gnuplot_file_name.

delay [Graphic option]

Default value: 5

This is the delay in 1/100 seconds of frames in animated gif files.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function draw.

Example:

```
(%i1) draw(
        delay     = 100,
        file_name = "zzz",
        terminal  = 'animated_gif,
        gr2d(explicit(x^2,x,-1,1)),
        gr2d(explicit(x^3,x,-1,1)),
        gr2d(explicit(x^4,x,-1,1)));
End of animation sequence
(%o2)          [gr2d(explicit), gr2d(explicit), gr2d(explicit)]
```

Option delay is only active in animated gif's; it is ignored in any other case.

See also terminal, and dimensions.

dimensions [Graphic option]

Default value: [600,500]

Dimensions of the output terminal. Its value is a list formed by the width and the height. The meaning of the two numbers depends on the terminal you are working with.

With terminals gif, animated_gif, png, jpg, svg, screen, wxt, qt, x11, windows and aquaterm, the integers represent the number of points in each direction. If they are not integers, they are rounded.

With terminals eps, epslatex, epslatex_standalone, eps_color, multipage_eps, multipage_eps_color, cairolatex_pdf, cairolatex_pdf_standalone, pdf, multipage_pdf, pdfcairo, multipage_pdfcairo, tikz, and tikz_standalone, both numbers represent hundredths of cm, which means that, by default, pictures in these formats are 6 cm in width and 5 cm in height.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.

Examples:

Option `dimensions` applied to file output and to wxt canvas.

```
(%i1) draw2d(
        dimensions = [300,300],
        terminal   = 'png,
        explicit(x^4,x,-1,1)) $
(%i2) draw2d(
        dimensions = [300,300],
        terminal   = 'wxt,
        explicit(x^4,x,-1,1)) $
```

Option `dimensions` applied to eps output. We want an eps file with A4 portrait dimensions.

```
(%i1) A4portrait: 100*[21, 29.7]$
(%i2) draw3d(
        dimensions = A4portrait,
        terminal   = 'eps,
        explicit(x^2-y^2,x,-2,2,y,-2,2)) $
```

`draw_realpart` [Graphic option]

Default value: `true`

When `true`, functions to be drawn are considered as complex functions whose real part value should be plotted; when `false`, nothing will be plotted when the function does not give a real value.

This option affects objects `explicit` and `parametric` in 2D and 3D, and `parametric_surface`.

Example:

```
(%i1) draw2d(
        draw_realpart = false,
        explicit(sqrt(x^2  - 4*x) - x, x, -1, 5),
        color         = red,
        draw_realpart = true,
        parametric(x,sqrt(x^2  - 4*x) - x + 1, x, -1, 5) );
```

`enhanced3d` [Graphic option]

Default value: `none`

If `enhanced3d` is `none`, surfaces are not colored in 3D plots. In order to get a colored surface, a list must be assigned to option `enhanced3d`, where the first element is an expression and the rest are the names of the variables or parameters used in that expression. A list such `[f(x,y,z), x, y, z]` means that point `[x,y,z]` of the surface is assigned number `f(x,y,z)`, which will be colored according to the actual `palette`. For those 3D graphic objects defined in terms of parameters, it is possible to define the color number in terms of the parameters, as in `[f(u), u]`, as in objects `parametric` and `tube`, or `[f(u,v), u, v]`, as in object `parametric_surface`. While

all 3D objects admit the model based on absolute coordinates, `[f(x,y,z), x, y, z]`, only two of them, namely `explicit` and `elevation_grid`, accept also models defined on the `[x,y]` coordinates, `[f(x,y), x, y]`. 3D graphic object `implicit` accepts only the `[f(x,y,z), x, y, z]` model. Object `points` accepts also the `[f(x,y,z), x, y, z]` model, but when points have a chronological nature, model `[f(k), k]` is also valid, being `k` an ordering parameter.

When `enhanced3d` is assigned something different to `none`, options `color` and `surface_hide` are ignored.

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

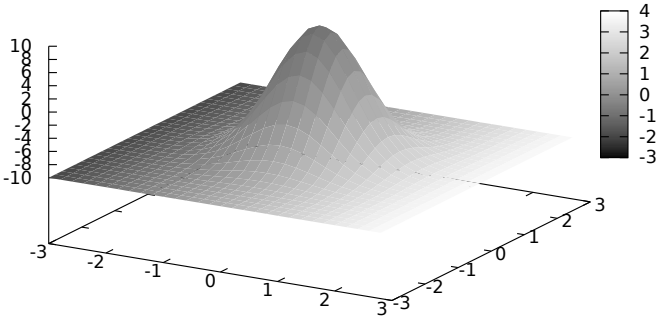
In order to maintain back compatibility, `enhanced3d = false` is equivalent to `enhanced3d = none`, and `enhanced3d = true` is equivalent to `enhanced3d = [z, x, y, z]`. If an expression is given to `enhanced3d`, its variables must be the same used in the surface definition. This is not necessary when using lists.

See option `palette` to learn how palettes are specified.

Examples:

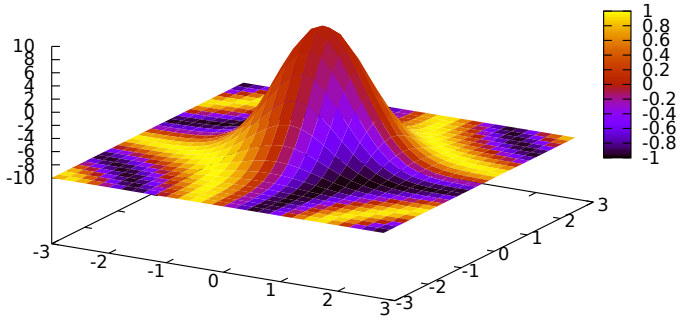
`explicit` object with coloring defined by the `[f(x,y,z), x, y, z]` model.

```
(%i1) draw3d(
          enhanced3d = [x-z/10,x,y,z],
          palette    = gray,
          explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
```



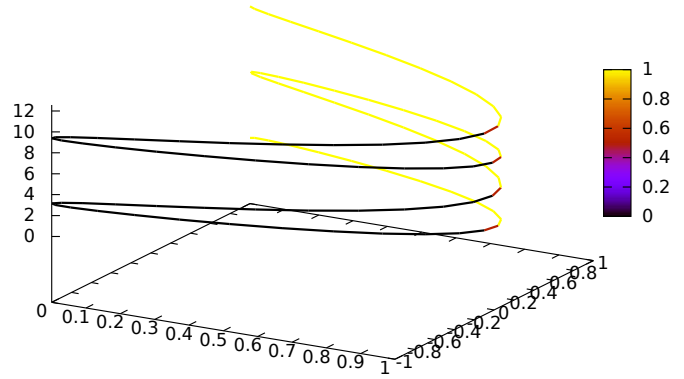
`explicit` object with coloring defined by the `[f(x,y), x, y]` model. The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects; in this case, `r` corresponds to `x`, and `s` to `y`.

```
(%i1) draw3d(
          enhanced3d = [sin(r*s),r,s],
          explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
```



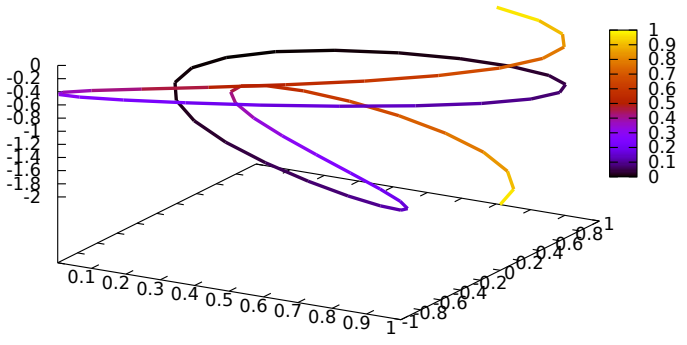
`parametric` object with coloring defined by the `[f(x,y,z), x, y, z]` model.

```
(%i1) draw3d(
        nticks = 100,
        line_width = 2,
        enhanced3d = [if y>= 0 then 1 else 0, x, y, z],
        parametric(sin(u)^2,cos(u),u,u,0,4*%pi)) $
```



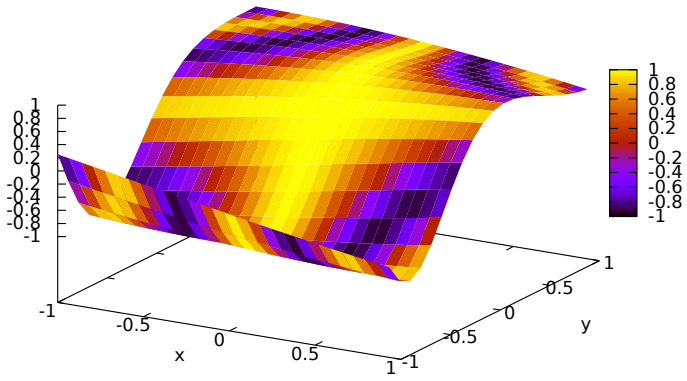
`parametric` object with coloring defined by the `[f(u), u]` model. In this case, `(u-1)^2` is a shortcut for `[(u-1)^2,u]`.

```
(%i1) draw3d(
        nticks = 60,
        line_width = 3,
        enhanced3d = (u-1)^2,
        parametric(cos(5*u)^2,sin(7*u),u-2,u,0,2))$
```



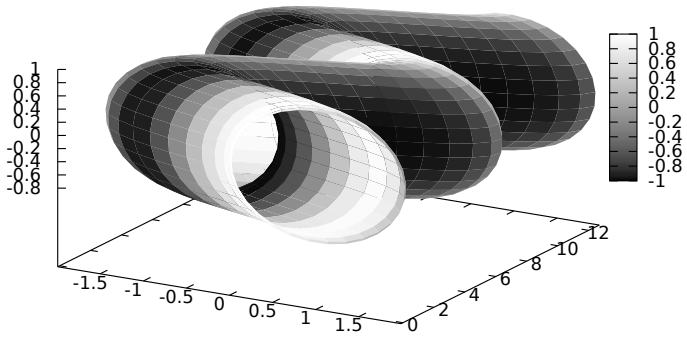
`elevation_grid` object with coloring defined by the `[f(x,y), x, y]` model.

```
(%i1) m: apply(
           matrix,
           makelist(makelist(cos(i^2/80-k/30),k,1,30),i,1,20)) $
(%i2) draw3d(
         enhanced3d = [cos(x*y*10),x,y],
         elevation_grid(m,-1,-1,2,2),
         xlabel = "x",
         ylabel = "y");
```



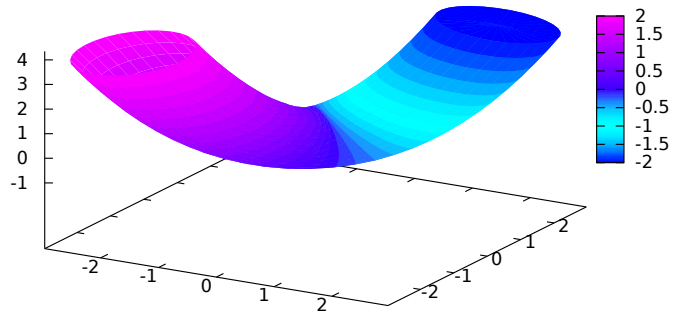
`tube` object with coloring defined by the `[f(x,y,z), x, y, z]` model.

```
(%i1) draw3d(
         enhanced3d = [cos(x-y),x,y,z],
         palette = gray,
         xu_grid = 50,
         tube(cos(a), a, 0, 1, a, 0, 4*%pi) )$
```



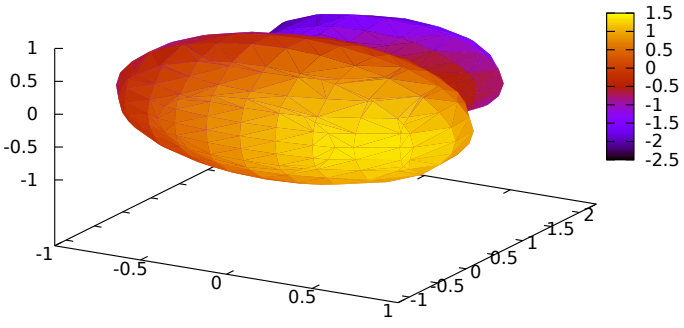
`tube` object with coloring defined by the `[f(u), u]` model. Here, `enhanced3d = -a` would be the shortcut for `enhanced3d = [-foo,foo]`.

```
(%i1) draw3d(
         capping = [true, false],
         palette = [26,15,-2],
         enhanced3d = [-foo, foo],
         tube(a, a, a^2, 1, a, -2, 2) )$
```

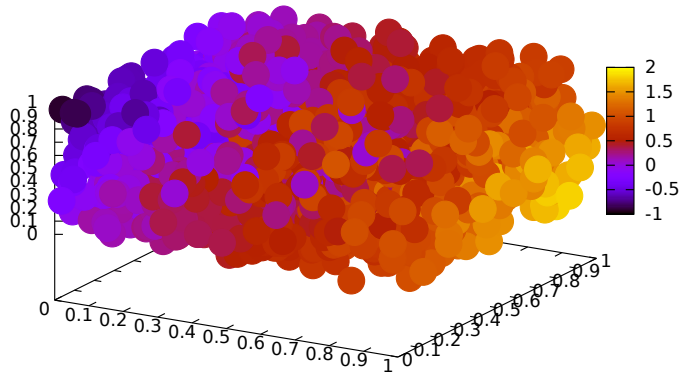


`implicit` and `points` objects with coloring defined by the `[f(x,y,z), x, y, z]` model.

```
(%i1) draw3d(
         enhanced3d = [x-y,x,y,z],
         implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015,
                  x,-1,1,y,-1.2,2.3,z,-1,1)) $
(%i2) m: makelist([random(1.0),random(1.0),random(1.0)],k,1,2000)$
```

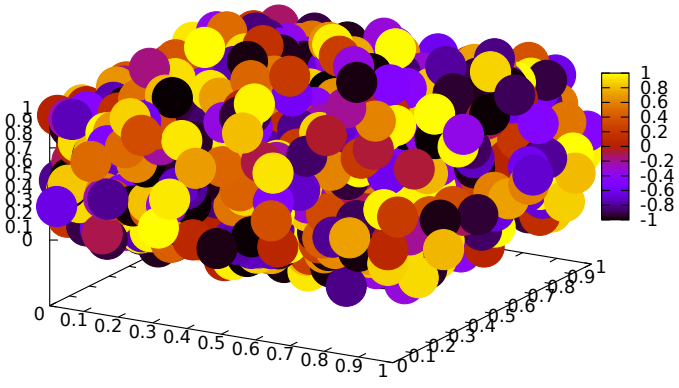



```
(%i3) draw3d(
        point_type = filled_circle,
        point_size = 2,
        enhanced3d = [u+v-w,u,v,w],
        points(m) ) $
```



When points have a chronological nature, model `[f(k), k]` is also valid, being `k` an ordering parameter.

```
(%i1) m:makelist([random(1.0), random(1.0), random(1.0)],k,1,5)$
(%i2) draw3d(
        enhanced3d = [sin(j), j],
        point_size = 3,
        point_type = filled_circle,
        points_joined = true,
        points(m)) $
```



`error_type` [Graphic option]

Default value: `y`

Depending on its value, which can be `x`, `y`, or `xy`, graphic object `errors` will draw points with horizontal, vertical, or both, error bars. When `error_type=boxes`, boxes will be drawn instead of crosses.

See also `errors`.

`file_name` [Graphic option]

Default value: `"maxima_out"`

This is the name of the file where terminals `png`, `jpg`, `gif`, `eps`, `eps_color`, `pdf`, `pdfcairo` and `svg` will save the graphic.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.

Example:

```
(%i1) draw2d(file_name = "myfile",
             explicit(x^2,x,-1,1),
             terminal  = 'png)$
```

See also `terminal`, `dimensions_draw`.

`fill_color` [Graphic option]

Default value: `"red"`

`fill_color` specifies the color for filling polygons and 2d `explicit` functions.

See `color` to learn how colors are specified.

`fill_density` [Graphic option]

Default value: 0

`fill_density` is a number between 0 and 1 that specifies the intensity of the `fill_color` in `bars` objects.

See `bars` for examples.

`filled_func` [Graphic option]

Default value: `false`

Option `filled_func` controls how regions limited by functions should be filled. When `filled_func` is `true`, the region bounded by the function defined with object `explicit` and the bottom of the graphic window is filled with `fill_color`. When `filled_func` contains a function expression, then the region bounded by this function and the function defined with object `explicit` will be filled. By default, explicit functions are not filled.

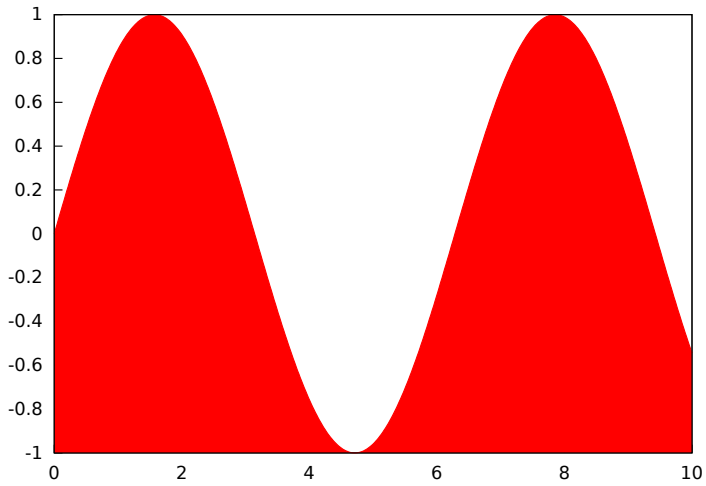
A useful special case is `filled_func=0`, which generates the region bond by the horizontal axis and the explicit function.

This option affects only the 2d graphic object `explicit`.

Example:

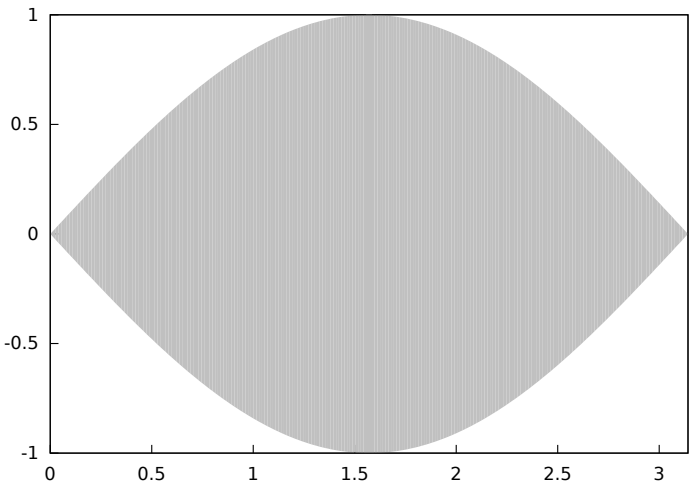
Region bounded by an `explicit` object and the bottom of the graphic window.

```
(%i1) draw2d(fill_color  = red,
             filled_func = true,
             explicit(sin(x),x,0,10) )$
```



Region bounded by an `explicit` object and the function defined by option `filled_func`. Note that the variable in `filled_func` must be the same as that used in `explicit`.

```
(%i1) draw2d(fill_color  = grey,
             filled_func = sin(x),
             explicit(-sin(x),x,0,%pi));
```



See also `fill_color` and `explicit`.

`font` [Graphic option]

Default value: `""` (empty string)

This option can be used to set the font face to be used by the terminal. Only one font face and size can be used throughout the plot.

Since this is a global graphics option, its position in the scene description does not matter.

See also `font_size`.

Gnuplot doesn't handle fonts by itself, it leaves this task to the support libraries of the different terminals, each one with its own philosophy about it. A brief summary follows:

- *x11*: Uses the normal x11 font server mechanism.

  Example:

```
(%i1) draw2d(font      = "Arial",
             font_size = 20,
             label(["Arial font, size 20",1,1]))$
```

- *windows*: The windows terminal doesn't support changing of fonts from inside the plot. Once the plot has been generated, the font can be changed right-clicking on the menu of the graph window.
- *png, jpeg, gif*: The *libgd* library uses the font path stored in the environment variable `GDFONTPATH`; in this case, it is only necessary to set option `font` to the font's name. It is also possible to give the complete path to the font file.

  Examples:

  Option `font` can be given the complete path to the font file:

```
(%i1) path: "/usr/share/fonts/truetype/freefont/" $
(%i2) file: "FreeSerifBoldItalic.ttf" $
(%i3) draw2d(
        font      = concat(path, file),
        font_size = 20,
        color     = red,
```

```
        label(["FreeSerifBoldItalic font, size 20",1,1]),
        terminal  = png)$
```

If environment variable `GDFONTPATH` is set to the path where font files are allocated, it is possible to set graphic option `font` to the name of the font.

```
(%i1) draw2d(
        font      = "FreeSerifBoldItalic",
        font_size = 20,
        color     = red,
        label(["FreeSerifBoldItalic font, size 20",1,1]),
        terminal  = png)$
```

- *Postscript*: Standard Postscript fonts are: `"Times-Roman"`, `"Times-Italic"`, `"Times-Bold"`, `"Times-BoldItalic"`, `"Helvetica"`, `"Helvetica-Oblique"`, `"Helvetica-Bold"`, `"Helvetic-BoldOblique"`, `"Courier"`, `"Courier-Oblique"`, `"Courier-Bold"`, and `"Courier-BoldOblique"`.

  Example:

```
(%i1) draw2d(
        font      = "Courier-Oblique",
        font_size = 15,
        label(["Courier-Oblique font, size 15",1,1]),
        terminal = eps)$
```

- *pdf*: Uses same fonts as *Postscript*.
- *pdfcairo*: Uses same fonts as *wxt*.
- *wxt*: The *pango* library finds fonts via the `fontconfig` utility.
- *aqua*: Default is `"Times-Roman"`.

The gnuplot documentation is an important source of information about terminals and fonts.

`font_size` [Graphic option]

Default value: 10

This option can be used to set the font size to be used by the terminal. Only one font face and size can be used throughout the plot. `font_size` is active only when option `font` is not equal to the empty string.

Since this is a global graphics option, its position in the scene description does not matter.

See also `font`.

`gnuplot_file_name` [Graphic option]

Default value: `"maxout_xxx.gnuplot"` with `"xxx"` being a number that is unique to each concurrently-running maxima process.

This is the name of the file with the necessary commands to be processed by Gnuplot.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.

Example:

```
(%i1) draw2d(
       file_name = "my_file",
       gnuplot_file_name = "my_commands_for_gnuplot",
       data_file_name    = "my_data_for_gnuplot",
       terminal          = png,
       explicit(x^2,x,-1,1)) $
```

See also data_file_name.

grid [Graphic option]

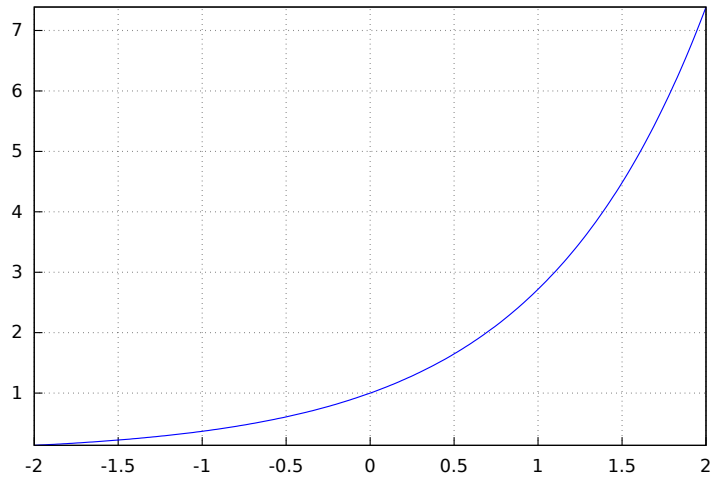
Default value: false

If grid is not false, a grid will be drawn on the *xy* plane. If grid is assigned true, one grid line per tick of each axis is drawn. If grid is assigned a list nx,ny with [nx,ny] > [0,0] instead nx lines per tick of the x axis and ny lines per tick of the y axis are drawn.

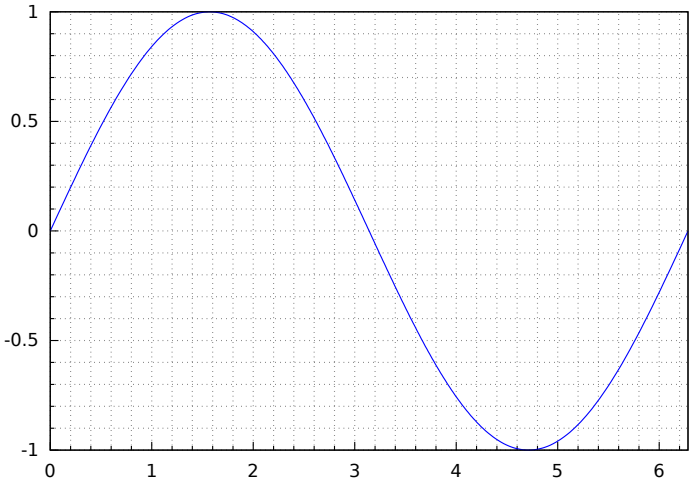
Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(grid = true,
             explicit(exp(u),u,-2,2))$
```




```
(%i1) draw2d(grid = [2,2],
             explicit(sin(x),x,0,2*%pi))$
```



`head_angle` [Graphic option]

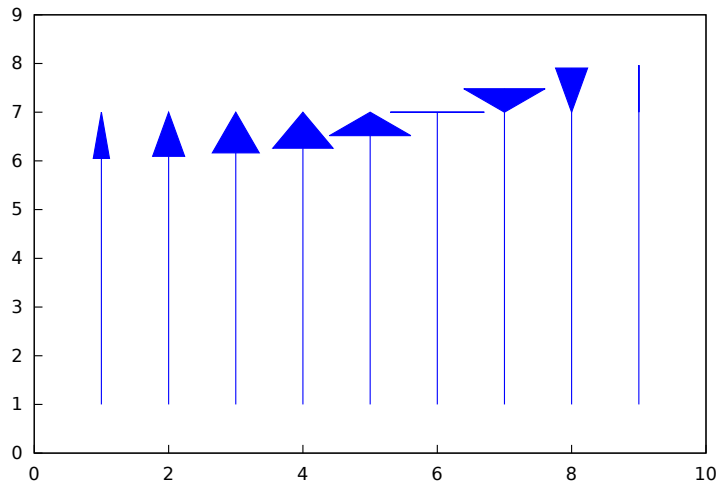
Default value: 45

`head_angle` indicates the angle, in degrees, between the arrow heads and the segment.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange      = [0,10],
             yrange      = [0,9],
             head_length = 0.7,
             head_angle  = 10,
             vector([1,1],[0,6]),
             head_angle  = 20,
             vector([2,1],[0,6]),
             head_angle  = 30,
             vector([3,1],[0,6]),
             head_angle  = 40,
             vector([4,1],[0,6]),
             head_angle  = 60,
             vector([5,1],[0,6]),
             head_angle  = 90,
             vector([6,1],[0,6]),
             head_angle  = 120,
             vector([7,1],[0,6]),
             head_angle  = 160,
             vector([8,1],[0,6]),
             head_angle  = 180,
             vector([9,1],[0,6]) )$
```



See also `head_both`, `head_length`, and `head_type`.

`head_both` [Graphic option]

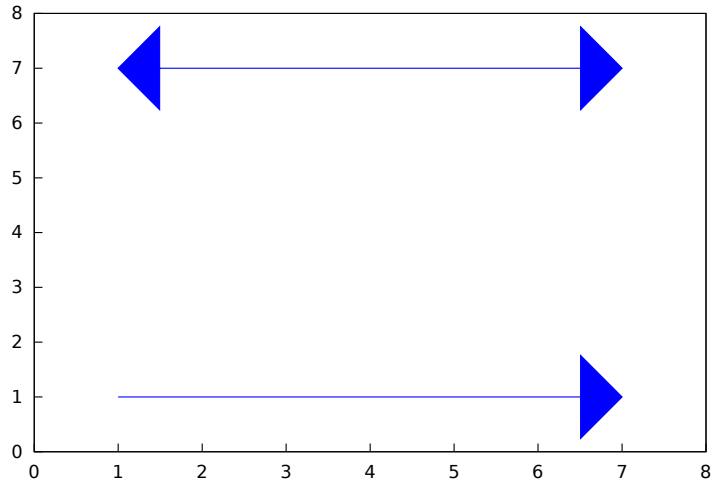
Default value: `false`

If `head_both` is `true`, vectors are plotted with two arrow heads. If `false`, only one arrow is plotted.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange      = [0,8],
             yrange      = [0,8],
             head_length = 0.7,
             vector([1,1],[6,0]),
             head_both   = true,
             vector([1,7],[6,0]) )$
```



See also `head_length`, `head_angle`, and `head_type`.

`head_length` [Graphic option]

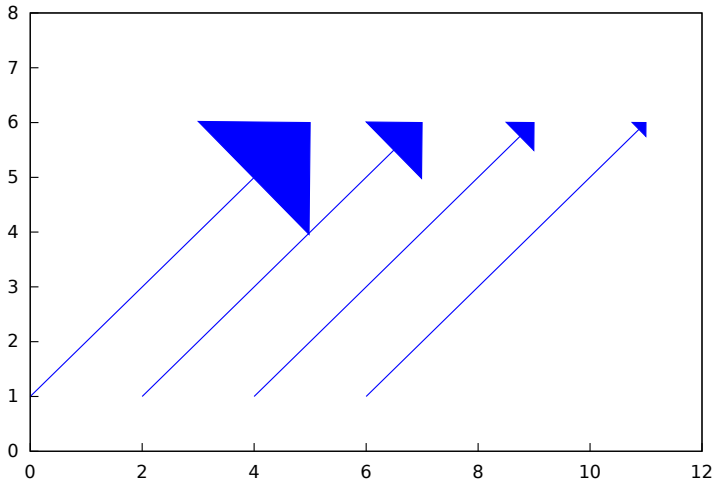
Default value: 2

`head_length` indicates, in *x*-axis units, the length of arrow heads.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange      = [0,12],
             yrange      = [0,8],
             vector([0,1],[5,5]),
             head_length = 1,
             vector([2,1],[5,5]),
             head_length = 0.5,
             vector([4,1],[5,5]),
             head_length = 0.25,
             vector([6,1],[5,5]))$
```



See also `head_both`, `head_angle`, and `head_type`.

`head_type` [Graphic option]

Default value: `filled`

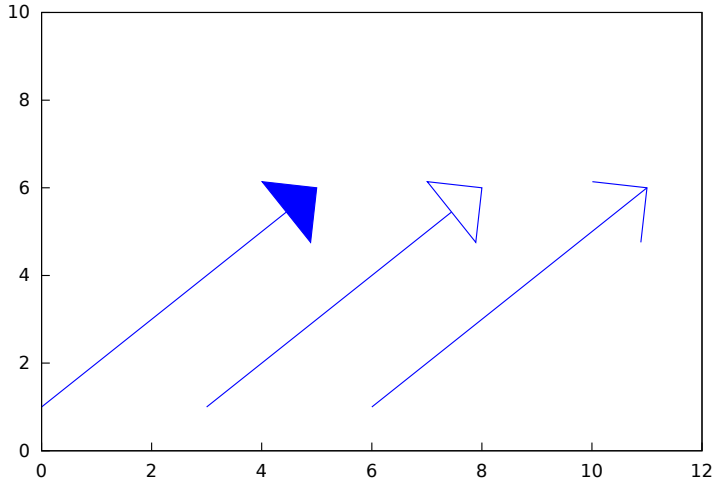
`head_type` is used to specify how arrow heads are plotted. Possible values are: `filled` (closed and filled arrow heads), `empty` (closed but not filled arrow heads), and `nofilled` (open arrow heads).

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange      = [0,12],
             yrange      = [0,10],
             head_length = 1,
             vector([0,1],[5,5]), /* default type */
             head_type = 'empty,
             vector([3,1],[5,5]),
```

```
head_type = 'nofilled,
vector([6,1],[5,5]))$
```



See also `head_both`, `head_angle`, and `head_length`.

`interpolate_color` [Graphic option]

Default value: `false`

This option is relevant only when `enhanced3d` is not `false`.

When `interpolate_color` is `false`, surfaces are colored with homogeneous quadrangles. When `true`, color transitions are smoothed by interpolation.

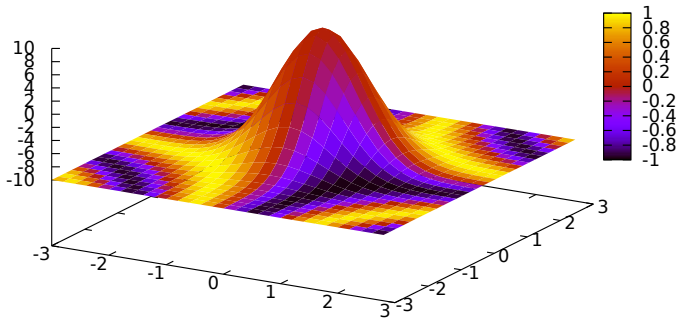
`interpolate_color` also accepts a list of two numbers, `[m,n]`. For positive m and n, each quadrangle or triangle is interpolated m times and n times in the respective direction. For negative m and n, the interpolation frequency is chosen so that there will be at least $|m|$ and $|n|$ points drawn; you can consider this as a special gridding function. Zeros, i.e. `interpolate_color=[0,0]`, will automatically choose an optimal number of interpolated surface points.

Also, `interpolate_color=true` is equivalent to `interpolate_color=[0,0]`.

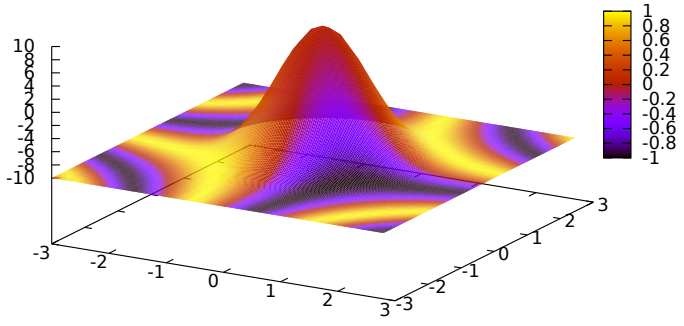
Examples:

Color interpolation with explicit functions.

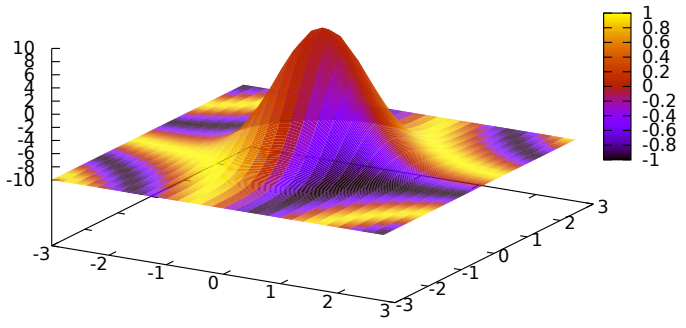
```
(%i1) draw3d(
        enhanced3d   = sin(x*y),
        explicit(20*exp(-x^2-y^2)-10, x ,-3, 3, y, -3, 3)) $
```




```
(%i2) draw3d(
        interpolate_color = true,
        enhanced3d   = sin(x*y),
        explicit(20*exp(-x^2-y^2)-10, x ,-3, 3, y, -3, 3)) $
```



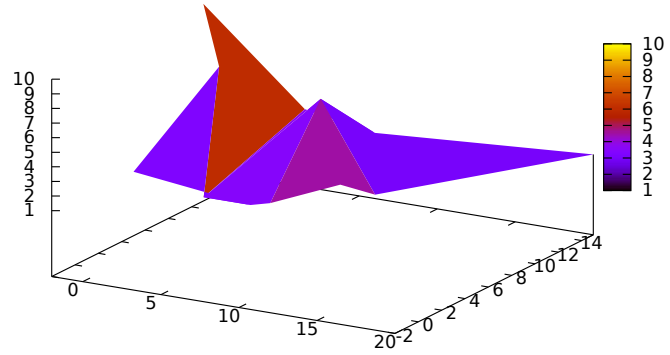

```
(%i3) draw3d(
        interpolate_color = [-10,0],
        enhanced3d   = sin(x*y),
        explicit(20*exp(-x^2-y^2)-10, x ,-3, 3, y, -3, 3)) $
```



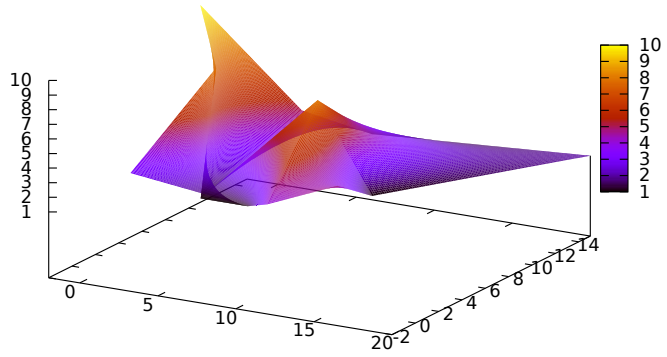
Color interpolation with the `mesh` graphic object.

Interpolating colors in parametric surfaces can give unexpected results.

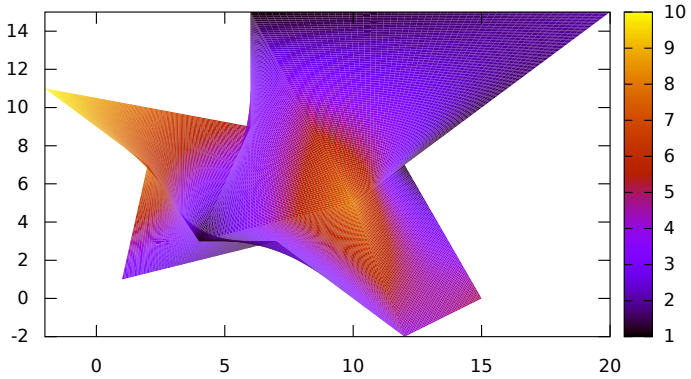
```
(%i1) draw3d(
        enhanced3d = true,
        mesh([[1,1,3],   [7,3,1],[12,-2,4],[15,0,5]],
             [[2,7,8],   [4,3,1],[10,5,8], [12,7,1]],
             [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $
```




```
(%i2) draw3d(
        enhanced3d        = true,
        interpolate_color = true,
        mesh([[1,1,3],   [7,3,1],[12,-2,4],[15,0,5]],
             [[2,7,8],   [4,3,1],[10,5,8], [12,7,1]],
             [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $
```



```
(%i3) draw3d(
        enhanced3d        = true,
        interpolate_color = true,
        view=map,
        mesh([[1,1,3],   [7,3,1],[12,-2,4],[15,0,5]],
             [[2,7,8],   [4,3,1],[10,5,8], [12,7,1]],
             [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $
```



See also `enhanced3d`.

`ip_grid` [Graphic option]

Default value: `[50, 50]`

`ip_grid` sets the grid for the first sampling in implicit plots.

This option is relevant only for `implicit` objects.

`ip_grid_in` [Graphic option]

Default value: `[5, 5]`

`ip_grid_in` sets the grid for the second sampling in implicit plots.

This option is relevant only for `implicit` objects.

`key` [Graphic option]

Default value: `""` (empty string)

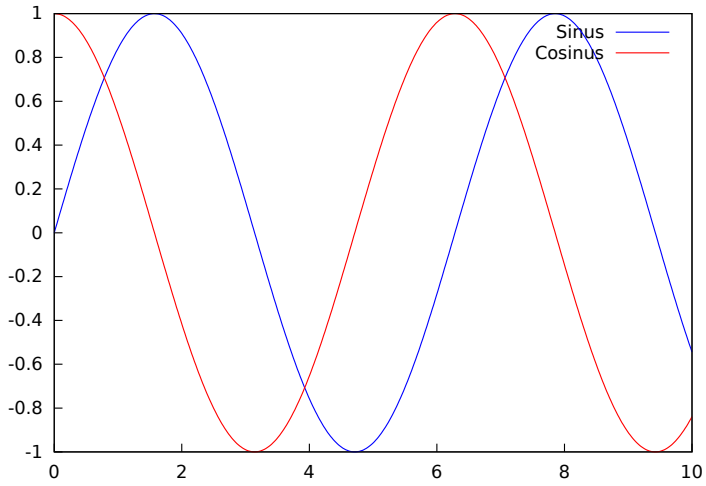
`key` is the name of a function in the legend. If `key` is an empty string, no key is assigned to the function.

This option affects the following graphic objects:

- `gr2d`: `points`, `polygon`, `rectangle`, `ellipse`, `vector`, `explicit`, `implicit`, `parametric` and `polar`.
- `gr3d`: `points`, `explicit`, `parametric` and `parametric_surface`.

Example:

```
(%i1) draw2d(key   = "Sinus",
             explicit(sin(x),x,0,10),
             key   = "Cosinus",
             color = red,
             explicit(cos(x),x,0,10) )$
```



`key_pos` [Graphic option]

Default value: `""` (empty string)

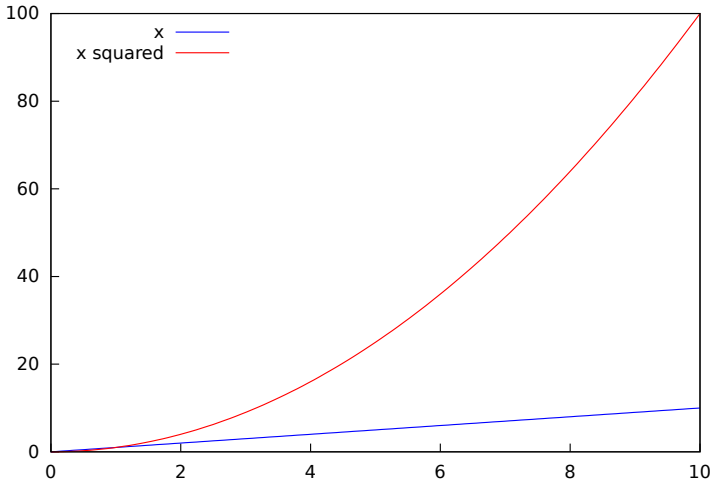
`key_pos` defines at which position the legend will be drawn. If `key` is an empty string, `"top_right"` is used. Available position specifiers are: `top_left`, `top_center`, `top_right`, `center_left`, `center`, `center_right`, `bottom_left`, `bottom_center`, and `bottom_right`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
        key_pos = top_left,
        key   = "x",
```

```
        explicit(x,  x,0,10),
        color= red,
        key   = "x squared",
        explicit(x^2,x,0,10))$
(%i3) draw3d(
        key_pos = center,
        key   = "x",
        explicit(x+y,x,0,10,y,0,10),
        color= red,
        key   = "x squared",
        explicit(x^2+y^2,x,0,10,y,0,10))$
```



`label_alignment` [Graphic option]

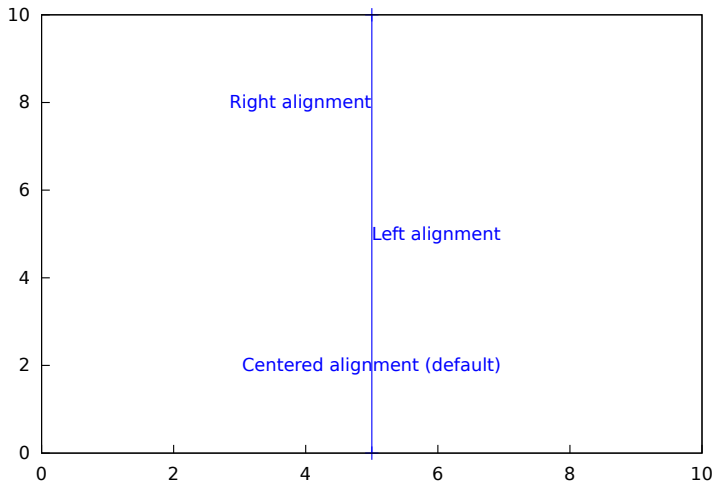
Default value: `center`

`label_alignment` is used to specify where to write labels with respect to the given coordinates. Possible values are: `center`, `left`, and `right`.

This option is relevant only for `label` objects.

Example:

```
(%i1) draw2d(xrange          = [0,10],
             yrange          = [0,10],
             points_joined   = true,
             points([[5,0],[5,10]]),
             color           = blue,
             label(["Centered alignment (default)",5,2]),
             label_alignment = 'left,
             label(["Left alignment",5,5]),
             label_alignment = 'right,
             label(["Right alignment",5,8]))$
```



See also `label_orientation`, and `color`

`label_orientation` [Graphic option]

Default value: `horizontal`

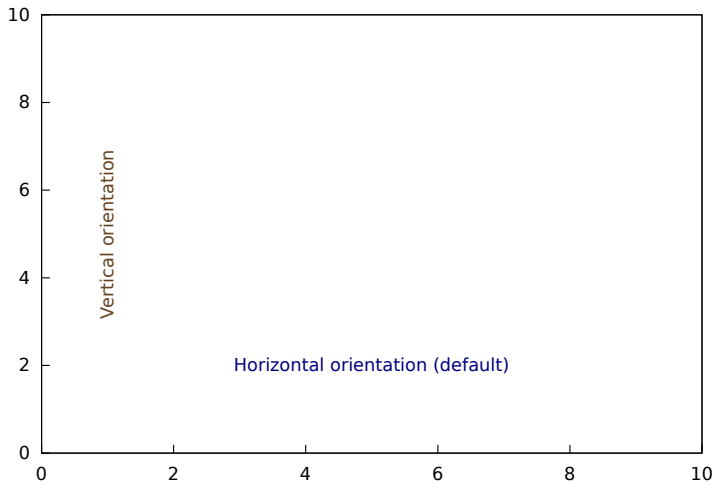
`label_orientation` is used to specify orientation of labels. Possible values are: `horizontal`, and `vertical`.

This option is relevant only for `label` objects.

Example:

In this example, a dummy point is added to get an image. Package `draw` needs always data to draw an scene.

```
(%i1) draw2d(xrange      = [0,10],
             yrange      = [0,10],
             point_size = 0,
             points([[5,5]]),
             color       = navy,
             label(["Horizontal orientation (default)",5,2]),
             label_orientation = 'vertical,
             color             = "#654321",
             label(["Vertical orientation",1,5]))$
```



See also `label_alignment` and `color`

`line_type` [Graphic option]

Default value: `solid`

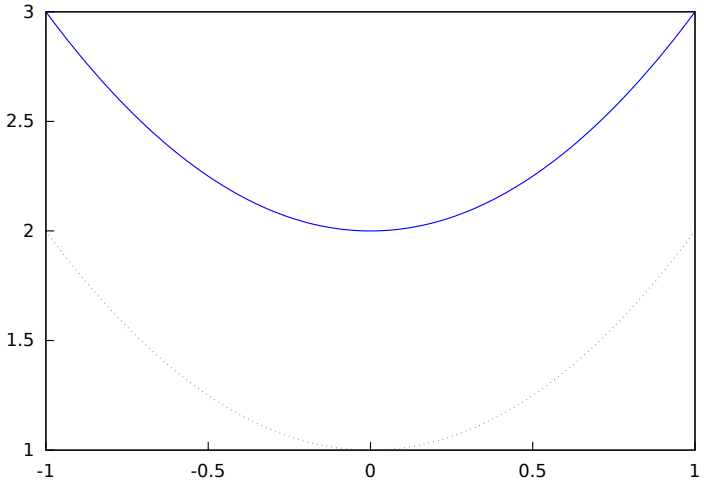
`line_type` indicates how lines are displayed; possible values are `solid` and `dots`, both available in all terminals, and `dashes`, `short_dashes`, `short_long_dashes`, `short_short_long_dashes`, and `dot_dash`, which are not available in `png`, `jpg`, and `gif` terminals.

This option affects the following graphic objects:

- `gr2d`: `points`, `polygon`, `rectangle`, `ellipse`, `vector`, `explicit`, `implicit`, `parametric` and `polar`.
- `gr3d`: `points`, `explicit`, `parametric` and `parametric_surface`.

Example:

```
(%i1) draw2d(line_type = dots,
             explicit(1 + x^2,x,-1,1),
             line_type = solid, /* default */
             explicit(2 + x^2,x,-1,1))$
```



See also `line_width`.

`line_width` [Graphic option]

Default value: 1

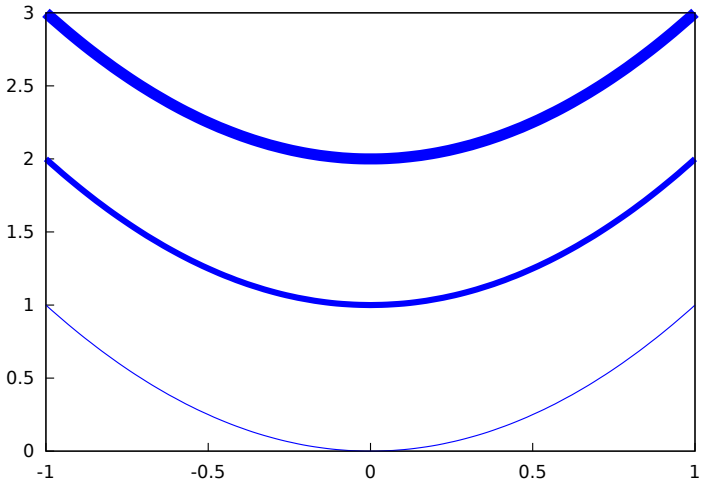
`line_width` is the width of plotted lines. Its value must be a positive number.

This option affects the following graphic objects:

- `gr2d`: `points`, `polygon`, `rectangle`, `ellipse`, `vector`, `explicit`, `implicit`, `parametric` and `polar`.
- `gr3d`: `points` and `parametric`.

Example:

```
(%i1) draw2d(explicit(x^2,x,-1,1), /* default width */
             line_width = 5.5,
             explicit(1 + x^2,x,-1,1),
             line_width = 10,
             explicit(2 + x^2,x,-1,1))$
```



See also `line_type`.

`logcb` [Graphic option]

Default value: `false`

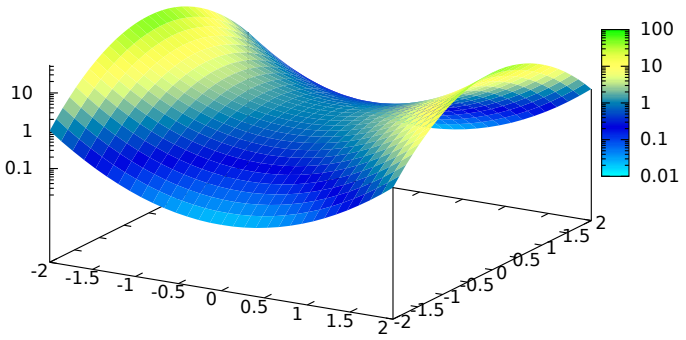
If `logcb` is `true`, the tics in the colorbox will be drawn in the logarithmic scale.

When `enhanced3d` or `colorbox` is `false`, option `logcb` has no effect.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d (
        enhanced3d = true,
        color      = green,
        logcb = true,
        logz  = true,
        palette = [-15,24,-9],
        explicit(exp(x^2-y^2), x,-2,2,y,-2,2)) $
```



See also `enhanced3d`, `colorbox` and `cbrange`.

`logx` [Graphic option]

Default value: `false`

If `logx` is `true`, the x axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter, with the exception that it should be written before any 2D `explicit` object, so that `draw` can produce a better plot.

Example:

```
(%i1) draw2d(logx = true,
             explicit(log(x),x,0.01,5))$
```

See also `logy`, `logx_secondary`, `logy_secondary`, and `logz`.

`logx_secondary` [Graphic option]

Default value: `false`

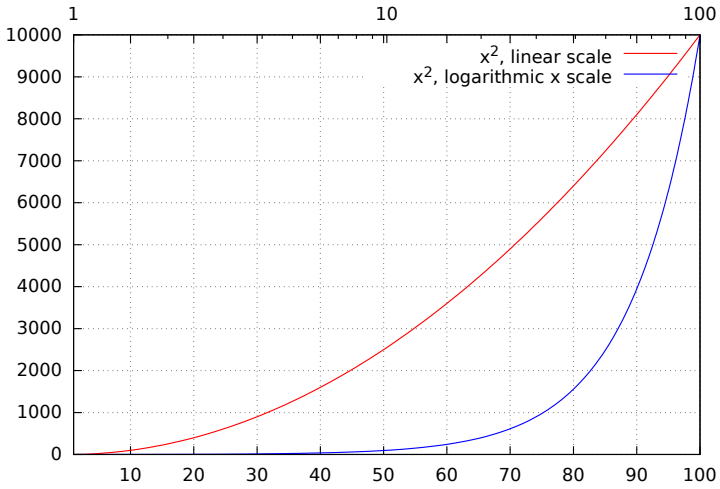
If `logx_secondary` is `true`, the secondary x axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
        grid = true,
        key="x^2, linear scale",
        color=red,
        explicit(x^2,x,1,100),
        xaxis_secondary = true,
        xtics_secondary = true,
        logx_secondary  = true,
        key = "x^2, logarithmic x scale",
        color = blue,
        explicit(x^2,x,1,100) )$
```



See also `logx_draw`, `logy_draw`, `logy_secondary`, and `logz`.

`logy` [Graphic option]

Default value: `false`

If `logy` is `true`, the y axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(logy = true,
             explicit(exp(x),x,0,5))$
```

See also `logx_draw`, `logx_secondary`, `logy_secondary`, and `logz`.

`logy_secondary` [Graphic option]

Default value: `false`

If `logy_secondary` is `true`, the secondary y axis will be drawn in the logarithmic scale.

This option is relevant only for 2d scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
        grid = true,
        key="x^2, linear scale",
        color=red,
        explicit(x^2,x,1,100),
        yaxis_secondary = true,
        ytics_secondary = true,
        logy_secondary  = true,
        key = "x^2, logarithmic y scale",
        color = blue,
        explicit(x^2,x,1,100) )$
```

See also `logx_draw`, `logy_draw`, `logx_secondary`, and `logz`.

`logz` [Graphic option]

Default value: `false`

If `logz` is `true`, the z axis will be drawn in the logarithmic scale.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(logz = true,
             explicit(exp(u^2+v^2),u,-2,2,v,-2,2))$
```

See also `logx_draw` and `logy_draw`.

`nticks` [Graphic option]

Default value: 29

In 2d, `nticks` gives the initial number of points used by the adaptive plotting routine for explicit objects. It is also the number of points that will be shown in parametric and polar curves.

This option affects the following graphic objects:

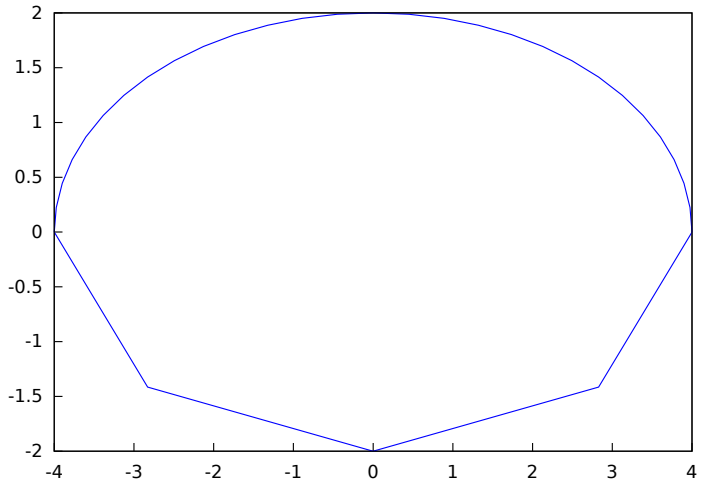
- `gr2d`: `ellipse`, `explicit`, `parametric` and `polar`.
- `gr3d`: `parametric`.

See also `adapt_depth`

Example:

```
(%i1) draw2d(transparent = true,
             ellipse(0,0,4,2,0,180),
```

```
nticks = 5,
ellipse(0,0,4,2,180,180) )$
```



`palette` [Graphic option]

Default value: `color`

`palette` indicates how to map gray levels onto color components. It works together with option `enhanced3d` in 3D graphics, who associates every point of a surfaces to a real number or gray level. It also works with gray images. With `palette`, levels are transformed into colors.

There are two ways for defining these transformations.

First, `palette` can be a vector of length three with components ranging from -36 to +36; each value is an index for a formula mapping the levels onto red, green and blue colors, respectively:

```
 0: 0               1: 0.5           2: 1
 3: x               4: x^2           5: x^3
 6: x^4             7: sqrt(x)       8: sqrt(sqrt(x))
 9: sin(90x)       10: cos(90x)     11: |x-0.5|
12: (2x-1)^2       13: sin(180x)    14: |cos(180x)|
15: sin(360x)      16: cos(360x)    17: |sin(360x)|
18: |cos(360x)|    19: |sin(720x)|  20: |cos(720x)|
21: 3x             22: 3x-1         23: 3x-2
24: |3x-1|         25: |3x-2|       26: (3x-1)/2
27: (3x-2)/2       28: |(3x-1)/2|   29: |(3x-2)/2|
30: x/0.32-0.78125 31: 2*x-0.84     32: 4x;1;-2x+1.84;x/0.08-11.5
33: |2*x - 0.5|    34: 2*x          35: 2*x - 0.5
36: 2*x - 1
```

negative numbers mean negative colour component. `palette = gray` and `palette = color` are short cuts for `palette = [3,3,3]` and `palette = [7,5,15]`, respectively.

Second, `palette` can be a user defined lookup table. In this case, the format for building a lookup table of length `n` is `palette=[color_1, color_2, ..., color_`

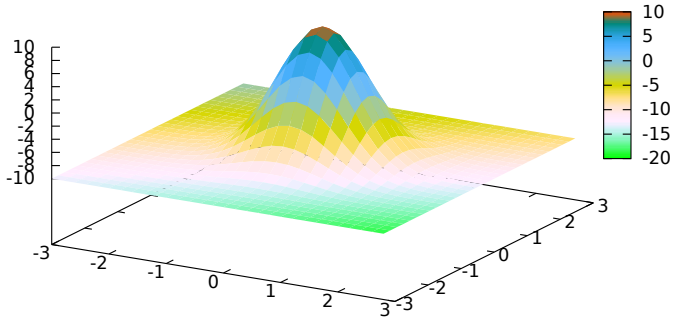
`n]`, where `color_i` is a well formed color (see option `color`) such that `color_1` is assigned to the lowest gray level and `color_n` to the highest. The rest of colors are interpolated.

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

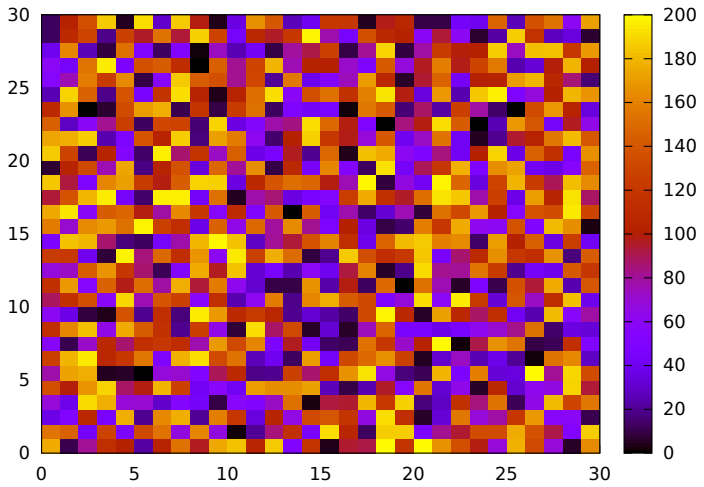
It works together with option `enhanced3d` in 3D graphics.

```
(%i1) draw3d(
        enhanced3d = [z-x+2*y,x,y,z],
        palette = [32, -8, 17],
        explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3))$
```



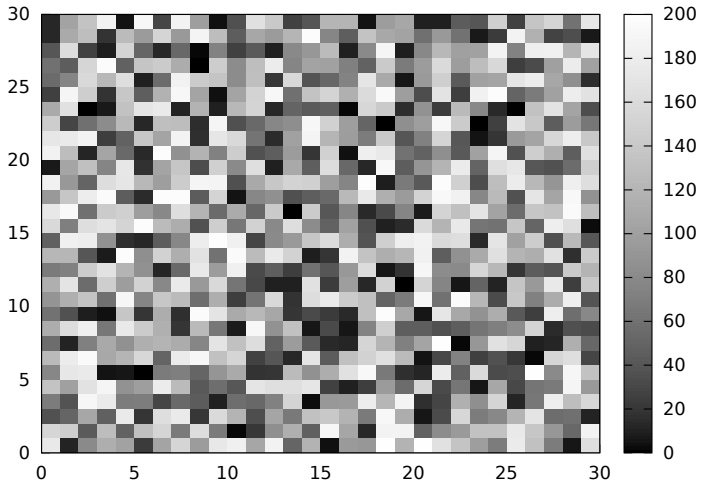
It also works with gray images.

```
(%i1) im: apply(
           'matrix,
            makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
      draw2d(image(im,0,0,30,30))$
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
(%i4) draw2d(palette = [15,20,-4],
             colorbox=false,
             image(im,0,0,30,30))$
```



`palette` can be a user defined lookup table. In this example, low values of `x` are colored in red, and higher values in yellow.

```
(%i1) draw3d(
        palette = [red, blue, yellow],
        enhanced3d = x,
        explicit(x^2+y^2,x,-1,1,y,-1,1)) $
```



See also `colorbox` and `enhanced3d`.

`point_size` [Graphic option]

Default value: 1

`point_size` sets the size for plotted points. It must be a non negative number.

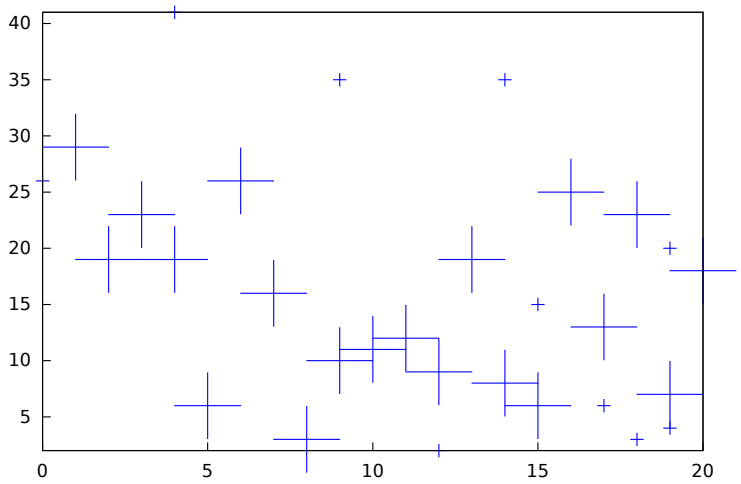
This option has no effect when graphic option `point_type` is set to `dot`.

This option affects the following graphic objects:

- `gr2d`: `points`.
- `gr3d`: `points`.

Example:

```
(%i1) draw2d(points(makelist([random(20),random(50)],k,1,10)),
        point_size = 5,
        points(makelist(k,k,1,20),makelist(random(30),k,1,20)))$
```



`point_type` [Graphic option]

Default value: 1

`point_type` indicates how isolated points are displayed; the value of this option can be any integer index greater or equal than -1, or the name of a point style: `$none` (-1), `dot` (0), `plus` (1), `multiply` (2), `asterisk` (3), `square` (4), `filled_square` (5), `circle` (6), `filled_circle` (7), `up_triangle` (8), `filled_up_triangle` (9), `down_triangle` (10), `filled_down_triangle` (11), `diamant` (12) and `filled_diamant` (13).

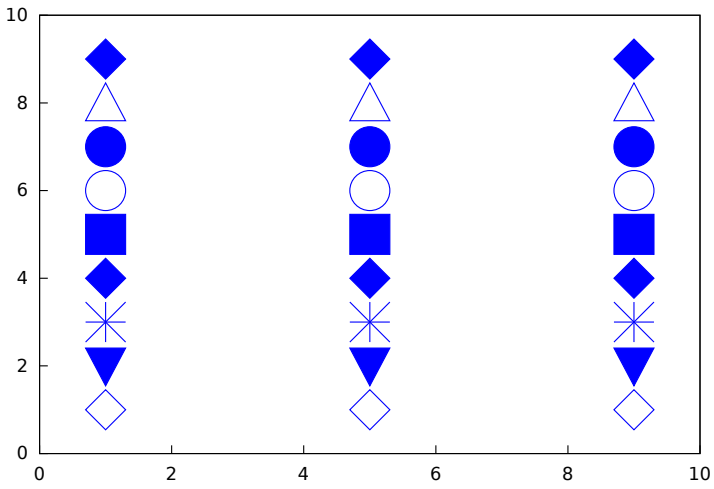
This option affects the following graphic objects:

- `gr2d`: `points`.
- `gr3d`: `points`.

Example:

```
(%i1) draw2d(xrange = [0,10],
             yrange = [0,10],
             point_size = 3,
             point_type = diamant,
             points([[1,1],[5,1],[9,1]]),
             point_type = filled_down_triangle,
             points([[1,2],[5,2],[9,2]]),
             point_type = asterisk,
             points([[1,3],[5,3],[9,3]]),
             point_type = filled_diamant,
             points([[1,4],[5,4],[9,4]]),
             point_type = 5,
             points([[1,5],[5,5],[9,5]]),
             point_type = 6,
```

```
                points([[1,6],[5,6],[9,6]]),
                point_type = filled_circle,
                points([[1,7],[5,7],[9,7]]),
                point_type = 8,
                points([[1,8],[5,8],[9,8]]),
                point_type = filled_diamant,
                points([[1,9],[5,9],[9,9]]) )$
```



`points_joined` [Graphic option]

Default value: `false`

When `points_joined` is `true`, points are joined by lines; when `false`, isolated points are drawn. A third possible value for this graphic option is `impulses`; in such case, vertical segments are drawn from points to the x-axis (2D) or to the xy-plane (3D).

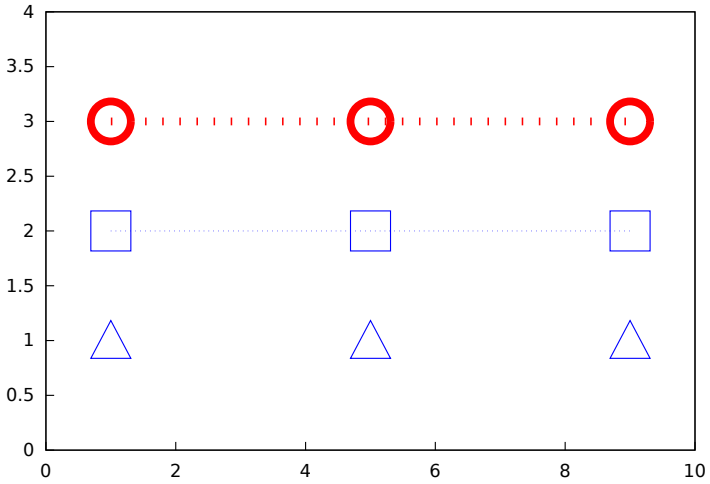
This option affects the following graphic objects:

- `gr2d`: `points`.
- `gr3d`: `points`.

Example:

```
(%i1) draw2d(xrange        = [0,10],
             yrange        = [0,4],
             point_size    = 3,
             point_type    = up_triangle,
             color         = blue,
             points([[1,1],[5,1],[9,1]]),
             points_joined = true,
             point_type    = square,
             line_type     = dots,
             points([[1,2],[5,2],[9,2]]),
             point_type    = circle,
             color         = red,
             line_width    = 7,
```

```
points([[1,3],[5,3],[9,3]]) )$
```



`proportional_axes` [Graphic option]

Default value: `none`

When `proportional_axes` is equal to `xy` or `xyz`, the aspect ratio of the axis units will be set to 1:1 resulting in a 2D or 3D scene that will be drawn with axes proportional to their relative lengths.

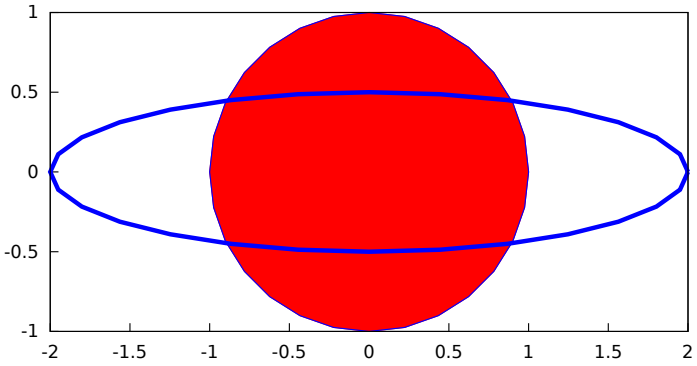
Since this is a global graphics option, its position in the scene description does not matter.

This option works with Gnuplot version 4.2.6 or greater.

Examples:

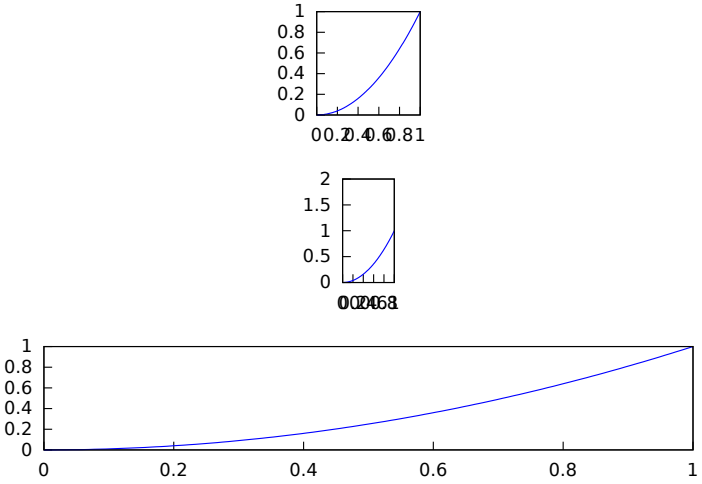
Single 2D plot.

```
(%i1) draw2d(
        ellipse(0,0,1,1,0,360),
        transparent=true,
        color = blue,
        line_width = 4,
        ellipse(0,0,2,1/2,0,360),
        proportional_axes = 'xy) $
```



Multiplot.

```
(%i1) draw(
        terminal = wxt,
        gr2d(proportional_axes = 'xy,
             explicit(x^2,x,0,1)),
        gr2d(explicit(x^2,x,0,1),
             xrange = [0,1],
             yrange = [0,2],
             proportional_axes='xy),
        gr2d(explicit(x^2,x,0,1)))$
```



`surface_hide` [Graphic option]

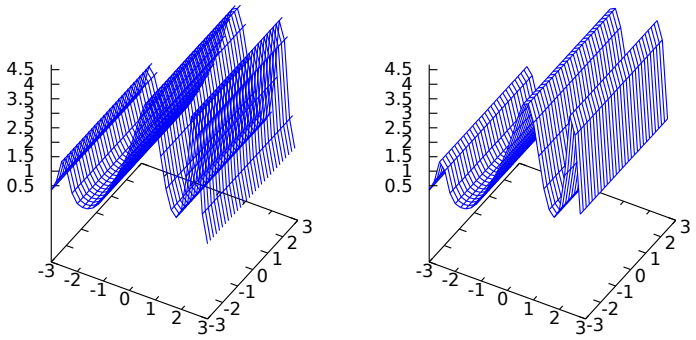
Default value: `false`

If `surface_hide` is `true`, hidden parts are not plotted in 3d surfaces.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw(columns=2,
           gr3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3)),
           gr3d(surface_hide = true,
                explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3)) )$
```



`terminal` [Graphic option]

Default value: `screen`

Selects the terminal to be used by Gnuplot; possible values are: `screen` (default), `png`, `pngcairo`, `jpg`, `gif`, `cairolatex_pdf`, `cairolatex_pdf_standalone`, `eps`, `eps_color`, `epslatex`, `epslatex_standalone`, `svg`, `canvas`, `dumb`, `dumb_file`, `pdf`, `pdfcairo`, `wxt`, `animated_gif`, `multipage_pdfcairo`, `multipage_pdf`, `multipage_eps`, `multipage_eps_color`, `tikz`, `tikz_standalone` and `aquaterm`.

Terminals `screen`, `wxt`, `windows` and `aquaterm` can be also defined as a list with two elements: the name of the terminal itself and a non negative integer number. In this form, multiple windows can be opened at the same time, each with its corresponding number. This feature does not work in Windows platforms.

Since this is a global graphics option, its position in the scene description does not matter. It can be also used as an argument of function `draw`.

N.B. pdfcairo requires Gnuplot 4.3 or newer. `pdf` requires Gnuplot to be compiled with the option `--enable-pdf` and libpdf must be installed. The pdf library is available from: http://www.pdflib.com/en/download/pdflib-family/pdflib-lite/

Examples:

```
(%i1) /* screen terminal (default) */
      draw2d(explicit(x^2,x,-1,1))$
(%i2) /* png file */
      draw2d(terminal  = 'png,
             explicit(x^2,x,-1,1))$
(%i3) /* jpg file */
      draw2d(terminal   = 'jpg,
             dimensions = [300,300],
```

```
              explicit(x^2,x,-1,1))$
(%i4) /* eps file */
      draw2d(file_name = "myfile",
              explicit(x^2,x,-1,1),
              terminal  = 'eps)$
(%i5) /* pdf file */
      draw2d(file_name = "mypdf",
              dimensions = 100*[12.0,8.0],
              explicit(x^2,x,-1,1),
              terminal  = 'pdf)$
(%i6) /* wxwidgets window */
      draw2d(explicit(x^2,x,-1,1),
              terminal  = 'wxt)$
(%i7) /* tikz file */
      draw2d(explicit(x^2,x,-1,1),
              file_name = "mytikz",
              dimensions = [8,8], /* in cms */
              terminal = 'tikz)$
```

Multiple windows.

```
(%i1) draw2d(explicit(x^5,x,-2,2), terminal=[screen, 3])$
(%i2) draw2d(explicit(x^2,x,-2,2), terminal=[screen, 0])$
```

An animated gif file.

```
(%i1) draw(
        delay     = 100,
        file_name = "zzz",
        terminal  = 'animated_gif,
        gr2d(explicit(x^2,x,-1,1)),
        gr2d(explicit(x^3,x,-1,1)),
        gr2d(explicit(x^4,x,-1,1)));
End of animation sequence
(%o1)          [gr2d(explicit), gr2d(explicit), gr2d(explicit)]
```

Option `delay` is only active in animated gif's; it is ignored in any other case.

Multipage output in eps format.

```
(%i1) draw(
        file_name = "parabol",
        terminal  = multipage_eps,
        dimensions = 100*[10,10],
        gr2d(explicit(x^2,x,-1,1)),
        gr3d(explicit(x^2+y^2,x,-1,1,y,-1,1))) $
```

See also `file_name`, `dimensions_draw` and `delay`.

`title` [Graphic option]

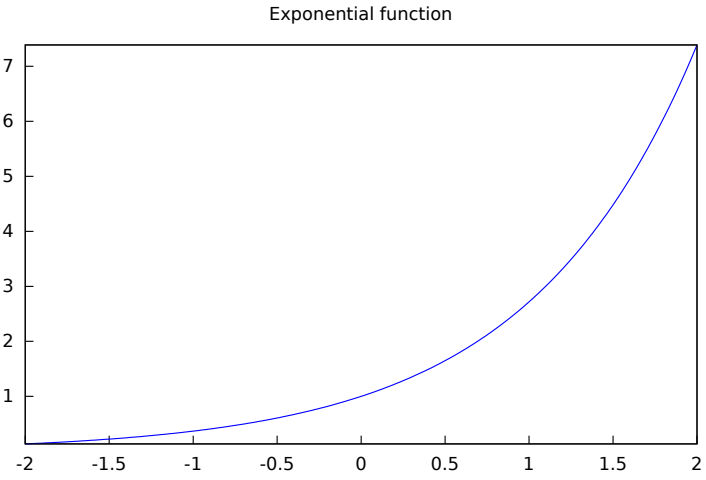
Default value: `""` (empty string)

Option `title`, a string, is the main title for the scene. By default, no title is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(exp(u),u,-2,2),
             title = "Exponential function")$
```



`transform` [Graphic option]

Default value: `none`

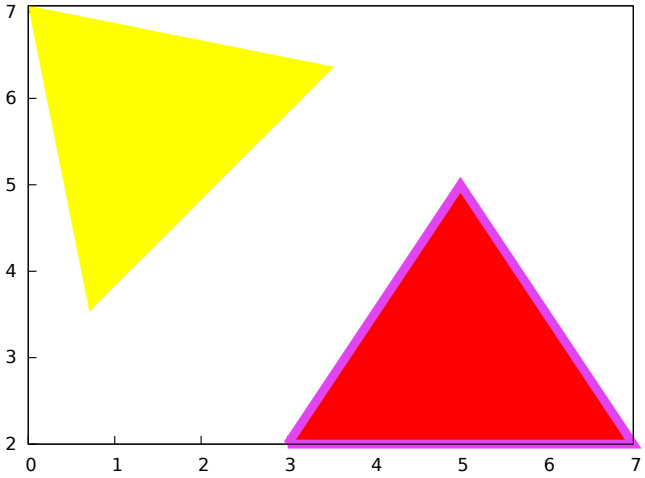
If `transform` is `none`, the space is not transformed and graphic objects are drawn as defined. When a space transformation is desired, a list must be assigned to option `transform`. In case of a 2D scene, the list takes the form `[f1(x,y), f2(x,y), x, y]`. In case of a 3D scene, the list is of the form `[f1(x,y,z), f2(x,y,z), f3(x,y,z), x, y, z]`.

The names of the variables defined in the lists may be different to those used in the definitions of the graphic objects.

Examples:

Rotation in 2D.

```
(%i1) th : %pi / 4$
(%i2) draw2d(
        color = "#e245f0",
        proportional_axes = 'xy,
        line_width = 8,
        triangle([3,2],[7,2],[5,5]),
        border     = false,
        fill_color = yellow,
        transform  = [cos(th)*x - sin(th)*y,
                      sin(th)*x + cos(th)*y, x, y],
        triangle([3,2],[7,2],[5,5]) )$
```



Translation in 3D.

```
(%i1) draw3d(
        color     = "#a02c00",
        explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
        transform = [x+10,y+10,z+10,x,y,z],
        color     = blue,
        explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3) )$
```

`transparent` [Graphic option]

Default value: `false`

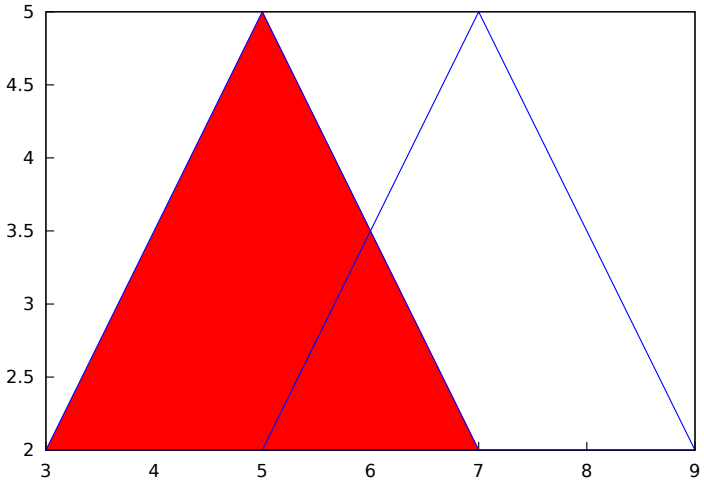
If `transparent` is `false`, interior regions of polygons are filled according to `fill_color`.

This option affects the following graphic objects:

- `gr2d`: `polygon`, `rectangle` and `ellipse`.

Example:

```
(%i1) draw2d(polygon([[3,2],[7,2],[5,5]]),
             transparent = true,
             color       = blue,
             polygon([[5,2],[9,2],[7,5]]) )$
```



`unit_vectors` [Graphic option]

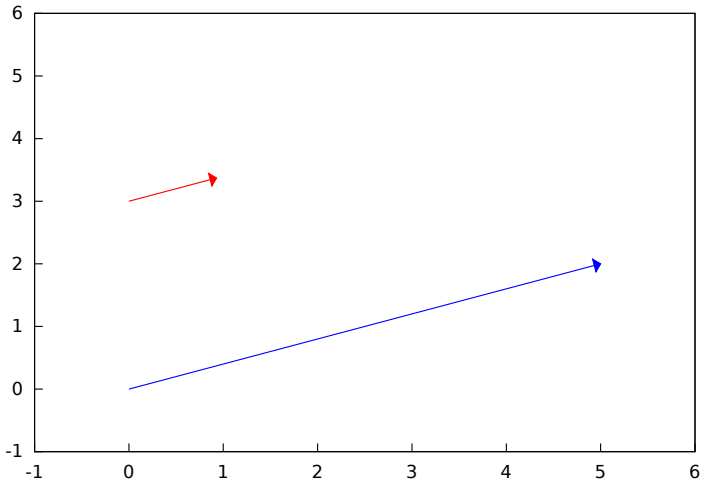
Default value: `false`

If `unit_vectors` is `true`, vectors are plotted with module 1. This is useful for plotting vector fields. If `unit_vectors` is `false`, vectors are plotted with its original length.

This option is relevant only for `vector` objects.

Example:

```
(%i1) draw2d(xrange      = [-1,6],
             yrange      = [-1,6],
             head_length = 0.1,
             vector([0,0],[5,2]),
             unit_vectors = true,
             color        = red,
             vector([0,3],[5,2]))$
```



`user_preamble` [Graphic option]

Default value: `""` (empty string)

Expert Gnuplot users can make use of this option to fine tune Gnuplot's behaviour by writing settings to be sent before the `plot` or `splot` command.

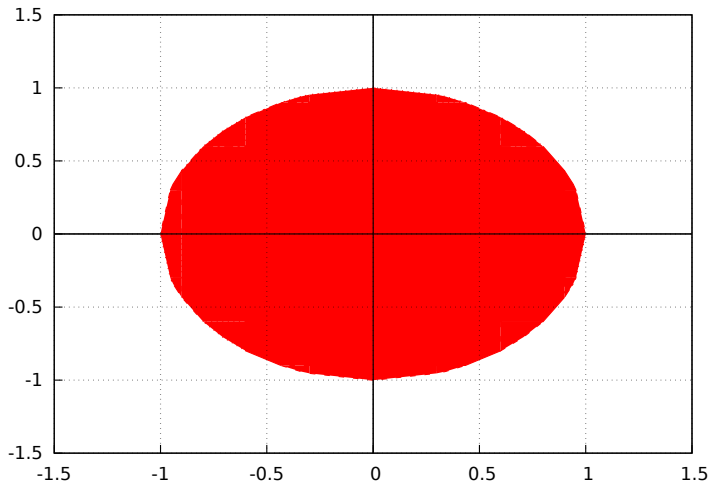
The value of this option must be a string or a list of strings (one per line).

Since this is a global graphics option, its position in the scene description does not matter.

Example:

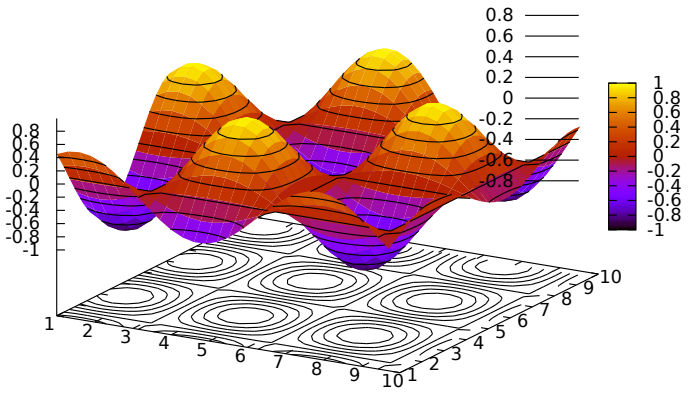
Tell Gnuplot to draw axes and grid on top of graphics objects,

```
(%i1) draw2d(
        xaxis =true, xaxis_type=solid,
        yaxis =true, yaxis_type=solid,
        user_preamble="set grid front",
        region(x^2+y^2<1 ,x,-1.5,1.5,y,-1.5,1.5))$
```



Tell gnuplot to draw all contour lines in black

```
(%i1) draw3d(
          contour=both,
          surface_hide=true,enhanced3d=true,wired_surface=true,
          contour_levels=10,
          user_preamble="set for [i=1:8] linetype i dashtype i linecolor 0",
          explicit(sin(x)*cos(y),x,1,10,y,1,10)
      );
```



`view` [Graphic option]

Default value: `[60,30]`

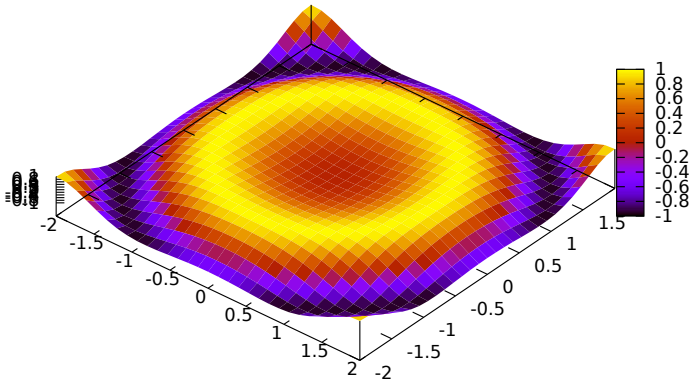
A pair of angles, measured in degrees, indicating the view direction in a 3D scene. The first angle is the vertical rotation around the x axis, in the range $[0, 360]$. The second one is the horizontal rotation around the z axis, in the range $[0, 360]$.

If option `view` is given the value `map`, the view direction is set to be perpendicular to the xy-plane.

Since this is a global graphics option, its position in the scene description does not matter.

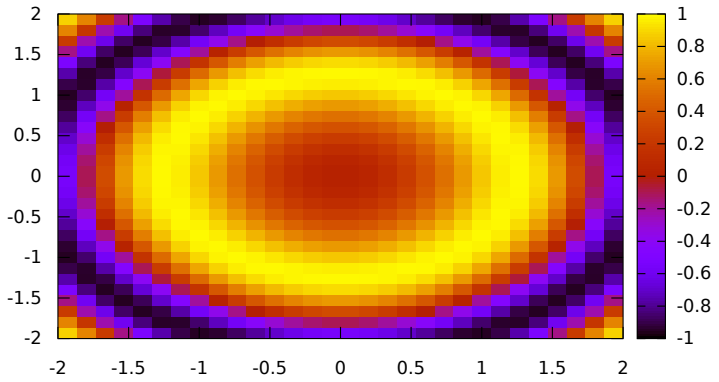
Example:

```
(%i1) draw3d(view = [170, 50],
             enhanced3d = true,
             explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$
```



```
(%i2) draw3d(view = map,
             enhanced3d = true,
```

```
explicit(sin(x^2+y^2),x,-2,2,y,-2,2) )$
```



`wired_surface` [Graphic option]

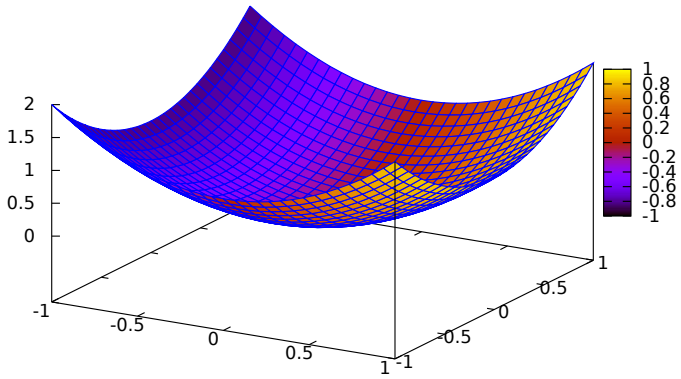
Default value: `false`

Indicates whether 3D surfaces in `enhanced3d` mode show the grid joining the points or not.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(
        enhanced3d    = [sin(x),x,y],
        wired_surface = true,
        explicit(x^2+y^2,x,-1,1,y,-1,1)) $
```



`x_voxel` [Graphic option]

Default value: 10

`x_voxel` is the number of voxels in the x direction to be used by the *marching cubes algorithm* implemented by the 3d `implicit` object. It is also used by graphic object `region`.

`xaxis` [Graphic option]

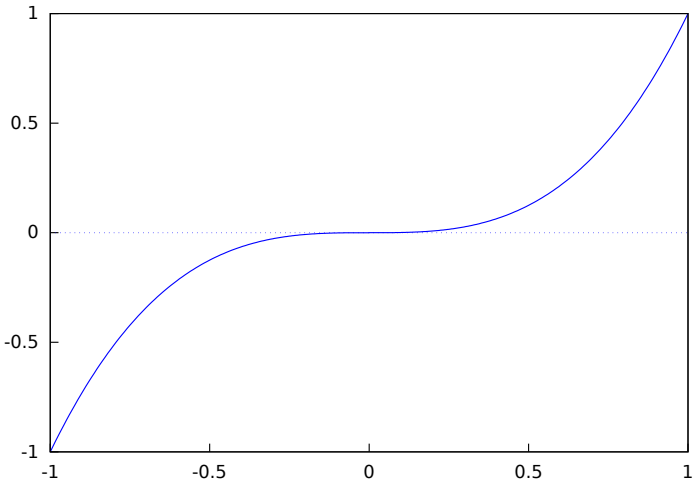
Default value: `false`

If `xaxis` is `true`, the x axis is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             xaxis       = true,
             xaxis_color = blue)$
```



See also `xaxis_width`, `xaxis_type` and `xaxis_color`.

`xaxis_color` [Graphic option]

Default value: `"black"`

`xaxis_color` specifies the color for the x axis. See `color` to know how colors are defined.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             xaxis       = true,
             xaxis_color = red)$
```

See also `xaxis`, `xaxis_width` and `xaxis_type`.

`xaxis_secondary` [Graphic option]

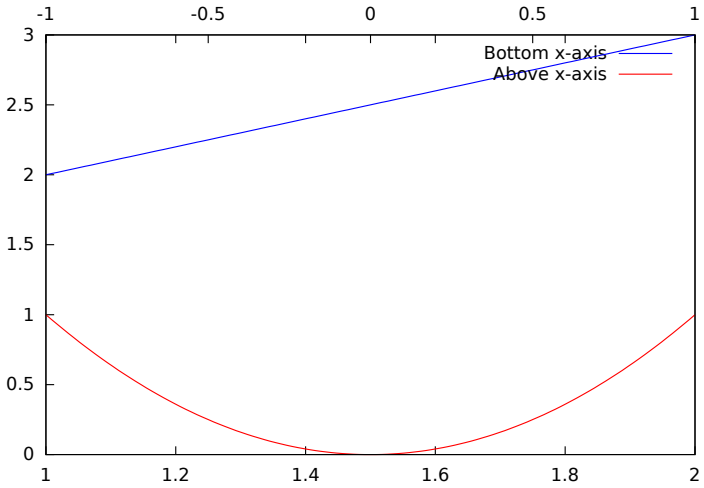
Default value: `false`

If `xaxis_secondary` is `true`, function values can be plotted with respect to the second x axis, which will be drawn on top of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

```
(%i1) draw2d(
        key   = "Bottom x-axis",
        explicit(x+1,x,1,2),
        color = red,
        key   = "Above x-axis",
        xtics_secondary = true,
        xaxis_secondary = true,
        explicit(x^2,x,-1,1)) $
```



See also `xrange_secondary`, `xtics_secondary`, `xtics_rotate_secondary`, `xtics_axis_secondary` and `xaxis_secondary`.

`xaxis_type` [Graphic option]

Default value: `dots`

`xaxis_type` indicates how the x axis is displayed; possible values are `solid` and `dots`

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             xaxis       = true,
             xaxis_type  = solid)$
```

See also `xaxis`, `xaxis_width` and `xaxis_color`.

`xaxis_width` [Graphic option]

Default value: 1

`xaxis_width` is the width of the x axis. Its value must be a positive number.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             xaxis       = true,
             xaxis_width = 3)$
```

See also xaxis, xaxis_type and xaxis_color.

xlabel [Graphic option]

Default value: ""

Option xlabel, a string, is the label for the x axis. By default, the axis is labeled with string "x".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xlabel = "Time",
             explicit(exp(u),u,-2,2),
             ylabel = "Population")$
```

See also xlabel_secondary, ylabel, ylabel_secondary and zlabel_draw.

xlabel_secondary [Graphic option]

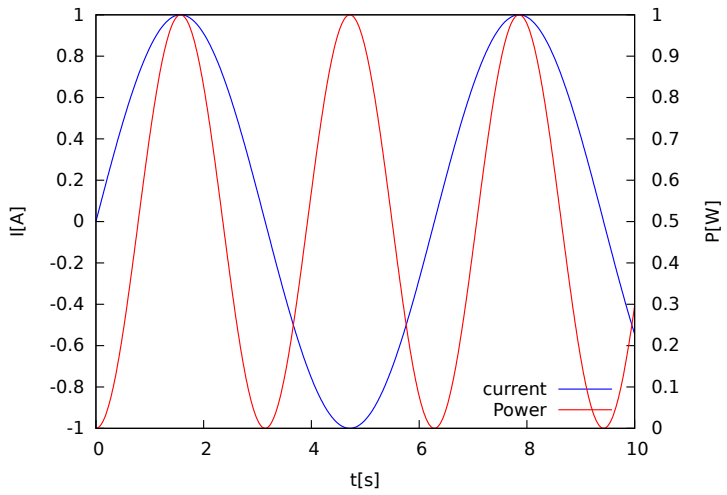
Default value: "" (empty string)

Option xlabel_secondary, a string, is the label for the secondary x axis. By default, no label is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
         xaxis_secondary=true,yaxis_secondary=true,
         xtics_secondary=true,ytics_secondary=true,
         xlabel_secondary="t[s]",
         ylabel_secondary="U[V]",
         explicit(sin(t),t,0,10) )$
```



See also `xlabel_draw`, `ylabel_draw`, `ylabel_secondary` and `zlabel_draw`.

`xrange` [Graphic option]

Default value: `auto`

If `xrange` is `auto`, the range for the x coordinate is computed automatically.

If the user wants a specific interval for x, it must be given as a Maxima list, as in `xrange=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xrange = [-3,5],
             explicit(x^2,x,-1,1))$
```

See also `yrange` and `zrange`.

`xrange_secondary` [Graphic option]

Default value: `auto`

If `xrange_secondary` is `auto`, the range for the second x axis is computed automatically.

If the user wants a specific interval for the second x axis, it must be given as a Maxima list, as in `xrange_secondary=[-2, 3]`.

Since this is a global graphics option, its position in the scene description does not matter.

See also `xrange`, `yrange`, `zrange` and `yrange_secondary`.

`xtics` [Graphic option]

Default value: `true`

This graphic option controls the way tic marks are drawn on the x axis.

- When option `xtics` is bounded to symbol *true*, tic marks are drawn automatically.
- When option `xtics` is bounded to symbol *false*, tic marks are not drawn.

- When option `xtics` is bounded to a positive number, this is the distance between two consecutive tic marks.
- When option `xtics` is bounded to a list of length three of the form `[start,incr,end]`, tic marks are plotted from `start` to `end` at intervals of length `incr`.
- When option `xtics` is bounded to a set of numbers of the form `{n1, n2, ...}`, tic marks are plotted at values `n1`, `n2`, ...
- When option `xtics` is bounded to a set of pairs of the form `{["label1", n1], ["label2", n2], ...}`, tic marks corresponding to values `n1`, `n2`, ... are labeled with `"label1"`, `"label2"`, ..., respectively.

Since this is a global graphics option, its position in the scene description does not matter.

Examples:

Disable tics.

```
(%i1) draw2d(xtics = 'false,
             explicit(x^3,x,-1,1)  )$
```

Tics every 1/4 units.

```
(%i1) draw2d(xtics = 1/4,
             explicit(x^3,x,-1,1)  )$
```

Tics from -3/4 to 3/4 in steps of 1/8.

```
(%i1) draw2d(xtics = [-3/4,1/8,3/4],
             explicit(x^3,x,-1,1)  )$
```

Tics at points -1/2, -1/4 and 3/4.

```
(%i1) draw2d(xtics = {-1/2,-1/4,3/4},
             explicit(x^3,x,-1,1)  )$
```

Labeled tics.

```
(%i1) draw2d(xtics = {["High",0.75],["Medium",0],["Low",-0.75]},
             explicit(x^3,x,-1,1)  )$
```

See also `ytics_draw`, and `ztics_draw`.

`xtics_axis` [Graphic option]

Default value: `false`

If `xtics_axis` is `true`, tic marks and their labels are plotted just along the x axis, if it is `false` tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

`xtics_rotate` [Graphic option]

Default value: `false`

If `xtics_rotate` is `true`, tic marks on the x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

`xtics_rotate_secondary` [Graphic option]

Default value: `false`

If `xtics_rotate_secondary` is `true`, tic marks on the secondary x axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

`xtics_secondary` [Graphic option]

Default value: `auto`

This graphic option controls the way tic marks are drawn on the second x axis.

See `xtics_draw` for a complete description.

`xtics_secondary_axis` [Graphic option]

Default value: `false`

If `xtics_secondary_axis` is `true`, tic marks and their labels are plotted just along the secondary x axis, if it is `false` tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

`xu_grid` [Graphic option]

Default value: 30

`xu_grid` is the number of coordinates of the first variable (`x` in explicit and `u` in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

- `gr3d`: `explicit` and `parametric_surface`.

Example:

```
(%i1) draw3d(xu_grid = 10,
             yv_grid = 50,
             explicit(x^2+y^2,x,-3,3,y,-3,3) )$
```

See also `yv_grid`.

`xy_file` [Graphic option]

Default value: `""` (empty string)

`xy_file` is the name of the file where the coordinates will be saved after clicking with the mouse button and hitting the 'x' key. By default, no coordinates are saved.

Since this is a global graphics option, its position in the scene description does not matter.

`xyplane` [Graphic option]

Default value: `false`

Allocates the xy-plane in 3D scenes. When `xyplane` is `false`, the xy-plane is placed automatically; when it is a real number, the xy-plane intersects the z-axis at this level. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(xyplane = %e-2,
             explicit(x^2+y^2,x,-1,1,y,-1,1))$
```

y_voxel [Graphic option]

Default value: 10

y_voxel is the number of voxels in the y direction to be used by the *marching cubes algorithm* implemented by the 3d implicit object. It is also used by graphic object region.

yaxis [Graphic option]

Default value: false

If yaxis is true, the y axis is drawn.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             yaxis       = true,
             yaxis_color = blue)$
```

See also yaxis_width, yaxis_type and yaxis_color.

yaxis_color [Graphic option]

Default value: "black"

yaxis_color specifies the color for the y axis. See color to know how colors are defined.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             yaxis       = true,
             yaxis_color = red)$
```

See also yaxis, yaxis_width and yaxis_type.

yaxis_secondary [Graphic option]

Default value: false

If yaxis_secondary is true, function values can be plotted with respect to the second y axis, which will be drawn on the right side of the scene.

Note that this is a local graphics option which only affects to 2d plots.

Example:

```
(%i1) draw2d(
         explicit(sin(x),x,0,10),
         yaxis_secondary = true,
         ytics_secondary = true,
         color = blue,
         explicit(100*sin(x+0.1)+2,x,0,10));
```

See also yrange_secondary, ytics_secondary, ytics_rotate_secondary and ytics_axis_secondary

yaxis_type [Graphic option]

Default value: dots

yaxis_type indicates how the y axis is displayed; possible values are solid and dots.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             yaxis       = true,
             yaxis_type  = solid)$
```

See also yaxis, yaxis_width and yaxis_color.

yaxis_width [Graphic option]

Default value: 1

yaxis_width is the width of the y axis. Its value must be a positive number.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(explicit(x^3,x,-1,1),
             yaxis       = true,
             yaxis_width = 3)$
```

See also yaxis, yaxis_type and yaxis_color.

ylabel [Graphic option]

Default value: ""

Option ylabel, a string, is the label for the y axis. By default, the axis is labeled with string "y".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(xlabel = "Time",
             ylabel = "Population",
             explicit(exp(u),u,-2,2) )$
```

See also xlabel_draw, xlabel_secondary, ylabel_secondary, and zlabel_draw.

ylabel_secondary [Graphic option]

Default value: "" (empty string)

Option ylabel_secondary, a string, is the label for the secondary y axis. By default, no label is written.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
        key_pos=bottom_right,
        key="current",
        xlabel="t[s]",
        ylabel="I[A]",ylabel_secondary="P[W]",
        explicit(sin(t),t,0,10),
        yaxis_secondary=true,
        ytics_secondary=true,
        color=red,key="Power",
        explicit((sin(t))^2,t,0,10)
      )$
```

See also xlabel_draw, xlabel_secondary, ylabel_draw and zlabel_draw.

yrange [Graphic option]

Default value: auto

If yrange is auto, the range for the y coordinate is computed automatically.

If the user wants a specific interval for y, it must be given as a Maxima list, as in yrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(yrange = [-2,3],
             explicit(x^2,x,-1,1),
             xrange = [-3,3])$
```

See also xrange, yrange_secondary and zrange.

yrange_secondary [Graphic option]

Default value: auto

If yrange_secondary is auto, the range for the second y axis is computed automatically.

If the user wants a specific interval for the second y axis, it must be given as a Maxima list, as in yrange_secondary=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw2d(
        explicit(sin(x),x,0,10),
        yaxis_secondary = true,
        ytics_secondary = true,
        yrange = [-3, 3],
        yrange_secondary = [-20, 20],
        color = blue,
        explicit(100*sin(x+0.1)+2,x,0,10)) $
```

See also xrange, yrange and zrange.

`ytics` [Graphic option]

Default value: `true`

This graphic option controls the way tic marks are drawn on the y axis.

See `xtics` for a complete description.

`ytics_axis` [Graphic option]

Default value: `false`

If `ytics_axis` is `true`, tic marks and their labels are plotted just along the y axis, if it is `false` tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

`ytics_rotate` [Graphic option]

Default value: `false`

If `ytics_rotate` is `true`, tic marks on the y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

`ytics_rotate_secondary` [Graphic option]

Default value: `false`

If `ytics_rotate_secondary` is `true`, tic marks on the secondary y axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

`ytics_secondary` [Graphic option]

Default value: `auto`

This graphic option controls the way tic marks are drawn on the second y axis.

See `xtics` for a complete description.

`ytics_secondary_axis` [Graphic option]

Default value: `false`

If `ytics_secondary_axis` is `true`, tic marks and their labels are plotted just along the secondary y axis, if it is `false` tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

`yv_grid` [Graphic option]

Default value: 30

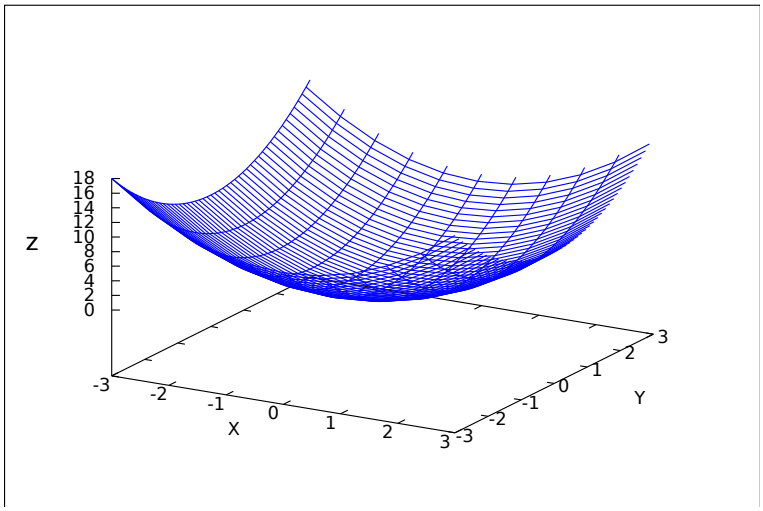
`yv_grid` is the number of coordinates of the second variable (`y` in explicit and `v` in parametric 3d surfaces) to build the grid of sample points.

This option affects the following graphic objects:

- `gr3d`: `explicit` and `parametric_surface`.

Example:

```
(%i1) draw3d(xu_grid = 10,
             yv_grid = 50,
             explicit(x^2+y^2,x,-3,3,y,-3,3) )$
```



See also `xu_grid`.

`z_voxel` [Graphic option]

Default value: 10

`z_voxel` is the number of voxels in the z direction to be used by the *marching cubes algorithm* implemented by the 3d `implicit` object.

`zaxis` [Graphic option]

Default value: `false`

If `zaxis` is `true`, the z axis is drawn in 3D plots. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
             zaxis       = true,
             zaxis_type  = solid,
             zaxis_color = blue)$
```

See also `zaxis_width`, `zaxis_type` and `zaxis_color`.

`zaxis_color` [Graphic option]

Default value: `"black"`

`zaxis_color` specifies the color for the z axis. See `color` to know how colors are defined. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
             zaxis       = true,
             zaxis_type  = solid,
             zaxis_color = red)$
```

See also zaxis, zaxis_width and zaxis_type.

zaxis_type [Graphic option]

Default value: dots

zaxis_type indicates how the z axis is displayed; possible values are solid and dots. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
             zaxis       = true,
             zaxis_type  = solid)$
```

See also zaxis, zaxis_width and zaxis_color.

zaxis_width [Graphic option]

Default value: 1

zaxis_width is the width of the z axis. Its value must be a positive number. This option has no effect in 2D scenes.

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(explicit(x^2+y^2,x,-1,1,y,-1,1),
             zaxis       = true,
             zaxis_type  = solid,
             zaxis_width = 3)$
```

See also zaxis, zaxis_type and zaxis_color.

zlabel [Graphic option]

Default value: ""

Option zlabel, a string, is the label for the z axis. By default, the axis is labeled with string "z".

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(zlabel = "Z variable",
             ylabel = "Y variable",
             explicit(sin(x^2+y^2),x,-2,2,y,-2,2),
             xlabel = "X variable" )$
```

See also xlabel_draw, ylabel_draw, and zlabel_rotate.

zlabel_rotate [Graphic option]

Default value: "auto"

This graphics option allows to choose if the z axis label of 3d plots is drawn horizontal (false), vertical (true) or if maxima automatically chooses an orientation based on the length of the label (auto).

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(
          explicit(sin(x)*sin(y),x,0,10,y,0,10),
          zlabel_rotate=false
      )$
```

See also zlabel_draw.

zrange [Graphic option]

Default value: auto

If zrange is auto, the range for the z coordinate is computed automatically.

If the user wants a specific interval for z, it must be given as a Maxima list, as in zrange=[-2, 3].

Since this is a global graphics option, its position in the scene description does not matter.

Example:

```
(%i1) draw3d(yrange = [-3,3],
             zrange = [-2,5],
             explicit(x^2+y^2,x,-1,1,y,-1,1),
             xrange = [-3,3])$
```

See also xrange and yrange.

ztics [Graphic option]

Default value: true

This graphic option controls the way tic marks are drawn on the z axis.

See xtics_draw for a complete description.

ztics_axis [Graphic option]

Default value: false

If ztics_axis is true, tic marks and their labels are plotted just along the z axis, if it is false tics are plotted on the border.

Since this is a global graphics option, its position in the scene description does not matter.

ztics_rotate [Graphic option]

Default value: false

If ztics_rotate is true, tic marks on the z axis are rotated 90 degrees.

Since this is a global graphics option, its position in the scene description does not matter.

55.2.4 Graphics objects

`bars` (`[x1,h1,w1]`, `[x2,h2,w2, ...]`) [Graphic object]

Draws vertical bars in 2D.

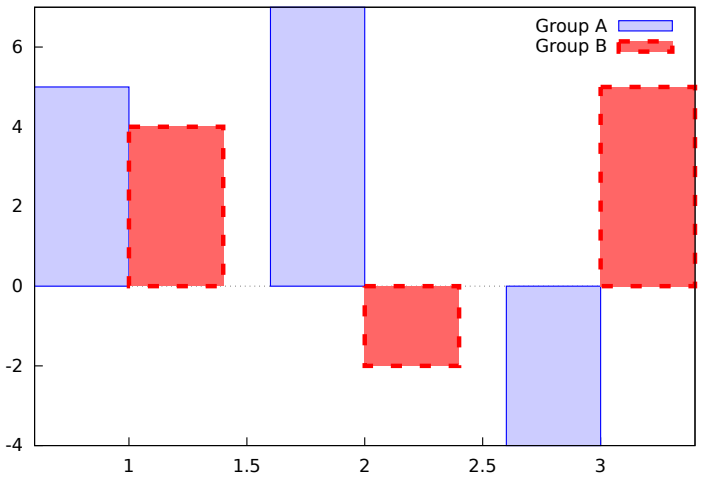
2D

`bars ([x1,h1,w1], [x2,h2,w2, ...])` draws bars centered at values *x1*, *x2*, ... with heights *h1*, *h2*, ... and widths *w1*, *w2*, ...

This object is affected by the following *graphic options*: `key`, `fill_color`, `fill_density` and `line_width`.

Example:

```
(%i1) draw2d(
        key          = "Group A",
        fill_color   = blue,
        fill_density = 0.2,
        bars([0.8,5,0.4],[1.8,7,0.4],[2.8,-4,0.4]),
        key          = "Group B",
        fill_color   = red,
        fill_density = 0.6,
        line_width   = 4,
        bars([1.2,4,0.4],[2.2,-2,0.4],[3.2,5,0.4]),
        xaxis = true);
```



`cylindrical` (*`radius`*, *`z`*, *`minz`*, *`maxz`*, *`azi`*, *`minazi`*, *`maxazi`*) [Graphic object]

Draws 3D functions defined in cylindrical coordinates.

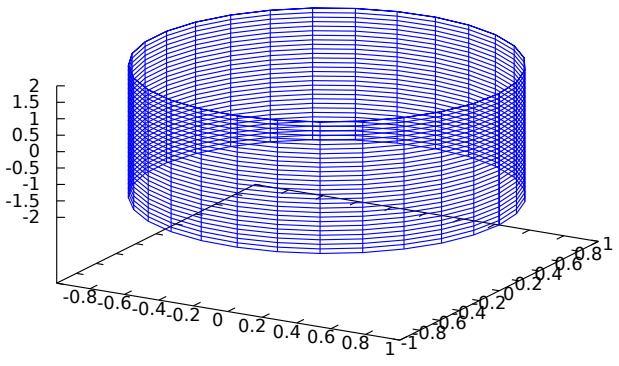
3D

`cylindrical(radius, z, minz, maxz, azi, minazi, maxazi)` plots the function `radius(z, azi)` defined in cylindrical coordinates, with variable *z* taking values from *minz* to *maxz* and *azimuth azi* taking values from *minazi* to *maxazi*.

This object is affected by the following *graphic options*: `xu_grid`, `yv_grid`, `line_type`, `key`, `wired_surface`, `enhanced3d` and `color`

Example:

```
(%i1) draw3d(cylindrical(1,z,-2,2,az,0,2*%pi))$
```



`elevation_grid` (*mat*,*x0*,*y0*,*width*,*height*) [Graphic object]

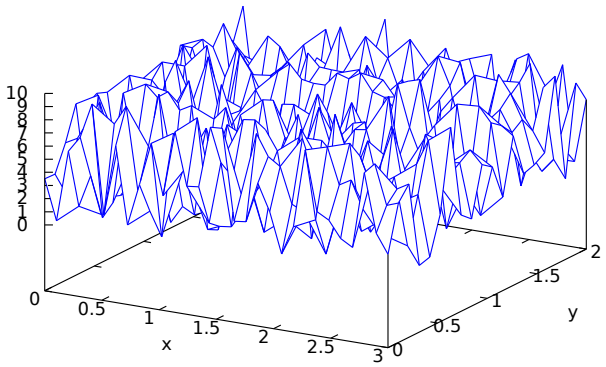
Draws matrix *mat* in 3D space. z values are taken from *mat*, the abscissas range from *x0* to $x0 + width$ and ordinates from *y0* to $y0 + height$. Element $a(1,1)$ is projected on point $(x0, y0 + height)$, $a(1,n)$ on $(x0 + width, y0 + height)$, $a(m,1)$ on $(x0, y0)$, and $a(m,n)$ on $(x0 + width, y0)$.

This object is affected by the following *graphic options*: `line_type`,, `line_width key`, `wired_surface`, `enhanced3d` and `color`

In older versions of Maxima, `elevation_grid` was called `mesh`. See also `mesh`.

Example:

```
(%i1) m: apply(
            matrix,
            makelist(makelist(random(10.0),k,1,30),i,1,20)) $
(%i2) draw3d(
          color = blue,
          elevation_grid(m,0,0,3,2),
          xlabel = "x",
          ylabel = "y",
          surface_hide = true);
```



`ellipse` (*xc*, *yc*, `a`, *b*, *ang1*, *ang2*) [Graphic object]

Draws ellipses and circles in 2D.

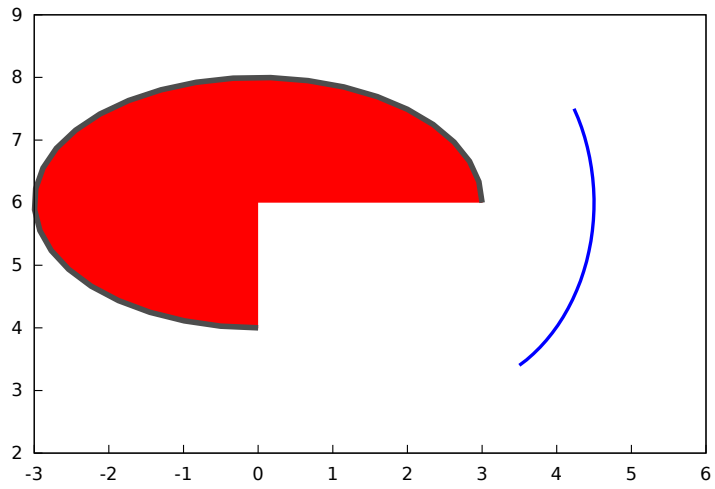
2D

`ellipse` (*xc*, *yc*, `a`, *b*, *ang1*, *ang2*) plots an ellipse centered at [*xc*, *yc*] with horizontal and vertical semi axis *a* and *b*, respectively, starting at angle *ang1* with an amplitude equal to angle *ang2*.

This object is affected by the following *graphic options*: `nticks`, `transparent`, `fill_color`, `fill_density`, `border`, `line_width`, `line_type`, `key` and `color`

Example:

```
(%i1) draw2d(transparent = false,
             fill_color  = red,
             color       = gray30,
             transparent = false,
             line_width  = 5,
             ellipse(0,6,3,2,270,-270),
             /* center (x,y), a, b, start & end in degrees */
             transparent = true,
             color       = blue,
             line_width  = 3,
             ellipse(2.5,6,2,3,30,-90),
             xrange      = [-3,6],
             yrange      = [2,9] )$
```



`errors` ([*x1*, *x2*, ...], [*y1*, *y2*, ...]) [Graphic object]

Draws points with error bars, horizontally, vertically or both, depending on the value of option `error_type`.

2D

If `error_type = x`, arguments to `errors` must be of the form `[x, y, xdelta]` or `[x, y, xlow, xhigh]`. If `error_type = y`, arguments must be of the form `[x, y, ydelta]` or `[x, y, ylow, yhigh]`. If `error_type = xy` or `error_type = boxes`, arguments to `errors` must be of the form `[x, y, xdelta, ydelta]` or `[x, y, xlow, xhigh, ylow, yhigh]`.

See also `error_type`.

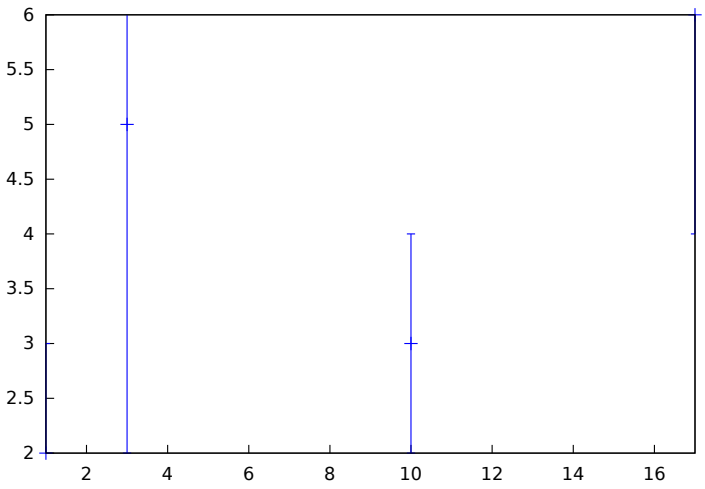
This object is affected by the following *graphic options*: `error_type`, `points_joined`, `line_width`, `key`, `line_type`, `color` `fill_density`, `xaxis_secondary` and `yaxis_secondary`.

Option `fill_density` is only relevant when `error_type=boxes`.

Examples:

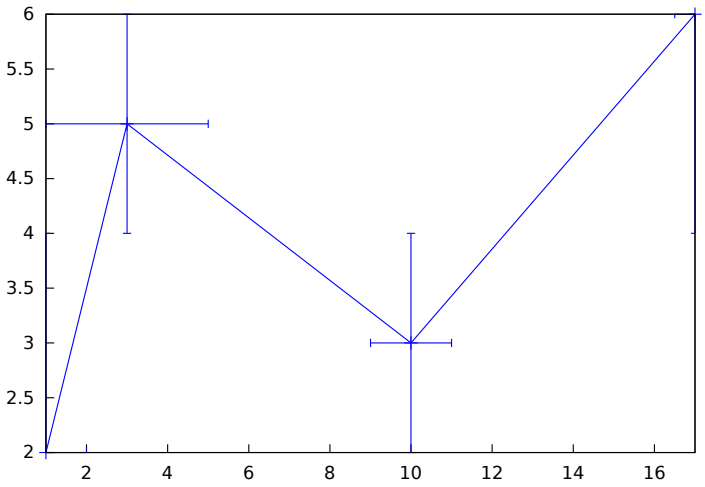
Horizontal error bars.

```
(%i1) draw2d(
        error_type = 'y,
        errors([[1,2,1], [3,5,3], [10,3,1], [17,6,2]]))$
```



Vertical and horizontal error bars.

```
(%i1) draw2d(
        error_type = 'xy,
        points_joined = true,
        color = blue,
        errors([[1,2,1,2], [3,5,2,1], [10,3,1,1], [17,6,1/2,2]]));
```



`explicit` [Graphic object]

`explicit` (*expr*,*var*,*minval*,*maxval*)
`explicit` (*fcn*,*var*,*minval*,*maxval*)
`explicit` (*expr*,*var1*,*minval1*,*maxval1*,*var2*,*minval2*,*maxval2*)
`explicit` (*fcn*,*var1*,*minval1*,*maxval1*,*var2*,*minval2*,*maxval2*)

Draws explicit functions in 2D and 3D.

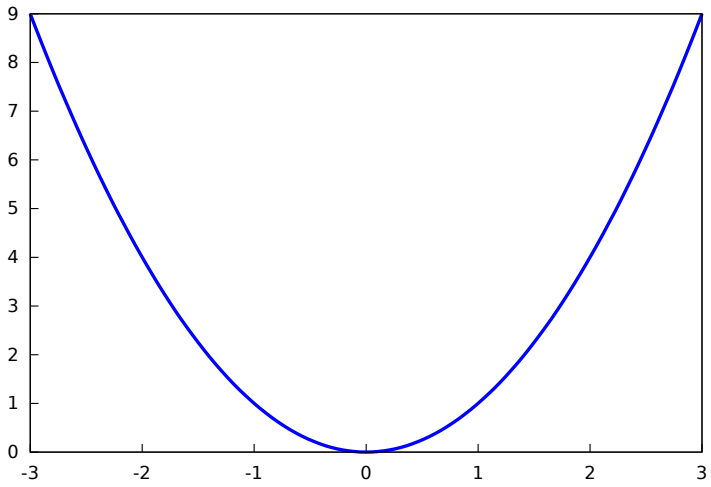
2D

`explicit(`*fcn*`,`*var*`,`*minval*`,`*maxval*`)` plots explicit function *fcn*, with variable *var* taking values from *minval* to *maxval*.

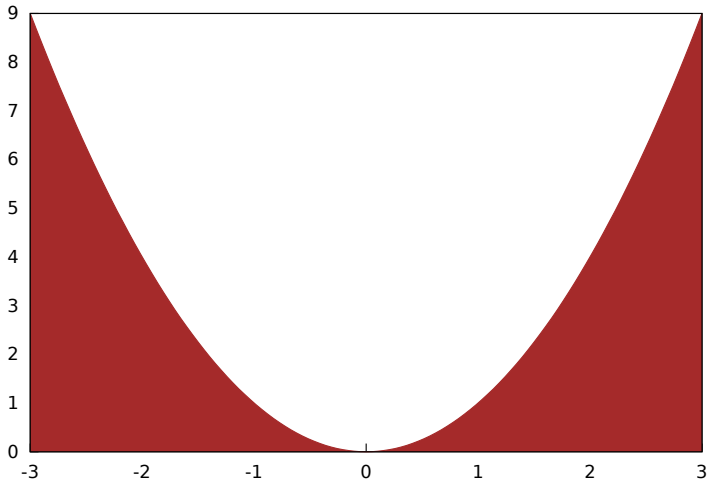
This object is affected by the following *graphic options*: nticks, adapt_depth, draw_realpart, line_width, line_type, key, filled_func, fill_color, fill_density, and color

Example:

```
(%i1) draw2d(line_width = 3,
             color      = blue,
             explicit(x^2,x,-3,3) )$
```




```
(%i2) draw2d(fill_color  = brown,
             filled_func = true,
             explicit(x^2,x,-3,3) )$
```



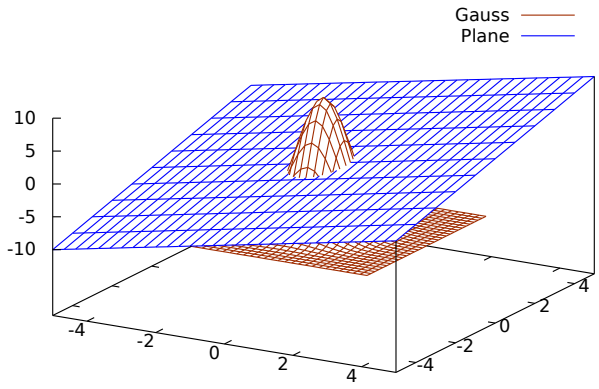
3D

explicit(*fcn*, *var1*, *minval1*, *maxval1*, *var2*, *minval2*, *maxval2*) plots the explicit function *fcn*, with variable *var1* taking values from *minval1* to *maxval1* and variable *var2* taking values from *minval2* to *maxval2*.

This object is affected by the following *graphic options*: `draw_realpart`, `xu_grid`, `yv_grid`, `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```
(%i1) draw3d(key   = "Gauss",
             color = "#a02c00",
             explicit(20*exp(-x^2-y^2)-10,x,-3,3,y,-3,3),
             yv_grid     = 10,
             color = blue,
             key   = "Plane",
             explicit(x+y,x,-5,5,y,-5,5),
             surface_hide = true)$
```



See also `filled_func` for filled functions.

`image` (*im*,*x0*,*y0*,*width*,*height*) [Graphic object]

Renders images in 2D.

2D

`image` (*im*,*x0*,*y0*,*width*,*height*) plots image *im* in the rectangular region from vertex (*x0*,*y0*) to (x0+*width*,y0+*height*) on the real plane. Argument *im* must be a matrix of real numbers, a matrix of vectors of length three or a *picture* object.

If *im* is a matrix of real numbers or a *levels picture* object, pixel values are interpreted according to graphic option `palette`, which is a vector of length three with components ranging from -36 to +36; each value is an index for a formula mapping the levels onto red, green and blue colors, respectively:

```
 0: 0              1: 0.5          2: 1
 3: x              4: x^2          5: x^3
 6: x^4            7: sqrt(x)      8: sqrt(sqrt(x))
 9: sin(90x)      10: cos(90x)    11: |x-0.5|
12: (2x-1)^2      13: sin(180x)   14: |cos(180x)|
15: sin(360x)     16: cos(360x)   17: |sin(360x)|
18: |cos(360x)|   19: |sin(720x)| 20: |cos(720x)|
```

```
21: 3x              22: 3x-1          23: 3x-2
24: |3x-1|          25: |3x-2|        26: (3x-1)/2
27: (3x-2)/2        28: |(3x-1)/2|    29: |(3x-2)/2|
30: x/0.32-0.78125                    31: 2*x-0.84
32: 4x;1;-2x+1.84;x/0.08-11.5
33: |2*x - 0.5|     34: 2*x           35: 2*x - 0.5
36: 2*x - 1
```

negative numbers mean negative colour component.

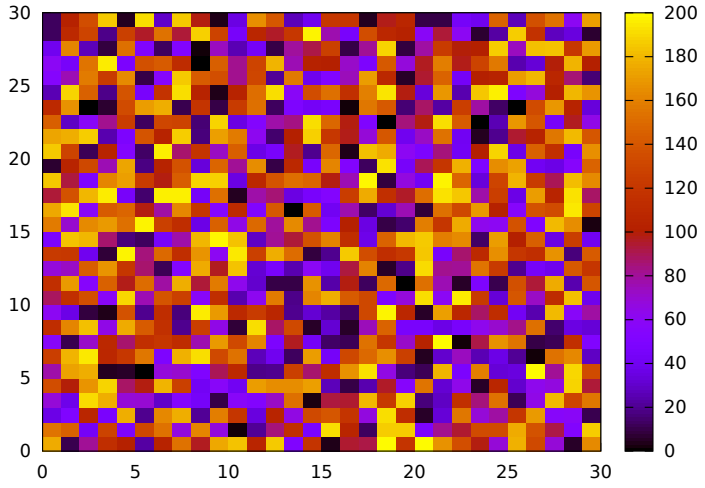
`palette = gray` and `palette = color` are short cuts for `palette = [3,3,3]` and `palette = [7,5,15]`, respectively.

If *im* is a matrix of vectors of length three or an *rgb picture* object, they are interpreted as red, green and blue color components.

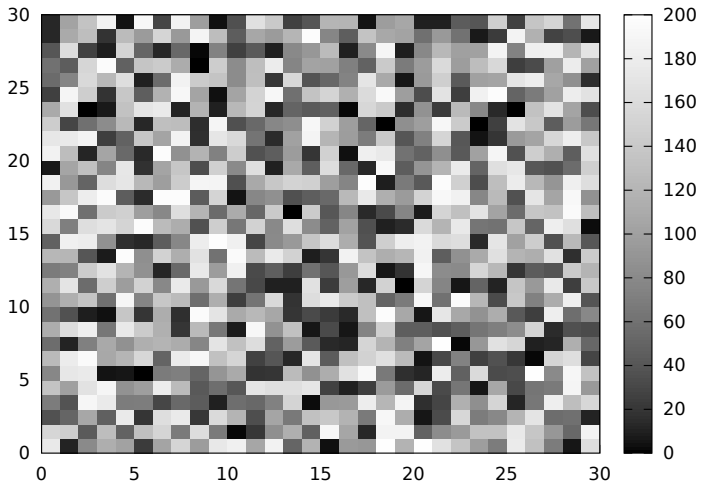
Examples:

If *im* is a matrix of real numbers, pixel values are interpreted according to graphic option `palette`.

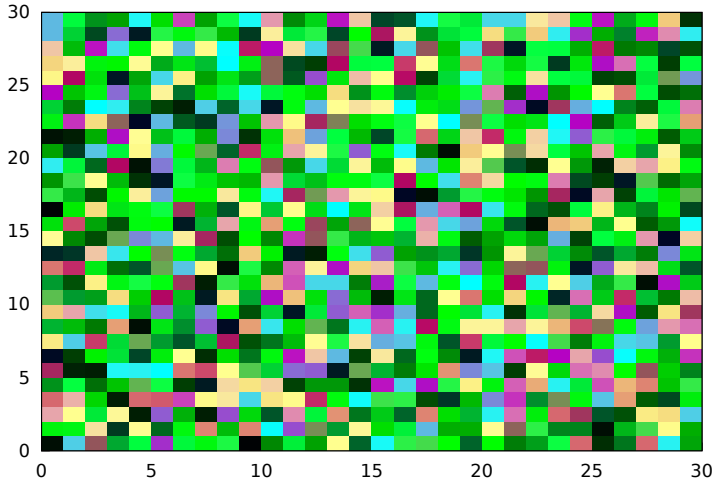
```
(%i1) im: apply(
           'matrix,
            makelist(makelist(random(200),i,1,30),i,1,30))$
(%i2) /* palette = color, default */
      draw2d(image(im,0,0,30,30))$
```




```
(%i3) draw2d(palette = gray, image(im,0,0,30,30))$
```



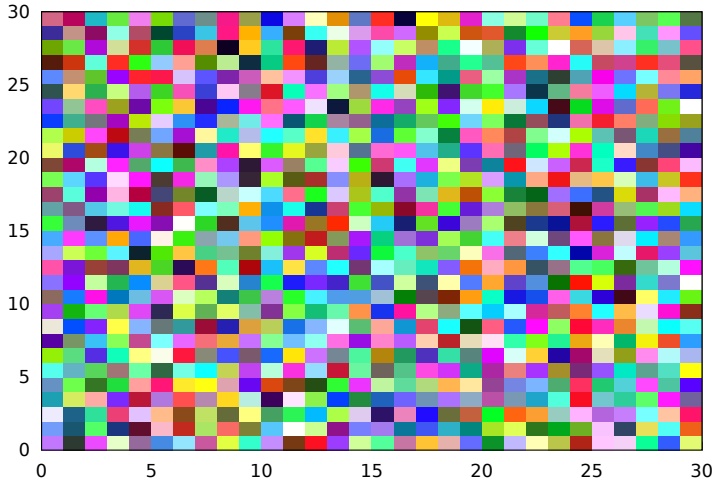

```
(%i4) draw2d(palette = [15,20,-4],
             colorbox=false,
             image(im,0,0,30,30))$
```



See also `colorbox`.

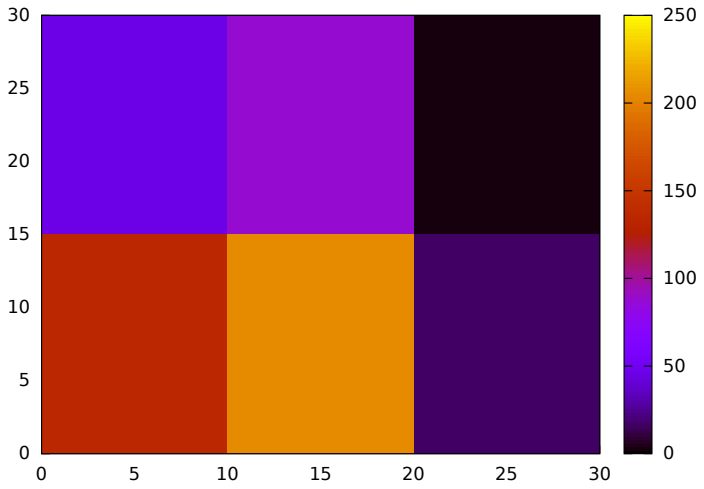
If *im* is a matrix of vectors of length three, they are interpreted as red, green and blue color components.

```
(%i1) im: apply(
            'matrix,
            makelist(
              makelist([random(300),
                        random(300),
                        random(300)],i,1,30),i,1,30))$
(%i2) draw2d(image(im,0,0,30,30))$
```

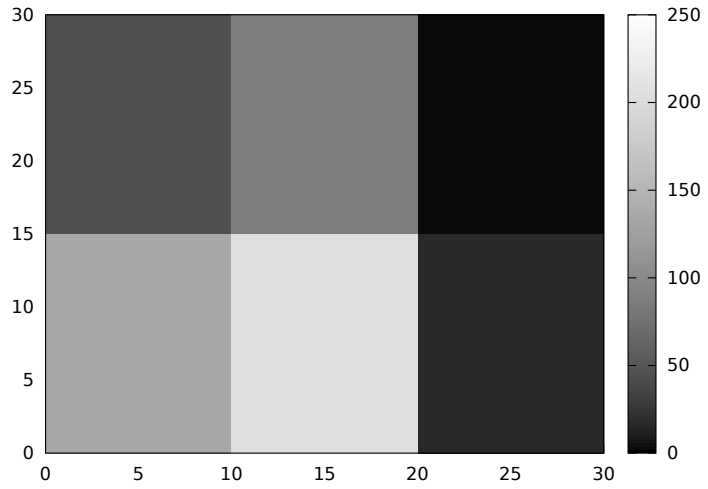


Package `draw` automatically loads package `picture`. In this example, a level picture object is built by hand and then rendered.

```
(%i1) im: make_level_picture([45,87,2,134,204,16],3,2);
(%o1)       picture(level, 3, 2, {Array:  #(45 87 2 134 204 16)})
(%i2) /* default color palette */
      draw2d(image(im,0,0,30,30))$
```




```
(%i3) /* gray palette */
      draw2d(palette = gray,
             image(im,0,0,30,30))$
```



An xpm file is read and then rendered.

```
(%i1) im: read_xpm("myfile.xpm")$
(%i2) draw2d(image(im,0,0,10,7))$
```

See also `make_level_picture`, `make_rgb_picture` and `read_xpm`.

http://www.telefonica.net/web2/biomates/maxima/gpdraw/image
contains more elaborated examples.

`implicit` [Graphic object]

`implicit` (*fcn*,*x*,*xmin*,*xmax*,*y*,*ymin*,*ymax*)
`implicit` (*fcn*,*x*,*xmin*,*xmax*,*y*,*ymin*,*ymax*,*z*,*zmin*,*zmax*)

Draws implicit functions in 2D and 3D.

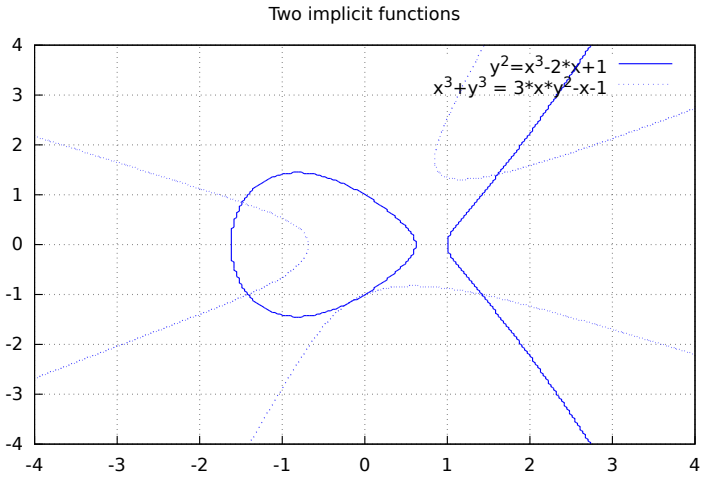
2D

`implicit(`*fcn*`,`*x*`,`*xmin*`,`*xmax*`,`*y*`,`*ymin*`,`*ymax*`)` plots the implicit function defined by *fcn*, with variable *x* taking values from *xmin* to *xmax*, and variable *y* taking values from *ymin* to *ymax*.

This object is affected by the following *graphic options*: `ip_grid`, `ip_grid_in`, `line_width`, `line_type`, `key` and `color`.

Example:

```
(%i1) draw2d(grid      = true,
             line_type = solid,
             key       = "y^2=x^3-2*x+1",
             implicit(y^2=x^3-2*x+1, x, -4,4, y, -4,4),
             line_type = dots,
             key       = "x^3+y^3 = 3*x*y^2-x-1",
             implicit(x^3+y^3 = 3*x*y^2-x-1, x,-4,4, y,-4,4),
             title     = "Two implicit functions" )$
```



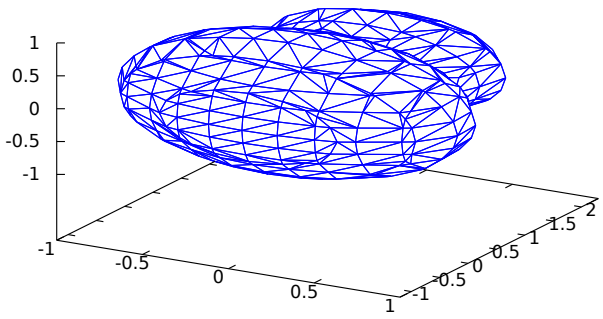
3D

`implicit (`*`fcn`*`,x,`*`xmin`*`,`*`xmax`*`, y,`*`ymin`*`,`*`ymax`*`, z,`*`zmin`*`,`*`zmax`*`)` plots the implicit surface defined by *fcn*, with variable *x* taking values from *xmin* to *xmax*, variable *y* taking values from *ymin* to *ymax* and variable *z* taking values from *zmin* to *zmax*. This object implements the *marching cubes algorithm*.

This object is affected by the following *graphic options*: `x_voxel`, `y_voxel`, `z_voxel`, `line_width`, `line_type`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```
(%i1) draw3d(
        color=blue,
        implicit((x^2+y^2+z^2-1)*(x^2+(y-1.5)^2+z^2-0.5)=0.015,
                 x,-1,1,y,-1.2,2.3,z,-1,1),
        surface_hide=true);
```



`label` [Graphic object]

`label` (`[`*string*`,`*x*`,`*y*`]`,...)
`label` (`[`*string*`,`*x*`,`*y*`,`*z*`]`,...)

Writes labels in 2D and 3D.

Colored labels work only with Gnuplot 4.3 and up.

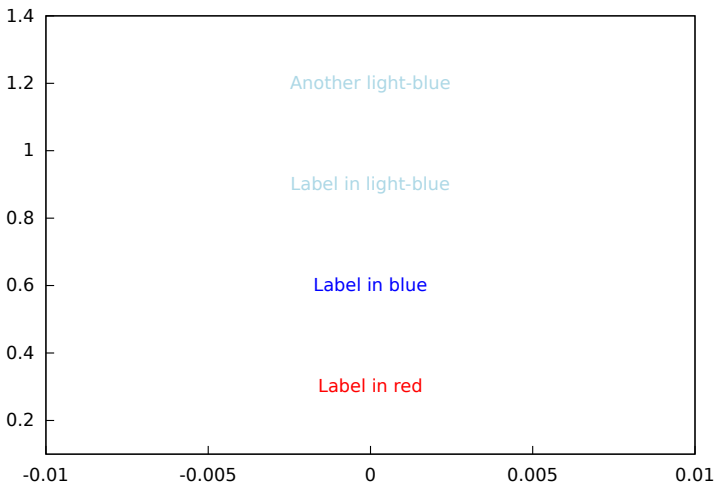
This object is affected by the following *graphic options*: `label_alignment`, `label_orientation` and `color`.

2D

`label([`*string*`,`*x*`,`*y*`])` writes the *string* at point `[`*x*`,`*y*`]`.

Example:

```
(%i1) draw2d(yrange = [0.1,1.4],
             color = red,
             label(["Label in red",0,0.3]),
             color = "#0000ff",
             label(["Label in blue",0,0.6]),
             color = light_blue,
             label(["Label in light-blue",0,0.9],
                   ["Another light-blue",0,1.2])  )$
```

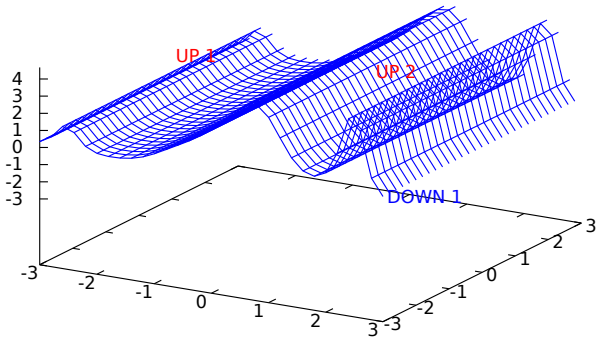


3D

`label([`*string*`,`*x*`,`*y*`,`*z*`])` writes the *string* at point `[`*x*`,`*y*`,`*z*`]`.

Example:

```
(%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),
             color = red,
             label(["UP 1",-2,0,3], ["UP 2",1.5,0,4]),
             color = blue,
             label(["DOWN 1",2,0,-3]) )$
```



`mesh` (*row_1*,*row_2*,...) [Graphic object]

Draws a quadrangular mesh in 3D.

3D

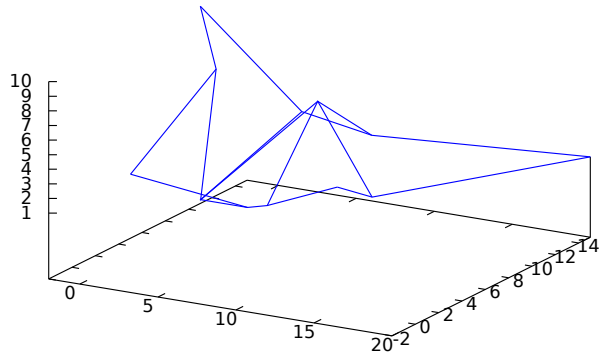
Argument *row_i* is a list of *n* 3D points of the form `[[x_i1,y_i1,z_i1], ...,[x_in,y_in,z_in]]`, and all rows are of equal length. All these points define an arbitrary surface in 3D and in some sense it's a generalization of the `elevation_grid` object.

This object is affected by the following *graphic options*: `line_type`, `line_width`, `color`, `key`, `wired_surface`, `enhanced3d` and `transform`.

Examples:

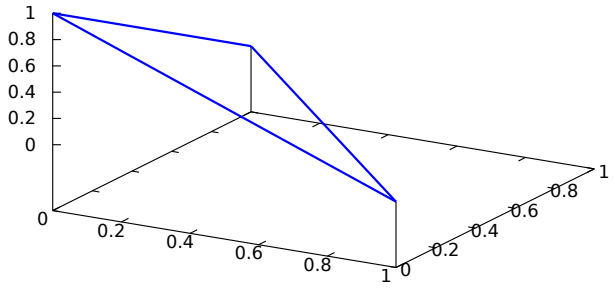
A simple example.

```
(%i1) draw3d(
        mesh([[1,1,3],   [7,3,1],[12,-2,4],[15,0,5]],
             [[2,7,8],   [4,3,1],[10,5,8], [12,7,1]],
             [[-2,11,10],[6,9,5],[6,15,1], [20,15,2]])) $
```



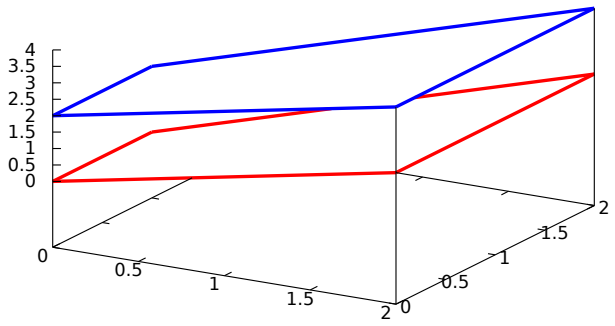
Plotting a triangle in 3D.

```
(%i1) draw3d(
        line_width = 2,
        mesh([[1,0,0],[0,1,0]],
             [[0,0,1],[0,0,1]])) $
```



Two quadrilaterals.

```
(%i1) draw3d(
        surface_hide = true,
        line_width   = 3,
        color = red,
        mesh([[0,0,0], [0,1,0]],
             [[2,0,2], [2,2,2]]),
        color = blue,
        mesh([[0,0,2], [0,1,2]],
             [[2,0,4], [2,2,4]])) $
```



`parametric` [Graphic object]

`parametric` (*xfun*,*yfun*,*par*,*parmin*,*parmax*)
`parametric` (*xfun*,*yfun*,*zfun*,*par*,*parmin*,*parmax*)

Draws parametric functions in 2D and 3D.

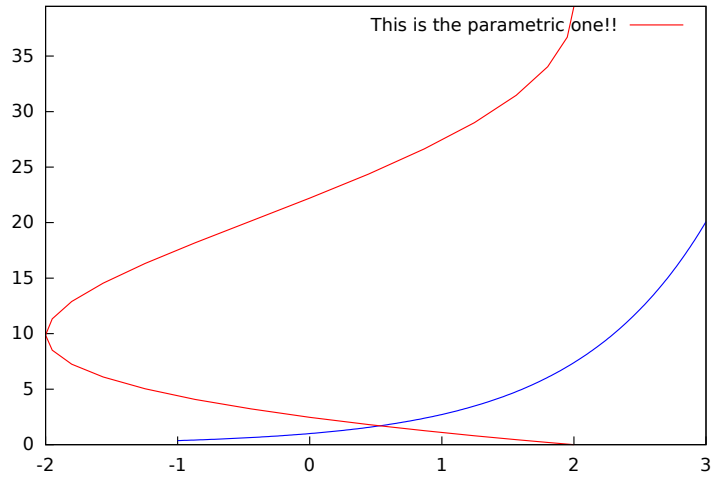
This object is affected by the following *graphic options*: `nticks`, `line_width`, `line_type`, `key`, `color` and `enhanced3d`.

2D

The command `parametric(`*xfun*`, `*yfun*`, `*par*`, `*parmin*`, `*parmax*`)` plots the parametric function `[`*xfun*`, `*yfun*`]`, with parameter *par* taking values from *parmin* to *parmax*.

Example:

```
(%i1) draw2d(explicit(exp(x),x,-1,3),
             color = red,
             key   = "This is the parametric one!!",
             parametric(2*cos(rrr),rrr^2,rrr,0,2*%pi))$
```

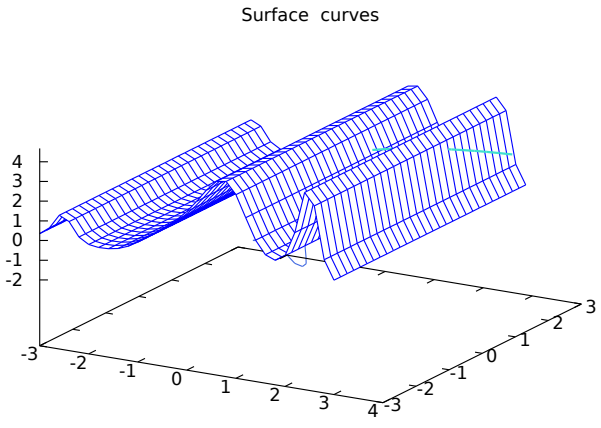


3D

`parametric(`*xfun*`, `*yfun*`, `*zfun*`, `*par*`, `*parmin*`, `*parmax*`)` plots the parametric curve `[`*xfun*`, `*yfun*`, `*zfun*`]`, with parameter *par* taking values from *parmin* to *parmax*.

Example:

```
(%i1) draw3d(explicit(exp(sin(x)+cos(x^2)),x,-3,3,y,-3,3),
             color = royalblue,
             parametric(cos(5*u)^2,sin(7*u),u-2,u,0,2),
             color      = turquoise,
             line_width = 2,
             parametric(t^2,sin(t),2+t,t,0,2),
             surface_hide = true,
             title = "Surface & curves" )$
```



`parametric_surface` (*xfun*, *yfun*, *zfun*, *par1*, *par1min*, *par1max*, *par2*, *par2min*, *par2max*) [Graphic object]

Draws parametric surfaces in 3D.

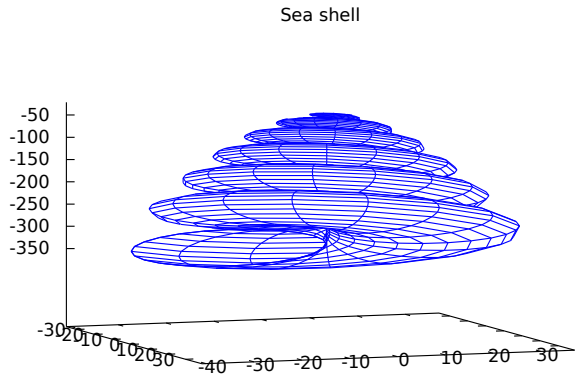
3D

The command `parametric_surface(`*xfun*`, `*yfun*`, `*zfun*`, `*par1*`, `*par1min*`, `*par1max*`, `*par2*`, `*par2min*`, `*par2max*`)` plots the parametric surface `[`*xfun*`, `*yfun*`, `*zfun*`]`, with parameter *par1* taking values from *par1min* to *par1max* and parameter *par2* taking values from *par2min* to *par2max*.

This object is affected by the following *graphic options*: `draw_realpart`, `xu_grid`, `yv_grid`, `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```
(%i1) draw3d(title          = "Sea shell",
             xu_grid        = 100,
             yv_grid        = 25,
             view           = [100,20],
             surface_hide   = true,
             parametric_surface(0.5*u*cos(u)*(cos(v)+1),
                                0.5*u*sin(u)*(cos(v)+1),
                                u*sin(v) - ((u+3)/8*%pi)^2 - 20,
                                u, 0, 13*%pi, v, -%pi, %pi) )$
```



`points` [Graphic object]

`points` ([[*x1*,*y1*], [*x2*,*y2*],...])
`points` ([*x1*,*x2*,...], [*y1*,*y2*,...])
`points` ([*y1*,*y2*,...])
`points` ([[*x1*,*y1*,*z1*], [*x2*,*y2*,*z2*],...])
`points` ([*x1*,*x2*,...], [*y1*,*y2*,...], [*z1*,*z2*,...])
`points` (*matrix*)
`points` (*1d_y_array*)
`points` (*1d_x_array*, *1d_y_array*)
`points` (*1d_x_array*, *1d_y_array*, *1d_z_array*)
`points` (*2d_xy_array*)
`points` (*2d_xyz_array*)

Draws points in 2D and 3D.

This object is affected by the following *graphic options*: `point_size`, `point_type`, `points_joined`, `line_width`, `key`, `line_type` and `color`. In 3D mode, it is also affected by `enhanced3d`

2D

`points ([[`*x1*`,`*y1*`], [`*x2*`,`*y2*`],...])` or `points ([`*x1*`,`*x2*`,...], [`*y1*`,`*y2*`,...])` plots points `[x1,y1]`, `[x2,y2]`, etc. If abscissas are not given, they are set to consecutive positive integers, so that `points ([`*y1*`,`*y2*`,...])` draws points `[1,`*y1*`]`, `[2,`*y2*`]`, etc. If *matrix* is a two-column or two-row matrix, `points (`*matrix*`)` draws the associated points. If *matrix* is an one-column or one-row matrix, abscissas are assigned automatically.

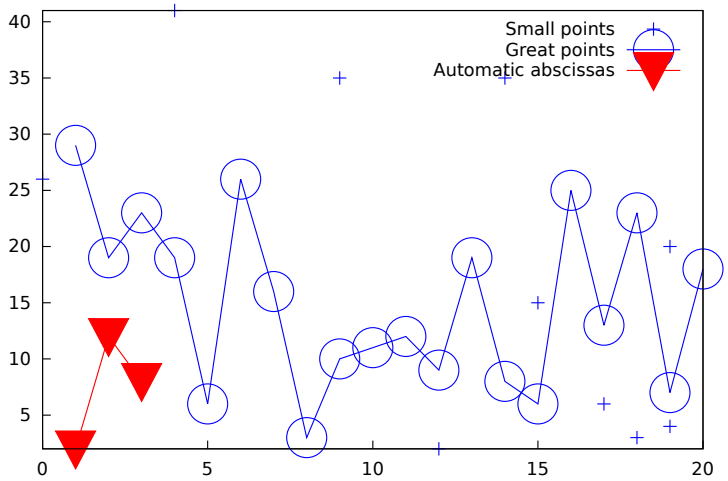
If *1d_y_array* is a 1D lisp array of numbers, `points (`*1d_y_array*`)` plots them setting abscissas to consecutive positive integers. `points (`*1d_x_array*`, `*1d_y_array*`)` plots points with their coordinates taken from the two arrays passed as arguments. If *2d_xy_array* is a 2D array with two columns, or with two rows, `points (`*2d_xy_array*`)` plots the corresponding points on the plane.

Examples:

Two types of arguments for `points`, a list of pairs and two lists of separate coordinates.

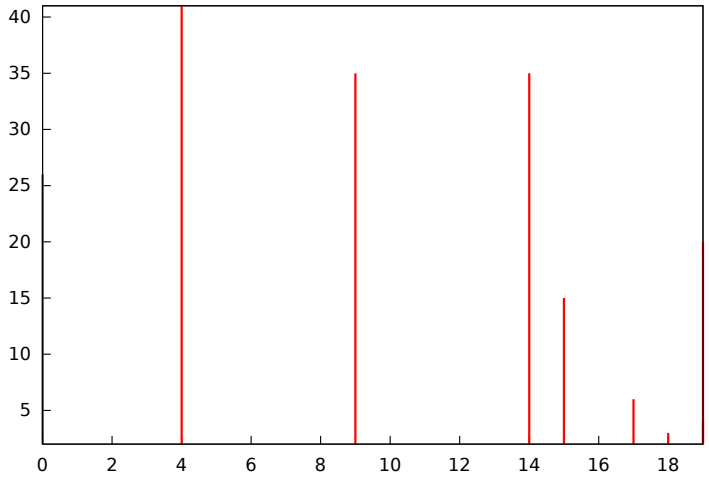
```
(%i1) draw2d(
```

```
        key = "Small points",
        points(makelist([random(20),random(50)],k,1,10)),
        point_type    = circle,
        point_size    = 3,
        points_joined = true,
        key           = "Great points",
        points(makelist(k,k,1,20),makelist(random(30),k,1,20)),
        point_type    = filled_down_triangle,
        key           = "Automatic abscissas",
        color         = red,
        points([2,12,8]))$
```



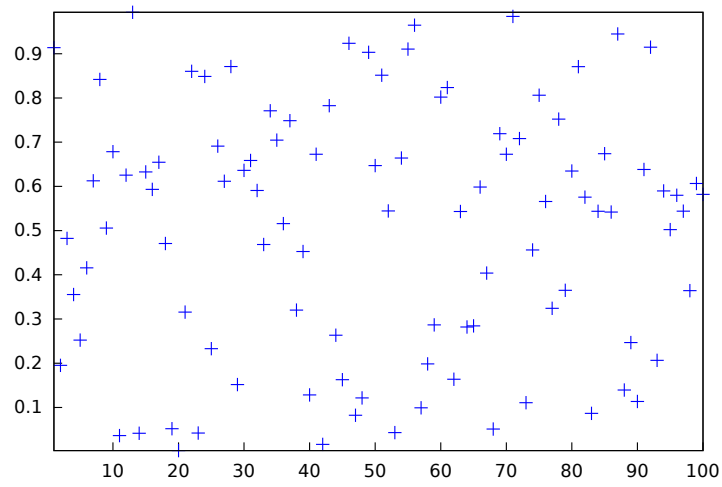
Drawing impulses.

```
(%i1) draw2d(
        points_joined = impulses,
        line_width    = 2,
        color         = red,
        points(makelist([random(20),random(50)],k,1,10)))$
```



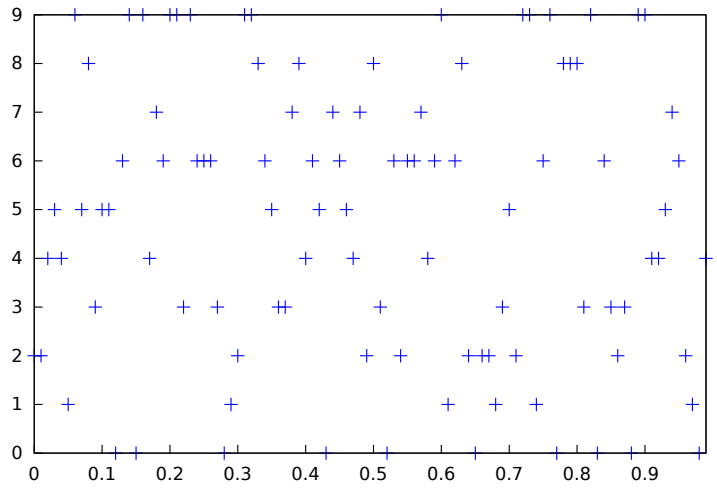
Array with ordinates.

```
(%i1) a: make_array (flonum, 100) $
(%i2) for i:0 thru 99 do a[i]: random(1.0) $
(%i3) draw2d(points(a)) $
```



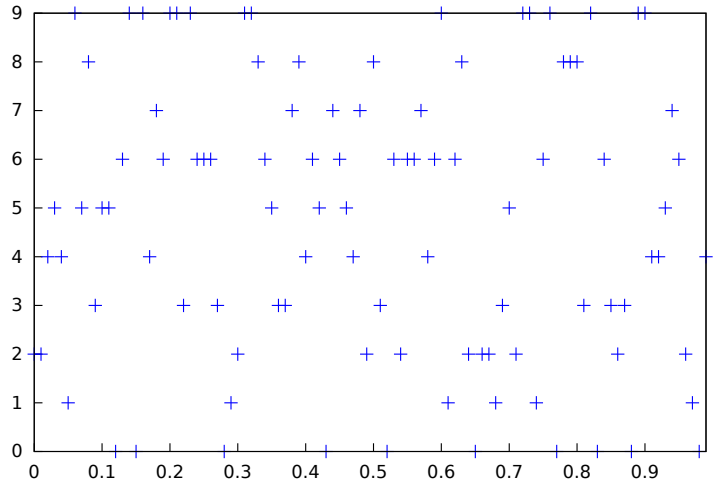
Two arrays with separate coordinates.

```
(%i1) x: make_array (flonum, 100) $
(%i2) y: make_array (fixnum, 100) $
(%i3) for i:0 thru 99 do (
        x[i]: float(i/100),
        y[i]: random(10) ) $
(%i4) draw2d(points(x, y)) $
```



A two-column 2D array.

```
(%i1) xy: make_array(flonum, 100, 2) $
(%i2) for i:0 thru 99 do (
        xy[i, 0]: float(i/100),
        xy[i, 1]: random(10) ) $
(%i3) draw2d(points(xy)) $
```



Drawing an array filled with function `read_array`.

```
(%i1) a: make_array(flonum,100) $
(%i2) read_array (file_search ("pidigits.data"), a) $
(%i3) draw2d(points(a)) $
```

3D

`points([[`*x1*`,` *y1*`,` *z1*`], [`*x2*`,` *y2*`,` *z2*`], ...])` or `points([`*x1*`,` *x2*`, ...], [`*y1*`,` *y2*`, ...], [`*z1*`,` *z2*`,...])` plots points `[`*x1*`,` *y1*`,` *z1*`]`, `[`*x2*`,` *y2*`,` *z2*`]`, etc. If *matrix* is a three-column or three-row matrix, `points (`*matrix*`)` draws the associated points.

When arguments are lisp arrays, `points (1d_x_array, 1d_y_array, 1d_z_array)` takes coordinates from the three 1D arrays. If *2d_xyz_array* is a 2D array with three columns, or with three rows, `points (2d_xyz_array)` plots the corresponding points.

Examples:

One tridimensional sample,

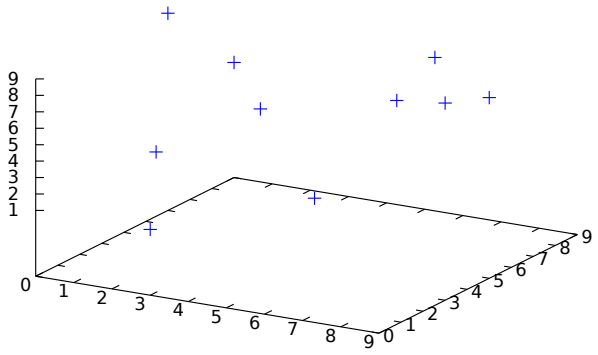
```
(%i1) load ("numericalio")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) draw3d(title = "Daily average wind speeds",
             point_size = 2,
             points(args(submatrix (s2, 4, 5))) )$
```

Two tridimensional samples,

```
(%i1) load ("numericalio")$
(%i2) s2 : read_matrix (file_search ("wind.data"))$
(%i3) draw3d(
         title = "Daily average wind speeds. Two data sets",
         point_size = 2,
         key        = "Sample from stations 1, 2 and 3",
         points(args(submatrix (s2, 4, 5))),
         point_type = 4,
         key        = "Sample from stations 1, 4 and 5",
         points(args(submatrix (s2, 2, 3))) )$
```

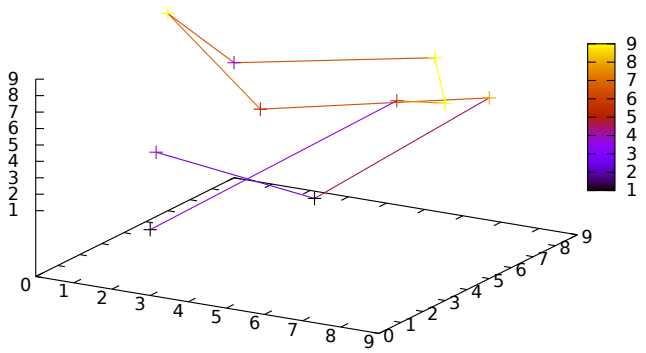
Unidimensional arrays,

```
(%i1) x: make_array (fixnum, 10) $
(%i2) y: make_array (fixnum, 10) $
(%i3) z: make_array (fixnum, 10) $
(%i4) for i:0 thru 9 do (
        x[i]: random(10),
        y[i]: random(10),
        z[i]: random(10) ) $
(%i5) draw3d(points(x,y,z)) $
```



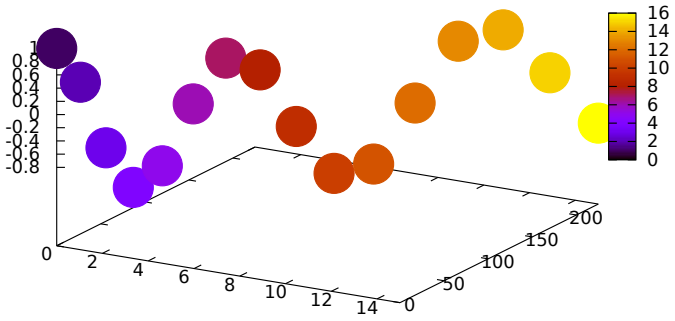
Bidimensional colored array,

```
(%i1) xyz: make_array(fixnum, 10, 3) $
(%i2) for i:0 thru 9 do (
        xyz[i, 0]: random(10),
        xyz[i, 1]: random(10),
        xyz[i, 2]: random(10) ) $
(%i3) draw3d(
         enhanced3d = true,
         points_joined = true,
         points(xyz)) $
```



Color numbers explicitly specified by the user.

```
(%i1) pts: makelist([t,t^2,cos(t)], t, 0, 15)$
(%i2) col_num: makelist(k, k, 1, length(pts))$
(%i3) draw3d(
        enhanced3d = ['part(col_num,k),k],
        point_size = 3,
        point_type = filled_circle,
        points(pts))$
```



`polar` (*radius*,*ang*,*minang*,*maxang*) [Graphic object]

Draws 2D functions defined in polar coordinates.

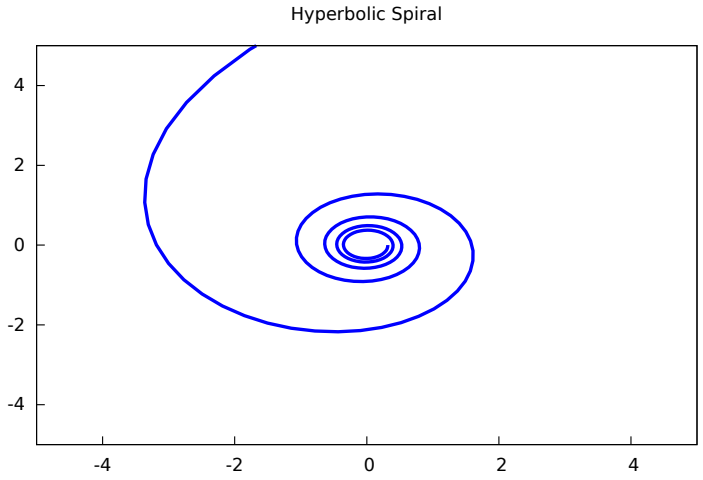
2D

`polar` (*radius*,*ang*,*minang*,*maxang*) plots function *radius*(*ang*) defined in polar coordinates, with variable *ang* taking values from *minang* to *maxang*.

This object is affected by the following *graphic options*: `nticks`, `line_width`, `line_type`, `key` and `color`.

Example:

```
(%i1) draw2d(user_preamble = "set grid polar",
             nticks        = 200,
             xrange        = [-5,5],
             yrange        = [-5,5],
             color         = blue,
             line_width    = 3,
             title         = "Hyperbolic Spiral",
             polar(10/theta,theta,1,10*%pi) )$
```



`polygon` [Graphic object]

`polygon` ([[*x1*, *y1*], [*x2*, *y2*], ...])
`polygon` ([*x1*, *x2*, ...], [*y1*, *y2*, ...])

Draws polygons in 2D.

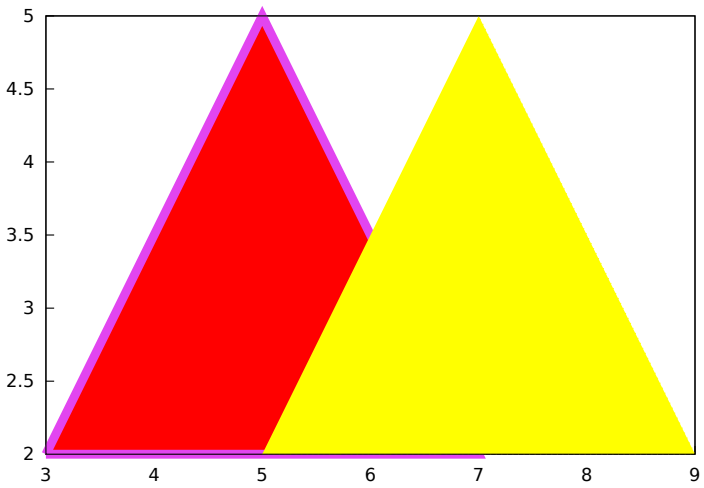
2D

The commands `polygon([[x1, y1], [x2, y2], ...])` or `polygon([x1, x2, ...], [y1, y2, ...])` plot on the plane a polygon with vertices `[x1, y1]`, `[x2, y2]`, etc.

This object is affected by the following *graphic options*: `transparent`, `fill_color`, `fill_density`, `border`, `line_width`, `key`, `line_type` and `color`.

Example:

```
(%i1) draw2d(color      = "#e245f0",
             line_width = 8,
             polygon([[3,2],[7,2],[5,5]]),
             border      = false,
             fill_color  = yellow,
             polygon([[5,2],[9,2],[7,5]]) )$
```



`quadrilateral` (*point_1*, *point_2*, *point_3*, *point_4*) [Graphic object]

Draws a quadrilateral.

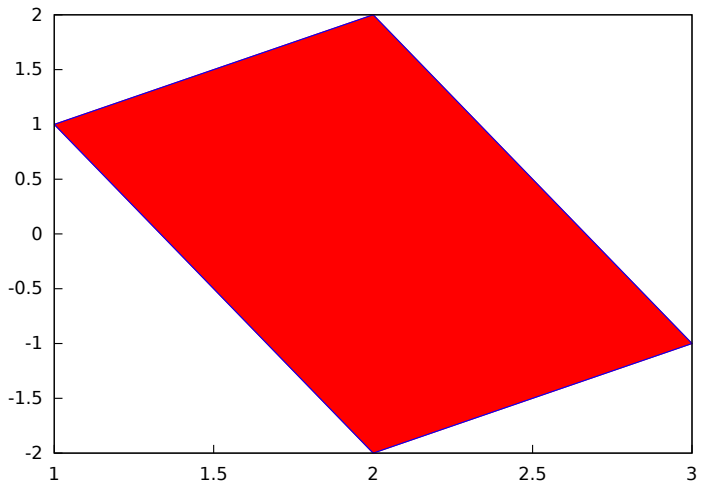
2D

`quadrilateral([x1, y1], [x2, y2], [x3, y3], [x4, y4])` draws a quadrilateral with vertices `[x1, y1]`, `[x2, y2]`, `[x3, y3]`, and `[x4, y4]`.

This object is affected by the following *graphic options*: `transparent`, `fill_color`, `border`, `line_width`, `key`, `xaxis_secondary`, `yaxis_secondary`, `line_type`, `transform` and `color`.

Example:

```
(%i1) draw2d(
        quadrilateral([1,1],[2,2],[3,-1],[2,-2]))$
```



3D

`quadrilateral([x1, y1, z1], [x2, y2, z2], [x3, y3, z3], [x4, y4, z4])` draws a quadrilateral with vertices `[x1, y1, z1]`, `[x2, y2, z2]`, `[x3, y3, z3]`, and `[x4, y4, z4]`.

This object is affected by the following *graphic options*: `line_type`, `line_width`, `color`, `key`, `enhanced3d` and `transform`.

`rectangle ([x1,y1], [x2,y2])` [Graphic object]

Draws rectangles in 2D.

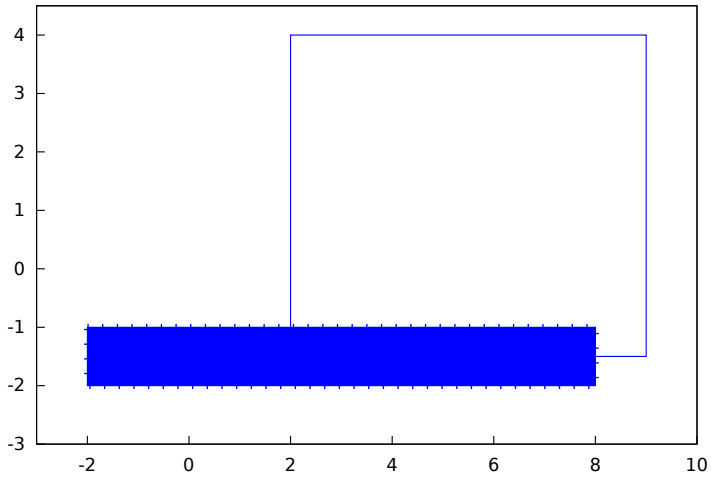
2D

`rectangle ([x1,y1], [x2,y2])` draws a rectangle with opposite vertices `[x1,y1]` and `[x2,y2]`.

This object is affected by the following *graphic options*: `transparent`, `fill_color`, `border`, `line_width`, `key`, `line_type` and `color`.

Example:

```
(%i1) draw2d(fill_color  = red,
             line_width  = 6,
             line_type   = dots,
             transparent = false,
             fill_color  = blue,
             rectangle([-2,-2],[8,-1]), /* opposite vertices */
             transparent = true,
             line_type   = solid,
             line_width  = 1,
             rectangle([9,4],[2,-1.5]),
             xrange      = [-3,10],
             yrange      = [-3,4.5] )$
```



`region` (*expr*,*var1*,*minval1*,*maxval1*,*var2*,*minval2*,*maxval2*) [Graphic object]

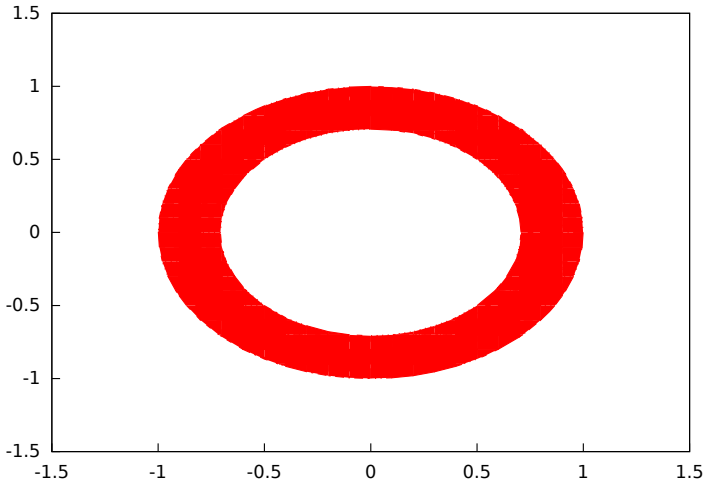
Plots a region on the plane defined by inequalities.

2D *expr* is an expression formed by inequalities and boolean operators `and`, `or`, and `not`. The region is bounded by the rectangle defined by [*minval1*, *maxval1*] and [*minval2*, *maxval2*].

This object is affected by the following *graphic options*: `fill_color`, `fill_density`, `key`, `x_voxel` and `y_voxel`.

Example:

```
(%i1) draw2d(
        x_voxel = 30,
        y_voxel = 30,
        region(x^2+y^2<1 and x^2+y^2 > 1/2,
               x, -1.5, 1.5, y, -1.5, 1.5));
```



`spherical` (*radius*, *azi*, *minazi*, *maxazi*, *zen*, *minzen*, *maxzen*) [Graphic object]

Draws 3D functions defined in spherical coordinates.

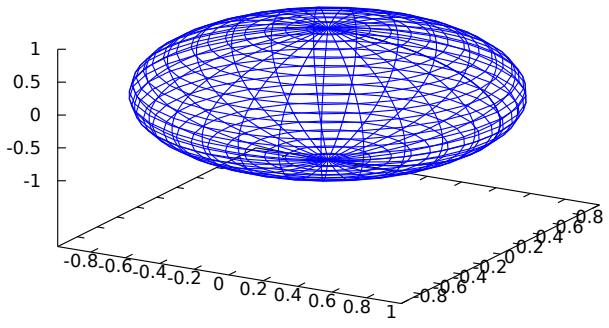
3D

`spherical(`*radius*`, azi, minazi, maxazi, zen, minzen, maxzen)` plots the function *radius*`(azi, zen)` defined in spherical coordinates, with *azimuth* *azi* taking values from *minazi* to *maxazi* and *zenith* *zen* taking values from *minzen* to *maxzen*.

This object is affected by the following *graphic options*: `xu_grid`, `yv_grid`, `line_type`, `key`, `wired_surface`, `enhanced3d` and `color`.

Example:

```
(%i1) draw3d(spherical(1,a,0,2*%pi,z,0,%pi))$
```



`triangle` (*point_1*, *point_2*, *point_3*) [Graphic object]

Draws a triangle.

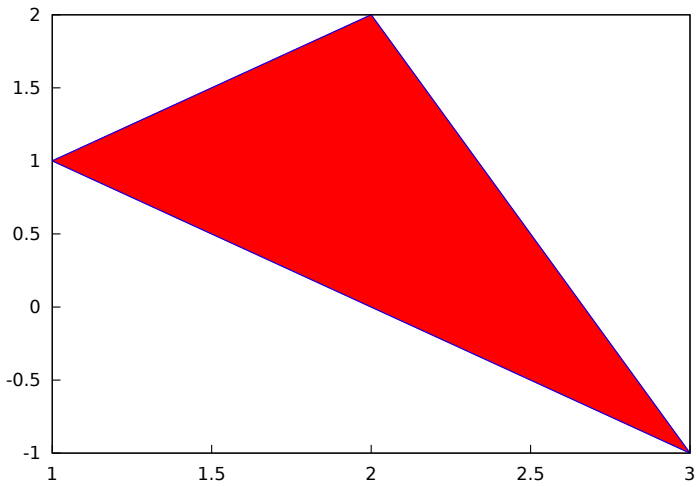
2D

`triangle ([`*x1*`,`*y1*`], [`*x2*`,`*y2*`], [`*x3*`,`*y3*`])` draws a triangle with vertices `[`*x1*`,`*y1*`]`, `[`*x2*`,`*y2*`]`, and `[`*x3*`,`*y3*`]`.

This object is affected by the following *graphic options*:
`transparent`, `fill_color`, `border`, `line_width`, `key`, `xaxis_secondary`, `yaxis_secondary`, `line_type`, `transform` and `color`.

Example:

```
(%i1) draw2d(
        triangle([1,1],[2,2],[3,-1]))$
```



3D

`triangle ([x1,y1,z1], [x2,y2,z2], [x3,y3,z3])` draws a triangle with vertices `[x1,y1,z1]`, `[x2,y2,z2]`, and `[x3,y3,z3]`.

This object is affected by the following *graphic options*: `line_type`, `line_width`, `color`, `key`, `enhanced3d` and `transform`.

`tube` (*xfun*,*yfun*,*zfun*,*rfun*,*p*,*pmin*,*pmax*) [Graphic object]

Draws a tube in 3D with varying diameter.

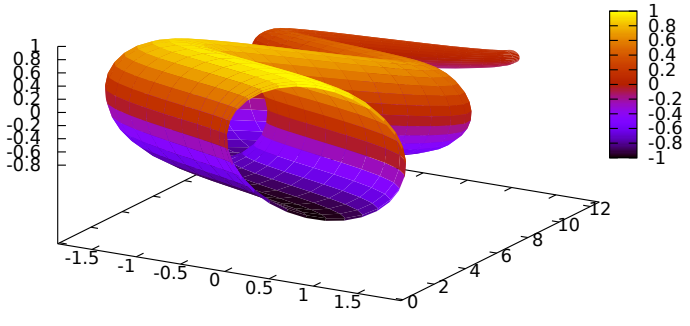
3D

`[xfun,yfun,zfun]` is the parametric curve with parameter *p* taking values from *pmin* to *pmax*. Circles of radius *rfun* are placed with their centers on the parametric curve and perpendicular to it.

This object is affected by the following *graphic options*: `xu_grid`, `yv_grid`, `line_type`, `line_width`, `key`, `wired_surface`, `enhanced3d`, `color` and `capping`.

Example:

```
(%i1) draw3d(
        enhanced3d = true,
        xu_grid = 50,
        tube(cos(a), a, 0, cos(a/10)^2,
             a, 0, 4*%pi) )$
```



`vector` [Graphic object]

`vector` ([*x*,*y*], [*dx*,*dy*])
`vector` ([*x*,*y*,*z*], [*dx*,*dy*,*dz*])

Draws vectors in 2D and 3D.

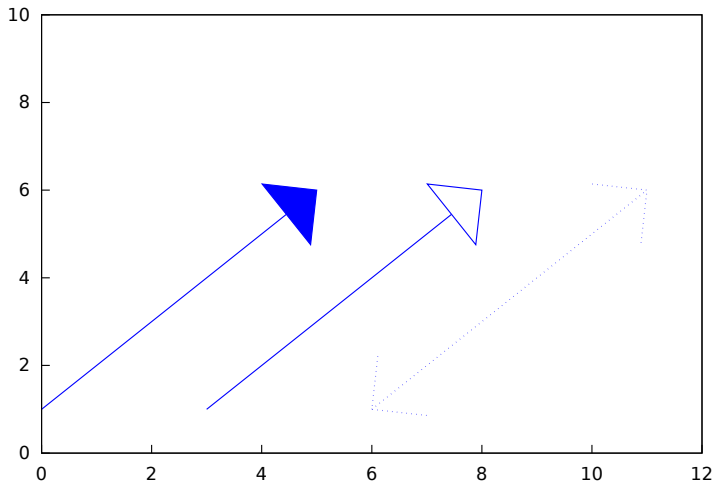
This object is affected by the following *graphic options*: `head_both`, `head_length`, `head_angle`, `head_type`, `line_width`, `line_type`, `key` and `color`.

2D

`vector([x,y], [dx,dy])` plots vector `[dx,dy]` with origin in `[x,y]`.

Example:

```
(%i1) draw2d(xrange      = [0,12],
             yrange      = [0,10],
             head_length = 1,
             vector([0,1],[5,5]), /* default type */
             head_type = 'empty,
             vector([3,1],[5,5]),
             head_both = true,
             head_type = 'nofilled,
             line_type = dots,
             vector([6,1],[5,5]))$
```

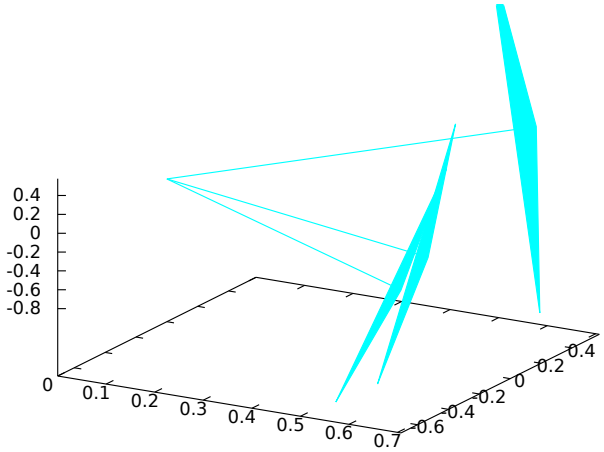


3D

`vector([x,y,z], [dx,dy,dz])` plots vector `[dx,dy,dz]` with origin in `[x,y,z]`.

Example:

```
(%i1) draw3d(color = cyan,
             vector([0,0,0],[1,1,1]/sqrt(3)),
             vector([0,0,0],[1,-1,0]/sqrt(2)),
             vector([0,0,0],[1,1,-2]/sqrt(6)) )$
```



`draw_renderer` [Variable]

Default value: `gnuplot_pipes`

The only permitted values are `gnuplot_pipes`, `gnuplot`, `vtk`, `vtk6` or `vtk7`. When `draw_renderer` is set to `vtk`, the VTK interface is used for draw.

55.3 Functions and Variables for pictures

`get_pixel` (*pic*,*x*,*y*) [Function]

Returns pixel from picture. Coordinates *x* and *y* range from 0 to `width-1` and `height-1`, respectively.

`make_level_picture` [Function]

`make_level_picture` (*data*)
`make_level_picture` (*data*,*width*,*height*)

Returns a levels *picture* object. `make_level_picture (`*data*`)` builds the *picture* object from matrix *data*. `make_level_picture (`*data*`,`*width*`,`*height*`)` builds the object from a list of numbers; in this case, both the *width* and the *height* must be given.

The returned *picture* object contains the following four parts:

1. symbol `level`
2. image width
3. image height
4. an integer array with pixel data ranging from 0 to 255. Argument *data* must contain only numbers ranged from 0 to 255; negative numbers are substituted by 0, and those which are greater than 255 are set to 255.

Example:

Level picture from matrix.

```
(%i1) make_level_picture(matrix([3,2,5],[7,-9,3000]));
(%o1)         picture(level, 3, 2, {Array:  #(3 2 5 7 0 255)})
```

Level picture from numeric list.

```
(%i1) make_level_picture([-2,0,54,%pi],2,2);
(%o1)            picture(level, 2, 2, {Array:  #(0 0 54 3)})
```

`make_rgb_picture` (*redlevel*,*greenlevel*,*bluelevel*) [Function]

Returns an rgb-coloured *picture* object. All three arguments must be levels picture; with red, green and blue levels.

The returned *picture* object contains the following four parts:

1. symbol `rgb`
2. image width
3. image height
4. an integer array of length *3*width*height* with pixel data ranging from 0 to 255. Each pixel is represented by three consecutive numbers (red, green, blue).

Example:

```
(%i1) red: make_level_picture(matrix([3,2],[7,260]));
(%o1)           picture(level, 2, 2, {Array:  #(3 2 7 255)})
(%i2) green: make_level_picture(matrix([54,23],[73,-9]));
(%o2)           picture(level, 2, 2, {Array:  #(54 23 73 0)})
(%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
(%o3)          picture(level, 2, 2, {Array:  #(123 82 45 33)})
```

```
(%i4) make_rgb_picture(red,green,blue);
(%o4) picture(rgb, 2, 2,
              {Array:  #(3 54 123 2 23 82 7 73 45 255 0 33)})
```

`negative_picture` (*pic*) [Function]
Returns the negative of a (*level* or *rgb*) picture.

`picture_equalp` (*x*,*y*) [Function]
Returns `true` in case of equal pictures, and `false` otherwise.

`picturep` (*x*) [Function]
Returns `true` if the argument is a well formed image, and `false` otherwise.

`read_xpm` (*xpm_file*) [Function]
Reads a file in xpm and returns a picture object.

`rgb2level` (*pic*) [Function]
Transforms an *rgb* picture into a *level* one by averaging the red, green and blue channels.

`take_channel` (*im*,*color*) [Function]
If argument *color* is `red`, `green` or `blue`, function `take_channel` returns the corresponding color channel of picture *im*. Example:

```
(%i1) red: make_level_picture(matrix([3,2],[7,260]));
(%o1)           picture(level, 2, 2, {Array:  #(3 2 7 255)})
(%i2) green: make_level_picture(matrix([54,23],[73,-9]));
(%o2)           picture(level, 2, 2, {Array:  #(54 23 73 0)})
(%i3) blue: make_level_picture(matrix([123,82],[45,32.5698]));
(%o3)          picture(level, 2, 2, {Array:  #(123 82 45 33)})
(%i4) make_rgb_picture(red,green,blue);
(%o4) picture(rgb, 2, 2,
              {Array:  #(3 54 123 2 23 82 7 73 45 255 0 33)})
(%i5) take_channel(%,'green);  /* simple quote!!! */
(%o5)           picture(level, 2, 2, {Array:  #(54 23 73 0)})
```

55.4 Functions and Variables for worldmap

55.4.1 Variables and Functions

`boundaries_array` [Global variable]
Default value: `false`

`boundaries_array` is where the graphic object `geomap` looks for boundaries coordinates.

Each component of `boundaries_array` is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.

See also `geomap`.

`numbered_boundaries` (*nlist*) [Function]

Draws a list of polygonal segments (boundaries), labeled by its numbers (`boundaries_array` coordinates). This is of great help when building new geographical entities.

Example:

Map of Europe labeling borders with their component number in `boundaries_array`.

```
(%i1) load("worldmap")$
(%i2) european_borders:
           region_boundaries(-31.81,74.92,49.84,32.06)$
(%i3) numbered_boundaries(european_borders)$
```

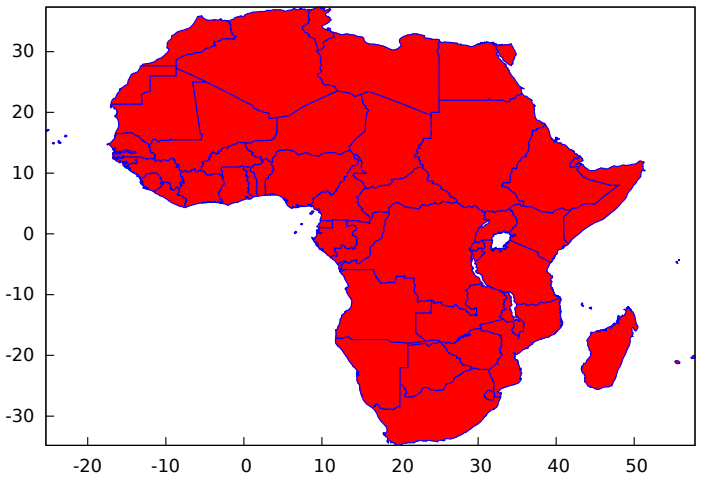
`make_poly_continent` [Function]

`make_poly_continent` (*continent_name*)
`make_poly_continent` (*country_list*)

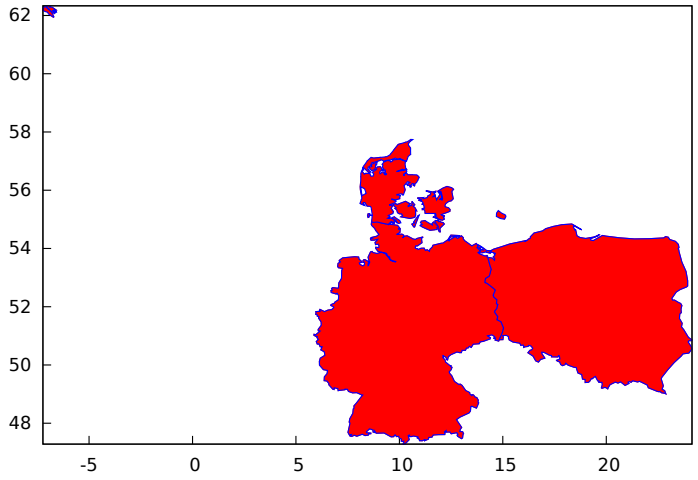
Makes the necessary polygons to draw a colored continent or a list of countries.

Example:

```
(%i1) load("worldmap")$
(%i2) /* A continent */
      make_poly_continent(Africa)$
(%i3) apply(draw2d, %)$
```




```
(%i4) /* A list of countries */
      make_poly_continent([Germany,Denmark,Poland])$
(%i5) apply(draw2d, %)$
```

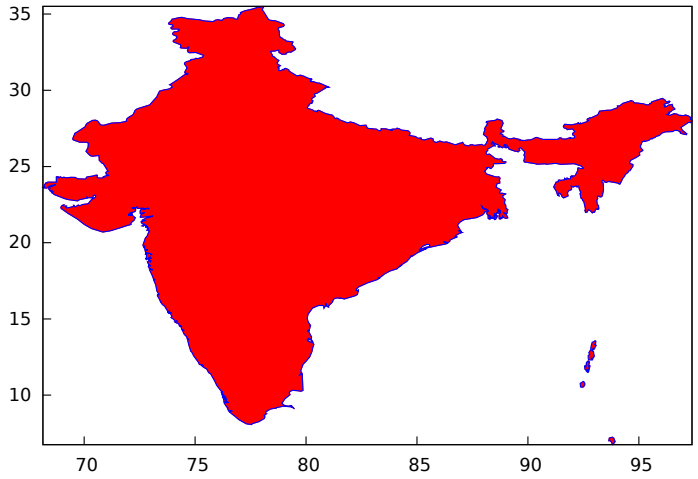


`make_poly_country` (*country_name*) [Function]

Makes the necessary polygons to draw a colored country. If islands exist, one country can be defined with more than just one polygon.

Example:

```
(%i1) load("worldmap")$
(%i2) make_poly_country(India)$
(%i3) apply(draw2d, %)$
```



`make_polygon` (*nlist*) [Function]

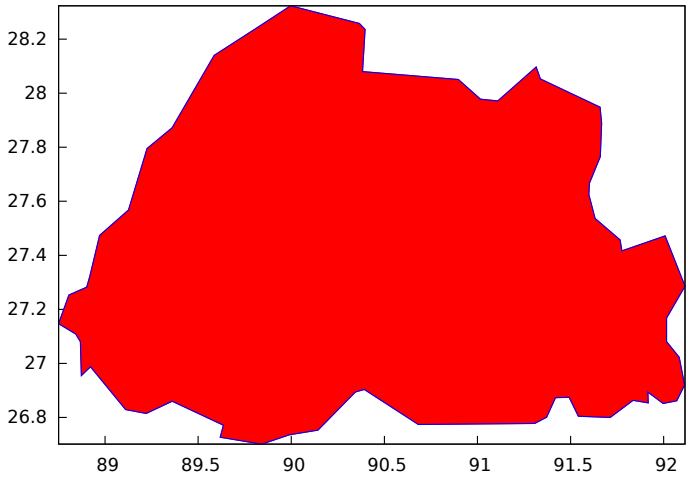
Returns a `polygon` object from boundary indices. Argument *nlist* is a list of components of `boundaries_array`.

Example:

Bhutan is defined by boundary numbers 171, 173 and 1143, so that `make_polygon([171,173,1143])` appends arrays of coordinates `boundaries_array[171]`, `boundaries_array[173]` and `boundaries_array[1143]` and returns a `polygon` object suited to be plotted by `draw`. To avoid an error message, arrays must be

compatible in the sense that any two consecutive arrays have two coordinates in the extremes in common. In this example, the two first components of `boundaries_array[171]` are equal to the last two coordinates of `boundaries_array[173]`, and the two first of `boundaries_array[173]` are equal to the two first of `boundaries_array[1143]`; in conclusion, boundary numbers 171, 173 and 1143 (in this order) are compatible and the colored polygon can be drawn.

```
(%i1) load("worldmap")$
(%i2) Bhutan;
(%o2)                          [[171, 173, 1143]]
(%i3) boundaries_array[171];
(%o3) {Array:
       #(88.750549 27.14727 88.806351 27.25305 88.901367 27.282221
         88.917877 27.321039)}
(%i4) boundaries_array[173];
(%o4) {Array:
       #(91.659554 27.76511 91.6008 27.66666 91.598022 27.62499
         91.631348 27.536381 91.765533 27.45694 91.775253 27.4161
         92.007751 27.471939 92.11441 27.28583 92.015259 27.168051
         92.015533 27.08083 92.083313 27.02277 92.112183 26.920271
         92.069977 26.86194 91.997192 26.85194 91.915253 26.893881
         91.916924 26.85416 91.8358 26.863331 91.712479 26.799999
         91.542191 26.80444 91.492188 26.87472 91.418854 26.873329
         91.371353 26.800831 91.307457 26.778049 90.682457 26.77417
         90.392197 26.903601 90.344131 26.894159 90.143044 26.75333
         89.98996 26.73583 89.841919 26.70138 89.618301 26.72694
         89.636093 26.771111 89.360786 26.859989 89.22081 26.81472
         89.110237 26.829161 88.921631 26.98777 88.873016 26.95499
         88.867737 27.080549 88.843307 27.108601 88.750549
         27.14727)}
(%i5) boundaries_array[1143];
(%o5) {Array:
       #(91.659554 27.76511 91.666924 27.88888 91.65831 27.94805
         91.338028 28.05249 91.314972 28.096661 91.108856 27.971109
         91.015808 27.97777 90.896927 28.05055 90.382462 28.07972
         90.396088 28.23555 90.366074 28.257771 89.996353 28.32333
         89.83165 28.24888 89.58609 28.139999 89.35997 27.87166
         89.225517 27.795 89.125793 27.56749 88.971077 27.47361
         88.917877 27.321039)}
(%i6) Bhutan_polygon: make_polygon([171,173,1143])$
(%i7) draw2d(Bhutan_polygon)$
```



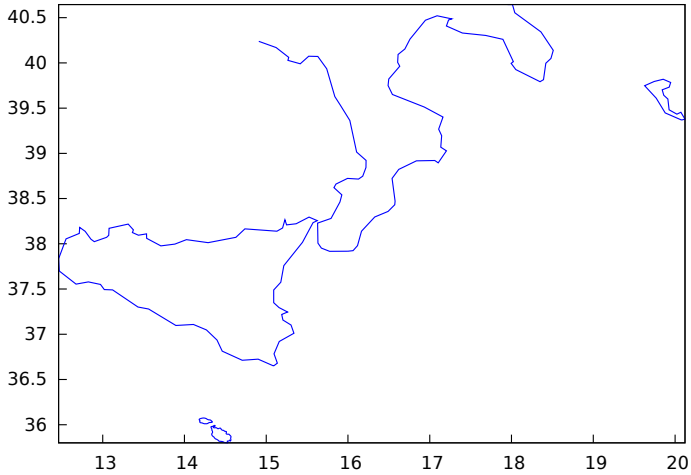
`region_boundaries` (*x1*,*y1*,*x2*,*y2*) [Function]

Detects polygonal segments of global variable `boundaries_array` fully contained in the rectangle with vertices (*x1*,*y1*) -upper left- and (*x2*,*y2*) -bottom right-.

Example:

Returns segment numbers for plotting southern Italy.

```
(%i1) load("worldmap")$
(%i2) region_boundaries(10.4,41.5,20.7,35.4);
(%o2)                  [1846, 1863, 1864, 1881, 1888, 1894]
(%i3) draw2d(geomap(%))$
```



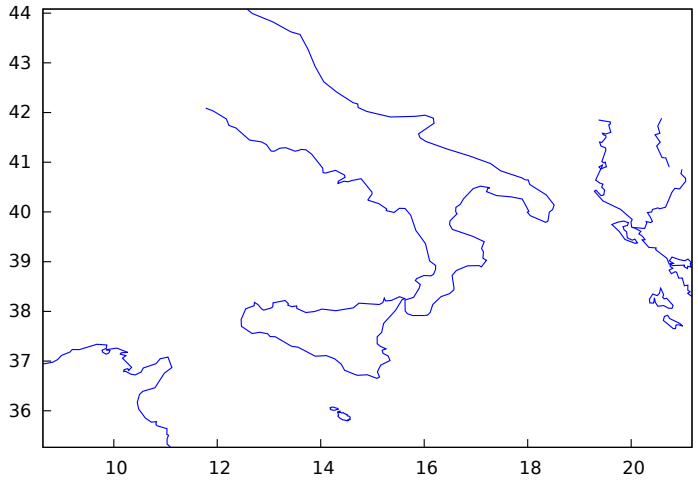
`region_boundaries_plus` (*x1*,*y1*,*x2*,*y2*) [Function]

Detects polygonal segments of global variable `boundaries_array` containing at least one vertex in the rectangle defined by vertices (*x1*,*y1*) -upper left- and (*x2*,*y2*) -bottom right-.

Example:

```
(%i1) load("worldmap")$
```

```
(%i2) region_boundaries_plus(10.4,41.5,20.7,35.4);
(%o2) [1060, 1062, 1076, 1835, 1839, 1844, 1846, 1858,
       1861, 1863, 1864, 1871, 1881, 1888, 1894, 1897]
(%i3) draw2d(geomap(%))$
```



55.4.2 Graphic objects

geomap [Graphic object]

geomap (*numlist*)
geomap (*numlist*,*3Dprojection*)

Draws cartographic maps in 2D and 3D.

2D

This function works together with global variable `boundaries_array`.

Argument *numlist* is a list containing numbers or lists of numbers. All these numbers must be integers greater or equal than zero, representing the components of global array `boundaries_array`.

Each component of `boundaries_array` is an array of floating point quantities, the coordinates of a polygonal segment or map boundary.

`geomap (numlist)` flattens its arguments and draws the associated boundaries in `boundaries_array`.

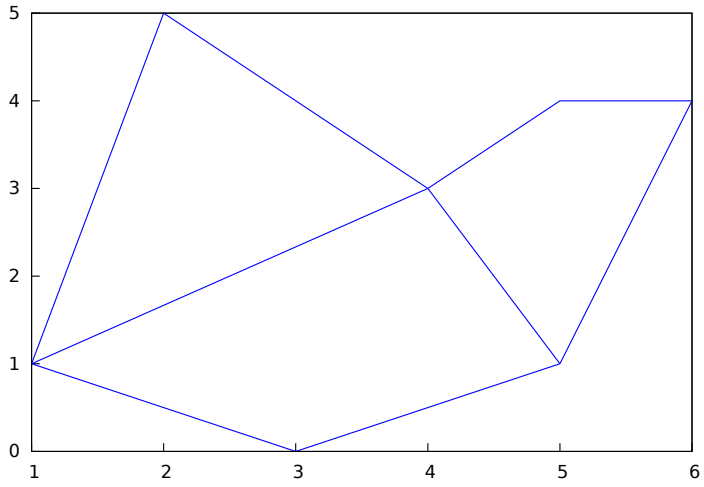
This object is affected by the following *graphic options*: `line_width`, `line_type` and `color`.

Examples:

A simple map defined by hand:

```
(%i1) load("worldmap")$
(%i2) /* Vertices of boundary #0: {(1,1),(2,5),(4,3)} */
   ( bnd0: make_array(flonum,6),
     bnd0[0]:1.0, bnd0[1]:1.0, bnd0[2]:2.0,
     bnd0[3]:5.0, bnd0[4]:4.0, bnd0[5]:3.0 )$
```

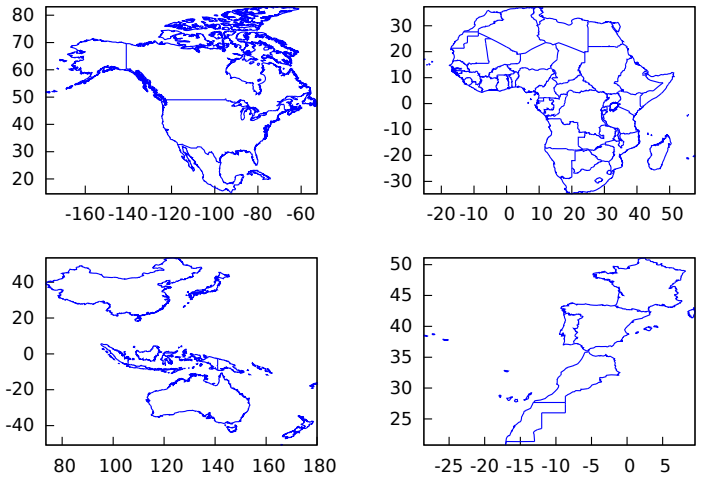
```
(%i3) /* Vertices of boundary #1: {(4,3),(5,4),(6,4),(5,1)} */
   ( bnd1: make_array(flonum,8),
     bnd1[0]:4.0, bnd1[1]:3.0, bnd1[2]:5.0, bnd1[3]:4.0,
     bnd1[4]:6.0, bnd1[5]:4.0, bnd1[6]:5.0, bnd1[7]:1.0)$
(%i4) /* Vertices of boundary #2: {(5,1), (3,0), (1,1)} */
   ( bnd2: make_array(flonum,6),
     bnd2[0]:5.0, bnd2[1]:1.0, bnd2[2]:3.0,
     bnd2[3]:0.0, bnd2[4]:1.0, bnd2[5]:1.0 )$
(%i5) /* Vertices of boundary #3: {(1,1), (4,3)} */
   ( bnd3: make_array(flonum,4),
     bnd3[0]:1.0, bnd3[1]:1.0, bnd3[2]:4.0, bnd3[3]:3.0)$
(%i6) /* Vertices of boundary #4: {(4,3), (5,1)} */
   ( bnd4: make_array(flonum,4),
     bnd4[0]:4.0, bnd4[1]:3.0, bnd4[2]:5.0, bnd4[3]:1.0)$
(%i7) /* Pack all together in boundaries_array */
   ( boundaries_array: make_array(any,5),
     boundaries_array[0]: bnd0, boundaries_array[1]: bnd1,
     boundaries_array[2]: bnd2, boundaries_array[3]: bnd3,
     boundaries_array[4]: bnd4 )$
(%i8) draw2d(geomap([0,1,2,3,4]))$
```



The auxiliary package `worldmap` sets the global variable `boundaries_array` to real world boundaries in (longitude, latitude) coordinates. These data are in the public domain and come from https://web.archive.org/web/20100310124019/http://www-cger.nies.go.jp/grid-e/gridtxt/grid19.html. Package `worldmap` defines also boundaries for countries, continents and coastlines as lists with the necessary components of `boundaries_array` (see file `share/draw/worldmap.mac` for more information). Package `worldmap` automatically loads package `worldmap`.

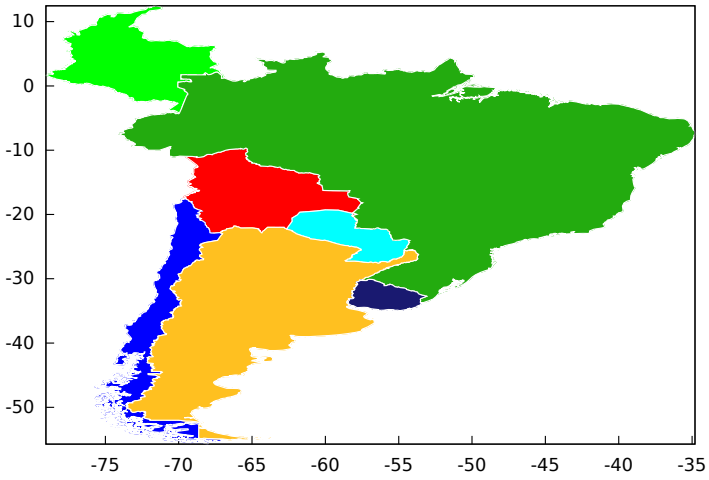
```
(%i1) load("worldmap")$
(%i2) c1: gr2d(geomap([Canada,United_States,
                       Mexico,Cuba]))$
```

```
(%i3) c2: gr2d(geomap(Africa))$
(%i4) c3: gr2d(geomap([Oceania,China,Japan]))$
(%i5) c4: gr2d(geomap([France,Portugal,Spain,
                       Morocco,Western_Sahara]))$
(%i6) draw(columns  = 2,
           c1,c2,c3,c4)$
```



Package `worldmap` is also useful for plotting countries as polygons. In this case, graphic object `geomap` is no longer necessary and the `polygon` object is used instead. Since lists are now used and not arrays, maps rendering will be slower. See also `make_poly_country` and `make_poly_continent` to understand the following code.

```
(%i1) load("worldmap")$
(%i2) mymap: append(
   [color      = white],  /* borders are white */
   [fill_color = red],             make_poly_country(Bolivia),
   [fill_color = cyan],            make_poly_country(Paraguay),
   [fill_color = green],           make_poly_country(Colombia),
   [fill_color = blue],            make_poly_country(Chile),
   [fill_color = "#23ab0f"],       make_poly_country(Brazil),
   [fill_color = goldenrod],       make_poly_country(Argentina),
   [fill_color = "midnight-blue"], make_poly_country(Uruguay))$
(%i3) apply(draw2d, mymap)$
```



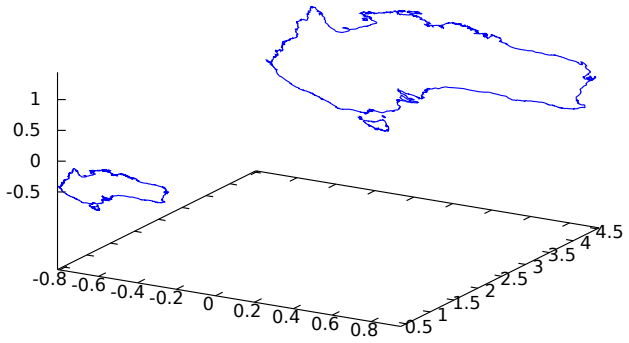
3D

`geomap (`*`numlist`*`)` projects map boundaries on the sphere of radius 1 centered at (0,0,0). It is possible to change the sphere or the projection type by using `geomap (`*`numlist`*`,`*`3Dprojection`*`)`.

Available 3D projections:

- `[spherical_projection,`*`x`*`,`*`y`*`,`*`z`*`,`*`r`*`]`: projects map boundaries on the sphere of radius *r* centered at (*x*,*y*,*z*).

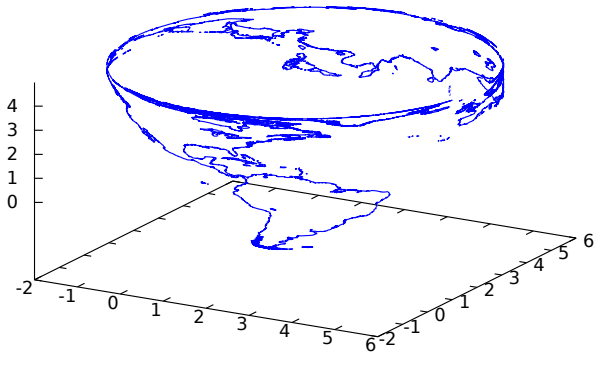
```
(%i1) load("worldmap")$
(%i2) draw3d(geomap(Australia), /* default projection */
             geomap(Australia,
                    [spherical_projection,2,2,2,3]))$
```



- `[cylindrical_projection,`*`x`*`,`*`y`*`,`*`z`*`,`*`r`*`,`*`rc`*`]`: re-projects spherical map boundaries on the cylinder of radius *rc* and axis passing through the poles of the globe of radius *r* centered at (*x*,*y*,*z*).

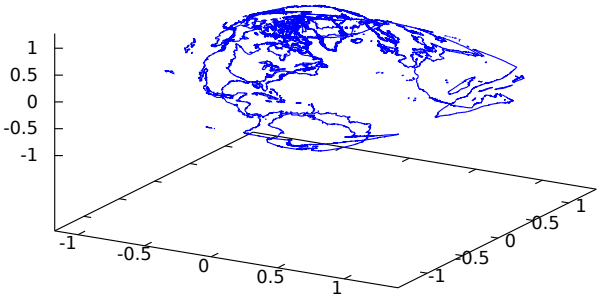
```
(%i1) load("worldmap")$
(%i2) draw3d(geomap([America_coastlines,Eurasia_coastlines],
```

```
                                [cylindrical_projection,2,2,2,3,4]))$
```



- `[conic_projection,`*x*`,`*y*`,`*z*`,`*r*`,`*alpha*`]`: re-projects spherical map boundaries on the cones of angle *alpha*, with axis passing through the poles of the globe of radius *r* centered at (*x*,*y*,*z*). Both the northern and southern cones are tangent to sphere.

```
(%i1) load("worldmap")$
(%i2) draw3d(geomap(World_coastlines,
                    [conic_projection,0,0,0,1,90]))$
```



See also https://riotorto.users.sourceforge.net/Maxima/gnuplot/geomap/ for more elaborated examples.

56 Package drawdf

56.1 Introduction to drawdf

The function `drawdf` draws the direction field of a first-order Ordinary Differential Equation (ODE) or a system of two autonomous first-order ODE's.

Since this is an additional package, in order to use it you must first load it with `load("drawdf")`. Drawdf is built upon the `draw` package, which requires Gnuplot 4.2.

To plot the direction field of a single ODE, the ODE must be written in the form:

$$\frac{dy}{dx} = F(x, y)$$

and the function F should be given as the argument for `drawdf`. If the independent and dependent variables are not x, and y, as in the equation above, then those two variables should be named explicitly in a list given as an argument to the drawdf command (see the examples).

To plot the direction field of a set of two autonomous ODE's, they must be written in the form

$$\frac{dx}{dt} = G(x, y) \qquad \frac{dy}{dt} = F(x, y)$$

and the argument for `drawdf` should be a list with the two functions G and F, in that order; namely, the first expression in the list will be taken to be the time derivative of the variable represented on the horizontal axis, and the second expression will be the time derivative of the variable represented on the vertical axis. Those two variables do not have to be x and y, but if they are not, then the second argument given to drawdf must be another list naming the two variables, first the one on the horizontal axis and then the one on the vertical axis.

If only one ODE is given, `drawdf` will implicitly admit `x=t`, and `G(x,y)=1`, transforming the non-autonomous equation into a system of two autonomous equations.

56.2 Functions and Variables for drawdf

56.2.1 Functions

`drawdf` [Function]

`drawdf` (*`dydx`*, *...options and objects...*)
`drawdf` (*`dvdu`*, [`u`,`v`], *...options and objects...*)
`drawdf` (*`dvdu`*, [`u`,*`umin`*,*`umax`*], [`v`,*`vmin`*,*`vmax`*], *...options and objects...*)
`drawdf` ([*`dxdt`*,*`dydt`*], *...options and objects...*)
`drawdf` ([*`dudt`*,*`dvdt`*], [`u`,`v`], *...options and objects...*)
`drawdf` ([*`dudt`*,*`dvdt`*], [`u`,*`umin`*,*`umax`*], [`v`,*`vmin`*,*`vmax`*], *...options and objects...*)

Function `drawdf` draws a 2D direction field with optional solution curves and other graphics using the `draw` package.

The first argument specifies the derivative(s), and must be either an expression or a list of two expressions. *dydx*, *dxdt* and *dydt* are expressions that depend on *x* and *y*. *dvdu*, *dudt* and *dvdt* are expressions that depend on *u* and *v*.

If the independent and dependent variables are not *x* and *y*, then their names must be specified immediately following the derivative(s), either as a list of two names `[`*u*,*v*`]`, or as two lists of the form `[`*u*,*umin*,*umax*`]` and `[`*v*,*vmin*,*vmax*`]`.

The remaining arguments are *graphic options*, *graphic objects*, or lists containing graphic options and objects, nested to arbitrary depth. The set of graphic options and objects supported by `drawdf` is a superset of those supported by `draw2d` and `gr2d` from the `draw` package.

The arguments are interpreted sequentially: *graphic options* affect all following *graphic objects*. Furthermore, *graphic objects* are drawn on the canvas in order specified, and may obscure graphics drawn earlier. Some *graphic options* affect the global appearance of the scene.

The additional *graphic objects* supported by `drawdf` include: `solns_at`, `points_at`, `saddles_at`, `soln_at`, `point_at`, and `saddle_at`.

The additional *graphic options* supported by `drawdf` include: `field_degree`, `soln_arrows`, `field_arrows`, `field_grid`, `field_color`, `show_field`, `tstep`, `nsteps`, `duration`, `direction`, `field_tstep`, `field_nsteps`, and `field_duration`.

Commonly used *graphic objects* inherited from the `draw` package include: `explicit`, `implicit`, `parametric`, `polygon`, `points`, `vector`, `label`, and all others supported by `draw2d` and `gr2d`.

Commonly used *graphic options* inherited from the `draw` package include: `points_joined`, `color`, `point_type`, `point_size`, `line_width`, `line_type`, `key`, `title`, `xlabel`, `ylabel`, `user_preamble`, `terminal`, `dimensions`, `file_name`, and all others supported by `draw2d` and `gr2d`.

See also `draw2d`, `rk`, `desolve` and `ode2`.

Users of wxMaxima or Imaxima may optionally use `wxdrawdf`, which is identical to `drawdf` except that the graphics are drawn within the notebook using `wxdraw`.

To make use of this function, write first `load("drawdf")`.

Examples:

```
(%i1) load("drawdf")$
(%i2) drawdf(exp(-x)+y)$        /* default vars: x,y */
(%i3) drawdf(exp(-t)+y, [t,y])$ /* default range: [-10,10] */
(%i4) drawdf([y,-9*sin(x)-y/5], [x,1,5], [y,-2,2])$
```

For backward compatibility, `drawdf` accepts most of the parameters supported by plotdf.

```
(%i5) drawdf(2*cos(t)-1+y, [t,y], [t,-5,10], [y,-4,9],
              [trajectory_at,0,0])$
```

`soln_at` and `solns_at` draw solution curves passing through the specified points, using a slightly enhanced 4th-order Runge Kutta numerical integrator.

```
(%i6) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
              solns_at([0,0.1],[0,-0.1]),
```

```
                color=blue, soln_at(0,0))$
```

`field_degree=2` causes the field to be composed of quadratic splines, based on the first and second derivatives at each grid point. `field_grid=[`*COLS*`,`*ROWS*`]` specifies the number of columns and rows in the grid.

```
(%i7) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
              field_degree=2, field_grid=[20,15],
              solns_at([0,0.1],[0,-0.1]),
              color=blue, soln_at(0,0))$
```

`soln_arrows=true` adds arrows to the solution curves, and (by default) removes them from the direction field. It also changes the default colors to emphasize the solution curves.

```
(%i8) drawdf(2*cos(t)-1+y, [t,-5,10], [y,-4,9],
              soln_arrows=true,
              solns_at([0,0.1],[0,-0.1],[0,0]))$
```

`duration=40` specifies the time duration of numerical integration (default 10). Integration will also stop automatically if the solution moves too far away from the plotted region, or if the derivative becomes complex or infinite. Here we also specify `field_degree=2` to plot quadratic splines. The equations below model a predator-prey system.

```
(%i9) drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1],
              field_degree=2, duration=40,
              soln_arrows=true, point_at(1/2,1/2),
              solns_at([0.1,0.2], [0.2,0.1], [1,0.8], [0.8,1],
                       [0.1,0.1], [0.6,0.05], [0.05,0.4],
                       [1,0.01], [0.01,0.75]))$
```

`field_degree='solns` causes the field to be composed of many small solution curves computed by 4th-order Runge Kutta, with better results in this case.

```
(%i10) drawdf([x*(1-x-y), y*(3/4-y-x/2)], [x,0,1.1], [y,0,1],
               field_degree='solns, duration=40,
               soln_arrows=true, point_at(1/2,1/2),
               solns_at([0.1,0.2], [0.2,0.1], [1,0.8],
                        [0.8,1], [0.1,0.1], [0.6,0.05],
                        [0.05,0.4], [1,0.01], [0.01,0.75]))$
```

`saddles_at` attempts to automatically linearize the equation at each saddle, and to plot a numerical solution corresponding to each eigenvector, including the separatrices. `tstep=0.05` specifies the maximum time step for the numerical integrator (the default is 0.1). Note that smaller time steps will sometimes be used in order to keep the x and y steps small. The equations below model a damped pendulum.

```
(%i11) drawdf([y,-9*sin(x)-y/5], tstep=0.05,
               soln_arrows=true, point_size=0.5,
               points_at([0,0], [2*%pi,0], [-2*%pi,0]),
               field_degree='solns,
               saddles_at([%pi,0], [-%pi,0]))$
```

`show_field=false` suppresses the field entirely.

```
(%i12) drawdf([y,-9*sin(x)-y/5], tstep=0.05,
```

```
                show_field=false, soln_arrows=true,
                point_size=0.5,
                points_at([0,0], [2*%pi,0], [-2*%pi,0]),
                saddles_at([3*%pi,0], [-3*%pi,0],
                           [%pi,0], [-%pi,0]))$
```

`drawdf` passes all unrecognized parameters to `draw2d` or `gr2d`, allowing you to combine the full power of the `draw` package with `drawdf`.

```
(%i13) drawdf(x^2+y^2, [x,-2,2], [y,-2,2], field_color=gray,
                key="soln 1", color=black, soln_at(0,0),
                key="soln 2", color=red, soln_at(0,1),
                key="isocline", color=green, line_width=2,
                nticks=100, parametric(cos(t),sin(t),t,0,2*%pi))$
```

`drawdf` accepts nested lists of graphic options and objects, allowing convenient use of makelist and other function calls to generate graphics.

```
(%i14) colors : ['red,'blue,'purple,'orange,'green]$
(%i15) drawdf([x-x*y/2, (x*y - 3*y)/4],
                [x,2.5,3.5], [y,1.5,2.5],
                field_color = gray,
                makelist([ key   = concat("soln",k),
                           color = colors[k],
                           soln_at(3, 2 + k/20) ],
                         k,1,5))$
```

57 Package dynamics

57.1 The dynamics package

Package `dynamics` includes functions for 3D visualization, animations, graphical analysis of differential and difference equations and numerical solution of differential equations. The functions for differential equations are described in the section on Chapter 24 [Numerical], page 451 and the functions to plot the Mandelbrot and Julia sets are described in the section on Chapter 14 Plotting.

All the functions in this package will be loaded automatically the first time they are used.

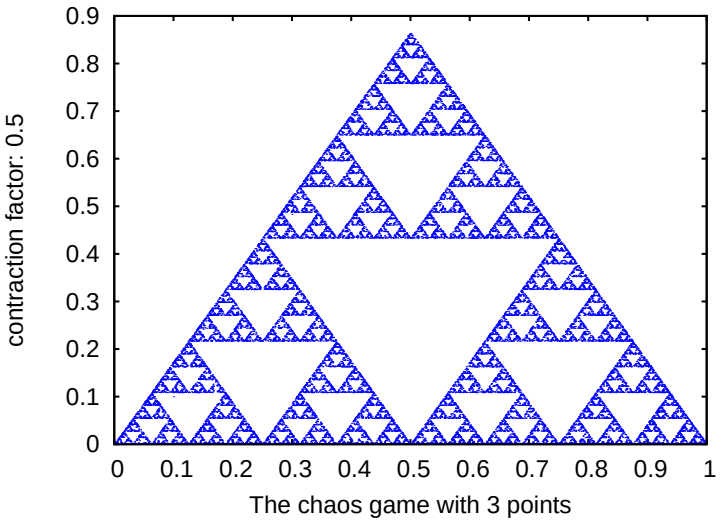
57.2 Graphical analysis of discrete dynamical systems

`chaosgame` (`[[`*x1*, *y1*`]...[`*xm*, *ym*`]]`, `[`*x0*, *y0*`]`, *b*, *n*, *options*, ...); [Function]

Implements the so-called chaos game: the initial point (*x0*, *y0*) is plotted and then one of the *m* points [*x1*, *y1*]...*xm*, *ym*] will be selected at random. The next point plotted will be on the segment from the previous point plotted to the point chosen randomly, at a distance from the random point which will be *b* times that segment's length. The procedure is repeated *n* times. The options are the same as for `plot2d`.

Example. A plot of Sierpinsky's triangle:

```
(%i1) chaosgame([[0, 0], [1, 0], [0.5, sqrt(3)/2]], [0.1, 0.1], 1/2,
                30000, [style, dots]);
```



`evolution` (*F*, *y0*, *n*, ..., *options*, ...); [Function]

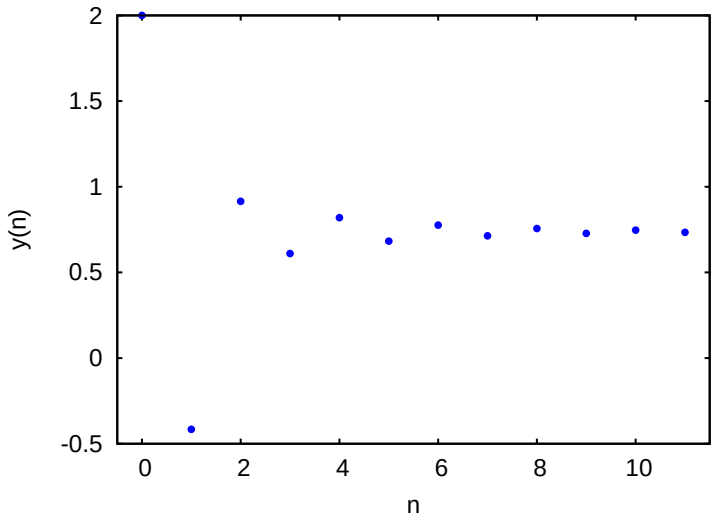
Draws *n+1* points in a two-dimensional graph, where the horizontal coordinates of the points are the integers 0, 1, 2, ..., *n*, and the vertical coordinates are the corresponding values *y(n)* of the sequence defined by the recurrence relation

$$y_{n+1} = F(y_n)$$

With initial value *y(0)* equal to *y0*. *F* must be an expression that depends only on one variable (in the example, it depend on *y*, but any other variable can be used), *y0* must be a real number and *n* must be a positive integer. This function accepts the same options as `plot2d`.

Example.

```
(%i1) evolution(cos(y), 2, 11);
```



`evolution2d` (*[F, G]*, *[u, v]*, *[u0, y0]*, *n*, *options*, ...); [Function]

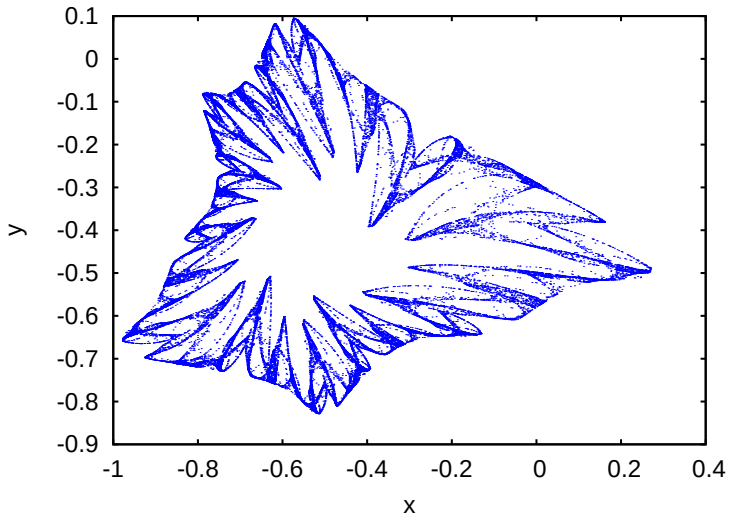
Shows, in a two-dimensional plot, the first *n+1* points in the sequence of points defined by the two-dimensional discrete dynamical system with recurrence relations

$$\begin{cases} u_{n+1} = F(u_n, v_n) \\ v_{n+1} = G(u_n, v_n) \end{cases}$$

With initial values *u0* and *v0*. *F* and *G* must be two expressions that depend only on two variables, *u* and *v*, which must be named explicitly in a list. The options are the same as for `plot2d`.

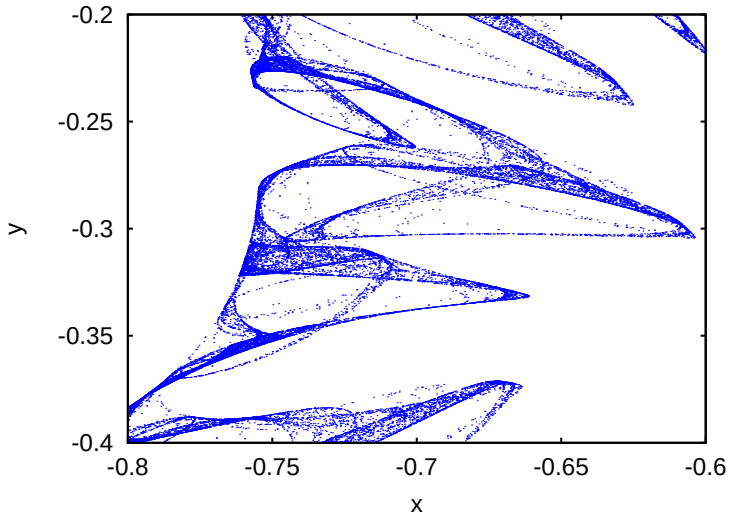
Example. Evolution of a two-dimensional discrete dynamical system:

```
(%i1) f: 0.6*x*(1+2*x)+0.8*y*(x-1)-y^2-0.9$
(%i2) g: 0.1*x*(1-6*x+4*y)+0.1*y*(1+9*y)-0.4$
(%i3) evolution2d([f,g], [x,y], [-0.5,0], 50000, [style,dots]);
```



And an enlargement of a small region in that fractal:

```
(%i9) evolution2d([f,g], [x,y], [-0.5,0], 300000, [x,-0.8,-0.6],
                  [y,-0.4,-0.2], [style, dots]);
```



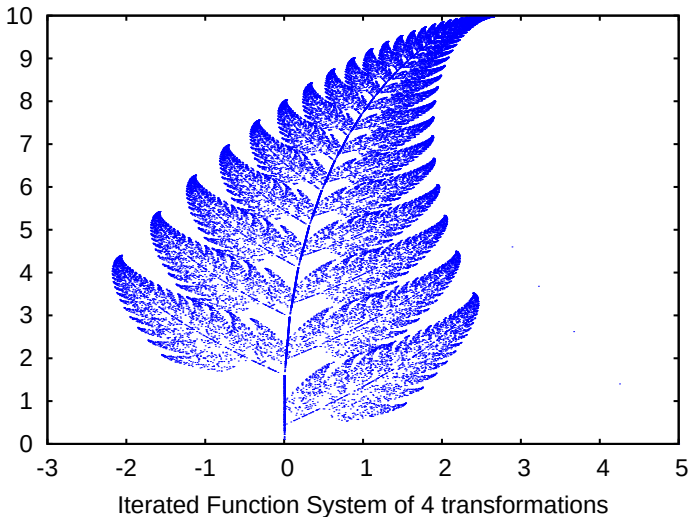
`ifs` (*[r1, ..., rm], [A1,..., Am], [[x1, y1], ..., [xm, ym]], [x0, y0], n, options, ...*); [Function]

Implements the Iterated Function System method. This method is similar to the method described in the function `chaosgame`. but instead of shrinking the segment from the current point to the randomly chosen point, the 2 components of that segment will be multiplied by the 2 by 2 matrix *Ai* that corresponds to the point chosen randomly.

The random choice of one of the *m* attractive points can be made with a non-uniform probability distribution defined by the weights *r1*,...,*rm*. Those weights are given in cumulative form; for instance if there are 3 points with probabilities 0.2, 0.5 and 0.3, the weights *r1*, *r2* and *r3* could be 2, 7 and 10. The options are the same as for `plot2d`.

Example. Barnsley's fern, obtained with 4 matrices and 4 points:

```
(%i1) a1: matrix([0.85,0.04],[-0.04,0.85])$
(%i2) a2: matrix([0.2,-0.26],[0.23,0.22])$
(%i3) a3: matrix([-0.15,0.28],[0.26,0.24])$
(%i4) a4: matrix([0,0],[0,0.16])$
(%i5) p1: [0,1.6]$
(%i6) p2: [0,1.6]$
(%i7) p3: [0,0.44]$
(%i8) p4: [0,0]$
(%i9) w: [85,92,99,100]$
(%i10) ifs(w, [a1,a2,a3,a4], [p1,p2,p3,p4], [5,0], 50000, [style,dots]);
```



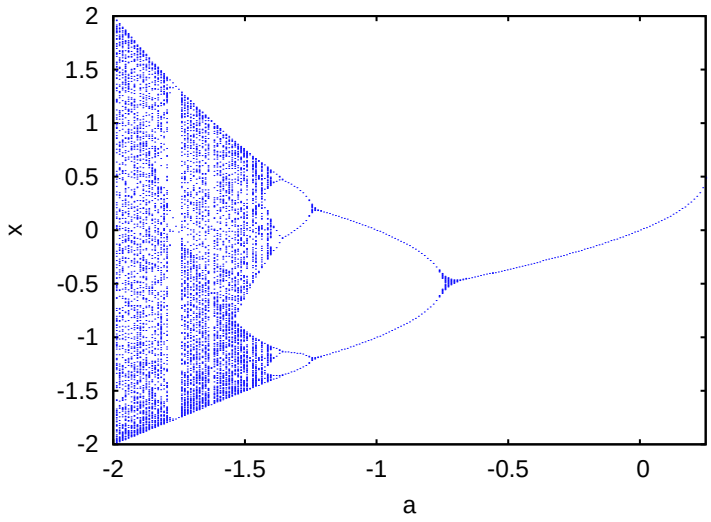
`orbits` (*F*, *y0*, *n1*, *n2*, [*x*, *x0*, *xf*, *xstep*], *options*, ...); [Function]

Draws the orbits diagram for a family of one-dimensional discrete dynamical systems, with one parameter *x*; that kind of diagram is used to study the bifurcations of an one-dimensional discrete system.

The function *F(y)* defines a sequence with a starting value of *y0*, as in the case of the function `evolution`, but in this case that function will also depend on a parameter *x* that will take values in the interval from *x0* to *xf* with increments of *xstep*. Each value used for the parameter *x* is shown on the horizontal axis. The vertical axis will show the *n2* values of the sequence *y(n1+1)*,..., *y(n1+n2+1)* obtained after letting the sequence evolve *n1* iterations. In addition to the options accepted by `plot2d`, it accepts an option *pixels* that sets up the maximum number of different points that will be represented in the vertical direction.

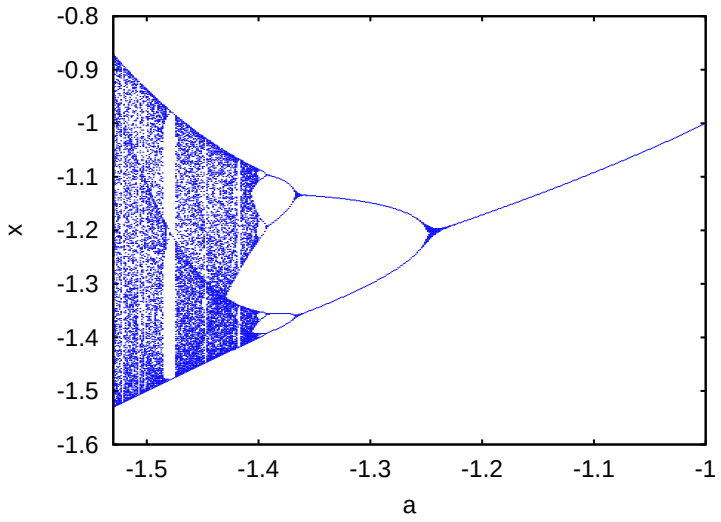
Example. Orbits diagram of the quadratic map, with a parameter *a*:

```
(%i1) orbits(x^2+a, 0, 50, 200, [a, -2, 0.25], [style, dots]);
```



To enlarge the region around the lower bifurcation near x = -1.25 use:

```
(%i2) orbits(x^2+a, 0, 100, 400, [a,-1,-1.53], [x,-1.6,-0.8],
                [nticks, 400], [style,dots]);
```



`staircase` (*F*, *y0*, *n*,*options*,...); [Function]

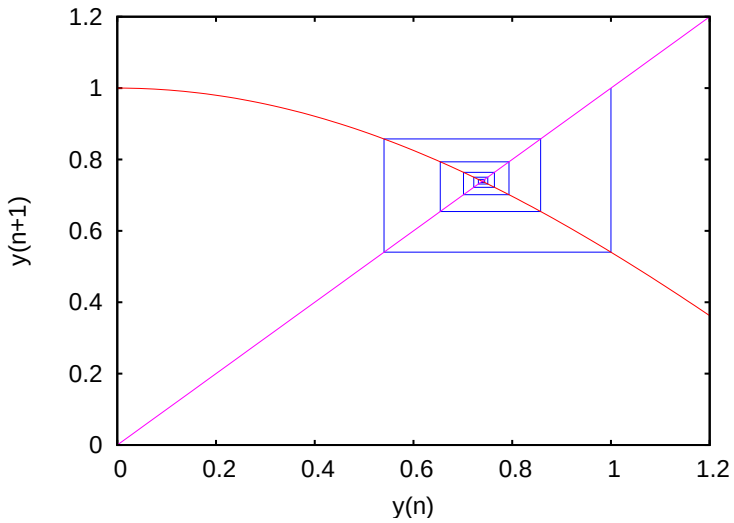
Draws a staircase diagram for the sequence defined by the recurrence relation

$$y_{n+1} = F(y_n)$$

The interpretation and allowed values of the input parameters is the same as for the function `evolution`. A staircase diagram consists of a plot of the function *F(y)*, together with the line *G(y)* = *y*. A vertical segment is drawn from the point (*y0*, *y0*) on that line until the point where it intersects the function *F*. From that point a horizontal segment is drawn until it reaches the point (*y1*, *y1*) on the line, and the procedure is repeated *n* times until the point (*yn*, *yn*) is reached. The options are the same as for `plot2d`.

Example.

```
(%i1) staircase(cos(y), 1, 11, [y, 0, 1.2]);
```



57.3 Visualization with VTK

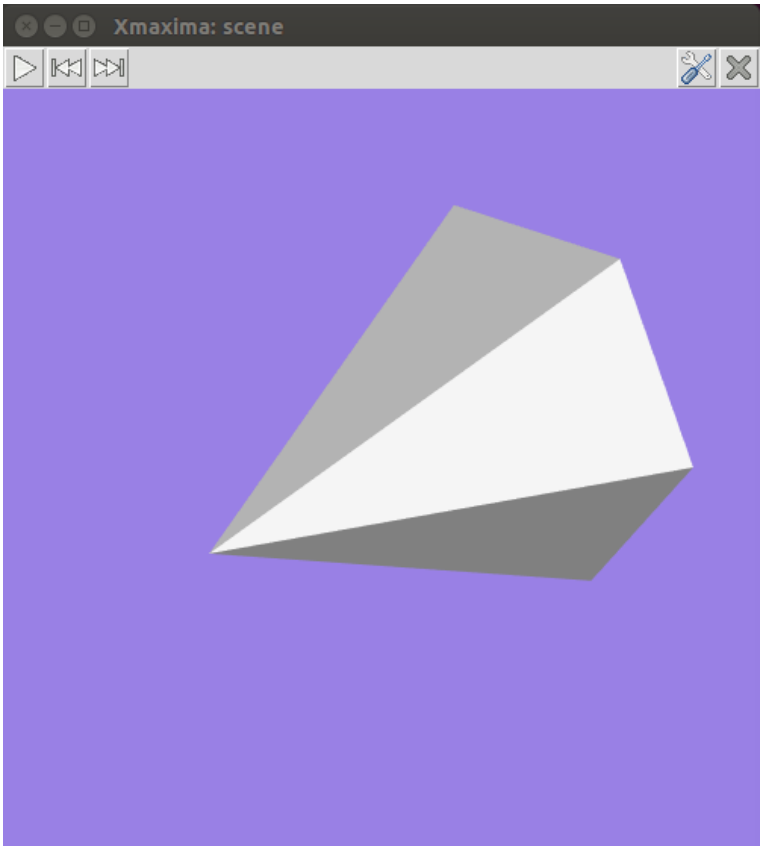
Function scene creates 3D images and animations using the *Visualization ToolKit* (VTK) software. In order to use that function, Xmaxima and VTK should be installed in your system (including the TCL bindings of VTK, which in some system might come in a separate package).

`scene` (*objects*, ..., *options*, ...); [Function]

Accepts an empty list or a list of several [scene_objects], page 977 and [scene_options], page 976. The program launches Xmaxima, which opens an external window representing the given objects in a 3-dimensional space and applying the options given. Each object must belong to one of the following 4 classes: sphere, cube, cylinder or cone (see [scene_objects], page 977). Objects are identified by giving their name or by a list in which the first element is the class name and the following elements are options for that object.

Example. A hexagonal pyramid with a blue background:

```
(%i1) scene(cone, [background,"#9980e5"])$
```



By holding down the left button of the mouse while it is moved on the graphics window, the camera can be rotated showing different views of the pyramid. The two plot options [scene_elevation], page 976 and [scene_azimuth], page 976 can also be used to change the initial orientation of the viewing camera. The camera can be moved by holding the middle mouse button while moving it and holding the right-side mouse button while moving it up or down will zoom in or out.

Each object option should be a list starting with the option name, followed by its value. The list of allowed options can be found in the [object_options], page 978 section.

Example. This will show a sphere falling to the ground and bouncing off without losing any energy. To start or pause the animation, press the play/pause button.

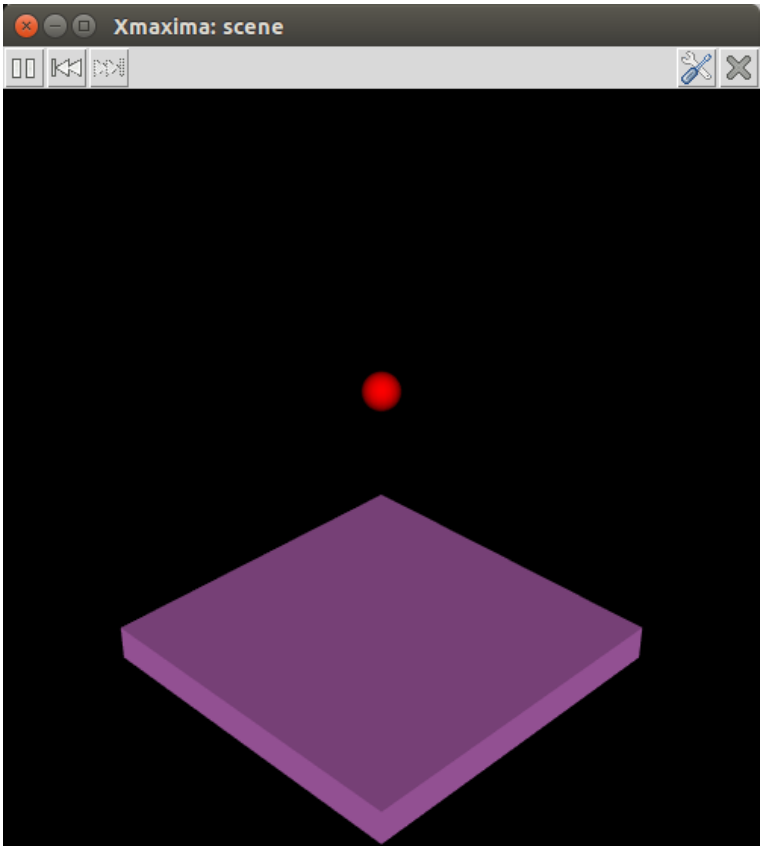
```
(%i1) p: makelist ([0,0,2.1- 9.8*t^2/2], t, 0, 0.64, 0.01)$

(%i2) p: append (p, reverse(p))$

(%i3) ball: [sphere, [radius,0.1], [thetaresolution,20],
  [phiresolution,20], [position,0,0,2.1], [color,red],
  [animate,position,p]]$

(%i4) ground: [cube, [xlength,2], [ylength,2], [zlength,0.2],
  [position,0,0,-0.1],[color,violet]]$

(%i5) scene (ball, ground, restart)$
```



The *restart* option was used to make the animation restart automatically every time the last point in the position list is reached. The accepted values for the colors are the same as for the `color` option of plot2d.

57.3.1 Scene options

`azimuth` [*azimuth*, `angle`] [Scene option]

Default value: `135`

The rotation of the camera on the horizontal (x, y) plane. *angle* must be a real number; an angle of 0 means that the camera points in the direction of the y axis and the x axis will appear on the right.

`background` [*background*, `color`] [Scene option]

Default value: `black`

The color of the graphics window's background. It accepts color names or hexadecimal red-green-blue strings (see the `color` option of plot2d).

`elevation` [*elevation*, `angle`] [Scene option]

Default value: `30`

The vertical rotation of the camera. The *angle* must be a real number; an angle of 0 means that the camera points on the horizontal, and the default angle of 30 means that the camera is pointing 30 degrees down from the horizontal.

`height` [*height*, `pixels`] [Scene option]

Default value: `500`

The height, in pixels, of the graphics window. *pixels* must be a positive integer number.

`restart` [*restart*, `value`] [Scene option]
Default value: `false`

A true value means that animations will restart automatically when the end of the list is reached. Writing just "restart" is equivalent to [restart, *true*].

`tstep` [*tstep*, `time`] [Scene option]
Default value: `10`

The amount of time, in mili-seconds, between iterations among consecutive animation frames. *time* must be a real number.

`width` [*width*, `pixels`] [Scene option]
Default value: `500`

The width, in pixels, of the graphics window. *pixels* must be a positive integer number.

`windowname` [*windowtitle*, `name`] [Scene option]
Default value: `.scene`

name must be a string that can be used as the name of the Tk window created by Xmaxima for the `scene` graphics. The default value `.scene` implies that a new top level window will be created.

`windowtitle` [*windowtitle*, `name`] [Scene option]
Default value: `Xmaxima: scene`

name must be a string that will be written in the title of the window created by `scene`.

57.3.2 Scene objects

`cone` [*cone*, `options`] [Scene object]
Creates a regular pyramid with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options [object_height], page 979 and [object_radius], page 980 can be used to change those defaults and option [object_resolution], page 980 can be used to change the number of edges of the base; higher values will make it look like a cone. By default, the axis will be along the x axis, the middle point of the axis will be at the origin and the vertex on the positive side of the x axis; use options [object_orientation], page 979 and [object_center], page 978 to change those defaults.

Example. This shows a pyramid that starts rotating around the z axis when the play button is pressed.

```
(%i1) scene([cone, [orientation,0,30,0], [tstep,100],
   [animate,orientation,makelist([0,30,i],i,5,360,5)]], restart)$
```

`cube` [*cube*, `options`] [Scene object]
A cube with edges of 1 unit and faces parallel to the xy, xz and yz planes. The lengths of the three edges can be changed with options [object_xlength], page 981,

[object_ylength], page 981 and [object_zlength], page 981, turning it into a rectangular box and the faces can be rotated with option [object_orientation], page 979.

`cylinder` [*cylinder*, *options*] [Scene object]

Creates a regular prism with height equal to 1 and a hexagonal base with vertices 0.5 units away from the axis. Options [object_height], page 979 and [object_radius], page 980 can be used to change those defaults and option [object_resolution], page 980 can be used to change the number of edges of the base; higher values will make it look like a cylinder. The default height can be changed with the option [object_height], page 979. By default, the axis will be along the x axis and the middle point of the axis will be at the origin; use options [object_orientation], page 979 and [object_center], page 978 to change those defaults.

`sphere` [*sphere*, *options*] [Scene object]

A sphere with default radius of 0.5 units and center at the origin.

57.3.3 Scene object's options

`animation` [*animation*, *property*, *positions*] [Object option]

property should be one of the following 4 object's properties: [object_origin], page 979, [object_scale], page 980, [object_position], page 980 or [object_orientation], page 979 and *positions* should be a list of points. When the play button is pressed, the object property will be changed sequentially through all the values in the list, at intervals of time given by the option [scene_tstep], page 977. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again.

See also [object_track], page 981.

`capping` [*capping*, *number*] [Object option]

Default value: `1`

In a cone or a cylinder, it defines whether the base (or bases) will be shown. A value of 1 for *number* makes the base visible and a value of 0 makes it invisible.

`center` [*center*, *point*] [Object option]

Default value: `[0, 0, 0]`

The coordinates of the object's geometric center, with respect to its [object_position], page 980. *point* can be a list with 3 real numbers, or 3 real numbers separated by commas. In a cylinder, cone or cube it will be at half its height and in a sphere at its center.

`color` [*color*, *colorname*] [Object option]

Default value: `white`

The color of the object. It accepts color names or hexadecimal red-green-blue strings (see the `color` option of plot2d).

`endphi` [*endphi*, `angle`] [Object option]

Default value: `180`

In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. *angle* must be a number between 0 and 180 that sets the final value of phi at which the surface will end. A value smaller than 180 will eliminate a part of the sphere's surface.

See also [object_startphi], page 980 and [object_phiresolution], page 979.

`endtheta` [*endtheta*, `angle`] [Object option]

Default value: `360`

In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. *angle* must be a number between 0 and 360 that sets the final value of theta at which the surface will end. A value smaller than 360 will eliminate a part of the sphere's surface.

See also [object_starttheta], page 980 and [object_thetaresolution], page 981.

`height` [*height*, `value`] [Object option]

Default value: `1`

value must be a positive number which sets the height of a cone or a cylinder.

`linewidth` [*linewidth*, `value`] [Object option]

Default value: `1`

The width of the lines, when option [object_wireframe], page 982 is used. *value* must be a positive number.

`opacity` [*opacity*, `value`] [Object option]

Default value: `1`

value must be a number between 0 and 1. The lower the number, the more transparent the object will become. The default value of 1 means a completely opaque object.

`orientation` [*orientation*, `angles`] [Object option]

Default value: `[0, 0, 0]`

Three angles by which the object will be rotated with respect to the three axis. *angles* can be a list with 3 real numbers, or 3 real numbers separated by commas. **Example**: `[0, 0, 90]` rotates the x axis of the object to the y axis of the reference frame.

`origin` [*origin*, `point`] [Object option]

Default value: `[0, 0, 0]`

The coordinates of the object's origin, with respect to which its other dimensions are defined. *point* can be a list with 3 real numbers, or 3 real numbers separated by commas.

`phiresolution` [*phiresolution*, `num`] [Object option]

Default value:

The number of sub-intervals into which the phi angle interval from [object_startphi], page 980 to [object_endphi], page 979 will be divided. *num* must be a positive integer.

See also [object_startphi], page 980 and [object_endphi], page 979.

`points` [*points*] [Object option]

Only the vertices of the triangulation used to render the surface will be shown. **Example**: `[sphere, [points]]`

See also `[object_surface], page 981` and `[object_wireframe], page 982`.

`pointsize` [*pointsize*, *value*] [Object option]

Default value: `1`

The size of the points, when option `[object_points], page 980` is used. *value* must be a positive number.

`position` [*position*, *point*] [Object option]

Default value: `[0, 0, 0]`

The coordinates of the object's position. *point* can be a list with 3 real numbers, or 3 real numbers separated by commas.

`radius` [*radius*, *value*] [Object option]

Default value: `0.5`

The radius or a sphere or the distance from the axis to the base's vertices in a cylinder or a cone. *value* must be a positive number.

`resolution` [*resolution*, *number*] [Object option]

Default value: `6`

number must be an integer greater than 2 that sets the number of edges in the base of a cone or a cylinder.

`scale` [*scale*, *factors*] [Object option]

Default value: `[1, 1, 1]`

Three numbers by which the object will be scaled with respect to the three axis. *factors* can be a list with 3 real numbers, or 3 real numbers separated by commas. **Example**: `[2, 0.5, 1]` enlarges the object to twice its size in the x direction, reduces the dimensions in the y direction to half and leaves the z dimensions unchanged.

`startphi` [*startphi*, *angle*] [Object option]

Default value: `0`

In a sphere phi is the angle on the vertical plane that passes through the z axis, measured from the positive part of the z axis. *angle* must be a number between 0 and 180 that sets the initial value of phi at which the surface will start. A value bigger than 0 will eliminate a part of the sphere's surface.

See also `[object_endphi], page 979` and `[object_phiresolution], page 979`.

`starttheta` [*starttheta*, *angle*] [Object option]

Default value: `0`

In a sphere theta is the angle on the horizontal plane (longitude), measured from the positive part of the x axis. *angle* must be a number between 0 and 360 that sets the initial value of theta at which the surface will start. A value bigger than 0 will eliminate a part of the sphere's surface.

See also `[object_endtheta], page 979` and `[object_thetaresolution], page 981`.

`surface` [*surface*] [Object option]

The surfaces of the object will be rendered and the lines and points of the triangulation used to build the surface will not be shown. This is the default behavior, which can be changed using either the option [object_points], page 980 or [object_wireframe], page 982.

`thetaresolution` [*thetaresolution*, *num*] [Object option]

Default value:

The number of sub-intervals into which the theta angle interval from [object_starttheta], page 980 to [object_endtheta], page 979 will be divided. *num* must be a positive integer.

See also [object_starttheta], page 980 and [object_endtheta], page 979.

`track` [*track*, *positions*] [Object option]

positions should be a list of points. When the play button is pressed, the object position will be changed sequentially through all the points in the list, at intervals of time given by the option [scene_tstep], page 977, leaving behind a track of the object's trajectory. The rewind button can be used to point at the start of the sequence making the animation restart after the play button is pressed again.

Example. This will show the trajectory of a ball thrown with speed of 5 m/s, at an angle of 45 degrees, when the air resistance can be neglected:

```
(%i1) p: makelist ([0,4*t,4*t- 9.8*t^2/2], t, 0, 0.82, 0.01)$

(%i2) ball: [sphere, [radius,0.1], [color,red], [track,p]]$

(%i3) ground: [cube, [xlength,2], [ylength,4], [zlength,0.2],
      [position,0,1.5,-0.2],[color,green]]$

(%i4) scene (ball, ground)$
```

See also [object_animation], page 978.

`xlength` [*xlength*, *length*] [Object option]

Default value: 1

The height of a cube in the x direction. *length* must be a positive number. See also [object_ylength], page 981 and [object_zlength], page 981.

`ylength` [*ylength*, *length*] [Object option]

Default value: 1

The height of a cube in the y direction. *length* must be a positive number. See also [object_xlength], page 981 and [object_zlength], page 981.

`zlength` [*zlength*, *length*] [Object option]

Default value: 1

The height of a cube in z the direction. *length* must be a positive number. See also [object_xlength], page 981 and [object_ylength], page 981.

`wireframe` [*wireframe*] [Object option]

Only the edges of the triangulation used to render the surface will be shown. **Example**: `[cube, [wireframe]]`

See also `[object_surface], page 981` and `[object_points], page 980`.

58 Package engineering-format

Engineering-format changes the way maxima outputs floating-point numbers to the notation engineers are used to: `a*10^b` with `b` dividable by three.

58.1 Functions and Variables for engineering-format

`engineering_format_floats` [Option variable]

Default value: `true`

This variable allows to temporarily switch off engineering-format.

```
(%i1) load("engineering-format");
(%o1) /home/gunter/src/maxima-code/share/contrib/engineering-for\
mat.lisp
(%i2) float(sin(10)/10000);
(%o2)                - 54.40211108893698e-6
(%i3) engineering_format_floats:false$
(%i4) float(sin(10)/10000);
(%o4)                - 5.440211108893698e-5
```

See also `fpprintprec` and `float`.

`engineering_format_min` [Option variable]

Default value: `0.0`

The minimum absolute value that isn't automatically converted to the engineering format. See also `engineering_format_max` and `engineering_format_floats`.

```
(%i1) lst: float([.05,.5,5,500,5000,500000]);
(%o1)       [0.05, 0.5, 5.0, 500.0, 5000.0, 500000.0]
(%i2) load("engineering-format");
(%o2) /home/gunter/src/maxima-code/share/contrib/engineering-for\
mat.lisp
(%i3) lst;
(%o3) [50.0e-3, 500.0e-3, 5.0e+0, 500.0e+0, 5.0e+3, 500.0e+3]
(%i4) engineering_format_min:.1$
(%i5) engineering_format_max:1000$
(%i6) lst;
(%o6)     [50.0e-3, 0.5, 5.0, 500.0, 5.0e+3, 500.0e+3]
```

`engineering_format_max` [Option variable]

Default value: `0.0`

The maximum absolute value that isn't automatically converted to the engineering format. See also `engineering_format_min` and `engineering_format_floats`.

59 Package ezunits

59.1 Introduction to ezunits

`ezunits` is a package for working with dimensional quantities, including some functions for dimensional analysis. `ezunits` can carry out arithmetic operations on dimensional quantities and unit conversions. The built-in units include Systeme Internationale (SI) and US customary units, and other units can be declared. See also `physical_constants`, a collection of physical constants.

`load("ezunits")` loads this package. `demo("ezunits")` displays several examples. The convenience function `known_units` returns a list of the built-in and user-declared units, while `display_known_unit_conversions` displays the set of known conversions in an easy-to-read format.

An expression a`` `b `` represents a dimensional quantity, with `a` indicating a nondimensional quantity and `b` indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression a`` `b `` can be extracted by the `qty` and `units` functions, respectively.

A symbol may be declared to be a dimensional quantity, with specified quantity or specified units or both.

An expression a`` `b ``` `c `` converts from unit `b` to unit `c`. `ezunits` has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to `ezunits` can be declared. The unit conversions known to `ezunits` are specified by the global variable `known_unit_conversions`, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

As Maxima generally prefers exact numbers (integers or rationals) to inexact (float or bigfloat), so `ezunits` preserves exact numbers when they appear in dimensional quantities. All built-in unit conversions are expressed in terms of exact numbers; inexact numbers in declared conversions are coerced to exact.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. `ezunits` recognizes the prefixes m-, k-, M, and G- (for milli-, kilo-, mega-, and giga-) as applied to SI base units and SI derived units, but such prefixes are applied only when indicated by an explicit conversion.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.

- $(x$`` `a ``$) * (y$`` `b ``$)$ is equal to $(x * y)$`` `(a * b) ``.
- $(x$`` `a ``$) + (y$`` `a ``$)$ is equal to $(x + y)$`` `a ``.
- $(x$`` `a ``$)^y$ is equal to x^y`` `a^y ``^{y} when `y` is nondimensional.

`ezunits` does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

`ezunits` includes functions for elementary dimensional analysis, namely the fundamental dimensions and fundamental units of a dimensional quantity, and computation of dimensionless quantities and natural units. The functions for dimensional analysis were adapted from similar functions in another package, written by Barton Willis.

For the purpose of dimensional analysis, a list of fundamental dimensions and an associated list of fundamental units are maintained; by default the fundamental dimensions are length, mass, time, charge, temperature, and quantity, and the fundamental units are the associated SI units, but other fundamental dimensions and units can be declared.

59.2 Introduction to physical_constants

`physical_constants` is a collection of physical constants, copied from CODATA 2006 recommended values (https://physics.nist.gov/cuu/Constants/). `load ("physical_constants")` loads this package, and loads `ezunits` also, if it is not already loaded.

A physical constant is represented as a symbol which has a property which is the constant value. The constant value is a dimensional quantity, as represented by `ezunits`. The function `constvalue` fetches the constant value; the constant value is not the ordinary value of the symbol, so symbols of physical constants persist in evaluated expressions until their values are fetched by `constvalue`.

`physical_constants` includes some auxiliary information, namely, a description string for each constant, an estimate of the error of its numerical value, and a property for TeX display. To identify physical constants, each symbol has the `physical_constant` property; `propvars(physical_constant)` therefore shows the list of all such symbols.

`physical_constants` comprises the following constants.

`%c`	speed of light in vacuum
`%mu_0`	magnetic constant
`%e_0`	electric constant
`%Z_0`	characteristic impedance of vacuum
`%G`	Newtonian constant of gravitation
`%h`	Planck constant
`%h_bar`	Planck constant
`%m_P`	Planck mass
`%T_P`	Planck temperature
`%l_P`	Planck length
`%t_P`	Planck time
`%%e`	elementary charge
`%Phi_0`	magnetic flux quantum
`%G_0`	conductance quantum
`%K_J`	Josephson constant
`%R_K`	von Klitzing constant
`%mu_B`	Bohr magneton
`%mu_N`	nuclear magneton

`%alpha` fine-structure constant

`%R_inf` Rydberg constant

`%a_0` Bohr radius

`%E_h` Hartree energy

`%ratio_h_me`
quantum of circulation

`%m_e` electron mass

`%N_A` Avogadro constant

`%m_u` atomic mass constant

`%F` Faraday constant

`%R` molar gas constant

`%%k` Boltzmann constant

`%V_m` molar volume of ideal gas

`%n_0` Loschmidt constant

`%ratio_S0_R`
Sackur-Tetrode constant (absolute entropy constant)

`%sigma` Stefan-Boltzmann constant

`%c_1` first radiation constant

`%c_1L` first radiation constant for spectral radiance

`%c_2` second radiation constant

`%b` Wien displacement law constant

`%b_prime` Wien displacement law constant

Reference: https://physics.nist.gov/cuu/Constants/

Examples:

The list of all symbols which have the `physical_constant` property.

```
(%i1) load ("physical_constants")$
(%i2) propvars (physical_constant);
(%o2) [%c, %mu_0, %e_0, %Z_0, %G, %h, %h_bar, %m_P, %T_P, %l_P,
%t_P, %%e, %Phi_0, %G_0, %K_J, %R_K, %mu_B, %mu_N, %alpha,
%R_inf, %a_0, %E_h, %ratio_h_me, %m_e, %N_A, %m_u, %F, %R, %%k,
%V_m, %n_0, %ratio_S0_R, %sigma, %c_1, %c_1L, %c_2, %b, %b_prime]
```

Properties of the physical constant `%c`.

```
(%i1) load ("physical_constants")$
(%i2) constantp (%c);
(%o2)                          true
(%i3) get (%c, description);
(%o3)               speed of light in vacuum
```

```
(%i4) constvalue (%c);
                                        m
(%o4)                     299792458 ' -
                                        s
(%i5) get (%c, RSU);
(%o5)                             0
(%i6) tex (%c);
$$c$$
(%o6)                           false
```

The energy equivalent of 1 pound-mass. The symbol `%c` persists until its value is fetched by `constvalue`.

```
(%i1) load ("physical_constants")$
(%i2) m * %c^2;
                                  2
(%o2)                            %c  m
(%i3) %, m = 1 ' lbm;
                                  2
(%o3)                           %c  ' lbm
(%i4) constvalue (%);
                                              2
                                           lbm m
(%o4)              89875517873681764 ' ------
                                             2
                                            s
(%i5) E : % '' J;
Computing conversions to base units; may take a moment.
                     366838848464007200
(%o5)                ------------------ ' J
                             9
(%i6) E '' GJ;
                      458548560580009
(%o6)                 --------------- ' GJ
                         11250000
(%i7) float (%);
(%o7)              4.0759872051556356e+7 ' GJ
```

59.3 Functions and Variables for ezunits

' [Operator]

The dimensional quantity operator. An expression a'b represents a dimensional quantity, with `a` indicating a nondimensional quantity and `b` indicating the dimensional units. A symbol can be used as a unit without declaring it as such; unit symbols need not have any special properties. The quantity and unit of an expression a'b can be extracted by the `qty` and `units` functions, respectively.

Arithmetic operations on dimensional quantities are carried out by conventional rules for such operations.

- $(x\text{`}a) * (y\text{`}b)$ is equal to $(x * y)\text{`}(a * b)$.
- $(x\text{`}a) + (y\text{`}a)$ is equal to $(x + y)\text{`}a$.
- $(x\text{`}a)^y$ is equal to $x^y\text{`}a^y$ when `y` is nondimensional.

`ezunits` does not require that units in a sum have the same dimensions; such terms are not added together, and no error is reported.

`load ("ezunits")` enables this operator.

Examples:

SI (Systeme Internationale) units.

```
(%i1) load ("ezunits")$
(%i2) foo : 10 ‘ m;
(%o2)                          10 ‘ m
(%i3) qty (foo);
(%o3)                            10
(%i4) units (foo);
(%o4)                             m
(%i5) dimensions (foo);
(%o5)                          length
```

"Customary" units.

```
(%i1) load ("ezunits")$
(%i2) bar : x ‘ acre;
(%o2)                         x ‘ acre
(%i3) dimensions (bar);
                                     2
(%o3)                          length
(%i4) fundamental_units (bar);
                                  2
(%o4)                            m
```

Units ad hoc.

```
(%i1) load ("ezunits")$
(%i2) baz : 3 ‘ sheep + 8 ‘ goat + 1 ‘ horse;
(%o2)           8 ‘ goat + 3 ‘ sheep + 1 ‘ horse
(%i3) subst ([sheep = 3*goat, horse = 10*goat], baz);
(%o3)                        27 ‘ goat
(%i4) baz2 : 1000‘gallon/fortnight;
                                   gallon
(%o4)                    1000 ‘ ---------
                                  fortnight
(%i5) subst (fortnight = 14*day, baz2);
                             500   gallon
(%o5)                        --- ‘ ------
                              7     day
```

Arithmetic operations on dimensional quantities.

```
(%i1) load ("ezunits")$
(%i2) 100 ‘ kg + 200 ‘ kg;
```

```
(%o2)                          300 ‘ kg
(%i3) 100 ‘ m^3 - 100 ‘ m^3;
                                    3
(%o3)                           0 ‘ m
(%i4) (10 ‘ kg) * (17 ‘ m/s^2);
                                  kg m
(%o4)                        170 ‘ ----
                                    2
                                   s
(%i5) (x ‘ m) / (y ‘ s);
                                 x   m
(%o5)                            - ‘ -
                                 y   s
(%i6) (a ‘ m)^2;
                                 2    2
(%o6)                           a  ‘ m
```

‘‘ [Operator]

The unit conversion operator. An expression a‘b‘‘c converts from unit `b` to unit `c`. `ezunits` has built-in conversions for SI base units, SI derived units, and some non-SI units. Unit conversions not already known to `ezunits` can be declared. The unit conversions known to `ezunits` are specified by the global variable `known_unit_conversions`, which comprises built-in and user-defined conversions. Conversions for products, quotients, and powers of units are derived from the set of known unit conversions.

There is no preferred system for display of units; input units are not converted to other units unless conversion is explicitly indicated. `ezunits` does not attempt to simplify units by prefixes (milli-, centi-, deci-, etc) unless such conversion is explicitly indicated.

`load ("ezunits")` enables this operator.

Examples:

The set of known unit conversions.

```
(%i1) load ("ezunits")$
(%i2) display2d : false$
(%i3) known_unit_conversions;
(%o3) {acre = 4840*yard^2,Btu = 1055*J,cfm = feet^3/minute,
       cm = m/100,day = 86400*s,feet = 381*m/1250,ft = feet,
       g = kg/1000,gallon = 757*l/200,GHz = 1000000000*Hz,
       GOhm = 1000000000*Ohm,GPa = 1000000000*Pa,
       GWb = 1000000000*Wb,Gg = 1000000*kg,Gm = 1000000000*m,
       Gmol = 1000000*mol,Gs = 1000000000*s,ha = hectare,
       hectare = 100*m^2,hour = 3600*s,Hz = 1/s,inch = feet/12,
       km = 1000*m,kmol = 1000*mol,ks = 1000*s,l = liter,
       lbf = pound_force,lbm = pound_mass,liter = m^3/1000,
       metric_ton = Mg,mg = kg/1000000,MHz = 1000000*Hz,
       microgram = kg/1000000000,micrometer = m/1000000,
```

```
micron = micrometer,microsecond = s/1000000,
mile = 5280*feet,minute = 60*s,mm = m/1000,
mmol = mol/1000,month = 2629800*s,MOhm = 1000000*Ohm,
MPa = 1000000*Pa,ms = s/1000,MWb = 1000000*Wb,
Mg = 1000*kg,Mm = 1000000*m,Mmol = 1000000000*mol,
Ms = 1000000*s,ns = s/1000000000,ounce = pound_mass/16,
oz = ounce,Ohm = s*J/C^2,
pound_force = 32*ft*pound_mass/s^2,
pound_mass = 200*kg/441,psi = pound_force/inch^2,
Pa = N/m^2,week = 604800*s,Wb = J/A,yard = 3*feet,
year = 31557600*s,C = s*A,F = C^2/J,GA = 1000000000*A,
GC = 1000000000*C,GF = 1000000000*F,GH = 1000000000*H,
GJ = 1000000000*J,GK = 1000000000*K,GN = 1000000000*N,
GS = 1000000000*S,GT = 1000000000*T,GV = 1000000000*V,
GW = 1000000000*W,H = J/A^2,J = m*N,kA = 1000*A,
kC = 1000*C,kF = 1000*F,kH = 1000*H,kHz = 1000*Hz,
kJ = 1000*J,kK = 1000*K,kN = 1000*N,kOhm = 1000*Ohm,
kPa = 1000*Pa,kS = 1000*S,kT = 1000*T,kV = 1000*V,
kW = 1000*W,kWb = 1000*Wb,mA = A/1000,mC = C/1000,
mF = F/1000,mH = H/1000,mHz = Hz/1000,mJ = J/1000,
mK = K/1000,mN = N/1000,mOhm = Ohm/1000,mPa = Pa/1000,
mS = S/1000,mT = T/1000,mV = V/1000,mW = W/1000,
mWb = Wb/1000,MA = 1000000*A,MC = 1000000*C,
MF = 1000000*F,MH = 1000000*H,MJ = 1000000*J,
MK = 1000000*K,MN = 1000000*N,MS = 1000000*S,
MT = 1000000*T,MV = 1000000*V,MW = 1000000*W,
N = kg*m/s^2,R = 5*K/9,S = 1/Ohm,T = J/(m^2*A),V = J/C,
W = J/s}
```

Elementary unit conversions.

```
(%i1) load ("ezunits")$
(%i2) 1 ‘ ft ‘‘ m;
Computing conversions to base units; may take a moment.
                              381
(%o2)                         ---- ‘ m
                              1250
(%i3) %, numer;
(%o3)                        0.3048 ‘ m
(%i4) 1 ‘ kg ‘‘ lbm;
                              441
(%o4)                         --- ‘ lbm
                              200
(%i5) %, numer;
(%o5)                        2.205 ‘ lbm
(%i6) 1 ‘ W ‘‘ Btu/hour;
                             720   Btu
(%o6)                        --- ‘ ----
```

```
                              211   hour
(%i7) %, numer;
                                          Btu
(%o7)                3.412322274881517 ` ----
                                          hour
(%i8) 100 ` degC `` degF;
(%o8)                        212 ` degF
(%i9) -40 ` degF `` degC;
(%o9)                       (- 40) ` degC
(%i10) 1 ` acre*ft `` m^3;
                         60228605349    3
(%o10)                   ----------- ` m
                          48828125
(%i11) %, numer;
                                            3
(%o11)                 1233.48183754752 ` m
```

Coercing quantities in feet and meters to one or the other.

```
(%i1) load ("ezunits")$
(%i2) 100 ` m + 100 ` ft;
(%o2)                   100 ` m + 100 ` ft
(%i3) (100 ` m + 100 ` ft) `` ft;
Computing conversions to base units; may take a moment.
                              163100
(%o3)                         ------ ` ft
                               381
(%i4) %, numer;
(%o4)                 428.0839895013123 ` ft
(%i5) (100 ` m + 100 ` ft) `` m;
                               3262
(%o5)                          ---- ` m
                                25
(%i6) %, numer;
(%o6)                         130.48 ` m
```

Dimensional analysis to find fundamental dimensions and fundamental units.

```
(%i1) load ("ezunits")$
(%i2) foo : 1 ` acre * ft;
(%o2)                         1 ` acre ft
(%i3) dimensions (foo);
                                      3
(%o3)                           length
(%i4) fundamental_units (foo);
                                   3
(%o4)                             m
(%i5) foo `` m^3;
Computing conversions to base units; may take a moment.
                         60228605349    3
```

```
(%o5)                     ----------- ‘ m
                           48828125
(%i6) %, numer;
                                                3
(%o6)                  1233.48183754752 ‘ m
```

Declared unit conversions.

```
(%i1) load ("ezunits")$
(%i2) declare_unit_conversion (MMBtu = 10^6*Btu, kW = 1000*W);
(%o2)                          done
(%i3) declare_unit_conversion (kWh = kW*hour, MWh = 1000*kWh,
                               bell = 1800*s);
(%o3)                          done
(%i4) 1 ‘ kW*s ‘‘ MWh;
Computing conversions to base units; may take a moment.
                                1
(%o4)                       ------- ‘ MWh
                            3600000
(%i5) 1 ‘ kW/m^2 ‘‘ MMBtu/bell/ft^2;
                         1306449      MMBtu
(%o5)                    ---------- ‘ --------
                         8242187500          2
                                    bell ft
```

`constvalue` (*x*) [Function]

Shows the value and the units of one of the constants declared by package `physical_constants`, which includes a list of physical constants, or of a new constant declared in package `ezunits` (see `declare_constvalue`).

Note that constant values as recognized by `constvalue` are separate from values declared by `numerval` and recognized by `constantp`.

Example:

```
(%i1) load ("physical_constants")$
(%i2) constvalue (%G);
                                      3
                                     m
(%o2)                  6.67428 ‘ -----
                                     2
                                    kg s
(%i3) get ('%G, 'description);
(%o3)          Newtonian constant of gravitation
```

`declare_constvalue` (*a*, *x*) [Function]

Declares the value of a constant to be used in package `ezunits`. This function should be loaded with `load ("ezunits")`.

Example:

```
(%i1) load ("ezunits")$
(%i2) declare_constvalue (FOO, 100 ‘ lbm / acre);
```

```
                                      lbm
(%o2)                        100 ‘ ----
                                      acre
(%i3) FOO * (50 ‘ acre);
(%o3)                       50 FOO ‘ acre
(%i4) constvalue (%);
(%o4)                        5000 ‘ lbm
```

`remove_constvalue` (`a`) [Function]

Reverts the effect of `declare_constvalue`. This function should be loaded with `load ("ezunits")`.

`units` (*x*) [Function]

Returns the units of a dimensional quantity *x*, or returns 1 if *x* is nondimensional.

x may be a literal dimensional expression *a*‘*b*, a symbol with declared units via `declare_units`, or an expression containing either or both of those.

This function should be loaded with `load ("ezunits")`.

Example:

```
(%i1) load ("ezunits")$
(%i2) foo : 100 ‘ kg;
(%o2)                          100 ‘ kg
(%i3) bar : x ‘ m/s;
                                     m
(%o3)                          x ‘ -
                                     s
(%i4) units (foo);
(%o4)                           kg
(%i5) units (bar);
                                 m
(%o5)                            -
                                 s
(%i6) units (foo * bar);
                                kg m
(%o6)                           ----
                                 s
(%i7) units (foo / bar);
                                kg s
(%o7)                           ----
                                 m
(%i8) units (foo^2);
                                   2
(%o8)                           kg
```

`declare_units` (`a`, *u*) [Function]

Declares that `units` should return units *u* for *a*, where *u* is an expression. This function should be loaded with `load ("ezunits")`.

Example:

```
(%i1) load ("ezunits")$
(%i2) units (aa);
(%o2)                           1
(%i3) declare_units (aa, J);
(%o3)                           J
(%i4) units (aa);
(%o4)                           J
(%i5) units (aa^2);
                                2
(%o5)                          J
(%i6) foo : 100 ‘ kg;
(%o6)                     100 ‘ kg
(%i7) units (aa * foo);
(%o7)                        kg J
```

qty (*x*) [Function]

Returns the nondimensional part of a dimensional quantity *x*, or returns *x* if *x* is nondimensional. *x* may be a literal dimensional expression *a*‘*b*, a symbol with declared quantity, or an expression containing either or both of those.

This function should be loaded with `load ("ezunits")`.

Example:

```
(%i1) load ("ezunits")$
(%i2) foo : 100 ‘ kg;
(%o2)                     100 ‘ kg
(%i3) qty (foo);
(%o3)                        100
(%i4) bar : v ‘ m/s;
                                m
(%o4)                         v ‘ -
                                s
(%i5) foo * bar;
                                kg m
(%o5)                  100 v ‘ ----
                                 s
(%i6) qty (foo * bar);
(%o6)                        100 v
```

declare_qty (*a*, *x*) [Function]

Declares that qty should return *x* for symbol *a*, where *x* is a nondimensional quantity. This function should be loaded with `load ("ezunits")`.

Example:

```
(%i1) load ("ezunits")$
(%i2) declare_qty (aa, xx);
(%o2)                          xx
(%i3) qty (aa);
```

```
(%o3)                           xx
(%i4) qty (aa^2);
                                  2
(%o4)                           xx
(%i5) foo : 100 ‘ kg;
(%o5)                       100 ‘ kg
(%i6) qty (aa * foo);
(%o6)                        100 xx
```

`unitp` (`x`) [Function]

Returns `true` if x is a literal dimensional expression, a symbol declared dimensional, or an expression in which the main operator is declared dimensional. `unitp` returns `false` otherwise.

`load ("ezunits")` loads this function.

Examples:

`unitp` applied to a literal dimensional expression.

```
(%i1) load ("ezunits")$
(%i2) unitp (100 ‘ kg);
(%o2)                           true
```

`unitp` applied to a symbol declared dimensional.

```
(%i1) load ("ezunits")$
(%i2) unitp (foo);
(%o2)                           false
(%i3) declare (foo, dimensional);
(%o3)                           done
(%i4) unitp (foo);
(%o4)                           true
```

`unitp` applied to an expression in which the main operator is declared dimensional.

```
(%i1) load ("ezunits")$
(%i2) unitp (bar (x, y, z));
(%o2)                           false
(%i3) declare (bar, dimensional);
(%o3)                           done
(%i4) unitp (bar (x, y, z));
(%o4)                           true
```

`declare_unit_conversion` ($u = v$, ...) [Function]

Appends equations $u = v$, ... to the list of unit conversions known to the unit conversion operator ‘‘. u and v are both multiplicative terms, in which any variables are units, or both literal dimensional expressions.

At present, it is necessary to express conversions such that the left-hand side of each equation is a simple unit (not a multiplicative expression) or a literal dimensional expression with the quantity equal to 1 and the unit being a simple unit. This limitation might be relaxed in future versions.

`known_unit_conversions` is the list of known unit conversions.

This function should be loaded with `load ("ezunits")`.

Examples:

Unit conversions expressed by equations of multiplicative terms.

```
(%i1) load ("ezunits")$
(%i2) declare_unit_conversion (nautical_mile = 1852 * m,
                               fortnight = 14 * day);
(%o2)                         done
(%i3) 100 ‘ nautical_mile / fortnight ‘‘ m/s;
Computing conversions to base units; may take a moment.
                             463    m
(%o3)                        ---- ‘ -
                             3024   s
```

Unit conversions expressed by equations of literal dimensional expressions.

```
(%i1) load ("ezunits")$
(%i2) declare_unit_conversion (1 ‘ fluid_ounce = 2 ‘ tablespoon);
(%o2)                         done
(%i3) declare_unit_conversion (1 ‘ tablespoon = 3 ‘ teaspoon);
(%o3)                         done
(%i4) 15 ‘ fluid_ounce ‘‘ teaspoon;
Computing conversions to base units; may take a moment.
(%o4)                     90 ‘ teaspoon
```

`declare_dimensions` (*a_1*, *d_1*, ..., *a_n*, *d_n*) [Function]

Declares *a_1*, ..., *a_n* to have dimensions *d_1*, ..., *d_n*, respectively.

Each *a_k* is a symbol or a list of symbols. If it is a list, then every symbol in *a_k* is declared to have dimension *d_k*.

`load ("ezunits")` loads these functions.

Examples:

```
(%i1) load ("ezunits") $
(%i2) declare_dimensions ([x, y, z], length, [t, u], time);
(%o2)                         done
(%i3) dimensions (y^2/u);
                                    2
                               length
(%o3)                          -------
                                time
(%i4) fundamental_units (y^2/u);
0 errors, 0 warnings
                                 2
                                m
(%o4)                            --
                                 s
```

`remove_dimensions` (*a_1*, ..., *a_n*) [Function]

Reverts the effect of `declare_dimensions`. This function should be loaded with `load ("ezunits")`.

`declare_fundamental_dimensions` (*d_1*, *d_2*, *d_3*, ...) [Function]
`remove_fundamental_dimensions` (*d_1*, *d_2*, *d_3*, ...) [Function]
`fundamental_dimensions` [Global variable]

`declare_fundamental_dimensions` declares fundamental dimensions. Symbols *d_1*, *d_2*, *d_3*, ... are appended to the list of fundamental dimensions, if they are not already on the list.

`remove_fundamental_dimensions` reverts the effect of `declare_fundamental_dimensions`.

`fundamental_dimensions` is the list of fundamental dimensions. By default, the list comprises several physical dimensions.

`load ("ezunits")` loads these functions.

Examples:

```
(%i1) load ("ezunits") $
(%i2) fundamental_dimensions;
(%o2) [length, mass, time, current, temperature, quantity]
(%i3) declare_fundamental_dimensions (money, cattle, happiness);
(%o3)                          done
(%i4) fundamental_dimensions;
(%o4) [length, mass, time, current, temperature, quantity,
                                          money, cattle, happiness]
(%i5) remove_fundamental_dimensions (cattle, happiness);
(%o5)                          done
(%i6) fundamental_dimensions;
(%o6) [length, mass, time, current, temperature, quantity, money]
```

`declare_fundamental_units` (*u_1*, *d_1*, ..., *u_n*, *d_n*) [Function]
`remove_fundamental_units` (*u_1*, ..., *u_n*) [Function]

`declare_fundamental_units` declares *u_1*, ..., *u_n* to have dimensions *d_1*, ..., *d_n*, respectively. All arguments must be symbols.

After calling `declare_fundamental_units`, `dimensions(u_k)` returns *d_k* for each argument *u_1*, ..., *u_n*, and `fundamental_units(d_k)` returns *u_k* for each argument *d_1*, ..., *d_n*.

`remove_fundamental_units` reverts the effect of `declare_fundamental_units`.

`load ("ezunits")` loads these functions.

Examples:

```
(%i1) load ("ezunits") $
(%i2) declare_fundamental_dimensions (money, cattle, happiness);
(%o2)                          done
(%i3) declare_fundamental_units (dollar, money, goat, cattle,
                                 smile, happiness);
(%o3)                 [dollar, goat, smile]
(%i4) dimensions (100 ' dollar/goat/km^2);
                              money
(%o4)                    --------------
                                  2
```

```
                              cattle length
(%i5) dimensions (x ‘ smile/kg);
                                 happiness
(%o5)                            ---------
                                   mass
(%i6) fundamental_units (money*cattle/happiness);
0 errors, 0 warnings
                               dollar goat
(%o6)                          -----------
                                  smile
```

`dimensions` (*x*) [Function]

`dimensions_as_list` (*x*) [Function]

`dimensions` returns the dimensions of the dimensional quantity *x* as an expression comprising products and powers of base dimensions.

`dimensions_as_list` returns the dimensions of the dimensional quantity *x* as a list, in which each element is an integer which indicates the power of the corresponding base dimension in the dimensions of *x*.

`load ("ezunits")` loads these functions.

Examples:

```
(%i1) load ("ezunits")$
(%i2) dimensions (1000 ‘ kg*m^2/s^3);
                                  2
                           length  mass
(%o2)                      ------------
                                 3
                              time
(%i3) declare_units (foo, acre*ft/hour);
                              acre ft
(%o3)                         -------
                               hour
(%i4) dimensions (foo);
                                    3
                              length
(%o4)                         -------
                               time
(%i1) load ("ezunits")$
(%i2) fundamental_dimensions;
(%o2)  [length, mass, time, charge, temperature, quantity]
(%i3) dimensions_as_list (1000 ‘ kg*m^2/s^3);
(%o3)                 [2, 1, - 3, 0, 0, 0]
(%i4) declare_units (foo, acre*ft/hour);
                              acre ft
(%o4)                         -------
                               hour
(%i5) dimensions_as_list (foo);
```

```
(%o5)                [3, 0, - 1, 0, 0, 0]
```

`fundamental_units` [Function]

`fundamental_units (x)`
`fundamental_units ()`

`fundamental_units(x)` returns the units associated with the fundamental dimensions of x. as determined by `dimensions(x)`.

x may be a literal dimensional expression $a`b$, a symbol with declared units via `declare_units`, or an expression containing either or both of those.

`fundamental_units()` returns the list of all known fundamental units, as declared by `declare_fundamental_units`.

`load ("ezunits")` loads this function.

Examples:

```
(%i1) load ("ezunits")$
(%i2) fundamental_units ();
(%o2)                [m, kg, s, A, K, mol]
(%i3) fundamental_units (100 ` mile/hour);
                                 m
(%o3)                            -
                                 s
(%i4) declare_units (aa, g/foot^2);
                                 g
(%o4)                          -----
                                   2
                               foot
(%i5) fundamental_units (aa);
                                kg
(%o5)                           --
                                 2
                                m
```

`dimensionless (L)` [Function]

Returns a basis for the dimensionless quantities which can be formed from a list L of dimensional quantities.

`load ("ezunits")` loads this function.

Examples:

```
(%i1) load ("ezunits") $
(%i2) dimensionless ([x ` m, y ` m/s, z ` s]);
0 errors, 0 warnings
0 errors, 0 warnings
                                y z
(%o2)                          [---]
                                x
```

Dimensionless quantities derived from fundamental physical quantities. Note that the first element on the list is proportional to the fine-structure constant.

```
(%i1) load ("ezunits") $
```

```
(%i2) load ("physical_constants") $
(%i3) dimensionless([%h_bar, %m_e, %m_P, %%e, %c, %e_0]);
0 errors, 0 warnings
0 errors, 0 warnings
                                  2
                               %%e        %m_e
(%o3)                  [--------------, ----]
                        %c %e_0 %h_bar  %m_P
```

`natural_unit` (*expr*, [*v_1*, ..., *v_n*]) [Function]

Finds exponents *e_1*, ..., *e_n* such that `dimension(`*expr*`) = dimension(`*v_1*`^`*e_1* `... `*v_n*`^`*e_n*`)`.

`load ("ezunits")` loads this function.

Examples:

60 Package f90

60.1 Package f90

`f90_output_line_length_max` [Option variable]

Default value: 65

`f90_output_line_length_max` is the maximum number of characters of Fortran code which are output by `f90` per line. Longer lines of code are divided, and printed with an ampersand `&` at the end of an output line and an ampersand at the beginning of the following line.

`f90_output_line_length_max` must be a positive integer.

Example:

```
(%i1) load ("f90")$
(%i2) foo : expand ((xxx + yyy + 7)^4);
         4            3         3        2    2             2
(%o2) yyy  + 4 xxx yyy  + 28 yyy  + 6 xxx  yyy  + 84 xxx yyy
          2        3             2
 + 294 yyy  + 4 xxx  yyy + 84 xxx  yyy + 588 xxx yyy + 1372 yyy
      4         3          2
 + xxx  + 28 xxx  + 294 xxx  + 1372 xxx + 2401
(%i3) f90_output_line_length_max;
(%o3)                          65
(%i4) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xxx**2*yyy**2+84*xxx*yyy**2&
&+294*yyy**2+4*xxx**3*yyy+84*xxx**2*yyy+588*xxx*yyy+1372*yyy+xxx**&
&4+28*xxx**3+294*xxx**2+1372*xxx+2401
(%o4)                          false
(%i5) f90_output_line_length_max : 40 $
(%i6) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xx&
&x**2*yyy**2+84*xxx*yyy**2+294*yyy**2+4*x&
&xx**3*yyy+84*xxx**2*yyy+588*xxx*yyy+1372&
&*yyy+xxx**4+28*xxx**3+294*xxx**2+1372*xx&
&x+2401
(%o6)                          false
```

`f90` (*expr_1*, ..., *expr_n*) [Function]

Prints one or more expressions *expr_1*, ..., *expr_n* as a Fortran 90 program. Output is printed to the standard output.

`f90` prints output in the so-called "free form" input format for Fortran 90: there is no special attention to column positions. Long lines are split at a fixed width with the ampersand `&` continuation character; the number of output characters per line, not including ampersands, is specified by `f90_output_line_length_max`. `f90` outputs an ampersand at the end of a split line and another at the beginning of the next line.

`load("f90")` loads this function. See also the function `fortran`.

Examples:

```
(%i1) load ("f90")$
(%i2) foo : expand ((xxx + yyy + 7)^4);
         4            3         3        2    2             2
(%o2) yyy  + 4 xxx yyy  + 28 yyy  + 6 xxx  yyy  + 84 xxx yyy
          2        3             2
 + 294 yyy  + 4 xxx  yyy + 84 xxx  yyy + 588 xxx yyy + 1372 yyy
      4         3          2
 + xxx  + 28 xxx  + 294 xxx  + 1372 xxx + 2401
(%i3) f90 ('foo = foo);
foo = yyy**4+4*xxx*yyy**3+28*yyy**3+6*xxx**2*yyy**2+84*xxx*yyy**2&
&+294*yyy**2+4*xxx**3*yyy+84*xxx**2*yyy+588*xxx*yyy+1372*yyy+xxx**&
&4+28*xxx**3+294*xxx**2+1372*xxx+2401
(%o3)                          false
```

Multiple expressions. Capture standard output into a file via the `with_stdout` function.

```
(%i1) load ("f90")$
(%i2) foo : sin (3*x + 1) - cos (7*x - 2);
(%o2)              sin(3 x + 1) - cos(7 x - 2)
(%i3) with_stdout ("foo.f90",
                   f90 (x=0.25, y=0.625, 'foo=foo, 'stop, 'end));
(%o3)                          false
(%i4) printfile ("foo.f90");
x = 0.25
y = 0.625
foo = sin(3*x+1)-cos(7*x-2)
stop
end
(%o4)                         foo.f90
```

61 Package finance

61.1 Introduction to finance

This is the Finance Package (Ver 0.1).

In all the functions, *rate* is the compound interest rate, *num* is the number of periods and must be positive and *flow* refers to cash flow so if you have an Output the flow is negative and positive for Inputs.

Note that before using the functions defined in this package, you have to load it writing `load("finance")$`.

Author: Nicolas Guarin Zapata.

61.2 Functions and Variables for finance

`days360` (*year1*,*month1*,*day1*,*year2*,*month2*,*day2*) [Function]

Calculates the distance between 2 dates, assuming 360 days years, 30 days months.

Example:

```
(%i1) load("finance")$
(%i2) days360(2008,12,16,2007,3,25);
(%o2)                        - 621
```

`fv` (*rate*,*PV*,*num*) [Function]

We can calculate the future value of a Present one given a certain interest rate. *rate* is the interest rate, *PV* is the present value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) fv(0.12,1000,3);
(%o2)                       1404.928
```

`pv` (*rate*,*FV*,*num*) [Function]

We can calculate the present value of a Future one given a certain interest rate. *rate* is the interest rate, *FV* is the future value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) pv(0.12,1000,3);
(%o2)                 711.7802478134108
```

`graph_flow` (*val*) [Function]

Plots the money flow in a time line, the positive values are in blue and upside; the negative ones are in red and downside. The direction of the flow is given by the sign of the value. *val* is a list of flow values.

Example:

```
(%i1) load("finance")$
(%i2) graph_flow([-5000,-3000,800,1300,1500,2000])$
```

annuity_pv (*rate*,*PV*,*num*) [Function]

We can calculate the annuity knowing the present value (like an amount), it is a constant and periodic payment. *rate* is the interest rate, PV is the present value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) annuity_pv(0.12,5000,10);
(%o2)                884.9208207992202
```

annuity_fv (*rate*,*FV*,*num*) [Function]

We can calculate the annuity knowing the desired value (future value), it is a constant and periodic payment. *rate* is the interest rate, FV is the future value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) annuity_fv(0.12,65000,10);
(%o2)                3703.970670389863
```

geo_annuity_pv (*rate*,*growing_rate*,*PV*,*num*) [Function]

We can calculate the annuity knowing the present value (like an amount), in a growing periodic payment. *rate* is the interest rate, *growing_rate* is the growing rate, PV is the present value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) geo_annuity_pv(0.14,0.05,5000,10);
(%o2)                802.6888176505123
```

geo_annuity_fv (*rate*,*growing_rate*,*FV*,*num*) [Function]

We can calculate the annuity knowing the desired value (future value), in a growing periodic payment. *rate* is the interest rate, *growing_rate* is the growing rate, FV is the future value and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) geo_annuity_fv(0.14,0.05,5000,10);
(%o2)                216.5203395312695
```

amortization (*rate*,*amount*,*num*) [Function]

Amortization table determined by a specific rate. *rate* is the interest rate, *amount* is the amount value, and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) amortization(0.05,56000,12)$
      "n"    "Balance"     "Interest"   "Amortization"  "Payment"
    0.000     56000.000         0.000         0.000         0.000
    1.000     52481.777      2800.000      3518.223      6318.223
    2.000     48787.643      2624.089      3694.134      6318.223
```

```
    3.000     44908.802      2439.382      3878.841      6318.223
    4.000     40836.019      2245.440      4072.783      6318.223
    5.000     36559.597      2041.801      4276.422      6318.223
    6.000     32069.354      1827.980      4490.243      6318.223
    7.000     27354.599      1603.468      4714.755      6318.223
    8.000     22404.106      1367.730      4950.493      6318.223
    9.000     17206.088      1120.205      5198.018      6318.223
   10.000     11748.170       860.304      5457.919      6318.223
   11.000      6017.355       587.408      5730.814      6318.223
   12.000         0.000       300.868      6017.355      6318.223
```

arit_amortization (*rate*,*increment*,*amount*,*num*) [Function]

The amortization table determined by a specific rate and with growing payment can be calculated by `arit_amortization`. Notice that the payment is not constant, it presents an arithmetic growing, increment is then the difference between two consecutive rows in the "Payment" column. *rate* is the interest rate, *increment* is the increment, *amount* is the amount value, and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) arit_amortization(0.05,1000,56000,12)$
      "n"    "Balance"     "Interest"   "Amortization"  "Payment"
    0.000     56000.000         0.000         0.000         0.000
    1.000     57403.679      2800.000     -1403.679      1396.321
    2.000     57877.541      2870.184      -473.863      2396.321
    3.000     57375.097      2893.877       502.444      3396.321
    4.000     55847.530      2868.755      1527.567      4396.321
    5.000     53243.586      2792.377      2603.945      5396.321
    6.000     49509.443      2662.179      3734.142      6396.321
    7.000     44588.594      2475.472      4920.849      7396.321
    8.000     38421.703      2229.430      6166.892      8396.321
    9.000     30946.466      1921.085      7475.236      9396.321
   10.000     22097.468      1547.323      8848.998     10396.321
   11.000     11806.020      1104.873     10291.448     11396.321
   12.000        -0.000       590.301     11806.020     12396.321
```

geo_amortization (*rate*,*growing_rate*,*amount*,*num*) [Function]

The amortization table determined by rate, amount, and number of periods can be found by `geo_amortization`. Notice that the payment is not constant, it presents a geometric growing, *growing_rate* is then the quotient between two consecutive rows in the "Payment" column. *rate* is the interest rate, *amount* is the amount value, and *num* is the number of periods.

Example:

```
(%i1) load("finance")$
(%i2) geo_amortization(0.05,0.03,56000,12)$
      "n"    "Balance"     "Interest"   "Amortization"  "Payment"
    0.000     56000.000         0.000         0.000         0.000
    1.000     53365.296      2800.000      2634.704      5434.704
```

```
   2.000     50435.816      2668.265      2929.480      5597.745
   3.000     47191.930      2521.791      3243.886      5765.677
   4.000     43612.879      2359.596      3579.051      5938.648
   5.000     39676.716      2180.644      3936.163      6116.807
   6.000     35360.240      1983.836      4316.475      6300.311
   7.000     30638.932      1768.012      4721.309      6489.321
   8.000     25486.878      1531.947      5152.054      6684.000
   9.000     19876.702      1274.344      5610.176      6884.520
  10.000     13779.481       993.835      6097.221      7091.056
  11.000      7164.668       688.974      6614.813      7303.787
  12.000         0.000       358.233      7164.668      7522.901
```

`saving` (*rate*,*amount*,*num*) [Function]

The table that represents the values in a constant and periodic saving can be found by `saving`. *amount* represents the desired quantity and num the number of periods to save.

Example:

```
(%i1) load("finance")$
(%i2) saving(0.15,12000,15)$
      "n"    "Balance"     "Interest"   "Payment"
    0.000         0.000         0.000         0.000
    1.000       252.205         0.000       252.205
    2.000       542.240        37.831       252.205
    3.000       875.781        81.336       252.205
    4.000      1259.352       131.367       252.205
    5.000      1700.460       188.903       252.205
    6.000      2207.733       255.069       252.205
    7.000      2791.098       331.160       252.205
    8.000      3461.967       418.665       252.205
    9.000      4233.467       519.295       252.205
   10.000      5120.692       635.020       252.205
   11.000      6141.000       768.104       252.205
   12.000      7314.355       921.150       252.205
   13.000      8663.713      1097.153       252.205
   14.000     10215.474      1299.557       252.205
   15.000     12000.000      1532.321       252.205
```

`npv` (*rate*,*val*) [Function]

Calculates the present value of a value series to evaluate the viability in a project. *val* is a list of varying cash flows.

Example:

```
(%i1) load("finance")$
(%i2) npv(0.25,[100,500,323,124,300]);
(%o2)                714.4703999999999
```

`irr` (*val*,*I0*) [Function]

IRR (Internal Rate of Return) is the value of rate which makes Net Present Value zero. *flowValues* is a list of varying cash flows, *I0* is the initial investment.

Example:

```
(%i1) load("finance")$
(%i2) res:irr([-5000,0,800,1300,1500,2000],0)$
(%i3) rhs(res[1][1]);
(%o3)                .03009250374237132
```

`benefit_cost` (*rate*,*input*,*output*) [Function]

Calculates the ratio Benefit/Cost. Benefit is the Net Present Value (NPV) of the inputs, and Cost is the Net Present Value (NPV) of the outputs. Notice that if there is not an input or output value in a specific period, the input/output would be a zero for that period. *rate* is the interest rate, *input* is a list of input values, and *output* is a list of output values.

Example:

```
(%i1) load("finance")$
(%i2) benefit_cost(0.24,[0,300,500,150],[100,320,0,180]);
(%o2)               1.427249324905784
```

62 Package format

62.1 Introduction to format

The `format` package was written by Bruce R. Miller, NIST (miller-at-cam.nist.gov).

`format` is a package for formatting algebraic expressions in Maxima. It provides facilities for user-directed hierarchical structuring of expressions, as well as for directing simplifications to selected subexpressions. It emphasizes a semantic rather than syntactic description of the desired form. The package also provides utilities for obtaining efficiently the coefficients of polynomials, trigonometric sums and power series.

In a general purpose Computer Algebra System (CAS), any particular mathematical expression can take on a variety of forms: expanded form, factored form or anything in between. Each form may have advantages; a given form may be more compact than another, or allow clear expression of certain algorithms. Or it may simply be more informative, particularly if it has physical significance. A CAS contains many tools for transforming expressions. However, most are like Maxima's `factor` and `expand`, operating only on the entire expression or its top level. At the other extreme are operations like `substpart` which extract a specific part of an expression, then transform and replace it. Unfortunately, the means of specifying the piece of interest is purely syntactic, requiring the user to keep close watch on the form of the arguments to avoid error.

The package described here gives users of Maxima more control over the structure of expressions, and it does so using a more semantic, almost algebraic, language describing the desired structure. It also provides a semantic means of addressing parts of an expression for particular simplifications. For example, to rearrange an expression into a series in `eps` through order 5, whose terms will be polynomials in `x` and `y`, whose coefficients, in turn, will be trigonometric sums in `l` and `g` with factored coefficients one uses the command:

```
format(foo; %series(eps; 5); %poly(x; y);%trig(l; g); %factor);
```

The principal tool, `format`, is described in section Section 62.2 [Functions and Variables for format], page 1011. It uses procedures in `coeflist` which obtain coefficients of polynomials, trigonometric sums and power series.

62.2 Functions and Variables for format

`format` (*expr*, *template*$_1$, ...) [Function]

Each `template` indicates the desired form for an expression; either the expected form or that into which it will be transformed. At the same time, the indicated form implies a set of *pieces*; the next template in the chain applies to those pieces. For example, `%poly(x)` specifies the transformation into a polynomial in `x`, with the pieces being the coefficients. The passive `%frac` treats the expression as a fraction; the pieces are the numerator and denominator. Whereas the next template formats all pieces of the previous layer, positional *subtemplates* may be used to specify formats for each piece individually. This is most useful when the pieces have unique roles and need to be treated differently, such as a fraction's numerator and denominator.

The full syntax of a template is

```
keyword (parameter ; ...)[subtemplate ; ...]
```

The recognized keywords are described below under [Template keywords], page 1012. The parameters (if not needed) and subtemplates (along with parentheses and brackets) are optional.

In addition to the keyword templates, arithmetic patterns are recognized. This is an expression involving addition, multiplication and exponentiation containing a single instance of a keyword template. In effect, the system 'solves' the expression to be formatted for the corresponding part, formats it accordingly and reinserts it. For example, `format(X,a+%factor)` is equivalent to `a+factor(X-a)`. Any other template is assumed to be a function to be applied to the expression; the result is then formatted according to the rest of the template chain.

Examples for general restructuring:

```
(%i1) load("format")$
(%i2) format((a+b*x)*(c-x)^2,%poly(x),factor);
           3               2                       2
(%o2)    b x  - (2 b c - a) x  + c (b c - 2 a) x + a c
(%i3) format((1+2*eps*(q+r*cos(g))^2)^4,%series(eps,2),%trig(g),factor);
                  2      2                2
(%o3) 1 + eps (4 (r  + 2 q ) + 4 cos(2 g) r  + 16 cos(g) q r)
      2        4       2  2      4                4
 + eps  (3 (3 r  + 24 q  r  + 8 q ) + 3 cos(4 g) r
                  3                      2      2
 + 24 cos(3 g) q r  + 24 cos(g) q r (3 r  + 4 q )
                2   2      2
 + 12 cos(2 g) r  (r  + 6 q )) + . . .
(%i4) format((1+2*a+a^2)*b + a*(1+2*b+b^2),%sum,%product,%factor);
                             2          2
(%o4)                a (b + 1)  + (a + 1)  b
(%i5) format(expand((a+x)^3-a^3),%f-a^3);
                                3    3
(%o5)                      (x + a)  - a
```

62.2.1 Template keywords

Keywords come in several *classes*: Algebraic, Sums, Products, Fractions, Complex, Bags, General, Targeting, Control, Subtemplate Aids, and Convenience.

A few remarks about keywords: A passive keyword does not transform the expression but treats it as a sum, fraction or whatever. The order of the pieces corresponds to the internal ordering; subtemplate usage may be awkward. See the documentation of `coerce_bag` for a description of the coercions used. Targeting templates are basically shorthand equivalents of structuring templates using subtemplates.

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Algebraic*		
`%poly(`x_1`,...)`, `%p`	polynomial in x_i	coefficients (ascending exps.)
`%series(`*eps*`,`*n*`)`, `%s`	series in *eps* through order *n*	coefficients (ascending exps.)
`%Taylor(`*eps*`,`*n*`)`	Taylor in *eps* through order *n*	coefficients (ascending exps.)
`%monicpoly(`x_1`,...),%mp`	monic polynomial in x_i	leading coef then coefs
`%trig(`x_1`,...)`, `%t`	trigonometric sum in x_i	`sin` coefs (ascending), then `cos`
`%coeff(`*v*`,`*n*`)`	polynomial in *v*	coefficient of v^n and remainder

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Sums*		
`%sum`	*passive*	terms (`inpart` order)
`%partfrac(`*x*`)`, `%pf`	partial fraction decomp in *x*	terms (`inpart` order)

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Products*		
`%product`, `%prod`	*passive*	factors (`inpart` order)
`%factor`, `%f`	factored form	factors (`inpart` order)
`%factor(`*minpoly*`)`, `%f`	factored with element adjoined	factors (`inpart` order)
`%sqfr`, `%sf`	square-free factored form	factors (`inpart` order)

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Fractions*		
`%frac`	*passive*	numerator and denominator
`%ratsimp`, `%r`	rationally simplified	numerator and denominator

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Complex*		
`%rectform`, `%g`	gaussian form	real and imaginary parts
`%polarform`	polar form	magnitude and phase

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *Bags*		
`%equation`, `%eq`	equation	l.h.s. and r.h.s.
`%relation(`*r*`)`, `%rel`	relation; *r* in `(=,>,>=,<,<=,!=)`	l.h.s. and r.h.s.

`%list`	list	elements
`%matrix`	matrix	rows (use `%list` for elements)

Template(w/abbrev.)	**Coersion to**	**Pieces and Ordering**
Class: *General*		
`%expression, %expr`	*passive*	the operands (`inpart` order)
`%preformat(`T_1`,...)`	format accord. to chain T_i	the result, not the parts

Template(w/abbrev.)	**Function**
Class: *Targeting*	
`%arg(n)`	formats the `n`-th argument
`%lhs(r)`	formats the l.h.s. of an eqn. or relation (default '=')
`%rhs(r)`	formats the l.h.s. of an eqn.
`%element(i,...), %el`	formats an element of a matrix
`%num, %denom`	formats the numerator or denominator of a fraction
`%match(P)`	formats all subexpressions for which *P(expr)* returns true

Template(w/abbrev.)	**Function**
Class: *Control*	
`%if(`P_1`,...)[`T_1`,...,`T_{n+1}`]`	Find first P_i*(expr)* $\rightarrow$ `true`, then format *expr* using T_i, else T_{n+1}

Template(w/abbrev.)	**Function**
Class: *Subtemplate Aids*	
`%noop`	does nothing; used to fill a subtemplate slot
`[`T_1`,`T_2`,...]`	creates a template chain where an individual template was expected
`%ditto(T)`	repeats the template so that it applies to following pices

Template(w/abbrev.)	**Function**
Class: *Convenience*	
`%subst(eqns,...)`	substitutes *eqns* into expression; result is formatted at next layer
`%ratsubst(eqns,...)`	`lratsubst`'s *eqns* into expression; result is formatted at next layer

Example with simplification on subexpression:

```
(%i1) load("format")$
(%i2) foo:x^2*sin(y)^4-2*x^2*sin(y)^2+x^4*cos(y)^4-2*x^4*cos(y)^2+x^4+x^2+1$
(%i3) trigsimp(foo);
                4    2     4         4    2       4
(%o3)        (x  + x ) cos (y) - 2 x  cos (y) + x  + 1
(%i4) format(foo,%p(x),trigsimp);
                     4    4       2    4
(%o4)               x  sin (y) + x  cos (y) + 1
(%i5) format([a=b,c=d,e=f],%equation);
(%o5)                  [a, c, e] = [b, d, f]
(%i6) format(%,%list);
(%o6)                  [a = b, c = d, e = f]
(%i7) m1:matrix([a^2+2*a+1=q,b^2+2*b+1=r], [c^2+2*c+1=s,d^2+2*d+1=t])$
(%i8) format(m1,%equation,%matrix[%noop,%list[%noop,%factor]]);
             [  2             2           ]
             [ a  + 2 a + 1  b  + 2 b + 1 ]   [ q  r ]
(%o8)        [                            ] = [     ]
             [  2                    2    ]   [ s  t ]
             [ c  + 2 c + 1    (d + 1)    ]
```

The following examples illustrate the usage with 'bags.'

```
(%i1) format([a=b,c=d,e=f],%equation);
(%o1) [a, c, e] = [b, d, f]
(%i2) format(%,%list);
(%o2) [a = b, c = d, e = f]
(%i3) m1:matrix([a^2+2*a+1=q,b^2+2*b+1=r], [c^2+2*c+1=s,d^2+2*d+1=t])$
(%i4) format(m1,%equation,%matrix[%noop,%list[%noop,%factor]]);
                   [  2             2           ]
                   [ a  + 2 a + 1  b  + 2 b + 1 ]   [ q  r ]
(%o4)              [                            ] = [     ]
                   [  2                    2    ]   [ s  t ]
                   [ c  + 2 c + 1    (d + 1)    ]
```

62.2.2 User defined templates

New templates can be defined by giving the template keyword the property `formatter`; the value should be a function (or lambda expression) of the expression to be formatted and any parameters for the template. For example, `%rectform` and `%if` could be defined as

```
put(%rectform,lambda([c],block([r:rectformlist(c)],
   format-piece(r[1]) +%I* format-piece(r[2]))),formatter)

put(%if, lambda([x,test],
   if test(x) then format-piece(x,1)
     else
       format-piece(x,2)),formatter)
```

Functions used for defining templates are the following.

`%format_piece` (*piece*,*nth*) [function]

`lratsubst`'s `eqns` into expression and the result is formatted at the next layer. Format a given piece of an expression, automatically accounting for subtemplates and the remaining template chain. A specific subtemplate, rather than the next one, can be selected by specifying *nth*.

`%coerce_bag` (*op*,*expr*) [function]

Attempts to coerce *expr* into an expression with *op* (one of `=, #, <, <=, >, >=, [` or matrix) as the top-level operator. It coerces the expression by swapping operands between layers but only if adjacent layers are also lists, matrices or relations. This model assumes that a list of equations, for example, can be viewed as an equation whose sides are lists. Certain combinations, particularly those involving inequalities may not be meaningful, however, so some caution is advised.

62.2.3 Determining coefficients

We define the 'algebras' of polynomials, trigonometric sums and power series to be those expressions that can be cast into the following forms.

$$\mathbf{P}(v_1,\cdots) = \{P \mid P = \sum_i c_i v_1^{p_{1,i}} v_2^{p_{2,i}} \cdots\}$$

$$\mathbf{T}(v_1,\cdots) = \{T \mid T = \sum_i [c_i \cos(m_{1,i} v_1 + \cdots) + s_i \sin(m'_{1,i} v_1 + \cdots)]\}$$

$$\mathbf{S}(v,O) = \{S \mid S = \sum_i c_i v^{p_i};\ p_n \le O\}$$

The variables v_i may be any atomic expression in the sense of `ratvars`. The shorthands `operator(op)` and `match(predicate)` may be used to specify all subexpressions having op as an operator, or that pass the predicate, respectively.

The coefficients c_i and s_i are general Maxima expressions. In principle they would be independent of the variables v_i, but in practice they may contain non-polynomial dependence (or non-trigonometric, in the trigonometric case). These non-polynomial cases would include expressions like `(1 + x)^n`, where `n` is symbolic. Likewise, `(x^a)^b` is, in general, multivalued; unless `a = 1` or `b` is a member of `Z`, or `radexpand=all`, it will not be interpreted as `x^ab` is a member of **P**. Furthermore, we extend the algebras to include lists, vectors, matrices and equations, by interpreting a list of polynomials, say, as a polynomial with lists as coefficients.

The exponents p_i in series are restricted to numbers, but the exponents $c_{j,i}$ and multiples $m_{j,i}$ for polynomials and trigonometric sums may be general expressions (excluding bags).

The following functions construct a list of the coefficients and 'keys', that is, the exponents or multiples. Note that these are sparse representations; no coefficients are zero.

```
coeffs(P,v1,...) → [[%poly,v1,...],[c1,p1,1,...],...]
trig_coeffs(T,v1,...) →
       [[%trig,v1,...],[[c1,m1,1,...],...],[[s1,m'1,1,...],...]]
series_coeffs(S,v,O) → [[%series,v,O],[c1,p1],...,[cn,pn]]
```

`Taylor_coeffs(S,v,O)` → `[[%Taylor,v,O],[`c_1`,`p_1`],...,[`c_n`,`p_n`]`

The latter two functions both expand an expression through order `O`, but the series version only carries expands arithmetic operations and is often considerably faster than `Taylor_coeffs`.

Examples:

```
(%i1) load("format")$
(%i2) cl1:coeffs((a+b*x)*(c-x)^2,x);
                        2          2
(%o2) [[%poly, x], [a c , 0], [b c  - 2 a c, 1], [a - 2 b c, 2],
                                                          [b, 3]]
(%i3) map('first,rest(coeffs((a+b*x)*(c-x)^2=q0+q1*x+q2*x^2+q3*x^3,x)));
            2          2
(%o3) [a c  = q0, b c  - 2 a c = q1, a - 2 b c = q2, b = q3]
(%i4) trig_coeffs(2*(a+cos(x))*cos(x+3*y),x,y);
(%o4) [[%trig, x, y], [], [[1, 0, 3], [2 a, 1, 3], [1, 2, 3]]]
(%i5) series_coeffs((a+b*x)*(c-x)^2,x,2);
                             2          2
(%o5) [[%series, x, 2], [a c , 0], [b c  - 2 a c, 1],
                                               [a - 2 b c, 2]]
(%i6) coeffs((a+b*x)*sin(x),x);
(%o6)      [[%poly, x], [a sin(x), 0], [b sin(x), 1]]
(%i7) coeffs((a+log(b)*x)*(c-log(x))^2,operator(log));
                                      2           2
(%o7) [[%poly, log(x), log(b)], [a c , 0, 0], [c  x, 0, 1],
          [- 2 a c, 1, 0], [- 2 c x, 1, 1], [a, 2, 0], [x, 2, 1]]
```

62.2.4 Related functions

`get_coef` (*clist*,k_1,...) [function]

Gets the coefficient from the coefficient list *clist* corresponding to the keys k_i. The keys are matched to variable powers when *clist* is a `%poly`, `%series` or `%Taylor` form. If *clist* is a `%trig` then k_1 should be `sin` or `cos` and the remaining keys are matched to multipliers.

`uncoef` (*clist*) [function]

Reconstructs the expression from a coefficient list *clist*. The coefficient list can be any of the coefficient list forms.

`partition_poly` (*expr*,*test*,v_1,...) [function]

Partitions *expr* into two polynomials; the first is made of those monomials for which the function test returns true and the second is the remainder. The test function is called on the powers of the v_i.

`partition_trig` (*expr*,*sintest*,*costest*,v_1,...) [function]

Trigonometric analog to partition poly; The functions *sintest* and *costest* select sine and cosine terms, respectively; each are called on the multipliers of the v_i.

`partition_series` (*expr*,*test*,*v*,*0*) [function]

`partition_Taylor` (*expr*,*test*,*v*,*0*) [function]

Analog to `partition_poly` for series.

Example:

```
(%i1) load("format")$
(%i2) partition_poly((a+b*x)*(c-x)^2,'evenp,x);
                       2      2     3       2
(%o2)   [(a - 2 b c) x  + a c , b x  + (b c  - 2 a c) x]
```

Support functions

`matching_parts` (*expr*,*predicate*,*args*...) [function]

Returns a list of all subexpressions of *expr* for which the application `predicate(piece,args ... )` returns `true`.

`function_calls` (*expr*,*functions*...) [function]

Returns a list of all calls in *expr* involving any of *functions*.

`function_arguments` (*expr*,*functions*...) [function]

Returns a list of all argument lists for calls to *functions* in *expr*.

Examples:

```
(%i1) load("format")$
(%i2) t2:(a+log(b)*x)*(c-log(x))^2$
(%i3) matching_parts(t2,constantp);
(%o3)                        [2, - 1]
(%i4) function_calls(t2,log);
(%o4)                    [log(x), log(b)]
```

62.2.5 Implementation

Original documentation is located in the share/contrib/format directory and contains an appendix describing the implementation algorithm in more detail.

63 Package fractals

63.1 Introduction to fractals

This package defines some well known fractals:

- with random IFS (Iterated Function System): the Sierpinsky triangle, a Tree and a Fern

- Complex Fractals: the Mandelbrot and Julia Sets

- the Koch snowflake sets

- Peano maps: the Sierpinski and Hilbert maps

Author: José Ramírez Labrador.

For questions, suggestions and bugs, please feel free to contact me at

pepe DOT ramirez AAATTT uca DOT es

63.2 Definitions for IFS fractals

Some fractals can be generated by iterative applications of contractive affine transformations in a random way; see

Hoggar S. G., "Mathematics for computer graphics", Cambridge University Press 1994.

We define a list with several contractive affine transformations, and we randomly select the transformation in a recursive way. The probability of the choice of a transformation must be related with the contraction ratio.

You can change the transformations and find another fractal

`sierpinskiale` (*n*) [Function]

Sierpinski Triangle: 3 contractive maps; .5 contraction constant and translations; all maps have the same contraction ratio. Argument *n* must be great enough, 10000 or greater.

Example:

```
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,sierpinskiale(n)], [style,dots])$
```

`treefale` (*n*) [Function]

3 contractive maps all with the same contraction ratio. Argument *n* must be great enough, 10000 or greater.

Example:

```
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,treefale(n)], [style,dots])$
```

`fernfale` (*n*) [Function]

4 contractive maps, the probability to choice a transformation must be related with the contraction ratio. Argument *n* must be great enough, 10000 or greater.

Example:

```
(%i1) load("fractals")$
(%i2) n: 10000$
(%i3) plot2d([discrete,fernfale(n)], [style,dots])$
```

63.3 Definitions for complex fractals

`mandelbrot_set` (*x*, *y*) [Function]

Mandelbrot set.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

```
(%i1) load("fractals")$
(%i2) plot3d (mandelbrot_set, [x, -2.5, 1], [y, -1.5, 1.5],
                [gnuplot_preamble, "set view map"],
                [gnuplot_pm3d, true],
                [grid, 150, 150])$
```

`julia_set` (*x*, *y*) [Function]

Julia sets.

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

Example:

```
(%i1) load("fractals")$
(%i2) plot3d (julia_set, [x, -2, 1], [y, -1.5, 1.5],
                [gnuplot_preamble, "set view map"],
                [gnuplot_pm3d, true],
                [grid, 150, 150])$
```

See also `julia_parameter`.

`julia_parameter` [Optional variable]

Default value: `%i`

Complex parameter for Julia fractals. Its default value is `%i`; we suggest the values `-.745+%i*.113002`, `-.39054-%i*.58679`, `-.15652+%i*1.03225`, `-.194+%i*.6557` and `.011031-%i*.67037`.

`julia_sin` (*x*, *y*) [Function]

While function `julia_set` implements the transformation `julia_parameter+z^2`, function `julia_sin` implements `julia_parameter*sin(z)`. See source code for more details.

This program runs slowly because it calculates a lot of sines.

Example:

This program is time consuming because it must make a lot of operations; the computing time is also related with the number of grid points.

```
(%i1) load("fractals")$
```

```
(%i2) julia_parameter:1+.1*%i$
(%i3) plot3d (julia_sin, [x, -2, 2], [y, -3, 3],
                [gnuplot_preamble, "set view map"],
                [gnuplot_pm3d, true],
                [grid, 150, 150])$
```

See also julia_parameter.

63.4 Definitions for Koch snowflakes

snowmap (*ent*, *nn*) [Function]

Koch snowflake sets. Function snowmap plots the snow Koch map over the vertex of an initial closed polygonal, in the complex plane. Here the orientation of the polygon is important. Argument *nn* is the number of recursive applications of Koch transformation; *nn* must be small (5 or 6).

Examples:

```
(%i1) load("fractals")$
(%i2) plot2d([discrete,
              snowmap([1,exp(%i*%pi*2/3),exp(-%i*%pi*2/3),1],4)])$
(%i3) plot2d([discrete,
              snowmap([1,exp(-%i*%pi*2/3),exp(%i*%pi*2/3),1],4)])$
(%i4) plot2d([discrete, snowmap([0,1,1+%i,%i,0],4)])$
(%i5) plot2d([discrete, snowmap([0,%i,1+%i,1,0],4)])$
```

63.5 Definitions for Peano maps

Continuous curves that cover an area. Warning: the number of points exponentially grows with *n*.

hilbertmap (*nn*) [Function]

Hilbert map. Argument *nn* must be small (5, for example). Maxima can crash if *nn* is 7 or greater.

Example:

```
(%i1) load("fractals")$
(%i2) plot2d([discrete,hilbertmap(6)])$
```

sierpinskimap (*nn*) [Function]

Sierpinski map. Argument *nn* must be small (5, for example). Maxima can crash if *nn* is 7 or greater.

Example:

```
(%i1) load("fractals")$
(%i2) plot2d([discrete,sierpinskimap(6)])$
```

64 Package gentran

64.1 Introduction to Gentran

Original Authors Barbara Gates and Paul Wang

Gentran is a powerful generator of foreign language code. Currently it can generate FORTRAN, 'C', and RATFOR code from Maxima language code. Gentran can translate mathematical expressions, iteration loops, conditional branching statements, data type information, function definitions, matrtices and arrays, and more.

64.2 Functions for Gentran

`gentran` (*`stmt1`*, *`stmt2`*, ... , *`stmtn`* , [*`f1, f2, ... , fm`*]) [Function]

Translates each stmt into formatted code in the target language. A substantial subset of expressions and statements in the Maxima programming language can be translated directly into numerical code. The **gentran** command translates Maxima statements or procedure definitions into code in the target language (**gentranlang:** fortran, c, or ratfor). Expressions may optionally be given to Maxima for evaluation prior to translation.

stmt1, stmt2, ... , stmtn is a sequence of one or more statements, each of which is any Maxima user level expression, (simple, group, or block) statement, or procedure definition that can be translated into the target language.

[f1, f2, ... , fm] is an optional list of output files to which translated output will be written. They can be any of the following:

string = the name of an output file in quotes

true (no quotes) = the terminal

false = the current output file(s)

all = all files currently open for output by gentran

If the files are not open they will be opened; if they are open, output will be appended to them. Filenames are given as quoted strings. If the optional variable **genoutpath** (string, including the final /) default **false** is set, it will be prepended to the output file names. If the output file list is omitted, output will be written to the current output, generally the terminal. **gentran** returns (a list of) the name(s) of file(s) to which code was written.

`gentranout` (*`f1`*, *`f2`*, ... , *`fn`*) [Function]

Gentran maintains a list of files currently open for output by gentran commands only. gentranout inserts each file name represented by *f1, f2,... , fn* into that list and opens each one for output. It also resets the current output file(s) to include all files in *f1, f2, ... , fn*. gentranout returns the list of files represented by *f1, f2, ... , fn*; i.e., the current output file(s) after the command has been executed.

`gentranshut` (*`f1`*, *`f2`*, ... , *`fn`*) [Function]

gentranshut creates a list of file names from *f1, f2, ... , fn*, deletes each from the output file list, and closes the corresponding files. If (all of) the current output

file(s) are closed, then the current output file is reset to the terminal. gentranshut returns (a list of) the current output file(s) after the command has been executed. **gentranshut**(**all**) will close all gentran output files.

`gentranpush` (*f1*, *f2*, ... , *fn*) [Function]

gentranpush pushes the file list onto the output stack. Each file in the list that is not already open for output is opened at this time. The current output file is reset to this new element on the top of the stack.

`gentranpop` (*f1*, *f2*, ... , *fn*) [Function]

gentranpop deletes the top-most occurrence of the single element containing the file name(s) represented by *f1, f2, ... , fn* from the output stack. Files whose names have been completely removed from the output stack are closed. The current output file is reset to the (new) element on the top of the output stack. gentranpop returns the current output file(s) after this command has been executed.

`gentranin` (*f1*, *f2*, ... , *fn*, [*f1*,*f2*, ... , *fm*]) [Function]

gentranin processes mixed-language template files consisting of active parts (delimited by `<<...>>`) containing Maxima statements, including calls to gentran, and passive parts, assumed to contain statements in the target language (including comments), which are transcribed verbatim. Input files are processed sequentially and the results appended to the output. The presence of `>>` in passive parts of the file (except for in comments) is interpreted as an end-of-file and terminates processing of that file. The optional list of output files *[f1,f2, ... , fm]* each receive a copy of the entire output. All filespecs are quoted strings. Input files may be given as (quoted string) filenames, which will be located by Maxima **file_search**. The optional variable **geninpath** (default **false**) must be a *list* of quoted strings describing the paths to be searched for the input files. If it is set, that list replaces the standard Maxima search paths.

Active parts may contain any number of Maxima expressions and statements. They are not copied directly to the output. Instead, they are given to Maxima for evaluation. All output generated by each evaluation is sent to the output file(s). Returned values are only printed on the terminal. Active parts will most likely contain calls to gentran to generate code. This means that the result of processing a template file will be the original template file with all active parts replaced by generated code. If *[f1, f2, ... , fm]* is not supplied, then generated code is simply written to the current output file(s). However, if it is given, then the current output file is temporarily overridden. Generated code is written to each file represented by *f1, f2, ... , fn* for this command only. Files which were open prior to the call to gentranin will remain open after the call, and files which did not exist prior to the call will be created, opened, written to, and closed. The output file stack will be exactly the same both before and after the call. gentranin returns (to the terminal) the name(s) of (all) file(s) written to by this command.

`gentraninshut` () [Function]

A cleanup function to close input files in case where gentranin hung due to error in template.

`tempvar (type)` [Function]
Generates temporary variable names by concatenating **tempvarname** (default **'t**) with sequence numbers. If *type* is non-false, *e.g.* "real*8" the corresponding type is assigned to the variable in the gentran symbol table, which may be used to generate declarations depending on the setting of the **gendecs** flag. It is the users responsibility to make sure temporary variable names do not conflict with the main program.

`markvar (vname)` [Function]
markvar "marks" variable name *vname* to indicate that it currently holds a significant value.

`unmarkvar (vname)` [Function]
unmarkvar "unmarks" variable name *vname* to indicate that it no longer holds a significant value.

`markedvarp (vname)` [Function]
markedvarp tests whether the variable name *vname* is currently marked.

`gendecs (name)` [Function]
The gendecs command can be called any time the gendecs flag is switched off to retrieve all type declarations from Gentran's symbol table for the given subprogram name (or the "current" subprogram if false is given as its argument).

`gentran_on (sw)` [Function]
Turns on the mode switch *sw*.

`gentran_off (sw)` [Function]
Turns the given switch, *sw*, off.

64.3 Gentran Mode Switches

`fortran` [Option variable]
`ratfor` [Option variable]
`c` [Option variable]
Default: off

These mode switches change the default mode of Maxima from evaluation to translation. They can be turned on and off with the gentran commands gentran_on and gentran_off. Each time a new Maxima session is started up, the system is in evaluation mode. It prints a prompt on the user's terminal screen and waits for an expression or statement to be entered. It then proceeds to evaluate the expression, prints a new prompt, and waits for the user to enter another expression or statement. This mode can be changed to translation mode by turning on either the fortran, ratfor or c switches. After one of these switches is turned on and until it is turned off, every expression or statement entered by the user is translated into the corresponding language just as if it had been given as an argument to the gentran command. Each of the special functions that can be used from within a call to gentran can be used at the top level until the switch is turned off.

`gendecs` [Option variable]

Default: on

When the gendecs switch is turned on, gentran generates type declarations whenever possible. When gendecs is switched off, type declarations are not generated. Instead, type information is stored in gentran's symbol table but is not retrieved in the form of declarations unless and until either the gendecs command is called or the gendecs flag is switched back. **Note**: Generated declarations may often be placed in an inappropriate place (*e.g.* in the middle of executable fortran code). Therefore the gendecs flag is turned off during processing of templates by **gentranin**.

64.4 Gentran Option Variables

`gentranlang` [Option variable]

Default: fortran

Selects the target numerical language. Currently, gentranlang must be fortran, ratfor, or c. Note that symbols entered in Maxima are case-sensitive. gentranlang should not be set to FORTRAN, RATFOR or C.

`fortlinelen` [Option variable]

default: 72

Maximum number of characters printed on each line of generated FORTRAN code.

`minfortlinelen` [Option variable]

Default: 40

Minimum number of characters printed on each line of generated FORTRAN code.

`fortcurrind` [Option variable]

Default: 0

Number of blank spaces printed at the beginning of each line of generated FORTRAN code (after column 6).

`ratlinelen` [Option variable]

Default: 80

Maximum number of characters printed on each line of generated Ratfor code.

`clinelen` [Option variable]

Default: 80

Maximum number of characters printed on each line of generated 'C' code.

`minclinelen` [Option variable]

Default: 40

Minimum number of characters printed on each line of generated 'C' code.

`ccurind` [Option variable]

Default: 0

Number of blank spaces printed at the beginning of each line of generated'C' code.

`tablen` [Option variable]

Default: 4

Number of blank spaces printed for each new level of indentation. (Automatic indentation can be turned off by setting this variable to 0.)

`genfloat` [Option variable]

Default: false

When set to true (or any non-false value), causes integers in generated numerical code to be converted to floating point numbers, except in the following places: array subscripts, exponents, and initial, final, and step values in do-loops. An exception (for compatibility with Macsyma 2.4) is that numbers in exponentials (with base %e only) are double-floated even when genfloat is false.

`dblfloat` [Option variable]

Default: **false** If dblfloat is set to true, floating point numbers in gentran output in implementations (such as Windows Maxima under CLISP) in which float and double-float are the same will be printed as *.d0. In implementations in which float and double-float are different, floats will be coerced to double-float before being printed.

`gentranseg` [Option variable]

Default: **true**

`maxexpprintlen` [Option variable]

Default: 800

When **gentranseg** is true (or any non-false value), causes Gentran to "segment" large expressions into subexpressions of manageable size. The segmentation facility generates a sequence of assignment statements, each of which assigns a subexpression to an automatically generated temporary variable name. This sequence is generated in such a way that temporary variables are re-used as soon as possible, thereby keeping the number of automatically generated variables to a minimum. The maximum allowable expression size can be controlled by setting the **maxexpprintlen** variable to the maximum number of characters allowed in an expression printed in the target numerical language (excluding spaces and other whitespace characters automatically printed by the formatter). When the segmentation routine generates temporary variables, it places type declarations in the symbol table for those variables if possible. It uses the following rules to determine their type:

1. If the type of the variable to which the large expression is being assigned is already known (i.e., has been declared by the user via a TYPE form), then the temporary variables will be declared to be of that same type. 2. If the global variable **tempvartype** has a non-false value, then the temporary variables are declared to be of that type. 3. Otherwise, the variables are not declared unless **implicit** has been set to **true**.

`gentranopt` [Option variable]

Default: **false**

When set to true (or any non-false value), causes Gentran to replace each block of straightline code by an optimized sequence of assignments obtained from the Maxima

optimize command. (The optimize command takes an expression and replaces common subexpressions by temporary variable names. It returns the resulting assignment statement, preceded by common-subexpression-to-temporary-variable assignments.

`tempvarname` [Option variable]

Default: **'t**

Name used as the prefix when generating temporary variable names.

`optimvarname` [Option variable]

default: **'u**

is the preface used to generate temporary file names produced by the optimizer when **gentranopt** is **true**. When both gentranseg and gentranopt are true, the optimizer generates temporary file names using **optimvarname** while the segmentation routine uses **tempvarname** preventing conflict.

`tempvarnum` [Option variable]

Default: 0

Number appended onto tempvarname to create a temporary variable name. If the temporary variable name resulting from appending tempvarnum onto the end of tempvarname has already been generated and still holds a useful value or has a different type than requested, then the number is incremented until one is found that was not previously generated or does not still hold a significant value or a different type.

`tempvartype` [Option variable]

Default: **false**

Target language variable type (e.g., INTEGER, REAL*8, FLOAT, etc.) used as a default for automatically generated variables whose type cannot be determined otherwise. If tempvartype is false, then generated temporary variables whose type cannot be determined are not automatically declared.

`implicit` [Option variable]

Default: **false**

If implicit is set to **true** temporary variables are assigned their implicit type according to Fortran rules based on the initial letter of the name. If gendecs is on, this results in printed type declarations.

`gentranparser` [Option variable]

Default: **false**

If gentranparser is set to **true** Maxima forms input to gentran will be parsed and an error will be produced if an expression cannot be translated. Otherwise, untranslatable expressions may generate anomalous output, sometimes containing explicit calls to Maxima functions.

`genstmtno` [Option variable]

Default: 25000

Number used when a statement number must be generated. Note: it is the user's responsibility to make sure this number will not clash with statement numbers in template files.

`genstmtincr` [Option variable]

Default: 1

number by which genstmtno is incremented each time a new statement number is generated.

`usefortcomplex` [Option variable]

Default: **false**

If usefortcomplex is true, real numbers in expressions declared to be complex by *type(complex,...)* will be printed in Fortran complex number format *(realpart,0.0)*. This is a purely syntactic device and does not carry out any complex calculations.

64.5 Gentran Evaluation Forms

The following special functions can be included in Maxima statements which are to be translated by the gentran command to indicate that they are to be partially or fully evaluated by Maxima before being translated into numerical code. Note that these functions have the described effect only when supplied in arguments to the gentran command.

`eval` (*exp*) [Function]

Where *exp* is any Maxima expression or statement which, after evaluation by Maxima, results in an expression that can be translated by gentran into the target language. When eval is called from an argument that is to be translated, it tells gentran to give the expression to Maxima for evaluation first, and then to translate the result of that evaluation.

`rsetq` (*var*, *exp*) [Function]

Where *var* is any Maxima variable, matrix or array element, and *exp* is any Maxima expression which, after evaluation by Maxima results in an expression that can be translated by Gentran into the target language. This is equivalent to VAR : EVAL(EXP) ;

`lsetq` (*var, exp*) [Function]

Where *var* is any Maxima user level matrix or array element with indices which, after evaluation by Maxima, will result in expressions that can be translated by Gentran, and *exp* is any Maxima user level expression that can be translated into the target language. This is equivalent to `var[eval(s1), eval(s2), ...]: exp` where *s1*, *s2*, ... are indices.

`lrsetq` (*var, exp*) [Function]

Where *var* is any Maxima matrix or array element with indices which, after evaluation by Maxima, will result in expressions that can be translated by Gentran; and *exp* is any user level expression which, after evaluation, will result in an expression that can be translated by Gentran into the target language. This is equivalent to `var[eval(s1), eval(s2), ...]: eval(exp);`.

`type` (*type,v1...vn*) [Function]

Places information in the gentran symbol table to assign *type* to variables *v1...vn*. This may result in type declarations printed by gentran depending on the setting of gendecs. **type** must be called from within gentran and does not evaluate its arguments unless **eval**() is used.

`literal` (*arg1, arg2, ... , argn*) [Function]

where arg1, arg2, ... , argn is an argument list containing one or more arg's, each of which either is, or evaluates to, an atom. The atoms *tab* and *cr* have special meanings. arg's are not evaluated unless given as arguments to eval. This function call is replaced by the character sequence resulting from concatenation of the given atoms. Double quotes are stripped from all string type arg's, and each occurrence of the reserved atom *tab* or *cr* is replaced by a tab to the current level of indentation, or an end-of-line character.

65 Package ggf

65.1 Functions and Variables for ggf

`GGFINFINITY` [Option variable]

Default value: 3

This is an option variable for function `ggf`.

When computing the continued fraction of the generating function, a partial quotient having a degree (strictly) greater than *GGFINFINITY* will be discarded and the current convergent will be considered as the exact value of the generating function; most often the degree of all partial quotients will be 0 or 1; if you use a greater value, then you should give enough terms in order to make the computation accurate enough.

See also `ggf`.

`GGFCFMAX` [Option variable]

Default value: 3

This is an option variable for function `ggf`.

When computing the continued fraction of the generating function, if no good result has been found (see the *GGFINFINITY* flag) after having computed *GGFCFMAX* partial quotients, the generating function will be considered as not being a fraction of two polynomials and the function will exit. Put freely a greater value for more complicated generating functions.

See also `ggf`.

`ggf` (*l*) [Function]

Compute the generating function (if it is a fraction of two polynomials) of a sequence, its first terms being given. *l* is a list of numbers.

The solution is returned as a fraction of two polynomials. If no solution has been found, it returns with `done`.

This function is controlled by global variables *GGFINFINITY* and *GGFCFMAX*. See also *GGFINFINITY* and *GGFCFMAX*.

To use this function write first `load("ggf")`.

```
(%i1) load("ggf")$
(%i2) makelist(fib(n),n,0,10);
(%o2)                [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
(%i3) ggf(%);
                                      x
(%o3)                          - ----------
                                     2
                                    x  + x - 1
(%i4) taylor(%,x,0,10);
                2      3      4      5      6       7       8       9       10
(%o4)/T/ x + x  + 2 x  + 3 x  + 5 x  + 8 x  + 13 x  + 21 x  + 34 x  + 55 x
```

```
                                                                      + . . .
(%i5) makelist(2*fib(n+1)-fib(n),n,0,10);
(%o5)              [2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
(%i6) ggf(%);
                                    x - 2
(%o6)                             ----------
                                   2
                                  x  + x - 1
(%i7) taylor(%,x,0,10);
                     2      3      4       5       6       7       8       9
(%o7)/T/ 2 + x + 3 x  + 4 x  + 7 x  + 11 x  + 18 x  + 29 x  + 47 x  + 76 x
                                                                        10
                                                                + 123 x   + . . .
```

As these examples show, the generating function does create a function whose Taylor series has coefficients that are the elements of the original list.

66 Package graphs

66.1 Introduction to graphs

The `graphs` package provides graph and digraph data structure for Maxima. Graphs and digraphs are simple (have no multiple edges nor loops), although digraphs can have a directed edge from u to v and a directed edge from v to u.

Internally graphs are represented by adjacency lists and implemented as a lisp structures. Vertices are identified by their ids (an id is an integer). Edges/arcs are represented by lists of length 2. Labels can be assigned to vertices of graphs/digraphs and weights can be assigned to edges/arcs of graphs/digraphs.

There is a `draw_graph` function for drawing graphs. Graphs are drawn using a force based vertex positioning algorithm. `draw_graph` can also use graphviz programs available from https://www.graphviz.org. `draw_graph` is based on the maxima `draw` package.

To use the `graphs` package, first load it with `load("graphs")`.

66.2 Functions and Variables for graphs

66.2.1 Building graphs

`create_graph` [Function]

`create_graph` (*v_list*, *e_list*)
`create_graph` (*n*, *e_list*)
`create_graph` (*v_list*, *e_list*, *directed*)

Creates a new graph on the set of vertices *v_list* and with edges *e_list*.

v_list is a list of vertices `[v1, v2, ..., vn]` or a list of vertices together with vertex labels `[[v1, l1], [v2 ,l2], ..., [vn, ln]]`. A vertex may be any integer, and *v_list* may contain vertices in any order. A label may be any Maxima expression, and two or more vertices may have the same label.

n is the number of vertices. Vertices will be identified by integers from 0 to n-1.

e_list is a list of edges `[e1, e2,..., em]` or a list of edges together with edge-weights `[[e1, w1], ..., [em, wm]]`.

If *directed* is not `false`, a directed graph will be returned.

Example 1: create a cycle on 3 vertices:

```
(%i1) load ("graphs")$
(%i2) g : create_graph([1,2,3], [[1,2], [2,3], [1,3]])$
(%i3) print_graph(g)$
Graph on 3 vertices with 3 edges.
Adjacencies:
  3 :  1  2
  2 :  3  1
  1 :  3  2
```

Example 2: create a cycle on 3 vertices with edge weights:

```
(%i1) load ("graphs")$
```

```
(%i2) g : create_graph([1,2,3], [[[1,2], 1.0], [[2,3], 2.0],
                                 [[1,3], 3.0]])$
(%i3) print_graph(g)$
Graph on 3 vertices with 3 edges.
Adjacencies:
  3 :  1  2
  2 :  3  1
  1 :  3  2
```

Example 3: create a directed graph:

```
(%i1) load ("graphs")$
(%i2) d : create_graph(
        [1,2,3,4],
        [
         [1,3], [1,4],
         [2,3], [2,4]
        ],
        'directed = true)$
(%i3) print_graph(d)$
Digraph on 4 vertices with 4 arcs.
Adjacencies:
  4 :
  3 :
  2 :  4  3
  1 :  4  3
```

`copy_graph` (*g*) [Function]

Returns a copy of the graph *g*.

`circulant_graph` (*n*, *d*) [Function]

Returns the circulant graph with parameters *n* and *d*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : circulant_graph(10, [1,3])$
(%i3) print_graph(g)$
Graph on 10 vertices with 20 edges.
Adjacencies:
  9 :  2  6  0  8
  8 :  1  5  9  7
  7 :  0  4  8  6
  6 :  9  3  7  5
  5 :  8  2  6  4
  4 :  7  1  5  3
  3 :  6  0  4  2
  2 :  9  5  3  1
  1 :  8  4  2  0
  0 :  7  3  9  1
```

`clebsch_graph` () [Function]
Returns the Clebsch graph.

`complement_graph` (*g*) [Function]
Returns the complement of the graph *g*.

`complete_bipartite_graph` (*n*, *m*) [Function]
Returns the complete bipartite graph on *n*+*m* vertices.

`complete_graph` (*n*) [Function]
Returns the complete graph on *n* vertices.

`cycle_digraph` (*n*) [Function]
Returns the directed cycle on *n* vertices.

`cycle_graph` (*n*) [Function]
Returns the cycle on *n* vertices.

`cuboctahedron_graph` (*n*) [Function]
Returns the cuboctahedron graph.

`cube_graph` (*n*) [Function]
Returns the *n*-dimensional cube.

`dodecahedron_graph` () [Function]
Returns the dodecahedron graph.

`empty_graph` (*n*) [Function]
Returns the empty graph on *n* vertices.

`flower_snark` (*n*) [Function]
Returns the flower graph on *4n* vertices.

Example:

```
(%i1) load ("graphs")$
(%i2) f5 : flower_snark(5)$
(%i3) chromatic_index(f5);
(%o3)                          4
```

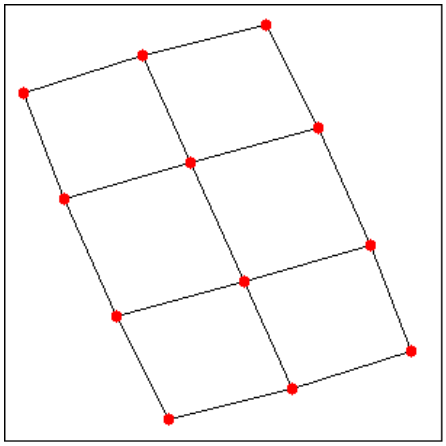
`from_adjacency_matrix` (*A*) [Function]
Returns the graph represented by its adjacency matrix *A*.

`frucht_graph` () [Function]
Returns the Frucht graph.

`graph_product` (*g1*, *g1*) [Function]
Returns the direct product of graphs *g1* and *g2*.

Example:

```
(%i1) load ("graphs")$
(%i2) grid : graph_product(path_graph(3), path_graph(4))$
(%i3) draw_graph(grid)$
```



`graph_union` (*g1*, *g1*) [Function]
Returns the union (sum) of graphs *g1* and *g2*.

`grid_graph` (*n*, *m*) [Function]
Returns the *n* x *m* grid.

`great_rhombicosidodecahedron_graph` () [Function]
Returns the great rhombicosidodecahedron graph.

`great_rhombicuboctahedron_graph` () [Function]
Returns the great rhombicuboctahedron graph.

`grotzch_graph` () [Function]
Returns the Grotzch graph.

`heawood_graph` () [Function]
Returns the Heawood graph.

`icosahedron_graph` () [Function]
Returns the icosahedron graph.

`icosidodecahedron_graph` () [Function]
Returns the icosidodecahedron graph.

`induced_subgraph` (*V*, *g*) [Function]
Returns the graph induced on the subset *V* of vertices of the graph *g*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) V : [0,1,2,3,4]$
(%i4) g : induced_subgraph(V, p)$
(%i5) print_graph(g)$
Graph on 5 vertices with 5 edges.
Adjacencies:
  4 :  3  0
  3 :  2  4
```

```
2 :  1  3
1 :  0  2
0 :  1  4
```

`line_graph` (*g*) [Function]

Returns the line graph of the graph *g*.

`make_graph` [Function]

`make_graph` (*vrt*, *f*)
`make_graph` (*vrt*, *f*, *oriented*)

Creates a graph using a predicate function *f*.

vrt is a list or set of vertices, or an integer.

When *vrt* is a list or set, its elements may be any integers, and, if a list, may be listed in any order.

When *vrt* is an integer, vertices of the graph will be integers from 1 to *vrt*.

f is a predicate function. Two vertices *a* and *b* will be connected if `f(a,b)=true`.

If *directed* is not *false*, then the graph will be directed.

Example 1:

```
(%i1) load("graphs")$
(%i2) g : make_graph(powerset({1,2,3,4,5}, 2), disjointp)$
(%i3) is_isomorphic(g, petersen_graph());
(%o3)                          true
(%i4) get_vertex_label(1, g);
(%o4)                        {1, 2}
```

Example 2:

```
(%i1) load("graphs")$
(%i2) f(i, j) := is (mod(j, i)=0)$
(%i3) g : make_graph(20, f, directed=true)$
(%i4) out_neighbors(4, g);
(%o4)                    [8, 12, 16, 20]
(%i5) in_neighbors(18, g);
(%o5)                    [1, 2, 3, 6, 9]
```

`mycielski_graph` (*g*) [Function]

Returns the mycielskian graph of the graph *g*.

`new_graph` () [Function]

Returns the graph with no vertices and no edges.

`path_digraph` (*n*) [Function]

Returns the directed path on *n* vertices.

`path_graph` (*n*) [Function]

Returns the path on *n* vertices.

`petersen_graph` [Function]

`petersen_graph` ()
`petersen_graph` (*n*, *d*)

Returns the petersen graph $P\_\{n,d\}$. The default values for *n* and *d* are `n=5` and `d=2`.

`random_bipartite_graph` (*a*, *b*, *p*) [Function]
Returns a random bipartite graph on `a+b` vertices. Each edge is present with probability *p*.

`random_digraph` (*n*, *p*) [Function]
Returns a random directed graph on *n* vertices. Each arc is present with probability *p*.

`random_regular_graph` [Function]
`random_regular_graph` (*n*)
`random_regular_graph` (*n*, *d*)
Returns a random *d*-regular graph on *n* vertices. The default value for *d* is `d=3`.

`random_graph` (*n*, *p*) [Function]
Returns a random graph on *n* vertices. Each edge is present with probability *p*.

`random_graph1` (*n*, *m*) [Function]
Returns a random graph on *n* vertices and random *m* edges.

`random_network` (*n*, *p*, *w*) [Function]
Returns a random network on *n* vertices. Each arc is present with probability *p* and has a weight in the range `[0,w]`. The function returns a list `[network, source, sink]`.

Example:

```
(%i1) load ("graphs")$
(%i2) [net, s, t] : random_network(50, 0.2, 10.0);
(%o2)                    [DIGRAPH, 50, 51]
(%i3) max_flow(net, s, t)$
(%i4) first(%);
(%o4)                    27.65981397932507
```

`random_tournament` (*n*) [Function]
Returns a random tournament on *n* vertices.

`random_tree` (*n*) [Function]
Returns a random tree on *n* vertices.

`small_rhombicosidodecahedron_graph` () [Function]
Returns the small rhombicosidodecahedron graph.

`small_rhombicuboctahedron_graph` () [Function]
Returns the small rhombicuboctahedron graph.

`snub_cube_graph` () [Function]
Returns the snub cube graph.

`snub_dodecahedron_graph` () [Function]
Returns the snub dodecahedron graph.

`truncated_cube_graph` () [Function]
Returns the truncated cube graph.

`truncated_dodecahedron_graph` () [Function]
Returns the truncated dodecahedron graph.

`truncated_icosahedron_graph` () [Function]
Returns the truncated icosahedron graph.

`truncated_tetrahedron_graph` () [Function]
Returns the truncated tetrahedron graph.

`tutte_graph` () [Function]
Returns the Tutte graph.

`underlying_graph` (*g*) [Function]
Returns the underlying graph of the directed graph *g*.

`wheel_graph` (*n*) [Function]
Returns the wheel graph on *n+1* vertices.

66.2.2 Graph properties

`adjacency_matrix` (*gr*) [Function]
Returns the adjacency matrix of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(4)$
(%i3) adjacency_matrix(c5);
                          [ 0  1  0  1 ]
                          [             ]
                          [ 1  0  1  0 ]
(%o3)                     [             ]
                          [ 0  1  0  1 ]
                          [             ]
                          [ 1  0  1  0 ]
```

`average_degree` (*gr*) [Function]
Returns the average degree of vertices in the graph *gr*.

Example:

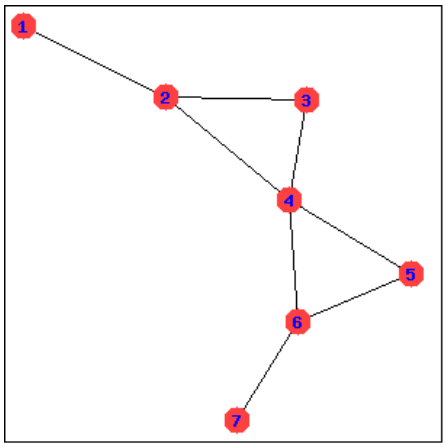
```
(%i1) load ("graphs")$
(%i2) average_degree(grotzch_graph());
                                 40
(%o2)                            --
                                 11
```

`biconnected_components` (*gr*) [Function]
Returns the (vertex sets of) 2-connected components of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph(
```

```
                [1,2,3,4,5,6,7],
                [
                 [1,2],[2,3],[2,4],[3,4],
                 [4,5],[5,6],[4,6],[6,7]
                ])$
(%i3) biconnected_components(g);
(%o3)         [[6, 7], [4, 5, 6], [1, 2], [2, 3, 4]]
```

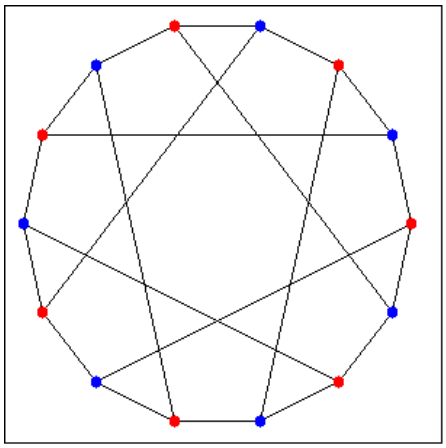


`bipartition` (*gr*) [Function]

Returns a bipartition of the vertices of the graph *gr* or an empty list if *gr* is not bipartite.

Example:

```
(%i1) load ("graphs")$
(%i2) h : heawood_graph()$
(%i3) [A,B]:bipartition(h);
(%o3)  [[8, 12, 6, 10, 0, 2, 4], [13, 5, 11, 7, 9, 1, 3]]
(%i4) draw_graph(h, show_vertices=A, program=circular)$
```



`chromatic_index` (*gr*) [Function]

Returns the chromatic index of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) chromatic_index(p);
(%o3)                          4
```

`chromatic_number` (*gr*) [Function]

Returns the chromatic number of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) chromatic_number(cycle_graph(5));
(%o2)                          3
(%i3) chromatic_number(cycle_graph(6));
(%o3)                          2
```

`clear_edge_weight` (*e*, *gr*) [Function]

Removes the weight of the edge *e* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph(3, [[[0,1], 1.5], [[1,2], 1.3]])$
(%i3) get_edge_weight([0,1], g);
(%o3)                         1.5
(%i4) clear_edge_weight([0,1], g)$
(%i5) get_edge_weight([0,1], g);
(%o5)                          1
```

`clear_vertex_label` (*v*, *gr*) [Function]

Removes the label of the vertex *v* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph([[0,"Zero"], [1, "One"]], [[0,1]])$
(%i3) get_vertex_label(0, g);
(%o3)                         Zero
(%i4) clear_vertex_label(0, g);
(%o4)                         done
(%i5) get_vertex_label(0, g);
(%o5)                         false
```

`connected_components` (*gr*) [Function]

Returns the (vertex sets of) connected components of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g: graph_union(cycle_graph(5), path_graph(4))$
(%i3) connected_components(g);
(%o3)           [[1, 2, 3, 4, 0], [8, 7, 6, 5]]
```

`diameter` (*gr*) [Function]

Returns the diameter of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) diameter(dodecahedron_graph());
(%o2)                          5
```

`edge_coloring` (*gr*) [Function]

Returns an optimal coloring of the edges of the graph *gr*.

The function returns the chromatic index and a list representing the coloring of the edges of *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) [ch_index, col] : edge_coloring(p);
(%o3) [4, [[[0, 5], 3], [[5, 7], 1], [[0, 1], 1], [[1, 6], 2],
[[6, 8], 1], [[1, 2], 3], [[2, 7], 4], [[7, 9], 2], [[2, 3], 2],
[[3, 8], 3], [[5, 8], 2], [[3, 4], 1], [[4, 9], 4], [[6, 9], 3],
[[0, 4], 2]]]
(%i4) assoc([0,1], col);
(%o4)                          1
(%i5) assoc([0,5], col);
(%o5)                          3
```

`degree_sequence` (*gr*) [Function]

Returns the list of vertex degrees of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) degree_sequence(random_graph(10, 0.4));
(%o2)           [2, 2, 2, 2, 2, 2, 3, 3, 3, 3]
```

`edge_connectivity` (*gr*) [Function]

Returns the edge-connectivity of the graph *gr*.

See also `min_edge_cut`.

`edges` (*gr*) [Function]

Returns the list of edges (arcs) in a (directed) graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) edges(complete_graph(4));
(%o2)   [[2, 3], [1, 3], [1, 2], [0, 3], [0, 2], [0, 1]]
```

`get_edge_weight` [Function]

`get_edge_weight` (*e*, *gr*)
`get_edge_weight` (*e*, *gr*, *ifnot*)

Returns the weight of the edge *e* in the graph *gr*.

If there is no weight assigned to the edge, the function returns 1. If the edge is not present in the graph, the function signals an error or returns the optional argument *ifnot*.

Example:

```
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) get_edge_weight([1,2], c5);
(%o3)                          1
(%i4) set_edge_weight([1,2], 2.0, c5);
(%o4)                         done
(%i5) get_edge_weight([1,2], c5);
(%o5)                          2.0
```

`get_vertex_label` (*v*, *gr*) [Function]

Returns the label of the vertex *v* in the graph *gr*.

If no label is assigned to vertex *v*, `get_vertex_label` returns `false`.

Example:

```
(%i1) load("graphs")$
(%i2) g: create_graph([[0, "Zero"], [1, "One"], 2, 3], [])$
(%i3) get_vertex_label(0, g);
(%o3)                         Zero
(%i4) get_vertex_label(2, g);
(%o4)                         false
```

`get_unique_vertex_by_label` (*l*, *gr*) [Function]

Returns the unique vertex which has the label *l* in graph *gr*.

If there is no such vertex, `get_unique_vertex_by_label` returns `false`.

If there are two or more vertices with label *l*, `get_unique_vertex_by_label` reports an error.

Example:

```
(%i1) load ("graphs")$
(%i2) g: create_graph ([[0, "Zero"], [1, "One"], [2, "Other"], [3, "Other"]], [])
(%i3) get_unique_vertex_by_label ("Zero", g);
(%o3)                           0
(%i4) get_unique_vertex_by_label ("Two", g);
(%o4)                         false
(%i5) errcatch (get_unique_vertex_by_label ("Other", g));
get_unique_vertex_by_label: two or more vertices have the same label "Other"
(%o5)                          []
```

`get_all_vertices_by_label` (*l*, *gr*) [Function]

Returns all vertices, if any, which have the label *l* in graph *gr*.

If there are no such vertices, `get_all_vertices_by_label` returns an empty list `[]`.

Example:

```
(%i1) load ("graphs")$
```

```
(%i2) g: create_graph ([[0, "Zero"], [1, "One"], [2, "Other"], [3, "Other"]], [])
(%i3) get_all_vertices_by_label ("Zero", g);
(%o3)                          [0]
(%i4) get_all_vertices_by_label ("Two", g);
(%o4)                          []
(%i5) get_all_vertices_by_label ("Other", g);
(%o5)                       [2, 3]
```

graph_charpoly (*gr*, *x*) [Function]

Returns the characteristic polynomial (in variable *x*) of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_charpoly(p, x), factor;
                                    5        4
(%o3)               (x - 3) (x - 1)  (x + 2)
```

graph_center (*gr*) [Function]

Returns the center of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_center(g);
(%o3)                         [12]
```

graph_eigenvalues (*gr*) [Function]

Returns the eigenvalues of the graph *gr*. The function returns eigenvalues in the same format as maxima `eigenvalues` function.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_eigenvalues(p);
(%o3)               [[3, - 2, 1], [1, 4, 5]]
```

graph_periphery (*gr*) [Function]

Returns the periphery of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : grid_graph(5,5)$
(%i3) graph_periphery(g);
(%o3)                    [24, 20, 4, 0]
```

graph_size (*gr*) [Function]

Returns the number of edges in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
```

```
(%i2) p : petersen_graph()$
(%i3) graph_size(p);
(%o3)                          15
```

`graph_order` (*gr*) [Function]

Returns the number of vertices in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) graph_order(p);
(%o3)                          10
```

`girth` (*gr*) [Function]

Returns the length of the shortest cycle in *gr*.

Example:

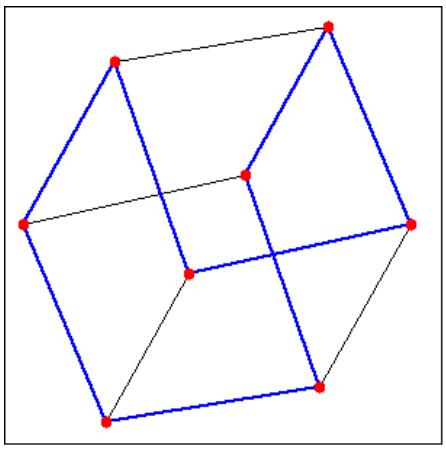
```
(%i1) load ("graphs")$
(%i2) g : heawood_graph()$
(%i3) girth(g);
(%o3)                           6
```

`hamilton_cycle` (*gr*) [Function]

Returns the Hamilton cycle of the graph *gr* or an empty list if *gr* is not hamiltonian.

Example:

```
(%i1) load ("graphs")$
(%i2) c : cube_graph(3)$
(%i3) hc : hamilton_cycle(c);
(%o3)               [7, 3, 2, 6, 4, 0, 1, 5, 7]
(%i4) draw_graph(c, show_edges=vertices_to_cycle(hc))$
```

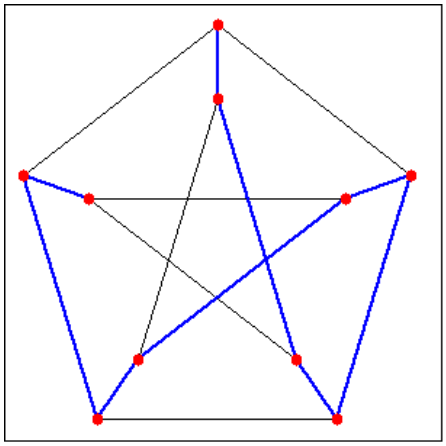


`hamilton_path` (*gr*) [Function]

Returns the Hamilton path of the graph *gr* or an empty list if *gr* does not have a Hamilton path.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) hp : hamilton_path(p);
(%o3)            [0, 5, 7, 2, 1, 6, 8, 3, 4, 9]
(%i4) draw_graph(p, show_edges=vertices_to_path(hp))$
```



`isomorphism` (*gr1*, *gr2*) [Function]

Returns a an isomorphism between graphs/digraphs *gr1* and *gr2*. If *gr1* and *gr2* are not isomorphic, it returns an empty list.

Example:

```
(%i1) load ("graphs")$
(%i2) clk5:complement_graph(line_graph(complete_graph(5)))$
(%i3) isomorphism(clk5, petersen_graph());
(%o3) [9 -> 0, 2 -> 1, 6 -> 2, 5 -> 3, 0 -> 4, 1 -> 5, 3 -> 6,
                                          4 -> 7, 7 -> 8, 8 -> 9]
```

`in_neighbors` (*v*, *gr*) [Function]

Returns the list of in-neighbors of the vertex *v* in the directed graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : path_digraph(3)$
(%i3) in_neighbors(2, p);
(%o3)                          [1]
(%i4) out_neighbors(2, p);
(%o4)                          []
```

`is_biconnected` (*gr*) [Function]

Returns `true` if *gr* is 2-connected and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_biconnected(cycle_graph(5));
```

```
(%o2)                          true
(%i3) is_biconnected(path_graph(5));
(%o3)                          false
```

`is_bipartite` (*gr*) [Function]

Returns `true` if *gr* is bipartite (2-colorable) and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_bipartite(petersen_graph());
(%o2)                          false
(%i3) is_bipartite(heawood_graph());
(%o3)                          true
```

`is_connected` (*gr*) [Function]

Returns `true` if the graph *gr* is connected and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_connected(graph_union(cycle_graph(4), path_graph(3)));
(%o2)                          false
```

`is_digraph` (*gr*) [Function]

Returns `true` if *gr* is a directed graph and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_digraph(path_graph(5));
(%o2)                          false
(%i3) is_digraph(path_digraph(5));
(%o3)                          true
```

`is_edge_in_graph` (*e*, *gr*) [Function]

Returns `true` if *e* is an edge (arc) in the (directed) graph *g* and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) c4 : cycle_graph(4)$
(%i3) is_edge_in_graph([2,3], c4);
(%o3)                          true
(%i4) is_edge_in_graph([3,2], c4);
(%o4)                          true
(%i5) is_edge_in_graph([2,4], c4);
(%o5)                          false
(%i6) is_edge_in_graph([3,2], cycle_digraph(4));
(%o6)                          false
```

`is_graph` (*gr*) [Function]

Returns `true` if *gr* is a graph and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
```

```
(%i2) is_graph(path_graph(5));
(%o2)                         true
(%i3) is_graph(path_digraph(5));
(%o3)                         false
```

`is_graph_or_digraph` (*gr*) [Function]

Returns `true` if *gr* is a graph or a directed graph and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_graph_or_digraph(path_graph(5));
(%o2)                         true
(%i3) is_graph_or_digraph(path_digraph(5));
(%o3)                         true
```

`is_isomorphic` (*gr1*, *gr2*) [Function]

Returns `true` if graphs/digraphs *gr1* and *gr2* are isomorphic and `false` otherwise.

See also `isomorphism`.

Example:

```
(%i1) load ("graphs")$
(%i2) clk5:complement_graph(line_graph(complete_graph(5)))$
(%i3) is_isomorphic(clk5, petersen_graph());
(%o3)                         true
```

`is_planar` (*gr*) [Function]

Returns `true` if *gr* is a planar graph and `false` otherwise.

The algorithm used is the Demoucron's algorithm, which is a quadratic time algorithm.

Example:

```
(%i1) load ("graphs")$
(%i2) is_planar(dodecahedron_graph());
(%o2)                         true
(%i3) is_planar(petersen_graph());
(%o3)                         false
(%i4) is_planar(petersen_graph(10,2));
(%o4)                         true
```

`is_sconnected` (*gr*) [Function]

Returns `true` if the directed graph *gr* is strongly connected and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_sconnected(cycle_digraph(5));
(%o2)                         true
(%i3) is_sconnected(path_digraph(5));
(%o3)                         false
```

`is_vertex_in_graph` (*v*, *gr*) [Function]

Returns `true` if *v* is a vertex in the graph *g* and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) c4 : cycle_graph(4)$
(%i3) is_vertex_in_graph(0, c4);
(%o3)                          true
(%i4) is_vertex_in_graph(6, c4);
(%o4)                          false
```

`is_tree` (*gr*) [Function]

Returns `true` if *gr* is a tree and `false` otherwise.

Example:

```
(%i1) load ("graphs")$
(%i2) is_tree(random_tree(4));
(%o2)                          true
(%i3) is_tree(graph_union(random_tree(4), random_tree(5)));
(%o3)                          false
```

`laplacian_matrix` (*gr*) [Function]

Returns the laplacian matrix of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) laplacian_matrix(cycle_graph(5));
                    [  2   - 1   0    0   - 1 ]
                    [                        ]
                    [ - 1   2   - 1   0    0  ]
                    [                        ]
(%o2)               [  0   - 1   2   - 1   0  ]
                    [                        ]
                    [  0    0   - 1   2   - 1 ]
                    [                        ]
                    [ - 1   0    0   - 1   2  ]
```

`max_clique` (*gr*) [Function]

Returns a maximum clique of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.5)$
(%i3) max_clique(g);
(%o3)          [6, 12, 31, 36, 52, 59, 62, 63, 80]
```

`max_degree` (*gr*) [Function]

Returns the maximal degree of vertices of the graph *gr* and a vertex of maximal degree.

Example:

```
(%i1) load ("graphs")$
```

```
(%i2) g : random_graph(100, 0.02)$
(%i3) max_degree(g);
(%o3)                          [6, 79]
(%i4) vertex_degree(95, g);
(%o4)                            2
```

max_flow (*net*, *s*, *t*) [Function]

Returns a maximum flow through the network *net* with the source *s* and the sink *t*.

The function returns the value of the maximal flow and a list representing the weights of the arcs in the optimal flow.

Example:

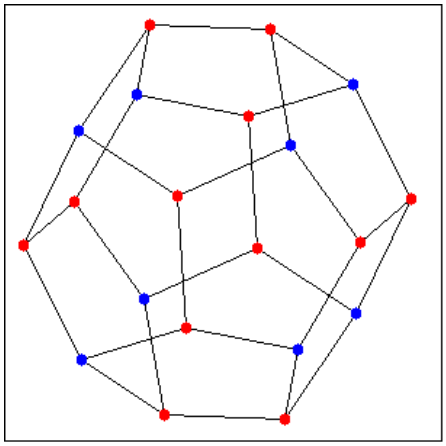
```
(%i1) load ("graphs")$
(%i2) net : create_graph(
  [1,2,3,4,5,6],
  [[[1,2], 1.0],
   [[1,3], 0.3],
   [[2,4], 0.2],
   [[2,5], 0.3],
   [[3,4], 0.1],
   [[3,5], 0.1],
   [[4,6], 1.0],
   [[5,6], 1.0]],
  directed=true)$
(%i3) [flow_value, flow] : max_flow(net, 1, 6);
(%o3) [0.7, [[[1, 2], 0.5], [[1, 3], 0.2], [[2, 4], 0.2],
[[2, 5], 0.3], [[3, 4], 0.1], [[3, 5], 0.1], [[4, 6], 0.3],
[[5, 6], 0.4]]]
(%i4) fl : 0$
(%i5) for u in out_neighbors(1, net)
     do fl : fl + assoc([1, u], flow)$
(%i6) fl;
(%o6)                           0.7
```

max_independent_set (*gr*) [Function]

Returns a maximum independent set of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) mi : max_independent_set(d);
(%o3)              [0, 3, 5, 9, 10, 11, 18, 19]
(%i4) draw_graph(d, show_vertices=mi)$
```

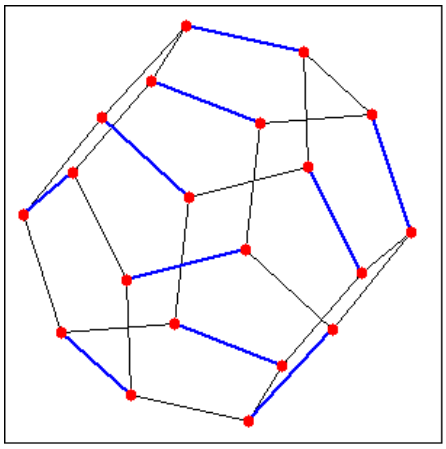


`max_matching` (*gr*) [Function]

Returns a maximum matching of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) m : max_matching(d);
(%o3) [[5, 7], [8, 9], [6, 10], [14, 19], [13, 18], [12, 17],
                                  [11, 16], [0, 15], [3, 4], [1, 2]]
(%i4) draw_graph(d, show_edges=m)$
```



`min_degree` (*gr*) [Function]

Returns the minimum degree of vertices of the graph *gr* and a vertex of minimum degree.

Example:

```
(%i1) load ("graphs")$
(%i2) g : random_graph(100, 0.1)$
(%i3) min_degree(g);
(%o3)                          [3, 49]
(%i4) vertex_degree(21, g);
(%o4)                            9
```

`min_edge_cut` (*gr*) [Function]

Returns the minimum edge cut in the graph *gr*.

See also `edge_connectivity`.

`min_vertex_cover` (*gr*) [Function]

Returns the minimum vertex cover of the graph *gr*.

`min_vertex_cut` (*gr*) [Function]

Returns the minimum vertex cut in the graph *gr*.

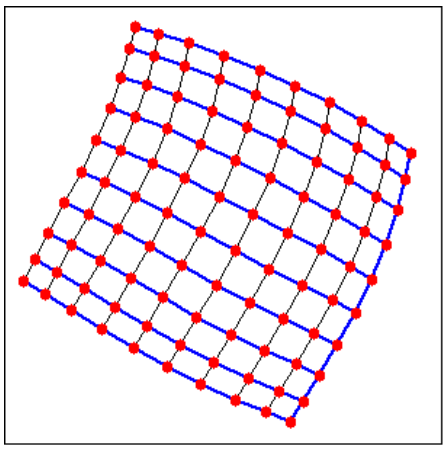
See also `vertex_connectivity`.

`minimum_spanning_tree` (*gr*) [Function]

Returns the minimum spanning tree of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : graph_product(path_graph(10), path_graph(10))$
(%i3) t : minimum_spanning_tree(g)$
(%i4) draw_graph(g, show_edges=edges(t))$
```



`neighbors` (*v*, *gr*) [Function]

Returns the list of neighbors of the vertex *v* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : petersen_graph()$
(%i3) neighbors(3, p);
(%o3)                         [4, 8, 2]
```

`odd_girth` (*gr*) [Function]

Returns the length of the shortest odd cycle in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : graph_product(cycle_graph(4), cycle_graph(7))$
(%i3) girth(g);
```

```
(%o3)                          4
(%i4) odd_girth(g);
(%o4)                          7
```

`out_neighbors` (*v*, *gr*) [Function]

Returns the list of out-neighbors of the vertex *v* in the directed graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : path_digraph(3)$
(%i3) in_neighbors(2, p);
(%o3)                         [1]
(%i4) out_neighbors(2, p);
(%o4)                         []
```

`planar_embedding` (*gr*) [Function]

Returns the list of facial walks in a planar embedding of *gr* and `false` if *gr* is not a planar graph.

The graph *gr* must be biconnected.

The algorithm used is the Demoucron's algorithm, which is a quadratic time algorithm.

Example:

```
(%i1) load ("graphs")$
(%i2) planar_embedding(grid_graph(3,3));
(%o2) [[3, 6, 7, 8, 5, 2, 1, 0], [4, 3, 0, 1], [3, 4, 7, 6],
                                       [8, 7, 4, 5], [1, 2, 5, 4]]
```

`print_graph` (*gr*) [Function]

Prints some information about the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) c5 : cycle_graph(5)$
(%i3) print_graph(c5)$
Graph on 5 vertices with 5 edges.
Adjacencies:
  4 :  0  3
  3 :  4  2
  2 :  3  1
  1 :  2  0
  0 :  4  1
(%i4) dc5 : cycle_digraph(5)$
(%i5) print_graph(dc5)$
Digraph on 5 vertices with 5 arcs.
Adjacencies:
  4 :  0
  3 :  4
  2 :  3
```

```
  1 :  2
  0 :  1
(%i6) out_neighbors(0, dc5);
(%o6)                          [1]
```

`radius` (*gr*) [Function]

Returns the radius of the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) radius(dodecahedron_graph());
(%o2)                           5
```

`set_edge_weight` (*e*, *w*, *gr*) [Function]

Assigns the weight *w* to the edge *e* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph([1, 2], [[[1,2], 1.2]])$
(%i3) get_edge_weight([1,2], g);
(%o3)                           1.2
(%i4) set_edge_weight([1,2], 2.1, g);
(%o4)                          done
(%i5) get_edge_weight([1,2], g);
(%o5)                           2.1
```

`set_vertex_label` (*v*, *l*, *gr*) [Function]

Assigns the label *l* to the vertex *v* in the graph *gr*.

A label may be any Maxima expression, and two or more vertices may have the same label.

Example:

```
(%i1) load ("graphs")$
(%i2) g : create_graph([[1, "One"], [2, "Two"]], [[1, 2]])$
(%i3) get_vertex_label(1, g);
(%o3)                           One
(%i4) set_vertex_label(1, "oNE", g);
(%o4)                          done
(%i5) get_vertex_label(1, g);
(%o5)                           oNE
(%i6) h : create_graph([[11, x], [22, y], [33, x + y]], [[11, 33], [22, 33]]) $
(%i7) get_vertex_label (33, h);
(%o7)                          y + x
```

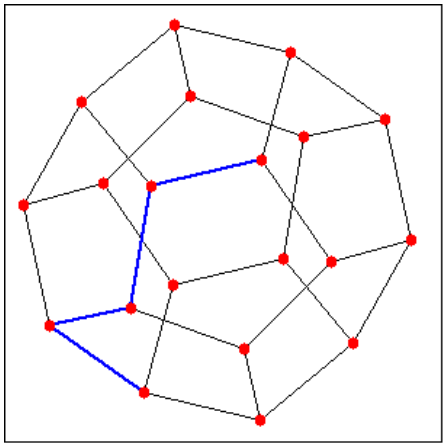
`shortest_path` (*u*, *v*, *gr*) [Function]

Returns the shortest path from *u* to *v* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
```

```
(%i3) path : shortest_path(0, 7, d);
(%o3)                    [0, 1, 19, 13, 7]
(%i4) draw_graph(d, show_edges=vertices_to_path(path))$
```



`shortest_weighted_path` (*u*, *v*, *gr*) [Function]

Returns the length of the shortest weighted path and the shortest weighted path from *u* to *v* in the graph *gr*.

The length of a weighted path is the sum of edge weights of edges in the path. If an edge has no weight, then it has a default weight 1.

Example:

```
(%i1) load ("graphs")$
(%i2) g: petersen_graph(20, 2)$
(%i3) for e in edges(g) do set_edge_weight(e, random(1.0), g)$
(%i4) shortest_weighted_path(0, 10, g);
(%o4) [2.575143920268482, [0, 20, 38, 36, 34, 32, 30, 10]]
```

`strong_components` (*gr*) [Function]

Returns the strong components of a directed graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) t : random_tournament(4)$
(%i3) strong_components(t);
(%o3)                  [[1], [0], [2], [3]]
(%i4) vertex_out_degree(3, t);
(%o4)                           3
```

`topological_sort` (*dag*) [Function]

Returns a topological sorting of the vertices of a directed graph *dag* or an empty list if *dag* is not a directed acyclic graph.

Example:

```
(%i1) load ("graphs")$
(%i2) g:create_graph(
```

```
                [1,2,3,4,5],
                [
                 [1,2], [2,5], [5,3],
                 [5,4], [3,4], [1,3]
                ],
                directed=true)$
(%i3) topological_sort(g);
(%o3)                     [1, 2, 5, 3, 4]
```

`vertex_connectivity` (*g*) [Function]

Returns the vertex connectivity of the graph *g*.

See also `min_vertex_cut`.

`vertex_degree` (*v*, *gr*) [Function]

Returns the degree of the vertex *v* in the graph *gr*.

`vertex_distance` (*u*, *v*, *gr*) [Function]

Returns the length of the shortest path between *u* and *v* in the (directed) graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) d : dodecahedron_graph()$
(%i3) vertex_distance(0, 7, d);
(%o3)                             4
(%i4) shortest_path(0, 7, d);
(%o4)                    [0, 1, 19, 13, 7]
```

`vertex_eccentricity` (*v*, *gr*) [Function]

Returns the eccentricity of the vertex *v* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g:cycle_graph(7)$
(%i3) vertex_eccentricity(0, g);
(%o3)                             3
```

`vertex_in_degree` (*v*, *gr*) [Function]

Returns the in-degree of the vertex *v* in the directed graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p5 : path_digraph(5)$
(%i3) print_graph(p5)$
Digraph on 5 vertices with 4 arcs.
Adjacencies:
  4 :
  3 :  4
  2 :  3
  1 :  2
  0 :  1
```

```
(%i4) vertex_in_degree(4, p5);
(%o4)                          1
(%i5) in_neighbors(4, p5);
(%o5)                         [3]
```

`vertex_out_degree` (*v*, *gr*) [Function]

Returns the out-degree of the vertex *v* in the directed graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) t : random_tournament(10)$
(%i3) vertex_out_degree(0, t);
(%o3)                          2
(%i4) out_neighbors(0, t);
(%o4)                       [7, 1]
```

`vertices` (*gr*) [Function]

Returns the list of vertices in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) vertices(complete_graph(4));
(%o2)                   [3, 2, 1, 0]
```

`vertex_coloring` (*gr*) [Function]

Returns an optimal coloring of the vertices of the graph *gr*.

The function returns the chromatic number and a list representing the coloring of the vertices of *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p:petersen_graph()$
(%i3) vertex_coloring(p);
(%o3) [3, [[0, 2], [1, 3], [2, 2], [3, 3], [4, 1], [5, 3],
                                  [6, 1], [7, 1], [8, 2], [9, 2]]]
```

`wiener_index` (*gr*) [Function]

Returns the Wiener index of the graph *gr*.

Example:

```
(%i2) wiener_index(dodecahedron_graph());
(%o2)                         500
```

66.2.3 Modifying graphs

`add_edge` (*e*, *gr*) [Function]

Adds the edge *e* to the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) p : path_graph(4)$
```

```
(%i3) neighbors(0, p);
(%o3)                          [1]
(%i4) add_edge([0,3], p);
(%o4)                         done
(%i5) neighbors(0, p);
(%o5)                       [3, 1]
```

add_edges (*e_list*, *gr*) [Function]

Adds all edges in the list *e_list* to the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : empty_graph(3)$
(%i3) add_edges([[0,1],[1,2]], g)$
(%i4) print_graph(g)$
Graph on 3 vertices with 2 edges.
Adjacencies:
  2 :  1
  1 :  2  0
  0 :  1
```

add_vertex (*v*, *gr*) [Function]

Adds the vertex *v* to the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g : path_graph(2)$
(%i3) add_vertex(2, g)$
(%i4) print_graph(g)$
Graph on 3 vertices with 1 edges.
Adjacencies:
  2 :
  1 :  0
  0 :  1
```

add_vertices (*v_list*, *gr*) [Function]

Adds all vertices in the list *v_list* to the graph *gr*. A vertex may be any integer, and *v_list* may contain vertices in any order.

connect_vertices (*v_list*, *u_list*, *gr*) [Function]

Connects all vertices from the list *v_list* with the vertices in the list *u_list* in the graph *gr*.

v_list and *u_list* can be single vertices or lists of vertices.

Example:

```
(%i1) load ("graphs")$
(%i2) g : empty_graph(4)$
(%i3) connect_vertices(0, [1,2,3], g)$
(%i4) print_graph(g)$
Graph on 4 vertices with 3 edges.
```

```
Adjacencies:
  3 :  0
  2 :  0
  1 :  0
  0 :  3  2  1
```

`contract_edge` (*e*, *gr*) [Function]

Contracts the edge *e* in the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) g: create_graph(
      8, [[0,3],[1,3],[2,3],[3,4],[4,5],[4,6],[4,7]])$
(%i3) print_graph(g)$
Graph on 8 vertices with 7 edges.
Adjacencies:
  7 :  4
  6 :  4
  5 :  4
  4 :  7  6  5  3
  3 :  4  2  1  0
  2 :  3
  1 :  3
  0 :  3
(%i4) contract_edge([3,4], g)$
(%i5) print_graph(g)$
Graph on 7 vertices with 6 edges.
Adjacencies:
  7 :  3
  6 :  3
  5 :  3
  3 :  5  6  7  2  1  0
  2 :  3
  1 :  3
  0 :  3
```

`remove_edge` (*e*, *gr*) [Function]

Removes the edge *e* from the graph *gr*.

Example:

```
(%i1) load ("graphs")$
(%i2) c3 : cycle_graph(3)$
(%i3) remove_edge([0,1], c3)$
(%i4) print_graph(c3)$
Graph on 3 vertices with 2 edges.
Adjacencies:
  2 :  0  1
  1 :  2
  0 :  2
```

`remove_vertex` (*v*, *gr*) [Function]
Removes the vertex *v* from the graph *gr*.

66.2.4 Reading and writing to files

`dimacs_export` [Function]
`dimacs_export` (*gr*, *fl*)
`dimacs_export` (*gr*, *fl*, *comment1*, ..., *commentn*)
Exports the graph into the file *fl* in the DIMACS format. Optional comments will be added to the top of the file.

`dimacs_import` (*fl*) [Function]
Returns the graph from file *fl* in the DIMACS format.

`graph6_decode` (*str*) [Function]
Returns the graph encoded in the graph6 format in the string *str*.

`graph6_encode` (*gr*) [Function]
Returns a string which encodes the graph *gr* in the graph6 format.

`graph6_export` (*gr_list*, *fl*) [Function]
Exports graphs in the list *gr_list* to the file *fl* in the graph6 format.

`graph6_import` (*fl*) [Function]
Returns a list of graphs from the file *fl* in the graph6 format.

`sparse6_decode` (*str*) [Function]
Returns the graph encoded in the sparse6 format in the string *str*.

`sparse6_encode` (*gr*) [Function]
Returns a string which encodes the graph *gr* in the sparse6 format.

`sparse6_export` (*gr_list*, *fl*) [Function]
Exports graphs in the list *gr_list* to the file *fl* in the sparse6 format.

`sparse6_import` (*fl*) [Function]
Returns a list of graphs from the file *fl* in the sparse6 format.

66.2.5 Visualization

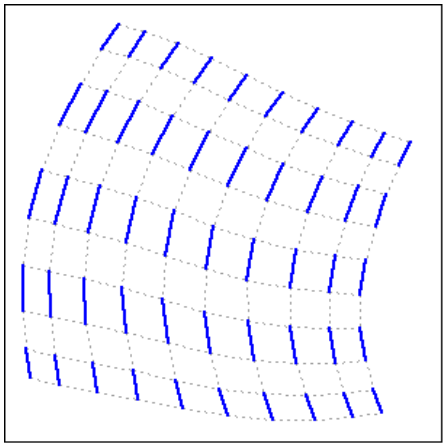
`draw_graph` [Function]
`draw_graph` (*graph*)
`draw_graph` (*graph*, *option1*, ..., *optionk*)
Draws the graph using the Chapter 55 [Package draw], page 847, package.

The algorithm used to position vertices is specified by the optional argument *program*. The default value is `program=spring_embedding`. *draw_graph* can also use the graphviz programs for positioning vertices, but graphviz must be installed separately.

Example 1:

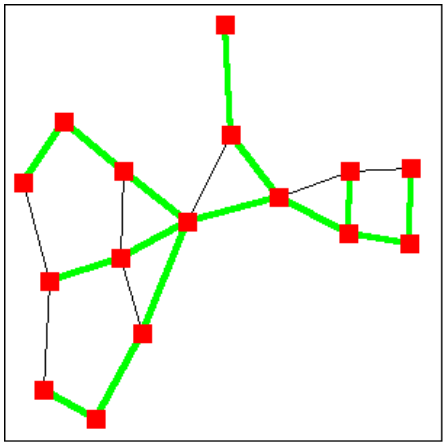
```
(%i1) load ("graphs")$
(%i2) g:grid_graph(10,10)$
```

```
(%i3) m:max_matching(g)$
(%i4) draw_graph(g,
   spring_embedding_depth=100,
   show_edges=m, edge_type=dots,
   vertex_size=0)$
```



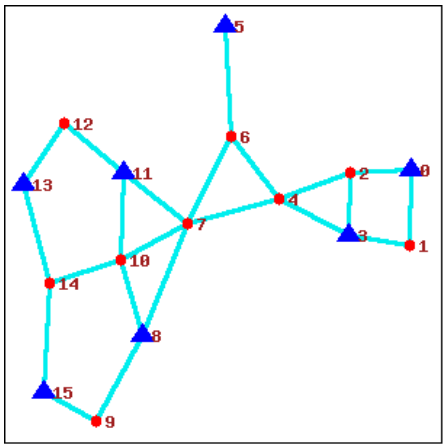
Example 2:

```
(%i1) load ("graphs")$
(%i2) g:create_graph(16,
    [
     [0,1],[1,3],[2,3],[0,2],[3,4],[2,4],
     [5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11],
     [8,10],[11,10],[8,9],[11,12],[9,15],[12,13],
     [10,14],[15,14],[13,14]
    ])$
(%i3) t:minimum_spanning_tree(g)$
(%i4) draw_graph(
    g,
    show_edges=edges(t),
    show_edge_width=4,
    show_edge_color=green,
    vertex_type=filled_square,
    vertex_size=2
    )$
```



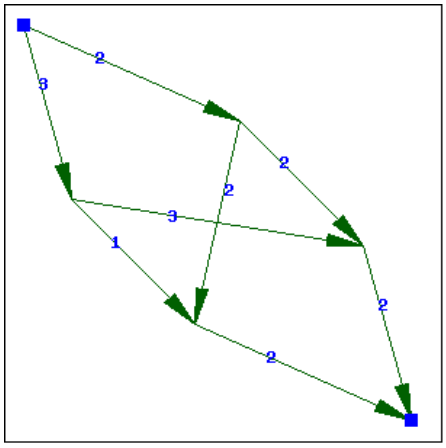
Example 3:

```
(%i1) load ("graphs")$
(%i2) g:create_graph(16,
    [
     [0,1],[1,3],[2,3],[0,2],[3,4],[2,4],
     [5,6],[6,4],[4,7],[6,7],[7,8],[7,10],[7,11],
     [8,10],[11,10],[8,9],[11,12],[9,15],[12,13],
     [10,14],[15,14],[13,14]
    ])$
(%i3) mi : max_independent_set(g)$
(%i4) draw_graph(
    g,
    show_vertices=mi,
    show_vertex_type=filled_up_triangle,
    show_vertex_size=2,
    edge_color=cyan,
    edge_width=3,
    show_id=true,
    text_color=brown
    )$
```



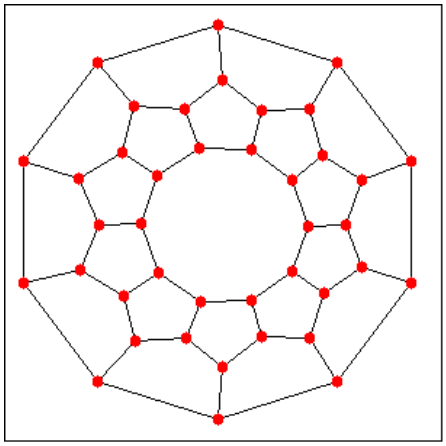
Example 4:

```
(%i1) load ("graphs")$
(%i2) net : create_graph(
    [0,1,2,3,4,5],
    [
     [[0,1], 3], [[0,2], 2],
     [[1,3], 1], [[1,4], 3],
     [[2,3], 2], [[2,4], 2],
     [[4,5], 2], [[3,5], 2]
    ],
    directed=true
    )$
(%i3) draw_graph(
    net,
    show_weight=true,
    vertex_size=0,
    show_vertices=[0,5],
    show_vertex_type=filled_square,
    head_length=0.2,
    head_angle=10,
    edge_color="dark-green",
    text_color=blue
    )$
```



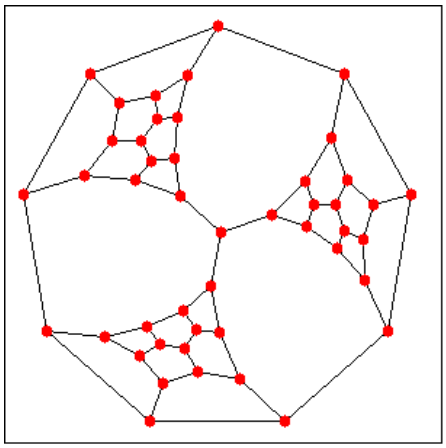
Example 5:

```
(%i1) load("graphs")$
(%i2) g: petersen_graph(20, 2);
(%o2)                          GRAPH
(%i3) draw_graph(g, redraw=true, program=planar_embedding);
(%o3)                          done
```



Example 6:

```
(%i1) load("graphs")$
(%i2) t: tutte_graph();
(%o2)                           GRAPH
(%i3) draw_graph(t, redraw=true,
                    fixed_vertices=[1,2,3,4,5,6,7,8,9]);
(%o3)                           done
```



`draw_graph_program` [Option variable]

Default value: *spring_embedding*

The default value for the program used to position vertices in `draw_graph` program.

`show_id` [draw_graph option]

Default value: *false*

If *true* then ids of the vertices are displayed.

`show_label` [draw_graph option]

Default value: *false*

If *true* then labels of the vertices are displayed.

`label_alignment` [draw_graph option]
Default value: *center*
Determines how to align the labels/ids of the vertices. Can be `left`, `center` or `right`.

`show_weight` [draw_graph option]
Default value: *false*
If *true* then weights of the edges are displayed.

`vertex_type` [draw_graph option]
Default value: *circle*
Defines how vertices are displayed. See the *point_type* option for the `draw` package for possible values.

`vertex_size` [draw_graph option]
The size of vertices.

`vertex_color` [draw_graph option]
The color used for displaying vertices.

`show_vertices` [draw_graph option]
Default value: []
Display selected vertices in the using a different color.

`show_vertex_type` [draw_graph option]
Defines how vertices specified in *show_vertices* are displayed. See the *point_type* option for the `draw` package for possible values.

`show_vertex_size` [draw_graph option]
The size of vertices in *show_vertices*.

`show_vertex_color` [draw_graph option]
The color used for displaying vertices in the *show_vertices* list.

`vertex_partition` [draw_graph option]
Default value: []
A partition `[[v1,v2,...],...,[vk,...,vn]]` of the vertices of the graph. The vertices of each list in the partition will be drawn in a different color.

`vertex_coloring` [draw_graph option]
Specifies coloring of the vertices. The coloring *col* must be specified in the format as returned by *vertex_coloring*.

`edge_color` [draw_graph option]
The color used for displaying edges.

`edge_width` [draw_graph option]
The width of edges.

`edge_type` [draw_graph option]
Defines how edges are displayed. See the *line_type* option for the `draw` package.

`show_edges` [draw_graph option]
Display edges specified in the list *e_list* using a different color.

`show_edge_color` [draw_graph option]
The color used for displaying edges in the *show_edges* list.

`show_edge_width` [draw_graph option]
The width of edges in *show_edges*.

`show_edge_type` [draw_graph option]
Defines how edges in *show_edges* are displayed. See the *line_type* option for the `draw` package.

`edge_partition` [draw_graph option]
A partition `[[e1,e2,...],...,[ek,...,em]]` of edges of the graph. The edges of each list in the partition will be drawn using a different color.

`edge_coloring` [draw_graph option]
The coloring of edges. The coloring must be specified in the format as returned by the function *edge_coloring*.

`redraw` [draw_graph option]
Default value: *false*

If `true`, vertex positions are recomputed even if the positions have been saved from a previous drawing of the graph.

`head_angle` [draw_graph option]
Default value: 15

The angle for the arrows displayed on arcs (in directed graphs).

`head_length` [draw_graph option]
Default value: 0.1

The length for the arrows displayed on arcs (in directed graphs).

`spring_embedding_depth` [draw_graph option]
Default value: 50

The number of iterations in the spring embedding graph drawing algorithm.

`terminal` [draw_graph option]
The terminal used for drawing (see the *terminal* option in the `draw` package).

`file_name` [draw_graph option]
The filename of the drawing if terminal is not screen.

`program` [draw_graph option]
Defines the program used for positioning vertices of the graph. Can be one of the graphviz programs (dot, neato, twopi, circ, fdp), *circular*, *spring_embedding* or *planar_embedding*. *planar_embedding* is only available for 2-connected planar graphs. When `program=spring_embedding`, a set of vertices with fixed position can be specified with the *fixed_vertices* option.

`fixed_vertices` [draw_graph option]
: Specifies a list of vertices which will have positions fixed along a regular polygon. Can be used when `program=spring_embedding`.

`vertices_to_path (`*`v_list`*`)` [Function]
: Converts a list *v_list* of vertices to a list of edges of the path defined by *v_list*.

`vertices_to_cycle (`*`v_list`*`)` [Function]
: Converts a list *v_list* of vertices to a list of edges of the cycle defined by *v_list*.

67 Package grobner

67.1 Introduction to grobner

`grobner` is a package for working with Groebner bases in Maxima.

To use the following functions you must load the `grobner.lisp` package.

```
load("grobner");
```

A demo can be started by

```
demo("grobner.demo");
```

or

```
batch("grobner.demo")
```

Some of the calculation in the demo will take a lot of time therefore the output `grobner-demo.output` of the demo can be found in the same directory as the demo file.

67.1.1 Notes on the grobner package

The package was written by

Marek Rychlik

http://alamos.math.arizona.edu

and is released 2002-05-24 under the terms of the General Public License(GPL) (see file `grobner.lisp`). This documentation was extracted from the files

`README`, `grobner.lisp`, `grobner.demo`, `grobner-demo.output`

by Günter Nowak. Suggestions for improvement of the documentation can be discussed at the *maxima*-mailing-list `maxima@math.utexas.edu`. The code is a little bit out of date now. Modern implementation use the fast *F4* algorithm described in

A new efficient algorithm for computing Gröbner bases (F4)
Jean-Charles Faugère
LIP6/CNRS Université Paris VI
January 20, 1999

67.1.2 Implementations of admissible monomial orders in grobner

- `lex`

  pure lexicographic, default order for monomial comparisons

- `grlex`

  total degree order, ties broken by lexicographic

- `grevlex`

  total degree, ties broken by reverse lexicographic

- `invlex`

  inverse lexicographic order

67.2 Functions and Variables for grobner

67.2.1 Global switches for grobner

`poly_monomial_order` [Option variable]

Default value: `lex`

This global switch controls which monomial order is used in polynomial and Groebner Bases calculations. If not set, `lex` will be used.

`poly_coefficient_ring` [Option variable]

Default value: `expression_ring`

This switch indicates the coefficient ring of the polynomials that will be used in grobner calculations. If not set, *maxima's* general expression ring will be used. This variable may be set to `ring_of_integers` if desired.

`poly_primary_elimination_order` [Option variable]

Default value: `false`

Name of the default order for eliminated variables in elimination-based functions. If not set, `lex` will be used.

`poly_secondary_elimination_order` [Option variable]

Default value: `false`

Name of the default order for kept variables in elimination-based functions. If not set, `lex` will be used.

`poly_elimination_order` [Option variable]

Default value: `false`

Name of the default elimination order used in elimination calculations. If set, it overrides the settings in variables `poly_primary_elimination_order` and `poly_secondary_elimination_order`. The user must ensure that this is a true elimination order valid for the number of eliminated variables.

`poly_return_term_list` [Option variable]

Default value: `false`

If set to `true`, all functions in this package will return each polynomial as a list of terms in the current monomial order rather than a *maxima* general expression.

`poly_grobner_debug` [Option variable]

Default value: `false`

If set to `true`, produce debugging and tracing output.

`poly_grobner_algorithm` [Option variable]

Default value: `buchberger`

Possible values:

- `buchberger`
- `parallel_buchberger`
- `gebauer_moeller`

The name of the algorithm used to find the Groebner Bases.

poly_top_reduction_only [Option variable]
Default value: false

If not false, use top reduction only whenever possible. Top reduction means that division algorithm stops after the first reduction.

67.2.2 Simple operators in grobner

poly_add, poly_subtract, poly_multiply and poly_expt are the arithmetical operations on polynomials. These are performed using the internal representation, but the results are converted back to the *maxima* general form.

poly_add (*poly1*, *poly2*, *varlist*) [Function]
Adds two polynomials *poly1* and *poly2*.

```
(%i1) poly_add(z+x^2*y,x-z,[x,y,z]);
                                    2
(%o1)                              x  y + x
```

poly_subtract (*poly1*, *poly2*, *varlist*) [Function]
Subtracts a polynomial *poly2* from *poly1*.

```
(%i1) poly_subtract(z+x^2*y,x-z,[x,y,z]);
                                      2
(%o1)                          2 z + x  y - x
```

poly_multiply (*poly1*, *poly2*, *varlist*) [Function]
Returns the product of polynomials *poly1* and *poly2*.

```
(%i2) poly_multiply(z+x^2*y,x-z,[x,y,z])-(z+x^2*y)*(x-z),expand;
(%o1)                                  0
```

poly_s_polynomial (*poly1*, *poly2*, *varlist*) [Function]
Returns the *syzygy polynomial* (*S-polynomial*) of two polynomials *poly1* and *poly2*.

poly_primitive_part (*poly1*, *varlist*) [Function]
Returns the polynomial *poly* divided by the GCD of its coefficients.

```
(%i1) poly_primitive_part(35*y+21*x,[x,y]);
(%o1)                              5 y + 3 x
```

poly_normalize (*poly*, *varlist*) [Function]
Returns the polynomial *poly* divided by the leading coefficient. It assumes that the division is possible, which may not always be the case in rings which are not fields.

67.2.3 Other functions in grobner

poly_expand (*poly*, *varlist*) [Function]
This function parses polynomials to internal form and back. It is equivalent to expand(*poly*) if *poly* parses correctly to a polynomial. If the representation is not compatible with a polynomial in variables *varlist*, the result is an error. It can be

used to test whether an expression correctly parses to the internal representation. The following examples illustrate that indexed and transcendental function variables are allowed.

```
(%i1) poly_expand((x-y)*(y+x),[x,y]);
                                      2    2
(%o1)                                 x  - y
(%i2) poly_expand((y+x)^2,[x,y]);
                                 2           2
(%o2)                           y  + 2 x y + x
(%i3) poly_expand((y+x)^5,[x,y]);
                  5      4         2  3       3  2      4      5
(%o3)            y  + 5 x y  + 10 x  y  + 10 x  y  + 5 x  y + x
(%i4) poly_expand(-1-x*exp(y)+x^2/sqrt(y),[x]);
                                          2
                                     y      x
(%o4)                       - x %e  + ------- - 1
                                       sqrt(y)

(%i5) poly_expand(-1-sin(x)^2+sin(x),[sin(x)]);
                                   2
(%o5)                       - sin (x) + sin(x) - 1
```

poly_expt (*poly*, *number*, *varlist*) [Function]

exponentitates *poly* by a positive integer *number*. If *number* is not a positive integer number an error will be raised.

```
(%i1) poly_expt(x-y,3,[x,y])-(x-y)^3,expand;
(%o1)                                  0
```

poly_content (*poly*. *varlist*) [Function]

`poly_content` extracts the GCD of its coefficients

```
(%i1) poly_content(35*y+21*x,[x,y]);
(%o1)                                  7
```

poly_pseudo_divide (*poly*, *polylist*, *varlist*) [Function]

Pseudo-divide a polynomial *poly* by the list of n polynomials *polylist*. Return multiple values. The first value is a list of quotients a. The second value is the remainder r. The third argument is a scalar coefficient c, such that $c * poly$ can be divided by *polylist* within the ring of coefficients, which is not necessarily a field. Finally, the fourth value is an integer count of the number of reductions performed. The resulting objects satisfy the equation:

$$c * poly = \sum_{i=1}^{n} (a_i * polylist_i) + r$$

`poly_exact_divide` (*poly1*, *poly2*, *varlist*) [Function]
Divide a polynomial *poly1* by another polynomial *poly2*. Assumes that exact division with no remainder is possible. Returns the quotient.

`poly_normal_form` (*poly*, *polylist*, *varlist*) [Function]
`poly_normal_form` finds the normal form of a polynomial *poly* with respect to a set of polynomials *polylist*.

`poly_buchberger_criterion` (*polylist*, *varlist*) [Function]
Returns `true` if *polylist* is a Groebner basis with respect to the current term order, by using the Buchberger criterion: for every two polynomials $h1$ and $h2$ in *polylist* the S-polynomial $S(h1, h2)$ reduces to 0 *modulo* *polylist*.

`poly_buchberger` (*polylist_fl varlist*) [Function]
`poly_buchberger` performs the Buchberger algorithm on a list of polynomials and returns the resulting Groebner basis.

67.2.4 Standard postprocessing of Groebner Bases

The *k-th elimination ideal* I_k of an ideal I over $K[x_1, ..., x_1]$ is $I \cap K[x_{k+1}, ..., x_n]$.

The *colon ideal* $I : J$ is the ideal $\{h | \forall w \in J : wh \in I\}$.

The ideal $I : p^\infty$ is the ideal $\{h | \exists n \in N : p^n h \in I\}$.

The ideal $I : J^\infty$ is the ideal $\{h | \exists n \in N, \exists p \in J : p^n h \in I\}$.

The *radical ideal* $\sqrt{I}$ is the ideal $\{h | \exists n \in N : h^n \in I\}$.

`poly_reduction` (*polylist*, *varlist*) [Function]
`poly_reduction` reduces a list of polynomials *polylist*, so that each polynomial is fully reduced with respect to the other polynomials.

`poly_minimization` (*polylist*, *varlist*) [Function]
Returns a sublist of the polynomial list *polylist* spanning the same monomial ideal as *polylist* but minimal, i.e. no leading monomial of a polynomial in the sublist divides the leading monomial of another polynomial.

`poly_normalize_list` (*polylist*, *varlist*) [Function]
`poly_normalize_list` applies `poly_normalize` to each polynomial in the list. That means it divides every polynomial in a list *polylist* by its leading coefficient.

`poly_grobner` (*polylist*, *varlist*) [Function]
Returns a Groebner basis of the ideal span by the polynomials *polylist*. Affected by the global flags.

`poly_reduced_grobner` (*polylist*, *varlist*) [Function]
Returns a reduced Groebner basis of the ideal span by the polynomials *polylist*. Affected by the global flags.

`poly_depends_p` (*poly*, *var*, *varlist*) [Function]
`poly_depends` tests whether a polynomial depends on a variable *var*.

poly_elimination_ideal (*polylist*, *number*, *varlist*) [Function]
: poly_elimination_ideal returns the grobner basis of the *number*-th elimination ideal of an ideal specified as a list of generating polynomials (not necessarily Groebner basis).

poly_colon_ideal (*polylist1*, *polylist2*, *varlist*) [Function]
: Returns the reduced Groebner basis of the colon ideal

  $I(polylist1) : I(polylist2)$

  where *polylist*1 and *polylist*2 are two lists of polynomials.

poly_ideal_intersection (*polylist1*, *polylist2*, *varlist*) [Function]
: poly_ideal_intersection returns the intersection of two ideals.

poly_lcm (*poly1*, *poly2*, *varlist*) [Function]
: Returns the lowest common multiple of *poly1* and *poly2*.

poly_gcd (*poly1*, *poly2*, *varlist*) [Function]
: Returns the greatest common divisor of *poly1* and *poly2*.

  See also ezgcd, gcd, gcdex, and gcdivide.

  Example:

```
(%i1) p1:6*x^3+19*x^2+19*x+6;
                          3       2
(%o1)                  6 x  + 19 x  + 19 x + 6
(%i2) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
                    5       4       3       2
(%o2)          6 x  + 13 x  + 12 x  + 13 x  + 6 x
(%i3) poly_gcd(p1, p2, [x]);
                             2
(%o3)                     6 x  + 13 x + 6
```

poly_grobner_equal (*polylist1*, *polylist2*, *varlist*) [Function]
: poly_grobner_equal tests whether two Groebner Bases generate the same ideal. Returns true if two lists of polynomials *polylist1* and *polylist2*, assumed to be Groebner Bases, generate the same ideal, and false otherwise. This is equivalent to checking that every polynomial of the first basis reduces to 0 modulo the second basis and vice versa. Note that in the example below the first list is not a Groebner basis, and thus the result is false.

```
(%i1) poly_grobner_equal([y+x,x-y],[x,y],[x,y]);
(%o1)                          false
```

poly_grobner_subsetp (*polylist1*, *polylist2*, *varlist*) [Function]
: poly_grobner_subsetp tests whether an ideal generated by *polylist1* is contained in the ideal generated by *polylist2*. For this test to always succeed, *polylist2* must be a Groebner basis.

poly_grobner_member (*poly*, *polylist*, *varlist*) [Function]
: Returns true if a polynomial *poly* belongs to the ideal generated by the polynomial list *polylist*, which is assumed to be a Groebner basis. Returns false otherwise.

`poly_grobner_member` tests whether a polynomial belongs to an ideal generated by a list of polynomials, which is assumed to be a Groebner basis. Equivalent to `normal_form` being 0.

`poly_ideal_saturation1` (*polylist*, *poly*, *varlist*) [Function]
Returns the reduced Groebner basis of the saturation of the ideal

$$I(polylist) : poly^{\infty}$$

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by *polylist* which do not identically vanish on the variety of *poly*.

`poly_ideal_saturation` (*polylist1*, *polylist2*, *varlist*) [Function]
Returns the reduced Groebner basis of the saturation of the ideal

$$I(polylist1) : I(polylist2)^{\infty}$$

Geometrically, over an algebraically closed field, this is the set of polynomials in the ideal generated by *polylist1* which do not identically vanish on the variety of *polylist2*.

`poly_ideal_polysaturation1` (*polylist1*, *polylist2*, *varlist*) [Function]
polylist2 is a list of n polynomials `[poly1,...,polyn]`. Returns the reduced Groebner basis of the ideal

$$I(polylist) : poly1^{\infty} : ... : polyn^{\infty}$$

obtained by a sequence of successive saturations in the polynomials of the polynomial list *polylist2* of the ideal generated by the polynomial list *polylist1*.

`poly_ideal_polysaturation` (*polylist*, *polylistlist*, *varlist*) [Function]
polylistlist is a list of n list of polynomials `[polylist1,...,polylistn]`. Returns the reduced Groebner basis of the saturation of the ideal

$$I(polylist) : I(polylist_1)^{\infty} : ... : I(polylist_n)^{\infty}$$

`poly_saturation_extension` (*poly*, *polylist*, *varlist1*, *varlist2*) [Function]
`poly_saturation_extension` implements the famous Rabinowitz trick.

`poly_polysaturation_extension` (*poly*, *polylist*, *varlist1*, *varlist2*) [Function]

68 Package hompack

68.1 Introduction to hompack

`Hompack` is a Common Lisp translation (via `f2cl`) of the Fortran library HOMPACK, as obtained from Netlib.

68.2 Functions and Variables for hompack

`hompack_polsys` (*eqnlist*, *varlist* [, *iflg1*, *epsbig*, *epssml*, *numrr*]) [Function]

Finds the roots of the system of polynomials in the variables *varlist* in the system of equations in *eqnlist*. The number of equations must match number of variables. Each equation must be a polynomial with variables in *varlist*. The coefficients must be real numbers.

The optional keyword arguments provide some control over the algorithm.

`epsbig` is the local error tolerance allowed by the path tracker, defaulting to 1e-4.

`epssml` is the accuracy desired for the final solution, defaulting to 1d-14.

`numrr` is the number of multiples of 1000 steps that will be tried before abandoning a path, defaulting to 10.

`iflg1` defaulting to 0, controls the algorithm as follows:

- `0` If the problem is to be solved without calling `polsys`' scaling routine, `sclgnp`, and without using the projective transformation.
- `1` If scaling but no projective transformation is to be used.
- `10` If no scaling but projective transformation is to be used.
- `11` If both scaling and projective transformation are to be used.

`hompack_polsys` returns a list. The elements of the list are:

`retcode` Indicates whether the solution is valid or not.

- `0` Solution found without problems
- `1` Solution succeeded but `iflg2` indicates some issues with a root. (That is, `iflg2` is not all ones.)
- `-1` `NN`, the declared dimension of the number of terms in the polynomials, is too small. (This should not happen.)
- `-2` `MMAXT`, the declared dimension for the internal coefficient and degree arrays, is too small. (This should not happen.)
- `-3` `TTOTDG`, the total degree of the equations, is too small. (This should not happen.)
- `-4` `LENWK`, the length of the internal real work array, is too small. (This should not happen.)

`-5` `LENIWK`, the length of the internal integer work array, is too small. (This should not happen.)

`-6` *iflg1* is not 0 or 1, or 10 or 11. (This should not happen; an error should be thrown before `polsys` is called.)

`roots` The roots of the system of equations. This is in the same format as `solve` would return.

`iflg2` A list containing information on how the path for the m'th root terminated:

`1` Normal return

`2` Specified error tolerance cannot be met. Increase *epsbig* and *epssml* and rerun.

`3` Maximum number of steps exceeded. To track the path further, increase *numrr* and rerun the path. However, the path may be diverging, if the lambda value is near 1 and the roots values are large.

`4` Jacobian matrix does not have full rank. The algorithm has failed (the zero curve of the homotopy map cannot be followed any further).

`5` The tracking algorithm has lost the zero curve of the homotopy map and is not making progress. The error tolerances *epsbig* and *epssml* were too lenient. The problem should be restarted with smaller error tolerances.

`6` The normal flow newton iteration in `stepnf` or `rootnf` failed to converge. The error tolerance *epsbig* may be too stringent.

`7` Illegal input parameters, a fatal error.

`lambda` A list of the final lambda value for the m-th root, where lambda is the continuation parameter.

`arclen` A list of the arc length of the m-th root.

`nfe` A list of the number of jacobian matrix evaluations required to track the m-th root.

Here are some examples of using `hompack_polsys`.

```
(%i1) load(hompack)$
```

```
(%i2) hompack_polsys([x1^2-1, x2^2-2],[x1,x2]);
(%o2) [0, [[x1 = 7.493410174972965e-17 %i - 1.0,
x2 = - 2.1199276810167172e-16 %i - 1.4142135623730951],
[x1 = 1.0 - 1.7812202259088373e-17 %i,
x2 = - 9.892128000334418e-17 %i - 1.4142135623730951],
[x1 = - 8.745710933584358e-17 %i - 1.0,
x2 = 1.4142135623730951 - 3.1543521174128613e-17 %i],
[x1 = 1.0 - 5.5487454344135625e-18 %i,
x2 = 9.617653810456737e-17 %i + 1.4142135623730951]],
[1, 1, 1, 1], [1.0, 1.0, 0.9999999999999999,
0.9999999999999999], [4.6126237693413445, 4.6126230108601,
4.612623872939582, 4.6126231144843475], [40, 40, 40, 40]]
```

The analytical solution can be obtained with solve:

```
(%i1) solve([x1^2-1, x2^2-2],[x1,x2]);
(%o1) [[x1 = 1, x2 = - sqrt(2)], [x1 = 1, x2 = sqrt(2)],
            [x1 = - 1, x2 = - sqrt(2)], [x1 = - 1, x2 = sqrt(2)]]
```

We see that `hompack_polsys` returned the correct answer except that the roots are in a different order and there is a small imaginary part.

Another example, with corresponding solution from solve:

```
(%i1) load(hompack)$
(%i2) hompack_polsys([x1^2 + 2*x2^2 + x1*x2 - 5, 2*x1^2 + x2^2 + x2-4],[x1,x2]);
(%o2) [0, [[x1 = 1.2015573017007832 - 6.51849608628558e-16 %i,
x2 = - 2.2667444298167825e-16 %i - 1.6672703634801431],
[x1 = - 5.901956169878514e-17 %i - 1.4285291895653132,
x2 = - 6.670215053038164e-17 %i - 0.9106199083334114],
[x1 = 1.942890293094024e-16 %i + 0.5920619420732687,
x2 = 1.383859154368197 - 1.942890293094024e-16 %i],
[x1 = 0.08945540033671631 - 8.534583737796769e-16 %i,
x2 = 2.415141536592239e-16 %i + 1.5576674810817213]],
[1, 1, 1, 1], [1.0000000000000004, 1.0000000000000002, 1.0,
0.9999999999999999], [6.205795654034981, 7.7222132593900525,
7.228287079174324, 5.6114742835849105], [35, 41, 48, 40]]
(%i3) solve([x1^2+2*x2^2+x1*x2 - 5, 2*x1^2+x2^2+x2-4],[x1,x2]);
(%o3) [[x1 = 0.08945540336850383, x2 = 1.5576673866090713],
[x1 = 0.5920619554695062, x2 = 1.3838592860838075],
[x1 = 1.2015573525007488, x2 = - 1.66727025803531],
[x1 = - 1.428529150636283, x2 = - 0.9106198942815954]]
```

Note that `hompack_polsys` can sometimes be very slow. Perhaps `solve` can be used. Or perhaps `eliminate` can be used to convert the system of polynomials into one polynomial for which `allroots` can find all the roots.

69 Package impdiff

69.1 Functions and Variables for impdiff

`implicit_derivative` (*f*,*indvarlist*,*orderlist*,*depvar*) [Function]

This subroutine computes implicit derivatives of multivariable functions. *f* is an array function, the indexes are the derivative degree in the *indvarlist* order; *indvarlist* is the independent variable list; *orderlist* is the order desired; and *depvar* is the dependent variable.

To use this function write first `load("impdiff")`.

70 Package interpol

70.1 Introduction to interpol

Package `interpol` defines the Lagrangian, the linear and the cubic splines methods for polynomial interpolation.

For comments, bugs or suggestions, please contact me at *'mario AT edu DOT xunta DOT es'*.

70.2 Functions and Variables for interpol

`lagrange` [Function]

`lagrange` (*points*)
`lagrange` (*points*, *option*)

Computes the polynomial interpolation by the Lagrangian method. Argument *points* must be either:

- a two column matrix, `p:matrix([2,4],[5,6],[9,3])`,
- a list of pairs, `p: [[2,4],[5,6],[9,3]]`,
- a list of numbers, `p: [4,6,3]`, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the *option* argument it is possible to select the name for the independent variable, which is `'x` by default; to define another one, write something like `varname='z`.

Note that when working with high degree polynomials, floating point evaluations are unstable.

See also `linearinterpol`, `cspline`, and `ratinterpol`.

Examples:

```
(%i1) load("interpol")$
(%i2) p:[[7,2],[8,2],[1,5],[3,2],[6,7]]$
(%i3) lagrange(p);
       (x - 7) (x - 6) (x - 3) (x - 1)
(%o3)  -------------------------------
                     35
   (x - 8) (x - 6) (x - 3) (x - 1)
 - -------------------------------
                  12
   7 (x - 8) (x - 7) (x - 3) (x - 1)
 + ---------------------------------
                   30
   (x - 8) (x - 7) (x - 6) (x - 1)
 - -------------------------------
                  60
   (x - 8) (x - 7) (x - 6) (x - 3)
```

```
 + ------------------------------
                 84
(%i4) f(x):=''%;
                (x - 7) (x - 6) (x - 3) (x - 1)
(%o4)  f(x) := ------------------------------
                              35
   (x - 8) (x - 6) (x - 3) (x - 1)
 - ------------------------------
                 12
   7 (x - 8) (x - 7) (x - 3) (x - 1)
 + --------------------------------
                  30
   (x - 8) (x - 7) (x - 6) (x - 1)
 - ------------------------------
                 60
   (x - 8) (x - 7) (x - 6) (x - 3)
 + ------------------------------
                 84
(%i5) /* Evaluate the polynomial at some points */
      expand(map(f,[2.3,5/7,%pi]));
                                 4          3           2
                    919062  73 %pi    701 %pi    8957 %pi
(%o5)  [- 1.567535, ------, ------- - -------- + ---------
                    84035     420       210         420
                                              5288 %pi   186
                                           - -------- + ---]
                                               105       5
(%i6) %,numer;
(%o6) [- 1.567535, 10.9366573451538, 2.89319655125692]
(%i7) load("draw")$  /* load draw package */
(%i8) /* Plot the polynomial together with points */
      draw2d(
        color      = red,
        key        = "Lagrange polynomial",
        explicit(f(x),x,0,10),
        point_size = 3,
        color      = blue,
        key        = "Sample points",
        points(p))$
(%i9) /* Change variable name */
      lagrange(p, varname=w);
       (w - 7) (w - 6) (w - 3) (w - 1)
(%o9)  ------------------------------
                     35
   (w - 8) (w - 6) (w - 3) (w - 1)
 - ------------------------------
                 12
```

$$\frac{7\,(w-8)\,(w-7)\,(w-3)\,(w-1)}{30}+\frac{(w-8)\,(w-7)\,(w-6)\,(w-1)}{60}-\frac{(w-8)\,(w-7)\,(w-6)\,(w-3)}{84}$$

`charfun2` (*x*, `a`, *b*) [Function]

The characteristic or indicator function on the half-open interval $[a, b)$, that is, including a and excluding b.

When $x >= a$ and $x < b$ evaluates to `true` or `false`, `charfun2` returns 1 or 0, respectively.

Otherwise, `charfun2` returns a partially-evaluated result in terms of `charfun`.

Package `interpol` loads this function.

See also `charfun`.

Examples:

When $x >= a$ and $x < b$ evaluates to `true` or `false`, `charfun2` returns 1 or 0, respectively.

```
(%i1) load ("interpol") $
(%i2) charfun2 (5, 0, 100);
(%o2)                          1
(%i3) charfun2 (-5, 0, 100);
(%o3)                          0
```

Otherwise, `charfun2` returns a partially-evaluated result in terms of `charfun`.

```
(%i1) load ("interpol") $
(%i2) charfun2 (t, 0, 100);
(%o2)           charfun((0 <= t) and (t < 100))
(%i3) charfun2 (5, u, v);
(%o3)            charfun((u <= 5) and (5 < v))
(%i4) assume (v > u, u > 5);
(%o4)                    [v > u, u > 5]
(%i5) charfun2 (5, u, v);
(%o5)                          0
```

`linearinterpol` [Function]

`linearinterpol` (*points*)
`linearinterpol` (*points*, *option*)

Computes the polynomial interpolation by the linear method. Argument *points* must be either:

- a two column matrix, `p:matrix([2,4],[5,6],[9,3])`,
- a list of pairs, `p: [[2,4],[5,6],[9,3]]`,

- a list of numbers, `p: [4,6,3]`, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

With the *option* argument it is possible to select the name for the independent variable, which is `'x` by default; to define another one, write something like `varname='z`.

See also `lagrange`, `cspline`, and `ratinterpol`.

Examples:

```
(%i1) load("interpol")$
(%i2) p: matrix([7,2],[8,3],[1,5],[3,2],[6,7])$
(%i3) linearinterpol(p);
        13   3 x
(%o3)  (-- - ---) charfun2(x, minf, 3)
        2     2
 + (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
    5 x
 + (--- - 3) charfun2(x, 3, 6)
     3

(%i4) f(x):=''%;
                13   3 x
(%o4)  f(x) := (-- - ---) charfun2(x, minf, 3)
                2     2
 + (x - 5) charfun2(x, 7, inf) + (37 - 5 x) charfun2(x, 6, 7)
    5 x
 + (--- - 3) charfun2(x, 3, 6)
     3
(%i5)  /* Evaluate the polynomial at some points */
       map(f,[7.3,25/7,%pi]);
                             62  5 %pi
(%o5)                 [2.3, --, ----- - 3]
                             21    3
(%i6) %,numer;
(%o6)  [2.3, 2.952380952380953, 2.235987755982989]
(%i7) load("draw")$  /* load draw package */
(%i8)  /* Plot the polynomial together with points */
       draw2d(
         color      = red,
         key        = "Linear interpolator",
         explicit(f(x),x,-5,20),
         point_size = 3,
         color      = blue,
         key        = "Sample points",
         points(args(p)))$
(%i9)  /* Change variable name */
```

```
       linearinterpol(p, varname='s);
       13   3 s
(%o9) (-- - ---) charfun2(s, minf, 3)
       2     2
 + (s - 5) charfun2(s, 7, inf) + (37 - 5 s) charfun2(s, 6, 7)
    5 s
 + (--- - 3) charfun2(s, 3, 6)
     3
```

cspline [Function]

cspline (*points*)
cspline (*points*, *option1*, *option2*, ...)

Computes the polynomial interpolation by the cubic splines method. Argument *points* must be either:

- a two column matrix, `p:matrix([2,4],[5,6],[9,3])`,
- a list of pairs, `p: [[2,4],[5,6],[9,3]]`,
- a list of numbers, `p: [4,6,3]`, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There are three options to fit specific needs:

- `'d1`, default `'unknown`, is the first derivative at x_1; if it is `'unknown`, the second derivative at x_1 is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.
- `'dn`, default `'unknown`, is the first derivative at x_n; if it is `'unknown`, the second derivative at x_n is made equal to 0 (natural cubic spline); if it is equal to a number, the second derivative is calculated based on this number.
- `'varname`, default `'x`, is the name of the independent variable.

See also lagrange, linearinterpol, and ratinterpol.

Examples:

```
(%i1) load("interpol")$
(%i2) p:[[7,2],[8,2],[1,5],[3,2],[6,7]]$
(%i3) /* Unknown first derivatives at the extremes
         is equivalent to natural cubic splines */
      cspline(p);
              3         2
       1159 x    1159 x    6091 x   8283
(%o3)  (------- - ------- - ------ + ----) charfun2(x, minf, 3)
        3288      1096      3288    1096
           3         2
      2587 x    5174 x    494117 x   108928
 + (- ------- + ------- - -------- + ------) charfun2(x, 7, inf)
       1644       137       1644      137
          3         2
```

```
   4715 x    15209 x    579277 x   199575
 + (------- - -------- + -------- - ------) charfun2(x, 6, 7)
    1644       274        1644      274
           3         2
     3287 x    2223 x    48275 x   9609
 + (- ------- + ------- - ------- + ----) charfun2(x, 3, 6)
      4932       274      1644     274

(%i4) f(x):=''%$
(%i5) /* Some evaluations */
      map(f,[2.3,5/7,%pi]), numer;
(%o5) [1.991460766423356, 5.823200187269903, 2.227405312429507]
(%i6) load("draw")$  /* load draw package */
(%i7) /* Plotting interpolating function */
      draw2d(
        color      = red,
        key        = "Cubic splines",
        explicit(f(x),x,0,10),
        point_size = 3,
        color      = blue,
        key        = "Sample points",
        points(p))$
(%i8) /* New call, but giving values at the derivatives */
      cspline(p,d1=0,dn=0);
             3          2
        1949 x    11437 x    17027 x   1247
(%o8)  (------- - -------- + ------- + ----) charfun2(x, minf, 3)
         2256       2256      2256     752
            3          2
      1547 x    35581 x    68068 x   173546
 + (- ------- + -------- - ------- + ------) charfun2(x, 7, inf)
        564       564        141      141
         3          2
    607 x    35147 x    55706 x   38420
 + (------ - -------- + ------- - -----) charfun2(x, 6, 7)
     188       564        141      47
           3         2
      3895 x    1807 x    5146 x   2148
 + (- ------- + ------- - ------ + ----) charfun2(x, 3, 6)
       5076       188      141      47
(%i8) /* Defining new interpolating function */
      g(x):=''%$
(%i9) /* Plotting both functions together */
      draw2d(
        color      = black,
        key        = "Cubic splines (default)",
        explicit(f(x),x,0,10),
```

```
                color      = red,
                key        = "Cubic splines (d1=0,dn=0)",
                explicit(g(x),x,0,10),
                point_size = 3,
                color      = blue,
                key        = "Sample points",
                points(p))$
```

ratinterpol [Function]

ratinterpol (*points*, *numdeg*)
ratinterpol (*points*, *numdeg*, *option1*)

Generates a rational interpolator for data given by *points* and the degree of the numerator being equal to *numdeg*; the degree of the denominator is calculated automatically. Argument *points* must be either:

- a two column matrix, `p:matrix([2,4],[5,6],[9,3])`,
- a list of pairs, `p: [[2,4],[5,6],[9,3]]`,
- a list of numbers, `p: [4,6,3]`, in which case the abscissas will be assigned automatically to 1, 2, 3, etc.

In the first two cases the pairs are ordered with respect to the first coordinate before making computations.

There is one option to fit specific needs:

- `'varname`, default `'x`, is the name of the independent variable.

See also `lagrange`, `linearinterpol`, `cspline`, `minpack_lsquares`, and Chapter 73 [Package lbfgs], page 1137,

Examples:

```
(%i1) load("interpol")$
(%i2) load("draw")$
(%i3) p:[[7.2,2.5],[8.5,2.1],[1.6,5.1],[3.4,2.4],[6.7,7.9]]$
(%i4) for k:0 thru length(p)-1 do
        draw2d(
          explicit(ratinterpol(p,k),x,0,9),
          point_size = 3,
          points(p),
          title = concat("Degree of numerator = ",k),
          yrange=[0,10])$
```

71 Package itensor

71.1 Introduction to itensor

Maxima implements symbolic tensor manipulation of two distinct types: component tensor manipulation (Chapter 51 [Package ctensor], page 723) and indicial tensor manipulation (Chapter 71 [Package itensor], page 1091).

Nota bene: Please see the note on 'new tensor notation' below.

Component tensor manipulation means that geometrical tensor objects are represented as arrays or matrices. Tensor operations such as contraction or covariant differentiation are carried out by actually summing over repeated (dummy) indices with `do` statements. That is, one explicitly performs operations on the appropriate tensor components stored in an array or matrix.

Indicial tensor manipulation is implemented by representing tensors as functions of their covariant, contravariant and derivative indices. Tensor operations such as contraction or covariant differentiation are performed by manipulating the indices themselves rather than the components to which they correspond.

These two approaches to the treatment of differential, algebraic and analytic processes in the context of Riemannian geometry have various advantages and disadvantages which reveal themselves only through the particular nature and difficulty of the user's problem. However, one should keep in mind the following characteristics of the two implementations:

The representation of tensors and tensor operations explicitly in terms of their components makes `ctensor` easy to use. Specification of the metric and the computation of the induced tensors and invariants is straightforward. Although all of Maxima's powerful simplification capacity is at hand, a complex metric with intricate functional and coordinate dependencies can easily lead to expressions whose size is excessive and whose structure is hidden. In addition, many calculations involve intermediate expressions which swell causing programs to terminate before completion. Through experience, a user can avoid many of these difficulties.

Because of the special way in which tensors and tensor operations are represented in terms of symbolic operations on their indices, expressions which in the component representation would be unmanageable can sometimes be greatly simplified by using the special routines for symmetrical objects in `itensor`. In this way the structure of a large expression may be more transparent. On the other hand, because of the special indicial representation in `itensor`, in some cases the user may find difficulty with the specification of the metric, function definition, and the evaluation of differentiated "indexed" objects.

The `itensor` package can carry out differentiation with respect to an indexed variable, which allows one to use the package when dealing with Lagrangian and Hamiltonian formalisms. As it is possible to differentiate a field Lagrangian with respect to an (indexed) field variable, one can use Maxima to derive the corresponding Euler-Lagrange equations in indicial form. These equations can be translated into component tensor (`ctensor`) programs using the `ic_convert` function, allowing us to solve the field equations in a particular coordinate representation, or to recast the equations of motion in Hamiltonian form. See `einhil.dem` and `bradic.dem` for two comprehensive examples. The first, `einhil.dem`, uses the Einstein-Hilbert action to derive the Einstein field tensor in

the homogeneous and isotropic case (Friedmann equations) and the spherically symmetric, static case (Schwarzschild solution.) The second, `bradic.dem`, demonstrates how to compute the Friedmann equations from the action of Brans-Dicke gravity theory, and also derives the Hamiltonian associated with the theory's scalar field.

71.1.1 New tensor notation

Earlier versions of the `itensor` package in Maxima used a notation that sometimes led to incorrect index ordering. Consider the following, for instance:

```
(%i2) imetric(g);
(%o2)                                done
(%i3) ishow(g([],[j,k])*g([],[i,l])*a([i,j],[]))$
                                  i l  j k
(%t3)                            g    g    a
                                                i j
(%i4) ishow(contract(%))$
                                   k l
(%t4)                              a
```

This result is incorrect unless `a` happens to be a symmetric tensor. The reason why this happens is that although `itensor` correctly maintains the order within the set of covariant and contravariant indices, once an index is raised or lowered, its position relative to the other set of indices is lost.

To avoid this problem, a new notation has been developed that remains fully compatible with the existing notation and can be used interchangeably. In this notation, contravariant indices are inserted in the appropriate positions in the covariant index list, but with a minus sign prepended. Functions like `contract_Itensor` and `ishow` are now aware of this new index notation and can process tensors appropriately.

In this new notation, the previous example yields a correct result:

```
(%i5) ishow(g([-j,-k],[])*g([-i,-l],[])*a([i,j],[]))$
                                  i l       j k
(%t5)                            g    a    g
                                        i j
(%i6) ishow(contract(%))$
                                        l k
(%t6)                              a
```

Presently, the only code that makes use of this notation is the `lc2kdt` function. Through this notation, it achieves consistent results as it applies the metric tensor to resolve Levi-Civita symbols without resorting to numeric indices.

Since this code is brand new, it probably contains bugs. While it has been tested to make sure that it doesn't break anything using the "old" tensor notation, there is a considerable chance that "new" tensors will fail to interoperate with certain functions or features. These bugs will be fixed as they are encountered... until then, caveat emptor!

71.1.2 Indicial tensor manipulation

The indicial tensor manipulation package may be loaded by `load("itensor")`. Demos are also available: try `demo("tensor")`.

In itensor a tensor is represented as an "indexed object" . This is a function of 3 groups of indices which represent the covariant, contravariant and derivative indices. The covariant indices are specified by a list as the first argument to the indexed object, and the contravariant indices by a list as the second argument. If the indexed object lacks either of these groups of indices then the empty list [] is given as the corresponding argument. Thus, g([a,b],[c]) represents an indexed object called g which has two covariant indices (a,b), one contravariant index (c) and no derivative indices.

The derivative indices, if they are present, are appended as additional arguments to the symbolic function representing the tensor. They can be explicitly specified by the user or be created in the process of differentiation with respect to some coordinate variable. Since ordinary differentiation is commutative, the derivative indices are sorted alphanumerically, unless iframe_flag is set to true, indicating that a frame metric is being used. This canonical ordering makes it possible for Maxima to recognize that, for example, t([a],[b],i,j) is the same as t([a],[b],j,i). Differentiation of an indexed object with respect to some coordinate whose index does not appear as an argument to the indexed object would normally yield zero. This is because Maxima would not know that the tensor represented by the indexed object might depend implicitly on the corresponding coordinate. By modifying the existing Maxima function diff in itensor, Maxima now assumes that all indexed objects depend on any variable of differentiation unless otherwise stated. This makes it possible for the summation convention to be extended to derivative indices. It should be noted that itensor does not possess the capabilities of raising derivative indices, and so they are always treated as covariant.

The following functions are available in the tensor package for manipulating indexed objects. At present, with respect to the simplification routines, it is assumed that indexed objects do not by default possess symmetry properties. This can be overridden by setting the variable allsym[false] to true, which will result in treating all indexed objects completely symmetric in their lists of covariant indices and symmetric in their lists of contravariant indices.

The itensor package generally treats tensors as opaque objects. Tensorial equations are manipulated based on algebraic rules, specifically symmetry and contraction rules. In addition, the itensor package understands covariant differentiation, curvature, and torsion. Calculations can be performed relative to a metric of moving frame, depending on the setting of the iframe_flag variable.

A sample session below demonstrates how to load the itensor package, specify the name of the metric, and perform some simple calculations.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2)                                done
(%i3) components(g([i,j],[]),p([i,j],[])*e([],[]))$
(%i4) ishow(g([k,l],[]))$
(%t4)                               e p
                                       k l
(%i5) ishow(diff(v([i],[]),t))$
(%t5)                                  0
(%i6) depends(v,t);
```

```
(%o6)                                [v(t)]
(%i7) ishow(diff(v([i],[]),t))$
                                     d
(%t7)                                -- (v )
                                     dt   i
(%i8) ishow(idiff(v([i],[]),j))$
(%t8)                                 v
                                       i,j
(%i9) ishow(extdiff(v([i],[]),j))$
(%t9)                              v    - v
                                    j,i    i,j
                                   -----------
                                       2
(%i10) ishow(liediff(v,w([i],[])))$
                                 %3          %3
(%t10)                          v   w     + v   w
                                     i,%3    ,i  %3
(%i11) ishow(covdiff(v([i],[]),j))$
                                                 %4
(%t11)                          v    - v   ichr2
                                 i,j    %4      i j
(%i12) ishow(ev(%,ichr2))$
                 %4 %5
(%t12) v    - (g      v   (e p       + e   p     - e p       - e    p
        i,j            %4     j %5,i    ,i  j %5      i j,%5    ,%5  i j

                                          + e p       + e   p    ))/2
                                               i %5,j    ,j  i %5
(%i13) iframe_flag:true;
(%o13)                                   true
(%i14) ishow(covdiff(v([i],[]),j))$
                                               %6
(%t14)                          v    - v   icc2
                                 i,j    %6     i j
(%i15) ishow(ev(%,icc2))$
                                               %6
(%t15)                          v    - v   ifc2
                                 i,j    %6     i j
(%i16) ishow(radcan(ev(%,ifc2,ifc1)))$
              %6 %7                      %6 %7
(%t16) - (ifg      v   ifb       + ifg      v   ifb       - 2 v
                    %6    j %7 i             %6    i j %7      i,j

                                               %6 %7
                                          - ifg      v   ifb      )/2
                                                      %6    %7 i j
(%i17) ishow(canform(s([i,j],[])-s([j,i])))$
```

```
(%t17)                              s    - s
                                     i j    j i
(%i18) decsym(s,2,0,[sym(all)],[]);
(%o18)                                done
(%i19) ishow(canform(s([i,j],[])-s([j,i])))$
(%t19)                                 0
(%i20) ishow(canform(a([i,j],[])+a([j,i])))$
(%t20)                              a    + a
                                     j i    i j
(%i21) decsym(a,2,0,[anti(all)],[]);
(%o21)                                done
(%i22) ishow(canform(a([i,j],[])+a([j,i])))$
(%t22)                                 0
```

71.2 Functions and Variables for itensor

71.2.1 Managing indexed objects

`dispcon` [Function]

`dispcon` (*tensor_1*, *tensor_2*, ...)
`dispcon` (*all*)

Displays the contraction properties of its arguments as were given to `defcon`. `dispcon (all)` displays all the contraction properties which were defined.

`entertensor` (*name*) [Function]

is a function which, by prompting, allows one to create an indexed object called *name* with any number of tensorial and derivative indices. Either a single index or a list of indices (which may be null) is acceptable input (see the example under `covdiff`).

`changename` (*old*, *new*, *expr*) [Function]

will change the name of all indexed objects called *old* to *new* in *expr*. *old* may be either a symbol or a list of the form `[name, m, n]` in which case only those indexed objects called *name* with *m* covariant and *n* contravariant indices will be renamed to *new*.

`listoftens` [Function]

Lists all tensors in a tensorial expression, complete with their indices. E.g.,

```
(%i6) ishow(a([i,j],[k])*b([u],[],v)+c([x,y],[])*d([],[])*e)$
                                              k
(%t6)                          d e c    + a    b
                                    x y    i j  u,v
(%i7) ishow(listoftens(%))$
                                 k
(%t7)                           [a   , b   , c   , d]
                                  i j   u,v   x y
```

ishow (*expr*) [Function]

displays *expr* with the indexed objects in it shown having their covariant indices as subscripts and contravariant indices as superscripts. The derivative indices are displayed as subscripts, separated from the covariant indices by a comma (see the examples throughout this document).

indices (*expr*) [Function]

Returns a list of two elements. The first is a list of the free indices in *expr* (those that occur only once). The second is the list of the dummy indices in *expr* (those that occur exactly twice) as the following example demonstrates.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(a([i,j],[k,l],m,n)*b([k,o],[j,m,p],q,r))$
                                k l      j m p
(%t2)                          a        b
                                i j,m n  k o,q r
(%i3) indices(%);
(%o3)                 [[l, p, i, n, o, q, r], [k, j, m]]
```

A tensor product containing the same index more than twice is syntactically illegal. `indices` attempts to deal with these expressions in a reasonable manner; however, when it is called to operate upon such an illegal expression, its behavior should be considered undefined.

rename [Function]

rename (*expr*)
rename (*expr*, *count*)

Returns an expression equivalent to *expr* but with the dummy indices in each term chosen from the set `[%1, %2, ...]`, if the optional second argument is omitted. Otherwise, the dummy indices are indexed beginning at the value of *count*. Each dummy index in a product will be different. For a sum, `rename` will operate upon each term in the sum resetting the counter with each term. In this way `rename` can serve as a tensorial simplifier. In addition, the indices will be sorted alphanumerically (if `allsym` is `true`) with respect to covariant or contravariant indices depending upon the value of `flipflag`. If `flipflag` is `false` then the indices will be renamed according to the order of the contravariant indices. If `flipflag` is `true` the renaming will occur according to the order of the covariant indices. It often happens that the combined effect of the two renamings will reduce an expression more than either one by itself.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) allsym:true;
(%o2)                                  true
(%i3) g([],[%4,%5])*g([],[%6,%7])*ichr2([%1,%4],[%3])*
ichr2([%2,%3],[u])*ichr2([%5,%6],[%1])*ichr2([%7,r],[%2])-
g([],[%4,%5])*g([],[%6,%7])*ichr2([%1,%2],[u])*
```

```
ichr2([%3,%5],[%1])*ichr2([%4,%6],[%3])*ichr2([%7,r],[%2]),noeval$
(%i4) expr:ishow(%)$
       %4 %5  %6 %7      %3         u          %1         %2
(%t4) g      g      ichr2      ichr2      ichr2      ichr2
                         %1 %4      %2 %3      %5 %6      %7 r

        %4 %5  %6 %7      u          %1         %3         %2
     - g      g      ichr2      ichr2      ichr2      ichr2
                         %1 %2      %3 %5      %4 %6      %7 r
(%i5) flipflag:true;
(%o5)                                 true
(%i6) ishow(rename(expr))$
       %2 %5  %6 %7      %4         u          %1         %3
(%t6) g      g      ichr2      ichr2      ichr2      ichr2
                         %1 %2      %3 %4      %5 %6      %7 r

        %4 %5  %6 %7      u          %1         %3         %2
     - g      g      ichr2      ichr2      ichr2      ichr2
                         %1 %2      %3 %4      %5 %6      %7 r
(%i7) flipflag:false;
(%o7)                                 false
(%i8) rename(%th(2));
(%o8)                                   0
(%i9) ishow(rename(expr))$
       %1 %2  %3 %4      %5         %6         %7        u
(%t9) g      g      ichr2      ichr2      ichr2     ichr2
                         %1 %6      %2 %3      %4 r      %5 %7

        %1 %2  %3 %4      %6         %5         %7        u
     - g      g      ichr2      ichr2      ichr2     ichr2
                         %1 %3      %2 %6      %4 r      %5 %7
```

`flipflag` [Option variable]

Default value: `false`

If `false` then the indices will be renamed according to the order of the contravariant indices, otherwise according to the order of the covariant indices.

If `flipflag` is `false` then `rename` forms a list of the contravariant indices as they are encountered from left to right (if `true` then of the covariant indices). The first dummy index in the list is renamed to `%1`, the next to `%2`, etc. Then sorting occurs after the `rename`-ing (see the example under `rename`).

`defcon` [Function]

`defcon` (*tensor_1*)
`defcon` (*tensor_1*, *tensor_2*, *tensor_3*)

gives *tensor_1* the property that the contraction of a product of *tensor_1* and *tensor_2* results in *tensor_3* with the appropriate indices. If only one argument, *tensor_1*, is given, then the contraction of the product of *tensor_1* with any indexed object having

the appropriate indices (say `my_tensor`) will yield an indexed object with that name, i.e. `my_tensor`, and with a new set of indices reflecting the contractions performed. For example, if `imetric:g`, then `defcon(g)` will implement the raising and lowering of indices through contraction with the metric tensor. More than one `defcon` can be given for the same indexed object; the latest one given which applies in a particular contraction will be used. `contractions` is a list of those indexed objects which have been given contraction properties with `defcon`.

`remcon` [Function]

`remcon` (*tensor_1*, ..., *tensor_n*)
`remcon` (*all*)

Removes all the contraction properties from the (*tensor_1*, ..., *tensor_n*). `remcon(all)` removes all contraction properties from all indexed objects.

`contract` (*expr*) [Function]

Carries out the tensorial contractions in *expr* which may be any combination of sums and products. This function uses the information given to the `defcon` function. For best results, `expr` should be fully expanded. `ratexpand` is the fastest way to expand products and powers of sums if there are no variables in the denominators of the terms. The `gcd` switch should be `false` if GCD cancellations are unnecessary.

`indexed_tensor` (*tensor*) [Function]

Must be executed before assigning components to a *tensor* for which a built in value already exists as with `ichr1`, `ichr2`, `icurvature`. See the example under `icurvature`.

`components` (*tensor*, *expr*) [Function]

permits one to assign an indicial value to an expression *expr* giving the values of the components of *tensor*. These are automatically substituted for the tensor whenever it occurs with all of its indices. The tensor must be of the form `t([...],[...])` where either list may be empty. *expr* can be any indexed expression involving other objects with the same free indices as *tensor*. When used to assign values to the metric tensor wherein the components contain dummy indices one must be careful to define these indices to avoid the generation of multiple dummy indices. Removal of this assignment is given to the function `remcomps`.

It is important to keep in mind that `components` cares only about the valence of a tensor, not about any particular index ordering. Thus assigning components to, say, `x([i,-j],[])`, `x([-j,i],[])`, or `x([i],[j])` all produce the same result, namely components being assigned to a tensor named `x` with valence `(1,1)`.

Components can be assigned to an indexed expression in four ways, two of which involve the use of the `components` command:

1) As an indexed expression. For instance:

```
(%i2) components(g([],[i,j]),e([],[i])*p([],[j]))$
(%i3) ishow(g([],[i,j]))$
                                        i  j
(%t3)                                  e  p
```

2) As a matrix:

```
(%i5) lg:-ident(4)$lg[1,1]:1$lg;
                              [ 1   0    0    0  ]
                              [                   ]
                              [ 0  - 1   0    0  ]
(%o5)                         [                   ]
                              [ 0   0   - 1   0  ]
                              [                   ]
                              [ 0   0    0   - 1 ]
(%i6) components(g([i,j],[]),lg);
(%o6)                                    done
(%i7) ishow(g([i,j],[]))$
(%t7)                                    g
                                          i j
(%i8) g([1,1],[]);
(%o8)                                     1
(%i9) g([4,4],[]);
(%o9)                                     - 1
```

3) As a function. You can use a Maxima function to specify the components of a tensor based on its indices. For instance, the following code assigns `kdelta` to `h` if `h` has the same number of covariant and contravariant indices and no derivative indices, and `g` otherwise:

```
(%i4) h(l1,l2,[l3]):=if length(l1)=length(l2) and length(l3)=0
  then kdelta(l1,l2) else apply(g,append([l1,l2], l3))$
(%i5) ishow(h([i],[j]))$
                                           j
(%t5)                                  kdelta
                                           i
(%i6) ishow(h([i,j],[k],l))$
                                          k
(%t6)                                     g
                                          i j,l
```

4) Using Maxima's pattern matching capabilities, specifically the `defrule` and `applyb1` commands:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) matchdeclare(l1,listp);
(%o2)                                    done
(%i3) defrule(r1,m(l1,[]),(i1:idummy(),
      g([l1[1],l1[2]],[])*q([i1],[])*e([],[i1])))$

(%i4) defrule(r2,m([],l1),(i1:idummy(),
      w([],[l1[1],l1[2]])*e([i1],[])*q([],[i1])))$
```

```
(%i5) ishow(m([i,n],[])*m([],[i,m]))$
                                    i m
(%t5)                              m    m
                                         i n
(%i6) ishow(rename(applyb1(%,r1,r2)))$
                             %1  %2  %3 m
(%t6)                       e   q   w     q   e   g
                                           %1  %2  %3 n
```

remcomps (*tensor*) [Function]

Unbinds all values from *tensor* which were assigned with the `components` function.

showcomps (*tensor*) [Function]

Shows component assignments of a tensor, as made using the `components` command. This function can be particularly useful when a matrix is assigned to an indicial tensor using `components`, as demonstrated by the following example:

```
(%i1) load("ctensor");
(%o1)       /share/tensor/ctensor.mac
(%i2) load("itensor");
(%o2)      /share/tensor/itensor.lisp
(%i3) lg:matrix([sqrt(r/(r-2*m)),0,0,0],[0,r,0,0],
                [0,0,sin(theta)*r,0],[0,0,0,sqrt((r-2*m)/r)]);
       [         r                                        ]
       [ sqrt(-------)  0       0              0         ]
       [      r - 2 m                                     ]
       [                                                  ]
       [       0        r       0              0         ]
(%o3)  [                                                  ]
       [       0        0  r sin(theta)        0         ]
       [                                                  ]
       [                                      r - 2 m     ]
       [       0        0       0        sqrt(-------)   ]
       [                                        r         ]
(%i4) components(g([i,j],[]),lg);
(%o4)                           done
(%i5) showcomps(g([i,j],[]));
             [         r                                        ]
             [ sqrt(-------)  0       0              0         ]
             [      r - 2 m                                     ]
             [                                                  ]
             [       0        r       0              0         ]
(%t5) g    = [                                                  ]
       i j   [       0        0  r sin(theta)        0         ]
             [                                                  ]
             [                                      r - 2 m     ]
```

```
                [       0        0       0        sqrt(-------) ]
                [                                         r     ]
(%o5)                                 false
```

The `showcomps` command can also display components of a tensor of rank higher than 2.

`idummy` () [Function]

Increments `icounter` and returns as its value an index of the form `%n` where n is a positive integer. This guarantees that dummy indices which are needed in forming expressions will not conflict with indices already in use (see the example under `indices`).

`idummyx` [Option variable]

Default value: `%`

Is the prefix for dummy indices (see the example under `indices`).

`icounter` [Option variable]

Default value: `1`

Determines the numerical suffix to be used in generating the next dummy index in the tensor package. The prefix is determined by the option `idummy` (default: `%`).

`kdelta` (*L1*, *L2*) [Function]

is the generalized Kronecker delta function defined in the `itensor` package with *L1* the list of covariant indices and *L2* the list of contravariant indices. `kdelta([i],[j])` returns the ordinary Kronecker delta. The command `ev(`*expr*`,kdelta)` causes the evaluation of an expression containing `kdelta([],[])` to the dimension of the manifold.

In what amounts to an abuse of this notation, `itensor` also allows `kdelta` to have 2 covariant and no contravariant, or 2 contravariant and no covariant indices, in effect providing a co(ntra)variant "unit matrix" capability. This is strictly considered a programming aid and not meant to imply that `kdelta([i,j],[])` is a valid tensorial object.

`kdels` (*L1*, *L2*) [Function]

Symmetrized Kronecker delta, used in some calculations. For instance:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) kdelta([1,2],[2,1]);
(%o2)                                - 1
(%i3) kdels([1,2],[2,1]);
(%o3)                                 1
(%i4) ishow(kdelta([a,b],[c,d]))$
                              c       d         d       c
(%t4)                 kdelta  kdelta  - kdelta  kdelta
                            a       b         a       b
```

```
(%i4) ishow(kdels([a,b],[c,d]))$
                              c       d         d       c
(%t4)                  kdelta  kdelta  + kdelta  kdelta
                              a       b         a       b
```

`levi_civita` (*L*) [Function]

is the permutation (or Levi-Civita) tensor which yields 1 if the list *L* consists of an even permutation of integers, -1 if it consists of an odd permutation, and 0 if some indices in *L* are repeated.

`lc2kdt` (*expr*) [Function]

Simplifies expressions containing the Levi-Civita symbol, converting these to Kronecker-delta expressions when possible. The main difference between this function and simply evaluating the Levi-Civita symbol is that direct evaluation often results in Kronecker expressions containing numerical indices. This is often undesirable as it prevents further simplification. The `lc2kdt` function avoids this problem, yielding expressions that are more easily simplified with `rename` or `contract`.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) expr:ishow('levi_civita([],[i,j])
                *'levi_civita([k,l],[])*a([j],[k]))$
                                    i j  k
(%t2)                  levi_civita    a  levi_civita
                                       j            k l
(%i3) ishow(ev(expr,levi_civita))$
                                    i j  k       1 2
(%t3)                       kdelta    a  kdelta
                                    1 2  j       k l
(%i4) ishow(ev(%,kdelta))$
              i       j         j       i   k
(%t4) (kdelta  kdelta  - kdelta  kdelta ) a
              1       2         1       2   j

                                 1       2         2       1
                          (kdelta  kdelta  - kdelta  kdelta )
                                 k       l         k       l
(%i5) ishow(lc2kdt(expr))$
                      k       i       j    k       j       i
(%t5)                a  kdelta  kdelta  - a  kdelta  kdelta
                      j       k       l    j       k       l
(%i6) ishow(contract(expand(%)))$
                                   i           i
(%t6)                             a  - a kdelta
                                   l           l
```

The lc2kdt function sometimes makes use of the metric tensor. If the metric tensor was not defined previously with imetric, this results in an error.

```
(%i7) expr:ishow('levi_civita([],[i,j])
                 *'levi_civita([],[k,l])*a([j,k],[]))$
                                  i j           k l
(%t7)                 levi_civita    levi_civita    a
                                                       j k
(%i8) ishow(lc2kdt(expr))$
Maxima encountered a Lisp error:

 Error in $IMETRIC [or a callee]:
 $IMETRIC [or a callee] requires less than two arguments.

Automatically continuing.
To re-enable the Lisp debugger set *debugger-hook* to nil.
(%i9) imetric(g);
(%o9)                                done
(%i10) ishow(lc2kdt(expr))$
         %3 i       k   %4 j       l     %3 i       l   %4 j
(%t10) (g     kdelta   g     kdelta   - g     kdelta   g
                    %3             %4              %3
              k
        kdelta  ) a
              %4   j k
(%i11) ishow(contract(expand(%)))$
                                  l i    l i  j
(%t11)                           a    - g    a
                                             j
```

lc_l [Function]

Simplification rule used for expressions containing the unevaluated Levi-Civita symbol (levi_civita). Along with lc_u, it can be used to simplify many expressions more efficiently than the evaluation of levi_civita. For example:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) el1:ishow('levi_civita([i,j,k],[])*a([],[i])*a([],[j]))$
                             i  j
(%t2)                       a  a  levi_civita
                                             i j k
(%i3) el2:ishow('levi_civita([],[i,j,k])*a([i])*a([j]))$
                                       i j k
(%t3)                       levi_civita      a  a
                                              i  j
(%i4) canform(contract(expand(applyb1(el1,lc_l,lc_u))));
(%t4)                                  0
```

```
(%i5) canform(contract(expand(applyb1(el2,lc_l,lc_u))));
(%t5)                                  0
```

lc_u [Function]
Simplification rule used for expressions containing the unevaluated Levi-Civita symbol (levi_civita). Along with lc_u, it can be used to simplify many expressions more efficiently than the evaluation of levi_civita. For details, see lc_l.

canten (*expr*) [Function]
Simplifies *expr* by renaming (see rename) and permuting dummy indices. rename is restricted to sums of tensor products in which no derivatives are present. As such it is limited and should only be used if canform is not capable of carrying out the required simplification.

The canten function returns a mathematically correct result only if its argument is an expression that is fully symmetric in its indices. For this reason, canten returns an error if allsym is not set to true.

concan (*expr*) [Function]
Similar to canten but also performs index contraction.

71.2.2 Tensor symmetries

allsym [Option variable]
Default value: false

If true then all indexed objects are assumed symmetric in all of their covariant and contravariant indices. If false then no symmetries of any kind are assumed in these indices. Derivative indices are always taken to be symmetric unless iframe_flag is set to true.

decsym (*tensor*, *m*, *n*, [*cov_1*, *cov_2*, ...], [*contr_1*, *contr_2*, ...]) [Function]
Declares symmetry properties for *tensor* of *m* covariant and *n* contravariant indices. The *cov_i* and *contr_i* are pseudofunctions expressing symmetry relations among the covariant and contravariant indices respectively. These are of the form symoper(*index_1*, *index_2*,...) where symoper is one of sym, anti or cyc and the *index_i* are integers indicating the position of the index in the *tensor*. This will declare *tensor* to be symmetric, antisymmetric or cyclic respectively in the *index_i*. symoper(all) is also an allowable form which indicates all indices obey the symmetry condition. For example, given an object b with 5 covariant indices, decsym(b,5,3,[sym(1,2),anti(3,4)],[cyc(all)]) declares b symmetric in its first and second and antisymmetric in its third and fourth covariant indices, and cyclic in all of its contravariant indices. Either list of symmetry declarations may be null. The function which performs the simplifications is canform as the example below illustrates.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) expr:contract( expand( a([i1, j1, k1], [])
```

```
           *kdels([i, j, k], [i1, j1, k1]))$
(%i3) ishow(expr)$
(%t3)         a      + a      + a      + a      + a      + a
               k j i    k i j    j k i    j i k    i k j    i j k
(%i4) decsym(a,3,0,[sym(all)],[]);
(%o4)                                done
(%i5) ishow(canform(expr))$
(%t5)                               6 a
                                       i j k
(%i6) remsym(a,3,0);
(%o6)                                done
(%i7) decsym(a,3,0,[anti(all)],[]);
(%o7)                                done
(%i8) ishow(canform(expr))$
(%t8)                                  0
(%i9) remsym(a,3,0);
(%o9)                                done
(%i10) decsym(a,3,0,[cyc(all)],[]);
(%o10)                               done
(%i11) ishow(canform(expr))$
(%t11)                        3 a      + 3 a
                                 i k j      i j k
(%i12) dispsym(a,3,0);
(%o12)                  [[cyc, [[1, 2, 3]], []]]
```

`remsym` (*tensor*, *m*, *n*) [Function]

Removes all symmetry properties from *tensor* which has *m* covariant indices and *n* contravariant indices.

`dispsym` (*tensor*, *m*, *n*) [Function]

Displays all of the defined symmetries from *tensor* which has *m* covariant indices and *n* contravariant indices. See `decsym` for an example.

`canform` [Function]

`canform` (*expr*)
`canform` (*expr*, *rename*)

Simplifies *expr* by renaming dummy indices and reordering all indices as dictated by symmetry conditions imposed on them. If `allsym` is `true` then all indices are assumed symmetric, otherwise symmetry information provided by `decsym` declarations will be used. The dummy indices are renamed in the same manner as in the `rename` function. When `canform` is applied to a large expression the calculation may take a considerable amount of time. This time can be shortened by calling `rename` on the expression first. Also see the example under `decsym`. Note: `canform` may not be able to reduce an expression completely to its simplest form although it will always return a mathematically correct result.

The optional second parameter *rename*, if set to `false`, suppresses renaming.

71.2.3 Indicial tensor calculus

`diff` (*expr*, *v_1*, [*n_1*, [*v_2*, *n_2*] ...]) [Function]
is the usual Maxima differentiation function which has been expanded in its abilities for `itensor`. It takes the derivative of *expr* with respect to *v_1* *n_1* times, with respect to *v_2* *n_2* times, etc. For the tensor package, the function has been modified so that the *v_i* may be integers from 1 up to the value of the variable `dim`. This will cause the differentiation to be carried out with respect to the *v_i*th member of the list `vect_coords`. If `vect_coords` is bound to an atomic variable, then that variable subscripted by *v_i* will be used for the variable of differentiation. This permits an array of coordinate names or subscripted names like `x[1]`, `x[2]`, ... to be used.

A further extension adds the ability to `diff` to compute derivatives with respect to an indexed variable. In particular, the tensor package knows how to differentiate expressions containing combinations of the metric tensor and its derivatives with respect to the metric tensor and its first and second derivatives. This capability is particularly useful when considering Lagrangian formulations of a gravitational theory, allowing one to derive the Einstein tensor and field equations from the action principle.

`idiff` (*expr*, *v_1*, [*n_1*, [*v_2*, *n_2*] ...]) [Function]
Indicial differentiation. Unlike `diff`, which differentiates with respect to an independent variable, `idiff)` can be used to differentiate with respect to a coordinate. For an indexed object, this amounts to appending the *v_i* as derivative indices. Subsequently, derivative indices will be sorted, unless `iframe_flag` is set to `true`.

`idiff` can also differentiate the determinant of the metric tensor. Thus, if `imetric` has been bound to `G` then `idiff(determinant(g),k)` will return `2 * determinant(g) * ichr2([%i,k],[%i])` where the dummy index `%i` is chosen appropriately.

`liediff` (*v*, *ten*) [Function]
Computes the Lie-derivative of the tensorial expression *ten* with respect to the vector field *v*. *ten* should be any indexed tensor expression; *v* should be the name (without indices) of a vector field. For example:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(liediff(v,a([i,j],[])*b([],[k],l)))$
       k    %2            %2          %2
(%t2) b   (v   a       + v   a     + v   a    )
       ,l       i j,%2    ,j  i %2    ,i  %2 j

                           %1  k        %1  k      %1  k
                      + (v   b      - b   v    + v   b   ) a
                             ,%1 l    ,l  ,%1    ,l  ,%1   i j
```

`rediff` (*ten*) [Function]
Evaluates all occurrences of the `idiff` command in the tensorial expression *ten*.

undiff (*expr*) [Function]

Returns an expression equivalent to *expr* but with all derivatives of indexed objects replaced by the noun form of the idiff function. Its arguments would yield that indexed object if the differentiation were carried out. This is useful when it is desired to replace a differentiated indexed object with some function definition resulting in *expr* and then carry out the differentiation by saying ev(*expr*, idiff).

evundiff (*expr*) [Function]

Equivalent to the execution of undiff, followed by ev and rediff.

The point of this operation is to easily evaluate expressions that cannot be directly evaluated in derivative form. For instance, the following causes an error:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) icurvature([i,j,k],[l],m);
Maxima encountered a Lisp error:

 Error in $ICURVATURE [or a callee]:
 $ICURVATURE [or a callee] requires less than three arguments.

Automatically continuing.
To re-enable the Lisp debugger set *debugger-hook* to nil.
```

However, if icurvature is entered in noun form, it can be evaluated using evundiff:

```
(%i3) ishow('icurvature([i,j,k],[l],m))$
                                      l
(%t3)                          icurvature
                                      i j k,m
(%i4) ishow(evundiff(%))$
             l              l         %1           l           %1
(%t4) - ichr2        - ichr2     ichr2      - ichr2       ichr2
             i k,j m        %1 j      i k,m        %1 j,m      i k

             l              l         %1           l           %1
      + ichr2        + ichr2     ichr2      + ichr2       ichr2
             i j,k m        %1 k      i j,m        %1 k,m      i j
```

Note: In earlier versions of Maxima, derivative forms of the Christoffel-symbols also could not be evaluated. This has been fixed now, so evundiff is no longer necessary for expressions like this:

```
(%i5) imetric(g);
(%o5)                              done
(%i6) ishow(ichr2([i,j],[k],l))$
       k %3
      g     (g         - g         + g        )
              j %3,i l    i j,%3 l    i %3,j l
(%t6) ----------------------------------------
                          2
```

```
                                  k %3
                                 g     (g       - g       + g      )
                                  ,l     j %3,i    i j,%3    i %3,j
                               + ------------------------------------
                                                  2
```

`flush` (*expr*, *tensor_1*, *tensor_2*, ...) [Function]
Set to zero, in *expr*, all occurrences of the *tensor_i* that have no derivative indices.

`flushd` (*expr*, *tensor_1*, *tensor_2*, ...) [Function]
Set to zero, in *expr*, all occurrences of the *tensor_i* that have derivative indices.

`flushnd` (*expr*, *tensor*, *n*) [Function]
Set to zero, in *expr*, all occurrences of the differentiated object *tensor* that have *n* or more derivative indices as the following example demonstrates.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(a([i],[J,r],k,r)+a([i],[j,r,s],k,r,s))$
                                  J r      j r s
(%t2)                            a      + a
                                  i,k r    i,k r s
(%i3) ishow(flushnd(%,a,3))$
                                       J r
(%t3)                                 a
                                       i,k r
```

`coord` (*tensor_1*, *tensor_2*, ...) [Function]
Gives *tensor_i* the coordinate differentiation property that the derivative of contravariant vector whose name is one of the *tensor_i* yields a Kronecker delta. For example, if `coord(x)` has been done then `idiff(x([],[i]),j)` gives `kdelta([i],[j])`. `coord` is a list of all indexed objects having this property.

`remcoord` [Function]
`remcoord` (*tensor_1*, *tensor_2*, ...)
`remcoord` (*all*)
Removes the coordinate differentiation property from the `tensor_i` that was established by the function `coord`. `remcoord(all)` removes this property from all indexed objects.

`makebox` (*expr*,*g*) [Function]
Display *expr* using the metric *g* such that any tensor d'Alembertian occurring in *expr* will be indicated using the symbol `[]`. For example, `[]p([m],[n])` represents `g([],[i,j])*p([m],[n],i,j)`.

`conmetderiv` (*expr*, *tensor*) [Function]
Simplifies expressions containing ordinary derivatives of both covariant and contravariant forms of the metric tensor (the current restriction). For example, `conmetderiv`

can relate the derivative of the contravariant metric tensor with the Christoffel symbols as seen from the following:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(g([],[a,b],c))$
                                        a b
(%t2)                                  g
                                        ,c
(%i3) ishow(conmetderiv(%,g))$
                            %1 b      a       %1 a      b
(%t3)                  - g     ichr2     - g     ichr2
                                       %1 c              %1 c
```

simpmetderiv [Function]

simpmetderiv (*expr*)
simpmetderiv (*expr*[, *stop*])

Simplifies expressions containing products of the derivatives of the metric tensor. Specifically, simpmetderiv recognizes two identities:

```
 ab        ab           ab                  a
g   g   + g   g     = (g   g  )   = (kdelta )   = 0
 ,d  bc        bc,d         bc ,d          c ,d
```

hence

```
 ab          ab
g   g   = - g   g
 ,d  bc          bc,d
```

and

```
 ab          ab
g   g     = g   g
 ,j  ab,i    ,i  ab,j
```

which follows from the symmetries of the Christoffel symbols.

The simpmetderiv function takes one optional parameter which, when present, causes the function to stop after the first successful substitution in a product expression. The simpmetderiv function also makes use of the global variable flipflag which determines how to apply a "canonical" ordering to the product indices.

Put together, these capabilities can be used to achieve powerful simplifications that are difficult or impossible to accomplish otherwise. This is demonstrated through the following example that explicitly uses the partial simplification features of simpmetderiv to obtain a contractible expression:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2)                                done
(%i3) ishow(g([],[a,b])*g([],[b,c])*g([a,b],[],d)*g([b,c],[],e))$
                              a b  b c
(%t3)                        g    g    g      g
                                       a b,d  b c,e
(%i4) ishow(canform(%))$

errexp1 has improper indices
 -- an error.  Quitting.  To debug this try debugmode(true);
(%i5) ishow(simpmetderiv(%))$
                              a b  b c
(%t5)                        g    g    g      g
                                       a b,d  b c,e
(%i6) flipflag:not flipflag;
(%o6)                                true
(%i7) ishow(simpmetderiv(%th(2)))$
                                a b  b c
(%t7)                          g    g    g    g
                                ,d   ,e   a b  b c
(%i8) flipflag:not flipflag;
(%o8)                                false
(%i9) ishow(simpmetderiv(%th(2),stop))$
                                a b  b c
(%t9)                        - g    g    g      g
                                      ,e   a b,d  b c
(%i10) ishow(contract(%))$
                                      b c
(%t10)                             - g    g
                                      ,e   c b,d
```

See also `weyl.dem` for an example that uses `simpmetderiv` and `conmetderiv` together to simplify contractions of the Weyl tensor.

`flush1deriv` (*expr*, *tensor*) [Function]

Set to zero, in `expr`, all occurrences of `tensor` that have exactly one derivative index.

71.2.4 Tensors in curved spaces

`imetric` (*g*) [Function]

`imetric` [System variable]

Specifies the metric by assigning the variable `imetric`:*g* in addition, the contraction properties of the metric *g* are set up by executing the commands `defcon(g)`, `defcon(g, g, kdelta)`. The variable `imetric` (unbound by default), is bound to the metric, assigned by the `imetric(g)` command.

idim (*n*) [Function]
Sets the dimensions of the metric. Also initializes the antisymmetry properties of the Levi-Civita symbols for the given dimension.

ichr1 ([*i*, *j*, *k*]) [Function]
Yields the Christoffel symbol of the first kind via the definition

```
(g      + g      - g     )/2 .
  ik,j     jk,i     ij,k
```

To evaluate the Christoffel symbols for a particular metric, the variable `imetric` must be assigned a name as in the example under `chr2`.

ichr2 ([*i*, *j*], [*k*]) [Function]
Yields the Christoffel symbol of the second kind defined by the relation

```
                          ks
ichr2([i,j],[k]) = g    (g      + g      - g     )/2
                           is,j     js,i     ij,s
```

icurvature ([*i*, *j*, *k*], [*h*]) [Function]
Yields the Riemann curvature tensor in terms of the Christoffel symbols of the second kind (`ichr2`). The following notation is used:

```
                h             h            h         %1         h
      icurvature     = - ichr2      - ichr2     ichr2    + ichr2
                i j k         i k,j        %1 j      i k        i j,k
                                h          %1
                          + ichr2      ichr2
                                %1 k       i j
```

covdiff (*expr*, *v_1*, *v_2*, ...) [Function]
Yields the covariant derivative of *expr* with respect to the variables *v_i* in terms of the Christoffel symbols of the second kind (`ichr2`). In order to evaluate these, one should use `ev(`*expr*`,ichr2)`.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) entertensor()$
Enter tensor name: a;
Enter a list of the covariant indices: [i,j];
Enter a list of the contravariant indices: [k];
Enter a list of the derivative indices: [];
                                      k
(%t2)                                a
                                      i j
(%i3) ishow(covdiff(%,s))$
             k         %1     k         %1     k
(%t3)     - a     ichr2    - a     ichr2    + a
             i %1      j s    %1 j      i s    i j,s
```

```
              k     %1
      + ichr2     a
              %1 s  i j
(%i4) imetric:g;
(%o4)                                   g
(%i5) ishow(ev(%th(2),ichr2))$
         %1 %4  k
        g      a     (g       - g       + g      )
                i %1   s %4,j    j s,%4    j %4,s
(%t5) - ----------------------------------------
                           2
    %1 %3  k
   g      a     (g       - g       + g      )
           %1 j   s %3,i    i s,%3    i %3,s
 - ----------------------------------------
                       2
    k %2  %1
   g     a    (g        - g        + g       )
          i j   s %2,%1    %1 s,%2    %1 %2,s     k
 + ---------------------------------------- + a
                       2                         i j,s
(%i6)
```

`lorentz_gauge` (*expr*) [Function]

Imposes the Lorentz condition by substituting 0 for all indexed objects in *expr* that have a derivative index identical to a contravariant index.

`igeodesic_coords` (*expr*, *name*) [Function]

Causes undifferentiated Christoffel symbols and first derivatives of the metric tensor vanish in *expr*. The *name* in the `igeodesic_coords` function refers to the metric *name* (if it appears in *expr*) while the connection coefficients must be called with the names `ichr1` and/or `ichr2`. The following example demonstrates the verification of the cyclic identity satisfied by the Riemann curvature tensor using the `igeodesic_coords` function.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(icurvature([r,s,t],[u]))$
              u            u         %1         u
(%t2) - ichr2      - ichr2     ichr2    + ichr2
              r t,s        %1 s      r t        r s,t

                                                 u         %1
                                            + ichr2     ichr2
                                                 %1 t      r s
(%i3) ishow(igeodesic_coords(%,ichr2))$
                                    u            u
```

```
(%t3)                          ichr2      - ichr2
                                    r s,t        r t,s
(%i4) ishow(igeodesic_coords(icurvature([r,s,t],[u]),ichr2)+
           igeodesic_coords(icurvature([s,t,r],[u]),ichr2)+
           igeodesic_coords(icurvature([t,r,s],[u]),ichr2))$
            u           u           u           u
(%t4) - ichr2      + ichr2      + ichr2      - ichr2
            t s,r        t r,s        s t,r        s r,t

                                                u           u
                                          - ichr2      + ichr2
                                                r t,s        r s,t
(%i5) canform(%);
(%o5)                                  0
```

71.2.5 Moving frames

Maxima now has the ability to perform calculations using moving frames. These can be orthonormal frames (tetrads, vielbeins) or an arbitrary frame.

To use frames, you must first set `iframe_flag` to `true`. This causes the Christoffel-symbols, `ichr1` and `ichr2`, to be replaced by the more general frame connection coefficients `icc1` and `icc2` in calculations. Specifically, the behavior of `covdiff` and `icurvature` is changed.

The frame is defined by two tensors: the inverse frame field (`ifri`, the dual basis tetrad), and the frame metric `ifg`. The frame metric is the identity matrix for orthonormal frames, or the Lorentz metric for orthonormal frames in Minkowski spacetime. The inverse frame field defines the frame base (unit vectors). Contraction properties are defined for the frame field and the frame metric.

When `iframe_flag` is true, many `itensor` expressions use the frame metric `ifg` instead of the metric defined by `imetric` for raising and lowerind indices.

IMPORTANT: Setting the variable `iframe_flag` to `true` does NOT undefine the contraction properties of a metric defined by a call to `defcon` or `imetric`. If a frame field is used, it is best to define the metric by assigning its name to the variable `imetric` and NOT invoke the `imetric` function.

Maxima uses these two tensors to define the frame coefficients (`ifc1` and `ifc2`) which form part of the connection coefficients (`icc1` and `icc2`), as the following example demonstrates:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) iframe_flag:true;
(%o2)                                true
(%i3) ishow(covdiff(v([],[i]),j))$
                              i        i     %1
(%t3)                        v   + icc2     v
```

```
                                  ,j       %1 j
(%i4) ishow(ev(%,icc2))$
                                 %1     i       i
(%t4)                           v   ifc2     + v
                                        %1 j    ,j
(%i5) ishow(ev(%,ifc2))$
                             %1    i %2                i
(%t5)                       v   ifg     ifc1        + v
                                           %1 j %2    ,j
(%i6) ishow(ev(%,ifc1))$
              %1    i %2
            v   ifg     (ifb        - ifb        + ifb       )
                           j %2 %1      %2 %1 j      %1 j %2     i
(%t6)       ------------------------------------------------- + v
                                  2                                ,j
(%i7) ishow(ifb([a,b,c]))$
                                                      %3    %4
(%t7)                 (ifri        - ifri       ) ifr   ifr
                           a %3,%4       a %4,%3     b     c
```

An alternate method is used to compute the frame bracket (`ifb`) if the `iframe_bracket_form` flag is set to `false`:

```
(%i8) block([iframe_bracket_form:false],ishow(ifb([a,b,c])))$
                                  %6    %5        %5      %6
(%t8)                ifri     (ifr   ifr     - ifr     ifr  )
                         a %5     b     c,%6      b,%6    c
```

`iframes ()` [Function]

Since in this version of Maxima, contraction identities for `ifr` and `ifri` are always defined, as is the frame bracket (`ifb`), this function does nothing.

`ifb` [Variable]

The frame bracket. The contribution of the frame metric to the connection coefficients is expressed using the frame bracket:

$$\mathtt{ifc1}_{abc} = \frac{-\mathtt{ifb}_{cab} + \mathtt{ifb}_{bca} + \mathtt{ifb}_{abc}}{2}$$

The frame bracket itself is defined in terms of the frame field and frame metric. Two alternate methods of computation are used depending on the value of `frame_bracket_form`. If true (the default) or if the `itorsion_flag` is `true`:

```
           d      e                                        f
ifb =  ifr    ifr   (ifri      - ifri      - ifri    itr   )
   abc    b      c       a d,e       a e,d       a f    d e
```

Otherwise:

```
              e      d        d      e
ifb    = (ifr    ifr    - ifr    ifr   ) ifri
   abc        b      c,e      b,e    c        a d
```

icc1 [Variable]

Connection coefficients of the first kind. In `itensor`, defined as

```
icc1    = ichr1    - ikt1    - inmc1
    abc        abc       abc       abc
```

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr1` is replaced with the frame connection coefficient `ifc1`. If `itorsion_flag` is `false`, `ikt1` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, of `inonmet_flag` is `false`, `inmc1` will not be present.

icc2 [Variable]

Connection coefficients of the second kind. In `itensor`, defined as

```
    c         c        c         c
icc2   = ichr2   - ikt2   - inmc2
    ab        ab       ab        ab
```

In this expression, if `iframe_flag` is true, the Christoffel-symbol `ichr2` is replaced with the frame connection coefficient `ifc2`. If `itorsion_flag` is `false`, `ikt2` will be omitted. It is also omitted if a frame base is used, as the torsion is already calculated as part of the frame bracket. Lastly, of `inonmet_flag` is `false`, `inmc2` will not be present.

ifc1 [Variable]

Frame coefficient of the first kind (also known as Ricci-rotation coefficients.) This tensor represents the contribution of the frame metric to the connection coefficient of the first kind. Defined as:

```
           - ifb      + ifb      + ifb
                c a b      b c a      a b c
ifc1    = --------------------------------
    abc                  2
```

ifc2 [Variable]

Frame coefficient of the second kind. This tensor represents the contribution of the frame metric to the connection coefficient of the second kind. Defined as a permutation of the frame bracket (ifb) with the appropriate indices raised and lowered as necessary:

$$\mathrm{ifc2}^{c}_{ab} = \mathrm{ifg}^{cd}\,\mathrm{ifc1}_{abd}$$

ifr [Variable]

The frame field. Contracts with the inverse frame field (ifri) to form the frame metric (ifg).

ifri [Variable]

The inverse frame field. Specifies the frame base (dual basis vectors). Along with the frame metric, it forms the basis of all calculations based on frames.

ifg [Variable]

The frame metric. Defaults to kdelta, but can be changed using components.

ifgi [Variable]

The inverse frame metric. Contracts with the frame metric (ifg) to kdelta.

iframe_bracket_form [Option variable]

Default value: true

Specifies how the frame bracket (ifb) is computed.

71.2.6 Torsion and nonmetricity

Maxima can now take into account torsion and nonmetricity. When the flag itorsion_flag is set to true, the contribution of torsion is added to the connection coefficients. Similarly, when the flag inonmet_flag is true, nonmetricity components are included.

inm [Variable]

The nonmetricity vector. Conformal nonmetricity is defined through the covariant derivative of the metric tensor. Normally zero, the metric tensor's covariant derivative will evaluate to the following when inonmet_flag is set to true:

$$g_{ij;k} =- g_{ij}\,\mathrm{inm}_{k}$$

inmc1 [Variable]

Covariant permutation of the nonmetricity vector components. Defined as

```
                g   inm  - inm  g   - g   inm
                 ab    c      a  bc    ac    b
inmc1    = ------------------------------
     abc                 2
```

(Substitute `ifg` in place of `g` if a frame metric is used.)

`inmc2` [Variable]

Contravariant permutation of the nonmetricity vector components. Used in the connection coefficients if `inonmet_flag` is `true`. Defined as:

```
                          c         c         cd
              -inm  kdelta  - kdelta  inm  + g   inm  g
     c         a       b         a    b          d  ab
inmc2   = ------------------------------------------
     ab                       2
```

(Substitute `ifg` in place of `g` if a frame metric is used.)

`ikt1` [Variable]

Covariant permutation of the torsion tensor (also known as contorsion). Defined as:

```
                  d           d       d
              -g   itr  - g    itr   - itr   g
                ad    cb    bd    ca      ab  cd
ikt1    = ---------------------------------
    abc                  2
```

(Substitute `ifg` in place of `g` if a frame metric is used.)

`ikt2` [Variable]

Contravariant permutation of the torsion tensor (also known as contorsion). Defined as:

```
    c     cd
ikt2   = g   ikt1
    ab           abd
```

(Substitute `ifg` in place of `g` if a frame metric is used.)

`itr` [Variable]

The torsion tensor. For a metric with torsion, repeated covariant differentiation on a scalar function will not commute, as demonstrated by the following example:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
```

```
(%i2) imetric:g;
(%o2)                                  g
(%i3) covdiff( covdiff( f( [], []), i), j)
                      - covdiff( covdiff( f( [], []), j), i)$
(%i4) ishow(%)$
                                     %4              %2
(%t4)                     f    ichr2    - f    ichr2
                           ,%4      j i    ,%2      i j
(%i5) canform(%);
(%o5)                                  0
(%i6) itorsion_flag:true;
(%o6)                                true
(%i7) covdiff( covdiff( f( [], []), i), j)
                      - covdiff( covdiff( f( [], []), j), i)$
(%i8) ishow(%)$
                            %8             %6
(%t8)              f    icc2    - f    icc2    - f     + f
                    ,%8     j i    ,%6     i j    ,j i    ,i j
(%i9) ishow(canform(%))$
                                     %1             %1
(%t9)                      f    icc2    - f    icc2
                            ,%1     j i    ,%1     i j
(%i10) ishow(canform(ev(%,icc2)))$
                                     %1             %1
(%t10)                     f    ikt2    - f    ikt2
                            ,%1     i j    ,%1     j i
(%i11) ishow(canform(ev(%,ikt2)))$
                       %2 %1                     %2 %1
(%t11)           f    g      ikt1       - f    g      ikt1
                  ,%2            i j %1    ,%2            j i %1
(%i12) ishow(factor(canform(rename(expand(ev(%,ikt1))))))$
                            %3 %2            %1       %1
                       f    g      g      (itr    - itr   )
                        ,%3         %2 %1     j i      i j
(%t12)                 ------------------------------------
                                     2
(%i13) decsym(itr,2,1,[anti(all)],[]);
(%o13)                               done
(%i14) defcon(g,g,kdelta);
(%o14)                               done
(%i15) subst(g,nounify(g),%th(3))$
(%i16) ishow(canform(contract(%)))$
                                          %1
(%t16)                           - f    itr
                                    ,%1    i j
```

71.2.7 Exterior algebra

The `itensor` package can perform operations on totally antisymmetric covariant tensor fields. A totally antisymmetric tensor field of rank (0,L) corresponds with a differential L-form. On these objects, a multiplication operation known as the exterior product, or wedge product, is defined.

Unfortunately, not all authors agree on the definition of the wedge product. Some authors prefer a definition that corresponds with the notion of antisymmetrization: in these works, the wedge product of two vector fields, for instance, would be defined as

```
                a a  - a a
                 i j    j i
a  /\ a  = -----------
 i     j         2
```

More generally, the product of a p-form and a q-form would be defined as

```
                          1     k1..kp l1..lq
A       /\ B       = ------ D               A       B
 i1..ip     j1..jq   (p+q)!  i1..ip j1..jq  k1..kp  l1..lq
```

where `D` stands for the Kronecker-delta.

Other authors, however, prefer a "geometric" definition that corresponds with the notion of the volume element:

```
a  /\ a  = a a  - a a
 i     j    i j    j i
```

and, in the general case

```
                          1    k1..kp l1..lq
A       /\ B       = ----- D              A       B
 i1..ip     j1..jq   p! q!  i1..ip j1..jq  k1..kp  l1..lq
```

Since `itensor` is a tensor algebra package, the first of these two definitions appears to be the more natural one. Many applications, however, utilize the second definition. To resolve this dilemma, a flag has been implemented that controls the behavior of the wedge product: if `igeowedge_flag` is `false` (the default), the first, `"tensorial"` definition is used, otherwise the second, `"geometric"` definition will be applied.

`~` [Operator]

The wedge product operator is denoted by the tilde `~`. This is a binary operator. Its arguments should be expressions involving scalars, covariant tensors of rank one, or covariant tensors of rank `1` that have been declared antisymmetric in all covariant indices.

The behavior of the wedge product operator is controlled by the `igeowedge_flag` flag, as in the following example:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) ishow(a([i])~b([j]))$
                                    a  b  - b  a
                                     i  j    i  j
(%t2)                               -------------
```

```
                                          2
(%i3) decsym(a,2,0,[anti(all)],[]);
(%o3)                                done
(%i4) ishow(a([i,j])~b([k]))$
                          a    b  + b  a    - a    b
                           i j  k    i  j k    i k  j
(%t4)                     --------------------------
                                       3
(%i5) igeowedge_flag:true;
(%o5)                                true
(%i6) ishow(a([i])~b([j]))$
(%t6)                              a  b  - b  a
                                    i  j    i  j
(%i7) ishow(a([i,j])~b([k]))$
(%t7)                     a    b  + b  a    - a    b
                           i j  k    i  j k    i k  j
```

| [Operator]

The vertical bar | denotes the "contraction with a vector" binary operation. When a totally antisymmetric covariant tensor is contracted with a contravariant vector, the result is the same regardless which index was used for the contraction. Thus, it is possible to define the contraction operation in an index-free manner.

In the `itensor` package, contraction with a vector is always carried out with respect to the first index in the literal sorting order. This ensures better simplification of expressions involving the | operator. For instance:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) decsym(a,2,0,[anti(all)],[]);
(%o2)                                done
(%i3) ishow(a([i,j],[])|v)$
                                     %1
(%t3)                               v   a
                                         %1 j
(%i4) ishow(a([j,i],[])|v)$
                                    %1
(%t4)                             - v   a
                                        %1 j
```

Note that it is essential that the tensors used with the | operator be declared totally antisymmetric in their covariant indices. Otherwise, the results will be incorrect.

`extdiff` (*expr*, *i*) [Function]

Computes the exterior derivative of *expr* with respect to the index *i*. The exterior derivative is formally defined as the wedge product of the partial derivative operator and a differential form. As such, this operation is also controlled by the setting of `igeowedge_flag`. For instance:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
```

```
(%i2) ishow(extdiff(v([i]),j))$
                                    v    - v
                                     j,i    i,j
(%t2)                               -----------
                                         2
(%i3) decsym(a,2,0,[anti(all)],[]);
(%o3)                                  done
(%i4) ishow(extdiff(a([i,j]),k))$
                              a      - a      + a
                               j k,i    i k,j    i j,k
(%t4)                         ------------------------
                                         3
(%i5) igeowedge_flag:true;
(%o5)                                  true
(%i6) ishow(extdiff(v([i]),j))$
(%t6)                               v    - v
                                     j,i    i,j
(%i7) ishow(extdiff(a([i,j]),k))$
(%t7)                     - (a      - a      + a     )
                             k j,i    k i,j    j i,k
```

hodge (*expr*) [Function]

Compute the Hodge-dual of *expr*. For instance:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) imetric(g);
(%o2)                             done
(%i3) idim(4);
(%o3)                             done
(%i4) icounter:100;
(%o4)                              100
(%i5) decsym(A,3,0,[anti(all)],[])$

(%i6) ishow(A([i,j,k],[]))$
(%t6)                            A
                                  i j k
(%i7) ishow(canform(hodge(%)))$
                            %1 %2 %3 %4
               levi_civita            g        A
                                       %1 %102  %2 %3 %4
(%t7)          ----------------------------------------
                                  6
(%i8) ishow(canform(hodge(%)))$
                  %1 %2 %3 %8            %4 %5 %6 %7
(%t8) levi_civita            levi_civita            g
```

```
                                                        %1 %106
                                 g        g        g      A          /6
                                  %2 %107  %3 %108  %4 %8  %5 %6 %7
(%i9) lc2kdt(%)$

(%i10) %,kdelta$

(%i11) ishow(canform(contract(expand(%))))$
(%t11)                      - A
                                %106 %107 %108
```

`igeowedge_flag` [Option variable]

Default value: `false`

Controls the behavior of the wedge product and exterior derivative. When set to `false` (the default), the notion of differential forms will correspond with that of a totally antisymmetric covariant tensor field. When set to `true`, differential forms will agree with the notion of the volume element.

71.2.8 Exporting TeX expressions

The `itensor` package provides limited support for exporting tensor expressions to TeX. Since `itensor` expressions appear as function calls, the regular Maxima `tex` command will not produce the expected output. You can try instead the `tentex` command, which attempts to translate tensor expressions into appropriately indexed TeX objects.

`tentex` (*expr*) [Function]

To use the `tentex` function, you must first load `tentex`, as in the following example:

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) load("tentex");
(%o2)       /share/tensor/tentex.lisp
(%i3) idummyx:m;
(%o3)                                  m
(%i4) ishow(icurvature([j,k,l],[i]))$
            m1       i           m1       i           i
(%t4)  ichr2    ichr2     - ichr2    ichr2     - ichr2
            j k      m1 l        j l      m1 k        j l,k

                                                        i
                                                + ichr2
                                                        j k,l
(%i5) tentex(%)$
$$\Gamma_{j\,k}^{m_1}\,\Gamma_{l\,m_1}^{i}-\Gamma_{j\,l}^{m_1}\,
 \Gamma_{k\,m_1}^{i}-\Gamma_{j\,l,k}^{i}+\Gamma_{j\,k,l}^{i}$$
```

Note the use of the `idummyx` assignment, to avoid the appearance of the percent sign in the TeX expression, which may lead to compile errors.

NB: This version of the `tentex` function is somewhat experimental.

71.2.9 Interfacing with ctensor

The itensor package has the ability to generate Maxima code that can then be executed in the context of the ctensor package. The function that performs this task is ic_convert.

ic_convert (*eqn*) [Function]

Converts the itensor equation *eqn* to a ctensor assignment statement. Implied sums over dummy indices are made explicit while indexed objects are transformed into arrays (the array subscripts are in the order of covariant followed by contravariant indices of the indexed objects). The derivative of an indexed object will be replaced by the noun form of diff taken with respect to ct_coords subscripted by the derivative index. The Christoffel symbols ichr1 and ichr2 will be translated to lcs and mcs, respectively and if metricconvert is true then all occurrences of the metric with two covariant (contravariant) indices will be renamed to lg (ug). In addition, do loops will be introduced summing over all free indices so that the transformed assignment statement can be evaluated by just doing ev. The following examples demonstrate the features of this function.

```
(%i1) load("itensor");
(%o1)      /share/tensor/itensor.lisp
(%i2) eqn:ishow(t([i,j],[k])=f([],[])*g([l,m],[])*a([],[m],j)
      *b([i],[l,k]))$
                              k        m   l k
(%t2)                        t    = f a   b    g
                              i j      ,j  i    l m
(%i3) ic_convert(eqn);
(%o3) for i thru dim do (for j thru dim do (
       for k thru dim do
        t        : f sum(sum(diff(a , ct_coords ) b
         i, j, k                  m           j   i, l, k

 g    , l, 1, dim), m, 1, dim)))
  l, m
(%i4) imetric(g);
(%o4)                                done
(%i5) metricconvert:true;
(%o5)                                true
(%i6) ic_convert(eqn);
(%o6) for i thru dim do (for j thru dim do (
       for k thru dim do
        t        : f sum(sum(diff(a , ct_coords ) b
         i, j, k                  m           j   i, l, k

 lg    , l, 1, dim), m, 1, dim)))
   l, m
```

71.2.10 Reserved words

The following Maxima words are used by the itensor package internally and should not be redefined:

Keyword	Comments
indices2()	Internal version of indices()
conti	Lists contravariant indices
covi	Lists covariant indices of an indexed object
deri	Lists derivative indices of an indexed object
name	Returns the name of an indexed object
concan	
irpmon	
lc0	
_lc2kdt0	
_lcprod	
_extlc	

72 Package lapack

72.1 Introduction to lapack

`lapack` is a Common Lisp translation (via the program `f2cl`) of the Fortran library LAPACK, as obtained from the SLATEC project.

72.2 Functions and Variables for lapack

`dgeev` [Function]

`dgeev (A)`
`dgeev (A, right_p, left_p)`

Computes the eigenvalues and, optionally, the eigenvectors of a matrix A. All elements of A must be integer or floating point numbers. A must be square (same number of rows and columns). A might or might not be symmetric.

To make use of this function, you must load the LaPack package via `load("lapack")`.

`dgeev(A)` computes only the eigenvalues of A. `dgeev(A, right_p, left_p)` computes the eigenvalues of A and the right eigenvectors when *right_p* = `true` and the left eigenvectors when *left_p* = `true`.

A list of three items is returned. The first item is a list of the eigenvalues. The second item is `false` or the matrix of right eigenvectors. The third item is `false` or the matrix of left eigenvectors.

The right eigenvector v_j (the j-th column of the right eigenvector matrix) satisfies

$$\mathbf{A}v_j = \lambda_j v_j$$

where λ_j is the corresponding eigenvalue. The left eigenvector u_j (the j-th column of the left eigenvector matrix) satisfies

$$u_j^{\mathbf{H}}\mathbf{A} = \lambda_j u_j^{\mathbf{H}}$$

where $u_j^{\mathbf{H}}$ denotes the conjugate transpose of u_j. For a Maxima function to compute the conjugate transpose, see [ctranspose], page 1153.

The computed eigenvectors are normalized to have Euclidean norm equal to 1, and largest component has imaginary part equal to zero.

Example:

```
(%i1) load ("lapack")$
(%i2) fpprintprec : 6;
(%o2)                           6
(%i3) M : matrix ([9.5, 1.75], [3.25, 10.45]);
                          [ 9.5   1.75  ]
(%o3)                     [             ]
                          [ 3.25  10.45 ]
(%i4) dgeev (M);
(%o4)          [[7.54331, 12.4067], false, false]
```

```
(%i5) [L, v, u] : dgeev (M, true, true);
                              [ - 0.666642  - 0.515792 ]
(%o5) [[7.54331, 12.4067], [                         ],
                              [  0.745378   - 0.856714 ]
                                         [ - 0.856714  - 0.745378 ]
                                         [                        ]]
                                         [  0.515792   - 0.666642 ]
(%i6) D : apply (diag_matrix, L);
                         [ 7.54331     0    ]
(%o6)                    [                  ]
                         [    0     12.4067 ]
(%i7) M . v - v . D;
                    [      0.0       - 8.88178e-16 ]
(%o7)               [                              ]
                    [ - 8.88178e-16       0.0      ]
(%i8) transpose (u) . M - D . transpose (u);
                        [ 0.0  - 4.44089e-16 ]
(%o8)                   [                    ]
                        [ 0.0       0.0      ]
```

dgeqrf (*A*) [Function]

Computes the QR decomposition of the matrix A. All elements of A must be integer or floating point numbers. A may or may not have the same number of rows and columns.

To make use of this function, you must load the LaPack package via `load("lapack")`.

The real square matrix **A** can be decomposed as

$$\mathbf{A} = \mathbf{QR}$$

where **Q** is a square orthogonal matrix with the same number of rows as **A** and **R** is an upper triangular matrix and is the same size as **A**.

A list of two items is returned. The first item is the matrix **Q**. The second item is the matrix **R**, The product $Q.R$, where "." is the noncommutative multiplication operator, is equal to A (ignoring floating point round-off errors).

```
(%i1) load ("lapack")$
(%i2) fpprintprec : 6;
(%o2)                           6
(%i3) M : matrix ([1, -3.2, 8], [-11, 2.7, 5.9]);
                         [  1    - 3.2   8  ]
(%o3)                    [                  ]
                         [ - 11   2.7   5.9 ]
```

```
(%i4) [q, r] : dgeqrf (M);
       [ - 0.0905357  0.995893  ]
(%o4) [[                        ],
       [  0.995893    0.0905357 ]
                                [ - 11.0454   2.97863   5.15148 ]
                                [                               ]]
                                [    0.0     - 2.94241  8.50131 ]
(%i5) q . r - M;
         [ - 7.77156e-16   1.77636e-15   - 8.88178e-16 ]
(%o5)    [                                            ]
         [      0.0       - 1.33227e-15   8.88178e-16  ]
(%i6) mat_norm (%, 1);
(%o6)                     3.10862e-15
```

`dgesv` (*A*, *b*) [Function]

Computes the solution x of the linear equation $\mathbf{A}x = b$, where $\mathbf{A}$ is a square matrix, and b is a matrix of the same number of rows as $\mathbf{A}$ and any number of columns. The return value x is the same size as b.

To make use of this function, you must load the LaPack package via `load("lapack")`.

The elements of A and b must evaluate to real floating point numbers via `float`; thus elements may be any numeric type, symbolic numerical constants, or expressions which evaluate to floats. The elements of x are always floating point numbers. All arithmetic is carried out as floating point operations.

`dgesv` computes the solution via the LU decomposition of A.

Examples:

`dgesv` computes the solution of the linear equation $Ax = b$.

```
(%i1) load("lapack")$
(%i2) A : matrix ([1, -2.5], [0.375, 5]);
                        [   1    - 2.5 ]
(%o2)                   [              ]
                        [ 0.375    5   ]
(%i3) b : matrix ([1.75], [-0.625]);
                           [  1.75   ]
(%o3)                      [         ]
                           [ - 0.625 ]
(%i4) x : dgesv (A, b);
                     [  1.2105263157894737   ]
(%o4)                [                       ]
                     [ - 0.21578947368421053 ]
(%i5) dlange (inf_norm, b - A . x);
(%o5)                          0.0
```

b is a matrix with the same number of rows as A and any number of columns. x is the same size as b.

```
(%i1) load ("lapack")$
```

```
(%i2) A : matrix ([1, -0.15], [1.82, 2]);
                          [  1    - 0.15 ]
(%o2)                     [              ]
                          [ 1.82    2    ]
(%i3) b : matrix ([3.7, 1, 8], [-2.3, 5, -3.9]);
                         [  3.7   1    8   ]
(%o3)                    [                 ]
                         [ - 2.3  5  - 3.9 ]
(%i4) x : dgesv (A, b);
(%o4)
 [ 3.1038275406951175   1.2098548174219095  6.7817861856577215  ]
 [                                                              ]
 [ - 3.974483062032557  1.3990321161460624  - 8.121425428948527 ]
(%i5) dlange (inf_norm, b - A . x);
(%o5)                1.1102230246251565e-15
```

The elements of A and b must evaluate to real floating point numbers.

```
(%i1) load ("lapack")$
(%i2) A : matrix ([5, -%pi], [1b0, 11/17]);
                          [   5    - %pi ]
                          [              ]
(%o2)                     [         11   ]
                          [ 1.0b0   --   ]
                          [         17   ]
(%i3) b : matrix ([%e], [sin(1)]);
                             [   %e   ]
(%o3)                        [        ]
                             [ sin(1) ]
(%i4) x : dgesv (A, b);
                        [ 0.6903756431559864  ]
(%o4)                   [                     ]
                        [ 0.23351098255295172 ]
(%i5) dlange (inf_norm, b - A . x);
(%o5)                 2.220446049250313e-16
```

`dgesvd` [Function]

`dgesvd` (*A*)
`dgesvd` (*A*, *left_p*, *right_p*)

Computes the singular value decomposition (SVD) of a matrix A, comprising the singular values and, optionally, the left and right singular vectors. All elements of A must be integer or floating point numbers. A might or might not be square (same number of rows and columns).

To make use of this function, you must load the LaPack package via `load("lapack")`.

Let m be the number of rows, and n the number of columns of A. The singular value decomposition of $\mathbf{A}$ comprises three matrices, $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$, such that

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$

where $\mathbf{U}$ is an m-by-m unitary matrix, $\mathbf{\Sigma}$ is an m-by-n diagonal matrix, and $\mathbf{V}$ is an n-by-n unitary matrix.

Let σ_i be a diagonal element of $\mathbf{\Sigma}$, that is, $\mathbf{\Sigma}_{ii} = \sigma_i$. The elements σ_i are the so-called singular values of $\mathbf{A}$; these are real and nonnegative, and returned in descending order. The first $\min(m, n)$ columns of $\mathbf{U}$ and $\mathbf{V}$ are the left and right singular vectors of $\mathbf{A}$. Note that `dgesvd` returns the transpose of $\mathbf{V}$, not $\mathbf{V}$ itself.

`dgesvd(A)` computes only the singular values of A. `dgesvd(A, left_p, right_p)` computes the singular values of A and the left singular vectors when *left_p* = `true` and the right singular vectors when *right_p* = `true`.

A list of three items is returned. The first item is a list of the singular values. The second item is `false` or the matrix of left singular vectors. The third item is `false` or the matrix of right singular vectors.

Example:

```
(%i1) load ("lapack")$
(%i2) fpprintprec : 6;
(%o2)                           6
(%i3) M: matrix([1, 2, 3], [3.5, 0.5, 8], [-1, 2, -3], [4, 9, 7]);
                        [  1    2    3  ]
                        [               ]
                        [ 3.5  0.5   8  ]
(%o3)                   [               ]
                        [ - 1   2   - 3 ]
                        [               ]
                        [  4    9    7  ]
(%i4) dgesvd (M);
(%o4)     [[14.4744, 6.38637, 0.452547], false, false]
(%i5) [sigma, U, VT] : dgesvd (M, true, true);
(%o5) [[14.4744, 6.38637, 0.452547],
[ - 0.256731  0.00816168   0.959029    - 0.119523 ]
[                                                ]
[ - 0.526456   0.672116   - 0.206236   - 0.478091 ]
[                                                ],
[  0.107997   - 0.532278  - 0.0708315  - 0.83666  ]
[                                                ]
[ - 0.803287  - 0.514659  - 0.180867    0.239046  ]
[ - 0.374486  - 0.538209  - 0.755044 ]
[                                    ]
[  0.130623   - 0.836799    0.5317   ]]
[                                    ]
[ - 0.917986   0.100488    0.383672  ]
(%i6) m : length (U);
(%o6)                           4
(%i7) n : length (VT);
(%o7)                           3
```

```
(%i8) Sigma:
  genmatrix(lambda ([i, j], if i=j then sigma[i] else 0),
            m, n);
                 [ 14.4744     0        0     ]
                 [                             ]
                 [    0     6.38637     0     ]
(%o8)            [                             ]
                 [    0        0     0.452547 ]
                 [                             ]
                 [    0        0        0     ]
(%i9) U . Sigma . VT - M;
           [  1.11022e-15        0.0       1.77636e-15 ]
           [                                           ]
           [  1.33227e-15    1.66533e-15       0.0     ]
(%o9)      [                                           ]
           [ - 4.44089e-16  - 8.88178e-16  4.44089e-16 ]
           [                                           ]
           [  8.88178e-16    1.77636e-15   8.88178e-16 ]
(%i10) transpose (U) . U;
       [     1.0      5.55112e-17    2.498e-16     2.77556e-17  ]
       [                                                        ]
       [ 5.55112e-17      1.0       5.55112e-17    4.16334e-17  ]
(%o10) [                                                        ]
       [  2.498e-16   5.55112e-17       1.0       - 2.08167e-16 ]
       [                                                        ]
       [ 2.77556e-17  4.16334e-17  - 2.08167e-16       1.0      ]
(%i11) VT . transpose (VT);
           [      1.0           0.0      - 5.55112e-17 ]
           [                                           ]
(%o11)     [      0.0           1.0       5.55112e-17  ]
           [                                           ]
           [ - 5.55112e-17  5.55112e-17       1.0      ]
```

`dlange` (*norm*, *A*) [Function]
`zlange` (*norm*, *A*) [Function]

Computes a norm or norm-like function of the matrix A. If A is a real matrix, use `dlange`. For a matrix with complex elements, use `zlange`.

To make use of this function, you must load the LaPack package via `load("lapack")`.

`norm` specifies the kind of norm to be computed:

`max` Compute $\max(|\mathbf{A}_{ij}|)$ where i and j range over the rows and columns, respectively, of $\mathbf{A}$. Note that this function is not a proper matrix norm.

`one_norm` Compute the L_1 norm of $\mathbf{A}$, that is, the maximum of the sum of the absolute value of elements in each column.

`inf_norm` Compute the L_∞ norm of $\mathbf{A}$, that is, the maximum of the sum of the absolute value of elements in each row.

`frobenius`
: Compute the Frobenius norm of **A**, that is, the square root of the sum of squares of the matrix elements.

`dgemm` [Function]

`dgemm` (*A*, *B*)
`dgemm` (*A*, *B*, *options*)

Compute the product of two matrices and optionally add the product to a third matrix.

In the simplest form, `dgemm(A, B)` computes the product of the two real matrices, *A* and *B*.

To make use of this function, you must load the LaPack package via `load("lapack")`.

In the second form, `dgemm` computes $\alpha\mathbf{AB} + \beta\mathbf{C}$ where **A**, **B**, and **C** are real matrices of the appropriate sizes and α and β are real numbers. Optionally, **A** and/or **B** can be transposed before computing the product. The extra parameters are specified by optional keyword arguments: The keyword arguments are optional and may be specified in any order. They all take the form `key=val`. The keyword arguments are:

`C`
: The matrix **C** that should be added. The default is `false`, which means no matrix is added.

`alpha`
: The product of **A** and **B** is multiplied by this value. The default is 1.

`beta`
: If a matrix **C** is given, this value multiplies **C** before it is added. The default value is 0, which implies that **C** is not added, even if **C** is given. Hence, be sure to specify a non-zero value for β.

`transpose_a`
: If `true`, the transpose of **A** is used instead of **A** for the product. The default is `false`.

`transpose_b`
: If `true`, the transpose of **B** is used instead of **B** for the product. The default is `false`.

```
(%i1) load ("lapack")$
(%i2) A : matrix([1,2,3],[4,5,6],[7,8,9]);
                            [ 1  2  3 ]
                            [         ]
(%o2)                       [ 4  5  6 ]
                            [         ]
                            [ 7  8  9 ]
(%i3) B : matrix([-1,-2,-3],[-4,-5,-6],[-7,-8,-9]);
                           [ - 1  - 2  - 3 ]
                           [               ]
(%o3)                      [ - 4  - 5  - 6 ]
                           [               ]
                           [ - 7  - 8  - 9 ]
```

```
(%i4) C : matrix([3,2,1],[6,5,4],[9,8,7]);
                            [ 3  2  1 ]
                            [         ]
(%o4)                       [ 6  5  4 ]
                            [         ]
                            [ 9  8  7 ]
(%i5) dgemm(A,B);
                   [ - 30.0   - 36.0   - 42.0  ]
                   [                           ]
(%o5)              [ - 66.0   - 81.0   - 96.0  ]
                   [                           ]
                   [ - 102.0  - 126.0  - 150.0 ]
(%i6) A . B;
                      [ - 30   - 36   - 42  ]
                      [                     ]
(%o6)                 [ - 66   - 81   - 96  ]
                      [                     ]
                      [ - 102  - 126  - 150 ]
(%i7) dgemm(A,B,transpose_a=true);
                   [ - 66.0  - 78.0   - 90.0  ]
                   [                          ]
(%o7)              [ - 78.0  - 93.0   - 108.0 ]
                   [                          ]
                   [ - 90.0  - 108.0  - 126.0 ]
(%i8) transpose(A) . B;
                      [ - 66  - 78   - 90  ]
                      [                    ]
(%o8)                 [ - 78  - 93   - 108 ]
                      [                    ]
                      [ - 90  - 108  - 126 ]
(%i9) dgemm(A,B,c=C,beta=1);
                   [ - 27.0  - 34.0   - 41.0  ]
                   [                          ]
(%o9)              [ - 60.0  - 76.0   - 92.0  ]
                   [                          ]
                   [ - 93.0  - 118.0  - 143.0 ]
(%i10) A . B + C;
                      [ - 27  - 34   - 41  ]
                      [                    ]
(%o10)                [ - 60  - 76   - 92  ]
                      [                    ]
                      [ - 93  - 118  - 143 ]
```

```
(%i11) dgemm(A,B,c=C,beta=1, alpha=-1);
                        [ 33.0   38.0   43.0  ]
                        [                     ]
(%o11)                  [ 72.0   86.0   100.0 ]
                        [                     ]
                        [ 111.0  134.0  157.0 ]
(%i12) -A . B + C;
                           [ 33   38   43  ]
                           [               ]
(%o12)                     [ 72   86   100 ]
                           [               ]
                           [ 111  134  157 ]
```

zgeev [Function]

zgeev (*A*)
zgeev (*A*, *right_p*, *left_p*)

Like dgeev, but the matrix **A** is complex.

To make use of this function, you must load the LaPack package via `load("lapack")`.

zheev [Function]

zheev (*A*)
zheev (*A*, *eigvec_p*)

Like dgeev, but the matrix **A** is assumed to be a square complex Hermitian matrix. If *eigvec_p* is `true`, then the eigenvectors of the matrix are also computed.

To make use of this function, you must load the LaPack package via `load("lapack")`.

No check is made that the matrix **A** is, in fact, Hermitian.

A list of two items is returned, as in `dgeev`: a list of eigenvalues, and `false` or the matrix of the eigenvectors.

```
(%i1) load("lapack")$
```

```
(%i2) M: matrix(
     [9.14 +%i*0.00 ,   -4.37 -%i*9.22 ,  -1.98 -%i*1.72 ,  -8.96 -%i*9.50],
     [-4.37 +%i*9.22 ,  -3.35 +%i*0.00 ,   2.25 -%i*9.51 ,   2.57 +%i*2.40],
     [-1.98 +%i*1.72 ,   2.25 +%i*9.51 ,  -4.82 +%i*0.00 ,  -3.24 +%i*2.04],
     [-8.96 +%i*9.50 ,   2.57 -%i*2.40 ,  -3.24 -%i*2.04 ,   8.44 +%i*0.00]);
               [      9.14      ]          [ - 9.22 %i - 4.37 ]
               [                ]          [                  ]
               [ 9.22 %i - 4.37 ]          [      - 3.35      ]
(%o2)  Col 1 = [                ] Col 2 = [                  ]
               [ 1.72 %i - 1.98 ]          [  9.51 %i + 2.25  ]
               [                ]          [                  ]
               [ 9.5 %i - 8.96  ]          [  2.57 - 2.4 %i   ]
                 [ - 1.72 %i - 1.98 ]          [ - 9.5 %i - 8.96 ]
                 [                  ]          [                 ]
                 [  2.25 - 9.51 %i  ]          [  2.4 %i + 2.57  ]
          Col 3 = [                  ] Col 4 = [                 ]
                 [      - 4.82      ]          [ 2.04 %i - 3.24  ]
                 [                  ]          [                 ]
                 [ - 2.04 %i - 3.24 ]          [      8.44       ]
(%i3) zheev(M);
(%o3) [[- 16.004746472094734, - 6.764970154793324,
                  6.6657114535070985, 25.51400517338097], false]
(%i4) E: zheev(M,true)$
(%i5) E[1];
(%o5) [- 16.004746472094737, - 6.764970154793325,
                          6.665711453507101, 25.514005173380962]
```

```
(%i6) E[2];
                [  0.26746505331727455 %i + 0.21754535866650165  ]
                [                                                ]
                [  0.002696730886619885 %i + 0.6968836773391712  ]
(%o6)  Col 1 = [                                                ]
                [ - 0.6082406376714117 %i - 0.012106142926979313 ]
                [                                                ]
                [              0.15930818580950368               ]
          [  0.26449374706674444 %i + 0.4773693349937472   ]
          [                                                ]
          [ - 0.28523890360316206 %i - 0.14143627420116733 ]
 Col 2 = [                                                ]
          [  0.2654607680986639 %i + 0.44678181171841735   ]
          [                                                ]
          [               0.5750762708542709               ]
          [  0.28106497673059216 %i - 0.13352639282451817  ]
          [                                                ]
          [  0.28663101328695556 %i - 0.4536971347853274   ]
 Col 3 = [                                                ]
          [ - 0.29336843237542953 %i - 0.49549724255410565 ]
          [                                                ]
          [               0.5325337537576771               ]
          [ - 0.5737316575503476 %i - 0.39661467994277055 ]
          [                                               ]
          [ 0.018265026190214573 %i + 0.35305577043870173 ]
 Col 4 = [                                               ]
          [ 0.16737009000854253 %i + 0.01476684746229564  ]
          [                                               ]
          [              0.6002632636961784               ]
```

73 Package lbfgs

73.1 Introduction to lbfgs

`lbfgs` is an implementation of the L-BFGS algorithm [1] to solve unconstrained minimization problems via a limited-memory quasi-Newton (BFGS) algorithm. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. The program was originally written in Fortran [2] by Jorge Nocedal, incorporating some functions originally written by Jorge J. Moré and David J. Thuente, and translated into Lisp automatically via the program `f2cl`. The Maxima package `lbfgs` comprises the translated code plus an interface function which manages some details.

References:

[1] D. Liu and J. Nocedal. "On the limited memory BFGS method for large scale optimization". *Mathematical Programming B* 45:503–528 (1989)

[2] https://www.netlib.org/opt/lbfgs_um.shar

73.2 Functions and Variables for lbfgs

`lbfgs` [Function]

`lbfgs` (*FOM*, *X*, *X0*, *epsilon*, *iprint*)
`lbfgs` ([*FOM*, *grad*] *X*, *X0*, *epsilon*, *iprint*)

Finds an approximate solution of the unconstrained minimization of the figure of merit *FOM* over the list of variables *X*, starting from initial estimates *X0*, such that $norm(grad(FOM)) < epsilon * max(1, norm(X))$.

grad, if present, is the gradient of *FOM* with respect to the variables *X*. *grad* may be a list or a function that returns a list, with one element for each element of *X*. If not present, the gradient is computed automatically by symbolic differentiation. If *FOM* is a function, the gradient *grad* must be supplied by the user.

The algorithm applied is a limited-memory quasi-Newton (BFGS) algorithm [1]. It is called a limited-memory method because a low-rank approximation of the Hessian matrix inverse is stored instead of the entire Hessian inverse. Each iteration of the algorithm is a line search, that is, a search along a ray in the variables *X*, with the search direction computed from the approximate Hessian inverse. The FOM is always decreased by a successful line search. Usually (but not always) the norm of the gradient of FOM also decreases.

iprint controls progress messages printed by `lbfgs`.

`iprint[1]`

iprint`[1]` controls the frequency of progress messages.

`iprint[1] < 0`

No progress messages.

`iprint[1] = 0`

Messages at the first and last iterations.

`iprint[1] > 0`
Print a message every *iprint*`[1]` iterations.

`iprint[2]`
iprint`[2]` controls the verbosity of progress messages.

`iprint[2] = 0`
Print out iteration count, number of evaluations of *FOM*, value of *FOM*, norm of the gradient of *FOM*, and step length.

`iprint[2] = 1`
Same as *iprint*`[2] = 0`, plus *X0* and the gradient of *FOM* evaluated at *X0*.

`iprint[2] = 2`
Same as *iprint*`[2] = 1`, plus values of *X* at each iteration.

`iprint[2] = 3`
Same as *iprint*`[2] = 2`, plus the gradient of *FOM* at each iteration.

The columns printed by `lbfgs` are the following.

`I` Number of iterations. It is incremented for each line search.

`NFN` Number of evaluations of the figure of merit.

`FUNC` Value of the figure of merit at the end of the most recent line search.

`GNORM` Norm of the gradient of the figure of merit at the end of the most recent line search.

`STEPLENGTH`
An internal parameter of the search algorithm.

Additional information concerning details of the algorithm are found in the comments of the original Fortran code [2].

See also `lbfgs_nfeval_max` and `lbfgs_ncorrections`.

References:

[1] D. Liu and J. Nocedal. "On the limited memory BFGS method for large scale optimization". *Mathematical Programming B* 45:503–528 (1989)

[2] https://www.netlib.org/opt/lbfgs_um.shar

Examples:

The same FOM as computed by FGCOMPUTE in the program sdrive.f in the LBFGS package from Netlib. Note that the variables in question are subscripted variables. The FOM has an exact minimum equal to zero at $u[k] = 1$ for $k = 1, ..., 8$.

```
(%i1) load ("lbfgs")$
(%i2) t1[j] := 1 - u[j];
(%o2)                       t1  := 1 - u
                              j         j
(%i3) t2[j] := 10*(u[j + 1] - u[j]^2);
                                                  2
```

```
(%o3)                t2  := 10 (u      - u )
                       j         j + 1    j
(%i4) n : 8;
(%o4)                           8
(%i5) FOM : sum (t1[2*j - 1]^2 + t2[2*j - 1]^2, j, 1, n/2);
                    2 2          2             2 2          2
(%o5) 100 (u  - u )  + (1 - u )  + 100 (u  - u )  + (1 - u )
           8    7           7           6    5           5
                        2 2           2             2 2           2
        + 100 (u  - u )  + (1 - u )  + 100 (u  - u )  + (1 - u )
               4    3           3           2    1           1
(%i6) lbfgs (FOM, '[u[1],u[2],u[3],u[4],u[5],u[6],u[7],u[8]],
       [-1.2, 1, -1.2, 1, -1.2, 1, -1.2, 1], 1e-3, [1, 0]);
************************************************
  N=    8   NUMBER OF CORRECTIONS=25
        INITIAL VALUES
 F=  9.680000000000000D+01   GNORM=  4.657353755084533D+02
************************************************
```

I	NFN	FUNC	GNORM	STEPLENGTH
1	3	1.651479526340304D+01	4.324359291335977D+00	7.926153934390631D-04
2	4	1.650209316638371D+01	3.575788161060007D+00	1.000000000000000D+00
3	5	1.645461701312851D+01	6.230869903601577D+00	1.000000000000000D+00
4	6	1.636867301275588D+01	1.177589920974980D+01	1.000000000000000D+00
5	7	1.612153014409201D+01	2.292797147151288D+01	1.000000000000000D+00
6	8	1.569118407390628D+01	3.687447158775571D+01	1.000000000000000D+00
7	9	1.510361958398942D+01	4.501931728123680D+01	1.000000000000000D+00
8	10	1.391077875774294D+01	4.526061463810632D+01	1.000000000000000D+00
9	11	1.165625686278198D+01	2.748348965356917D+01	1.000000000000000D+00
10	12	9.859422687859137D+00	2.111494974231644D+01	1.000000000000000D+00
11	13	7.815442521732281D+00	6.110762325766556D+00	1.000000000000000D+00
12	15	7.346380905773160D+00	2.165281166714631D+01	1.285316401779533D-01
13	16	6.330460634066370D+00	1.401220851762050D+01	1.000000000000000D+00
14	17	5.238763939851439D+00	1.702473787613255D+01	1.000000000000000D+00
15	18	3.754016790406701D+00	7.981845727704576D+00	1.000000000000000D+00
16	20	3.001238402309352D+00	3.925482944716691D+00	2.333129631296807D-01
17	22	2.794390709718290D+00	8.243329982546473D+00	2.503577283782332D-01
18	23	2.563783562918759D+00	1.035413426521790D+01	1.000000000000000D+00
19	24	2.019429976377856D+00	1.065187312346769D+01	1.000000000000000D+00
20	25	1.428003167670903D+00	2.475962450826961D+00	1.000000000000000D+00
21	27	1.197874264861340D+00	8.441707983493810D+00	4.303451060808756D-01
22	28	9.023848941942773D-01	1.113189216635162D+01	1.000000000000000D+00
23	29	5.508226405863770D-01	2.380830600326308D+00	1.000000000000000D+00
24	31	3.902893258815567D-01	5.625595816584421D+00	4.834988416524465D-01
25	32	3.207542206990315D-01	1.149444645416472D+01	1.000000000000000D+00
26	33	1.874468266362791D-01	3.632482152880997D+00	1.000000000000000D+00
27	34	9.575763380706598D-02	4.816497446154354D+00	1.000000000000000D+00

```
28   35  4.085145107543406D-02  2.087009350166495D+00  1.000000000000000D+00
29   36  1.931106001379290D-02  3.886818608498966D+00  1.000000000000000D+00
30   37  6.894000721499670D-03  3.198505796342214D+00  1.000000000000000D+00
31   38  1.443296033051864D-03  1.590265471025043D+00  1.000000000000000D+00
32   39  1.571766603154336D-04  3.098257063980634D-01  1.000000000000000D+00
33   40  1.288011776581970D-05  1.207784183577257D-02  1.000000000000000D+00
34   41  1.806140173752971D-06  4.587890233385193D-02  1.000000000000000D+00
35   42  1.769004645459358D-07  1.790537375052208D-02  1.000000000000000D+00
36   43  3.312164100763217D-10  6.782068426119681D-04  1.000000000000000D+00

 THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
 IFLAG = 0
(%o6) [u  = 1.000005339816132, u  = 1.000009942840108,
        1                        2
u  = 1.000005339816132, u  = 1.000009942840108,
 3                       4
u  = 1.000005339816132, u  = 1.000009942840108,
 5                       6
u  = 1.000005339816132, u  = 1.000009942840108]
 7                       8
```

A regression problem. The FOM is the mean square difference between the predicted value $F(X[i])$ and the observed value $Y[i]$. The function F is a bounded monotone function (a so-called "sigmoidal" function). In this example, `lbfgs` computes approximate values for the parameters of F and `plot2d` displays a comparison of F with the observed data.

```
(%i1) load ("lbfgs")$
(%i2) FOM : '((1/length(X))*sum((F(X[i]) - Y[i])^2, i, 1,
                                                    length(X)));
                                  2
                 sum((F(X ) - Y ) , i, 1, length(X))
                         i     i
(%o2)            ----------------------------------
                              length(X)
(%i3) X : [1, 2, 3, 4, 5];
(%o3)                     [1, 2, 3, 4, 5]
(%i4) Y : [0, 0.5, 1, 1.25, 1.5];
(%o4)                 [0, 0.5, 1, 1.25, 1.5]
(%i5) F(x) := A/(1 + exp(-B*(x - C)));
                                     A
(%o5)             F(x) := ----------------------
                          1 + exp((- B) (x - C))
(%i6) ''FOM;
                  A               2            A                2
(%o6) ((----------------- - 1.5)  + (----------------- - 1.25)
          - B (5 - C)                  - B (4 - C)
         %e            + 1            %e            + 1
```

```
                A              2            A               2
 + (----------------- - 1)  + (----------------- - 0.5)
      - B (3 - C)                - B (2 - C)
    %e            + 1          %e            + 1
                2
                A
 + --------------------)/5
      - B (1 - C)     2
   (%e            + 1)
(%i7) estimates : lbfgs (FOM, '[A, B, C], [1, 1, 1], 1e-4, [1, 0]);
************************************************
  N=    3   NUMBER OF CORRECTIONS=25
       INITIAL VALUES
 F=  1.348738534246918D-01   GNORM=  2.000215531936760D-01
************************************************

I  NFN  FUNC                   GNORM                  STEPLENGTH

1    3  1.177820636622582D-01  9.893138394953992D-02  8.554435968992371D-01
2    6  2.302653892214013D-02  1.180098521565904D-01  2.100000000000000D+01
3    8  1.496348495303004D-02  9.611201567691624D-02  5.257340567840710D-01
4    9  7.900460841091138D-03  1.325041647391314D-02  1.000000000000000D+00
5   10  7.314495451266914D-03  1.510670810312226D-02  1.000000000000000D+00
6   11  6.750147275936668D-03  1.914964958023037D-02  1.000000000000000D+00
7   12  5.850716021108202D-03  1.028089194579382D-02  1.000000000000000D+00
8   13  5.778664230657800D-03  3.676866074532179D-04  1.000000000000000D+00
9   14  5.777818823650780D-03  3.010740179797108D-04  1.000000000000000D+00

 THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
 IFLAG = 0
(%o7) [A = 1.461933911464101, B = 1.601593973254801,
                                                C = 2.528933072164855]
(%i8) plot2d ([F(x), [discrete, X, Y]], [x, -1, 6]), ''estimates;
(%o8)
```

Gradient of FOM is specified (instead of computing it automatically). Both the FOM and its gradient are passed as functions to `lbfgs`.

```
(%i1) load ("lbfgs")$
(%i2) F(a, b, c) := (a - 5)^2 + (b - 3)^4 + (c - 2)^6$
(%i3) define(F_grad(a, b, c),
             map (lambda ([x], diff (F(a, b, c), x)), [a, b, c]))$
(%i4) estimates : lbfgs ([F, F_grad],
                   [a, b, c], [0, 0, 0], 1e-4, [1, 0]);
************************************************
  N=    3   NUMBER OF CORRECTIONS=25
       INITIAL VALUES
 F=  1.700000000000000D+02   GNORM=  2.205175729958953D+02
```

```
          ************************************************

 I  NFN  FUNC                    GNORM                   STEPLENGTH

 1    2  6.632967565917637D+01  6.498411132518770D+01  4.534785987412505D-03
 2    3  4.368890936228036D+01  3.784147651974131D+01  1.000000000000000D+00
 3    4  2.685298972775191D+01  1.640262125898520D+01  1.000000000000000D+00
 4    5  1.909064767659852D+01  9.733664001790506D+00  1.000000000000000D+00
 5    6  1.006493272061515D+01  6.344808151880209D+00  1.000000000000000D+00
 6    7  1.215263596054292D+00  2.204727876126877D+00  1.000000000000000D+00
 7    8  1.080252896385329D-02  1.431637116951845D-01  1.000000000000000D+00
 8    9  8.407195124830860D-03  1.126344579730008D-01  1.000000000000000D+00
 9   10  5.022091686198525D-03  7.750731829225275D-02  1.000000000000000D+00
10   11  2.277152808939775D-03  5.032810859286796D-02  1.000000000000000D+00
11   12  6.489384688303218D-04  1.932007150271009D-02  1.000000000000000D+00
12   13  2.075791943844547D-04  6.964319310814365D-03  1.000000000000000D+00
13   14  7.349472666162258D-05  4.017449067849554D-03  1.000000000000000D+00
14   15  2.293617477985238D-05  1.334590390856715D-03  1.000000000000000D+00
15   16  7.683645404048675D-06  6.011057038099202D-04  1.000000000000000D+00

          THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
          IFLAG = 0
         (%o4) [a = 5.000086823042934, b = 3.052395429705181,
                                                 c = 1.927980629919583]
```

`lbfgs_nfeval_max` [Variable]

Default value: 100

`lbfgs_nfeval_max` is the maximum number of evaluations of the figure of merit (FOM) in `lbfgs`. When `lbfgs_nfeval_max` is reached, `lbfgs` returns the result of the last successful line search.

`lbfgs_ncorrections` [Variable]

Default value: 25

`lbfgs_ncorrections` is the number of corrections applied to the approximate inverse Hessian matrix which is maintained by `lbfgs`.

74 Package levin

74.1 Introduction to levin

Package levin provides functions that implement the Levin u-transformation ([Levin 1973], page 1146). This transformation may accelerate convergence of the sum of infinite sequences using a small number of terms.

Methods to accelerate the convergence of infinite series do not usually return the exact sum. There are two major sources of error: truncation error, due to the finite number of terms used; and, rounding error, due to the finite precision arithmetic used in the calculations. Truncation error dominates for a small number of terms (perhaps 5 to 10 terms). Rounding error dominates as the number of terms increases (perhaps above 10 to 20 terms for IEEE-754 double precision arithmetic).

This package has two functions that address these sources of error.

Function `levin_u_sum` (by default) uses rational arithmetic with a fixed number of terms. This eliminates rounding error if the terms are known exactly.

Function `bflevin_u_sum` uses bigfloat precision. It uses error estimates to increase the number of terms and the bigfloat arithmetic precision to meet an accuracy target.

See Section 74.3 Examples for levin for examples. See Section 74.4 References for levin for further information.

74.2 Functions and Variables for levin

`levin_u_sum` [Function]

`levin_u_sum (a, n, n_0, nterms, mode)`
`levin_u_sum (a, n, n_0, nterms)`

Estimate `sum(a(n), n, n_0, inf)` using at most *nterms* terms using the Levin u-transform see [Levin 1973], page 1146.

The following values are recognized for the optional argument *mode*. If *mode* is not supplied, it is assumed to be `levin_algebraic`.

`levin_algebraic`
: The calculation is performed in exact arithmetic. `levin_u_sum` returns the result.

`levin_numeric`
: The calculation is performed in bigfloat arithmetic. The return value is a list `[result, variance]` where *result* is the result of the bigfloat calculation, and *variance* is in units of `10^(-2*fpprec)`.

`load("levin")` loads this function.

See Section 74.3 Examples for levin for examples.

`bflevin_u_sum (a, n, n_0)` [Function]

Estimate `sum(a(n), n, n_0, inf)` using the Levin u-transform in bigfloat arithmetic.

`bflevin_u_sum` attempts to return the sum of the infinite series with a precision given by the global variable `fpprec` using bigfloat arithmetic.

See levin_options for options to control this function. bflevin_u_sum uses an adaptive algorithm to increase both the number of terms and the bigfloat precision used for internal calculations until the estimated error is acceptable.

load("levin") loads this function.

See Section 74.3 Examples for levin for examples.

levin_options [Variable]

Function bflevin_u_sum attempts to return the sum of an infinite series with a precision given by the global variable fpprec using bigfloat arithmetic. bflevin_u_sum uses an adaptive algorithm to increase both the number of terms used and the bigfloat precision used for internal calculations until the estimated error is acceptable.

The undeclared array levin_options contains options for controlling bflevin_u_sum. Note that the subscript values for levin_options are strings.

levin_options["debug"]
When true, bflevin_u_sum generates additional output. Default: false

levin_options["min_terms"]
Minimum number of terms used by bflevin_u_sum. Default: 5

levin_options["max_terms"]
Maximum number of terms used by bflevin_u_sum. Default: 640 (equal to 5*2^7)

levin_options["min_precision"]
Initial bigfloat precision for bflevin_u_sum. Default: 16

levin_options["max_precision"]
Maximum bigfloat precision for bflevin_u_sum. Default: 1000

74.3 Examples for levin

74.3.1 Example 1: The Basel problem

The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers. Leonhard Euler found, in 1734,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449340668482262...$$

The sum is evaluated using both: levin_u_sum using modes levin_algebraic and levin_numeric; and bflevin_u_sum with two values of bigfloat precision fpprec.

```
(%i1) (load("levin"), exact: %pi^2/6, done);
(%o1)                          done
(%i2) s: levin_u_sum(1/n^2, n, 1, 10);
                             3899836039
(%o2)                        ----------
                             2370816000
(%i3) float(s);
(%o3)                   1.644934081345832
```

```
(%i4) float(s - exact);
(%o4)                  1.4497605782537448e-8
(%i5) s: levin_u_sum(1/n^2, n, 1, 10, 'levin_numeric);
(%o5)     [1.644934081345756b0, 1.882785043290232b-12]
(%i6) bfloat(s[1] - exact);
(%o6)                  1.449752928817105b-8
(%i7) s: bflevin_u_sum(1/n^2, n, 1);
(%o7)                   1.644934066848226b0
(%i8) s - bfloat(exact);
(%o8)                  2.775557561562891b-17
(%i9) /* Now increase fpprec and try the same example again. */
 fpprec: 32;
(%o9)                           32
(%i10) s: bflevin_u_sum(1/n^2, n, 1);
(%o10)         1.6449340668482264364724151666646b0
(%i11) s - bfloat(exact);
(%o11)       - 3.0814879110195773648895647081359b-33
```

Using 10 terms in the series we were able to extrapolate to the exact sum with an error of approximately 10^{-8}. This would require around 10^8 terms by direct summation. In this case the two modes of *levin_u_sum* returned results of similar accuracy.

The effect of the number of terms *nterms* on the accuracy of *levin_u_sum* is shown in the following example. The sum of the series and the approximation error is evaluated for increasing values of argument *nterms* for both values of the optional argument *mode*: *levin_algebraic* and *levin_numeric*. The numeric calculations are performed with bigfloat precision *fpprec* of 16. Errors are calculated with *fpprec* equal to 64.

The results are reported in three columns: *nterms*, the number of terms used; *errora*, the difference between the algebraic result and the exact sum; and *errorn*, the difference between numeric result and the exact sum. For small values of *nterms* the two modes return similar results. As *nterms* increases above 12, the error for mode *levin_algebraic* continues to decrease. However, for mode *levin_numeric* the error increases with *nterms* to due to roundoff. This behavior is commonly encountered, and can be addressed by increasing *fpprec*.

```
(%i1) (load("levin"), fpprec: 32, fpprintprec: 3, exact: %pi^2/6, done);
(%o1)                          done
```

```
(%i2) for nterms: 6 step 2 thru 24 do (
     sa: levin_u_sum(1/n^2, n, 1, nterms),
     sn: block([fpprec: 16], levin_u_sum(1/n^2, n, 1, nterms, 'levin_numeric)),
     errora: abs(bfloat(sa - exact)),
     errorn: abs(bfloat(sn[1] - exact)),
     print(nterms, "     ", errora, "   ", errorn));
6       2.58b-4     2.58b-4
8       3.51b-6     3.51b-6
10       1.45b-8     1.45b-8
12       2.44b-10     2.44b-10
14       5.16b-12     3.78b-11
16       3.54b-14     4.5b-10
18       3.75b-16     3.21b-11
20       1.36b-17     8.66b-9
22       1.73b-19     4.48b-7
24       4.45b-22     4.25b-6
(%o2)                          done
```

74.3.2 Example 2: Divergent Taylor series for log(1+x) when x=2

The Levin u-transformation can find the anti-limit of certain divergent series. In this example we calculate the anti-limit of the divergent Taylor series for log(1+x) when x=2.

$$A_n = -\sum_{m=1}^{n} \frac{(-2)^m}{m}$$

with anti-limit equal to log(3) = 1.0986122886681098...

```
(%i1) (load("levin"), done);
(%o1)                           done
(%i2) levin_u_sum(-(-2)^n/n, n, 1, 10);
                              3641179708
(%o2)                         ----------
                              3314344635
(%i3) s: float(%);
(%o3)                    1.0986122775370342
(%i4) exact: log(3.0);
(%o4)                    1.0986122886681096
(%i5) s - exact;
(%o5)                 - 1.1131075394743561e-8
```

74.4 References for levin

The t, u and v Levin transformations are introduced in:

- (Levin 1973) Levin, David. Development of Non-Linear Transformations for Improving Convergence of Sequences, International Journal of Computer Mathematics 3 (1973): 371–88. doi:10.1080/00207167308803075

- Levin, David, and Avram Sidi. Two New Classes of Nonlinear Transformations for Accelerating the Convergence of Infinite Integrals and Series, Applied Mathematics and Computation 9, no. 3 (October 1981): 175–215. doi:10.1016/0096-3003(81)90028-X

Smith and Ford compared the performance of series acceleration algorithms in a series of papers. In conjunction with Fessler they published algorithms and code that implement the u-transformation with estimates of truncation and round-off errors.

- (Smith and Ford 1979) Smith, David A., and William F. Ford. Acceleration of Linear and Logarithmic Convergence, SIAM Journal on Numerical Analysis 16, no. 2 (1979): 223. doi:10.1137/0716017
- (Smith and Ford 1982) Smith, David A., and William F. Ford. Numerical Comparisons of Nonlinear Convergence Accelerators, Mathematics of Computation 38, no. 158 (1 May 1982): 481–481. doi:10.1090/S0025-5718-1982-0645665-1
- Fessler, Theodore, William F. Ford, and David A. Smith. HURRY: An Acceleration Algorithm for Scalar Sequences and Series. ACM Transactions on Mathematical Software 9, no. 3 (1 September 1983): 346–54. doi:10.1145/356044.356051
- Fessler, Theodore, David A. Smith, and William Ford. Algorithm 602: HURRY: An Acceleration Algorithm for Scalar Sequences and Series. ACM Transactions on Mathematical Software (TOMS) 9, no. 3 (September 1983): 355–57. doi:10.1145/356044.356052

Sidi (2003) discusses u transformations and more recent extensions.

- (Sidi 2003) Sidi, Avram. Practical Extrapolation Methods: Theory and Applications, Cambridge University Press, 2003, ISBN 9780521661591

Bender and Orszag (1978), Chapter 8 Summation of Series, provides general background on motivation and methods, but does not cover Levin transformations.

- (Bender and Orszag, 1978) Bender, Carl, and Steven Orszag. Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, 1978, ISBN 978-0-07-004452-4

75 Package lindstedt

75.1 Functions and Variables for lindstedt

`Lindstedt` (*eq*,*pvar*,*torder*,*ic*) [Function]

This is a first pass at a Lindstedt code. It can solve problems with initial conditions entered, which can be arbitrary constants, (just not *%k1* and *%k2*) where the initial conditions on the perturbation equations are $z[i] = 0, z'[i] = 0$ for $i > 0$. *ic* is the list of initial conditions.

Problems occur when initial conditions are not given, as the constants in the perturbation equations are the same as the zero order equation solution. Also, problems occur when the initial conditions for the perturbation equations are not $z[i] = 0, z'[i] = 0$ for $i > 0$, such as the Van der Pol equation.

Example:

```
(%i1) load("makeOrders")$

(%i2) load("lindstedt")$

(%i3) Lindstedt('diff(x,t,2)+x-(e*x^3)/6,e,2,[1,0]);
           2
          e  (cos(5 T) - 24 cos(3 T) + 23 cos(T))
(%o3) [[[--------------------------------------
                          36864
    e (cos(3 T) - cos(T))
  - --------------------- + cos(T)],
             192
           2
        7 e    e
T = (- ---- - -- + 1) t]]
        3072   16
```

To use this function write first `load("makeOrders")` and `load("lindstedt")`.

76 Package linearalgebra

76.1 Introduction to linearalgebra

`linearalgebra` is a collection of functions for linear algebra.

Example:

```
(%i1) M : matrix ([1, 2], [1, 2]);
                              [ 1  2 ]
(%o1)                         [      ]
                              [ 1  2 ]
(%i2) nullspace (M);
                                 [  1  ]
                                 [     ]
(%o2)                       span([   1 ])
                                 [ - - ]
                                 [   2 ]
(%i3) columnspace (M);
                                  [ 1 ]
(%o3)                          span([   ])
                                  [ 1 ]
(%i4) ptriangularize (M - z*ident(2), z);
                             [ 1   2 - z   ]
(%o4)                        [             ]
                             [           2 ]
                             [ 0  3 z - z  ]
(%i5) M : matrix ([1, 2, 3], [4, 5, 6], [7, 8, 9]) - z*ident(3);
                        [ 1 - z    2      3   ]
                        [                     ]
(%o5)                   [   4    5 - z    6   ]
                        [                     ]
                        [   7      8    9 - z ]
(%i6) MM : ptriangularize (M, z);
                 [ 4  5 - z            6           ]
                 [                                 ]
                 [                2                ]
                 [     66        z    102 z   132   ]
                 [ 0   --      - -- + ----- + ---   ]
(%o6)            [     49        7     49     49    ]
                 [                                 ]
                 [               3        2        ]
                 [           49 z    245 z    147 z ]
                 [ 0    0    ----- - ------ - ----- ]
                 [            264      88      44   ]
(%i7) algebraic : true;
(%o7)                          true
(%i8) tellrat (MM [3, 3]);
```

```
                          3       2
(%o8)                   [z  - 15 z  - 18 z]
(%i9) MM : ratsimp (MM);
               [ 4  5 - z           6          ]
               [                               ]
               [                2              ]
(%o9)          [     66      7 z  - 102 z - 132 ]
               [ 0   --    - ------------------ ]
               [     49              49        ]
               [                               ]
               [ 0    0             0          ]
(%i10) nullspace (MM);
                         [        1         ]
                         [                  ]
                         [   2              ]
                         [  z  - 14 z - 16  ]
                         [  --------------  ]
(%o10)              span([        8         ])
                         [                  ]
                         [    2             ]
                         [   z  - 18 z - 12 ]
                         [ - -------------- ]
                         [         12       ]
(%i11) M : matrix ([1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12],
                   [13, 14, 15, 16]);
                        [ 1   2   3   4  ]
                        [                ]
                        [ 5   6   7   8  ]
(%o11)                  [                ]
                        [ 9   10  11  12 ]
                        [                ]
                        [ 13  14  15  16 ]
(%i12) columnspace (M);
                             [ 1  ]  [ 2  ]
                             [    ]  [    ]
                             [ 5  ]  [ 6  ]
(%o12)                 span([    ], [    ])
                             [ 9  ]  [ 10 ]
                             [    ]  [    ]
                             [ 13 ]  [ 14 ]
(%i13) apply ('orthogonal_complement, args (nullspace (transpose (M))));
                             [ 0 ]  [  1  ]
                             [   ]  [     ]
                             [ 1 ]  [  0  ]
(%o13)                 span([   ], [     ])
                             [ 2 ]  [ - 1 ]
                             [   ]  [     ]
```

[3] [- 2]

76.2 Functions and Variables for linearalgebra

addmatrices (*f*, *M_1*, ..., *M_n*) [Function]

Using the function *f* as the addition function, return the sum of the matrices *M_1*, ..., *M_n*. The function *f* must accept any number of arguments (a Maxima nary function).

Examples:

```
(%i1) m1 : matrix([1,2],[3,4])$
(%i2) m2 : matrix([7,8],[9,10])$
(%i3) addmatrices('max,m1,m2);
(%o3) matrix([7,8],[9,10])
(%i4) addmatrices('max,m1,m2,5*m1);
(%o4) matrix([7,10],[15,20])
```

blockmatrixp (*M*) [Function]

Return true if and only if *M* is a matrix and every entry of *M* is a matrix.

columnop (*M*, *i*, *j*, *theta*) [Function]

If *M* is a matrix, return the matrix that results from doing the column operation `C_i <- C_i - theta * C_j`. If *M* doesn't have a row *i* or *j*, signal an error.

columnswap (*M*, *i*, *j*) [Function]

If *M* is a matrix, swap columns *i* and *j*. If *M* doesn't have a column *i* or *j*, signal an error.

columnspace (*M*) [Function]

If *M* is a matrix, return `span (v_1, ..., v_n)`, where the set `{v_1, ..., v_n}` is a basis for the column space of *M*. The span of the empty set is `{0}`. Thus, when the column space has only one member, return `span ()`.

cholesky [Function]

cholesky (*M*)
cholesky (*M*, *field*)

Return the Cholesky factorization of the matrix selfadjoint (or hermitian) matrix *M*. The second argument defaults to 'generalring.' For a description of the possible values for *field*, see `lu_factor`.

ctranspose (*M*) [Function]

Return the complex conjugate transpose of the matrix *M*. The function `ctranspose` uses `matrix_element_transpose` to transpose each matrix element.

diag_matrix (*d_1*, *d_2*, ..., *d_n*) [Function]

Return a diagonal matrix with diagonal entries *d_1*, *d_2*, ..., *d_n*. When the diagonal entries are matrices, the zero entries of the returned matrix are zero matrices of the appropriate size; for example:

```
(%i1) diag_matrix(diag_matrix(1,2),diag_matrix(3,4));
```

```
                                  [ [ 1  0 ]  [ 0  0 ] ]
                                  [ [      ]  [      ] ]
                                  [ [ 0  2 ]  [ 0  0 ] ]
(%o1)                             [                    ]
                                  [ [ 0  0 ]  [ 3  0 ] ]
                                  [ [      ]  [      ] ]
                                  [ [ 0  0 ]  [ 0  4 ] ]
(%i2) diag_matrix(p,q);

                                        [ p  0 ]
(%o2)                                   [      ]
                                        [ 0  q ]
```

`dotproduct` (*u*, *v*) [Function]

Return the dotproduct of vectors *u* and *v*. This is the same as `conjugate (transpose (u)) . v`. The arguments *u* and *v* must be column vectors.

`eigens_by_jacobi` [Function]

`eigens_by_jacobi` (*A*)
`eigens_by_jacobi` (*A*, *field_type*)

Computes the eigenvalues and eigenvectors of *A* by the method of Jacobi rotations. *A* must be a symmetric matrix (but it need not be positive definite nor positive semidefinite). *field_type* indicates the computational field, either `floatfield` or `bigfloatfield`. If *field_type* is not specified, it defaults to `floatfield`.

The elements of *A* must be numbers or expressions which evaluate to numbers via `float` or `bfloat` (depending on *field_type*).

Examples:

```
(%i1) S: matrix([1/sqrt(2), 1/sqrt(2)],[-1/sqrt(2), 1/sqrt(2)]);
                      [     1         1    ]
                      [  -------   ------- ]
                      [  sqrt(2)   sqrt(2) ]
(%o1)                 [                    ]
                      [      1        1    ]
                      [ - -------  ------- ]
                      [   sqrt(2)  sqrt(2) ]
(%i2) L : matrix ([sqrt(3), 0], [0, sqrt(5)]);
                       [ sqrt(3)     0    ]
(%o2)                  [                  ]
                       [    0     sqrt(5) ]
(%i3) M : S . L . transpose (S);
             [ sqrt(5)   sqrt(3)  sqrt(5)   sqrt(3) ]
             [ ------- + -------  ------- - ------- ]
             [    2         2        2         2    ]
(%o3)        [                                      ]
             [ sqrt(5)   sqrt(3)  sqrt(5)   sqrt(3) ]
             [ ------- - -------  ------- + ------- ]
             [    2         2        2         2    ]
```

```
(%i4) eigens_by_jacobi (M);
The largest percent change was 0.1454972243679
The largest percent change was 0.0
number of sweeps: 2
number of rotations: 1
(%o4) [[1.732050807568877, 2.23606797749979],
                        [  0.70710678118655   0.70710678118655 ]
                        [                                      ]]
                        [ - 0.70710678118655  0.70710678118655 ]
(%i5) float ([[sqrt(3), sqrt(5)], S]);
(%o5) [[1.732050807568877, 2.23606797749979],
                        [  0.70710678118655   0.70710678118655 ]
                        [                                      ]]
                        [ - 0.70710678118655  0.70710678118655 ]
(%i6) eigens_by_jacobi (M, bigfloatfield);
The largest percent change was 1.454972243679028b-1
The largest percent change was 0.0b0
number of sweeps: 2
number of rotations: 1
(%o6) [[1.732050807568877b0, 2.23606797749979b0],
                [  7.071067811865475b-1   7.071067811865475b-1 ]
                [                                              ]]
                [ - 7.071067811865475b-1  7.071067811865475b-1 ]
```

`get_lu_factors` (*x*) [Function]

When *x* = `lu_factor (A)`, then `get_lu_factors` returns a list of the form `[P, L, U]`, where P is a permutation matrix, L is lower triangular with ones on the diagonal, and U is upper triangular, and *A* = *P L U*.

`hankel` [Function]

`hankel` (*col*)
`hankel` (*col*, *row*)

Return a Hankel matrix H. The first column of H is *col*; except for the first entry, the last row of H is *row*. The default for *row* is the zero vector with the same length as *col*.

`hessian` (*f*, *x*) [Function]

Returns the Hessian matrix of f with respect to the list of variables x. The `(i, j)`-th element of the Hessian matrix is `diff(f, x[i], 1, x[j], 1)`.

Examples:

```
(%i1) hessian (x * sin (y), [x, y]);
                      [   0       cos(y)   ]
(%o1)                 [                    ]
                      [ cos(y)  - x sin(y) ]
(%i2) depends (F, [a, b]);
(%o2)                       [F(a, b)]
(%i3) hessian (F, [a, b]);
```

```
                                [   2      2   ]
                                [  d F    d F  ]
                                [  ---   -----]
                                [    2   da db ]
                                [  da          ]
(%o3)                           [              ]
                                [   2      2   ]
                                [  d F    d F  ]
                                [ -----   ---  ]
                                [ da db     2  ]
                                [         db   ]
```

`hilbert_matrix` (*n*) [Function]

Return the *n* by *n* Hilbert matrix. When *n* isn't a positive integer, signal an error.

`identfor` [Function]

`identfor` (*M*)

`identfor` (*M*, *fld*)

Return an identity matrix that has the same shape as the matrix *M*. The diagonal entries of the identity matrix are the multiplicative identity of the field *fld*; the default for *fld* is *generalring*.

The first argument *M* should be a square matrix or a non-matrix. When *M* is a matrix, each entry of *M* can be a square matrix – thus *M* can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.

See also `zerofor`

`invert_by_lu` (*M*, (*rng generalring*)) [Function]

Invert a matrix *M* by using the LU factorization. The LU factorization is done using the ring *rng*.

`jacobian` (*f*, *x*) [Function]

Returns the Jacobian matrix of the list of functions *f* with respect to the list of variables *x*. The `(i, j)`-th element of the Jacobian matrix is `diff(f[i], x[j])`.

Examples:

```
(%i1) jacobian ([sin (u - v), sin (u * v)], [u, v]);
                    [ cos(v - u)  - cos(v - u) ]
(%o1)               [                          ]
                    [ v cos(u v)   u cos(u v)  ]
(%i2) depends ([F, G], [y, z]);
(%o2)                    [F(y, z), G(y, z)]
(%i3) jacobian ([F, G], [y, z]);
                              [ dF  dF ]
                              [ --  -- ]
                              [ dy  dz ]
(%o3)                         [        ]
                              [ dG  dG ]
                              [ --  -- ]
                              [ dy  dz ]
```

`kronecker_product` (*A*, *B*) [Function]

Return the Kronecker product of the matrices *A* and *B*.

`locate_matrix_entry` (*M*, *r_1*, *c_1*, *r_2*, *c_2*, *f*, *rel*) [Function]

The first argument must be a matrix; the arguments *r_1* through *c_2* determine a sub-matrix of *M* that consists of rows *r_1* through *r_2* and columns *c_1* through *c_2*.

Find an entry in the sub-matrix *M* that satisfies some property. Three cases:

(1) *rel* = `'bool` and *f* a predicate:

Scan the sub-matrix from left to right then top to bottom, and return the index of the first entry that satisfies the predicate *f*. If no matrix entry satisfies *f*, return `false`.

(2) *rel* = `'max` and *f* real-valued:

Scan the sub-matrix looking for an entry that maximizes *f*. Return the index of a maximizing entry.

(3) *rel* = `'min` and *f* real-valued:

Scan the sub-matrix looking for an entry that minimizes *f*. Return the index of a minimizing entry.

`lu_backsub` (*M*, *b*) [Function]

When *M* = `lu_factor (A, field)`, then `lu_backsub (M, b)` solves the linear system *A x* = *b*.

The *n* by *m* matrix *b*, with *n* the number of rows of the matrix *A*, contains one right hand side per column. If there is only one right hand side then *b* must be a *n* by 1 matrix.

Each column of the matrix *x*=`lu_backsub (M, b)` is the solution corresponding to the respective column of *b*.

Examples:

```
(%i1) A : matrix ([1 - z, 3], [3, 8 - z]);
                                [ 1 - z    3   ]
(%o1)                           [              ]
                                [   3    8 - z ]
(%i2) M : lu_factor (A,generalring);
                [ 1 - z          3         ]
                [                          ]
(%o2)          [[   3              9       ], [1, 2], generalring]
                [ -----  (- z) - ----- + 8 ]
                [ 1 - z          1 - z     ]
(%i3) b : matrix([a],[c]);
                                      [ a ]
(%o3)                                 [   ]
                                      [ c ]
(%i4) x : lu_backsub(M,b);
                              [               3 a     ]
                              [       3 (c - -----)   ]
                              [              1 - z    ]
                              [ a - ----------------- ]
```

```
                                [               9       ]
                                [     (- z) - ----- + 8 ]
                                [             1 - z     ]
                                [ --------------------- ]
(%o4)                           [         1 - z         ]
                                [                       ]
                                [            3 a        ]
                                [       c - -----       ]
                                [           1 - z       ]
                                [   -----------------   ]
                                [             9         ]
                                [   (- z) - ----- + 8   ]
                                [           1 - z       ]
(%i5) ratsimp(A . x - b);
                                       [ 0 ]
(%o5)                                  [   ]
                                       [ 0 ]
(%i6) B : matrix([a,d],[c,f]);
                                     [ a  d ]
(%o6)                                [      ]
                                     [ c  f ]
(%i7) x : lu_backsub(M,B);
                [                 3 a                     3 d     ]
                [       3 (c - -----)           3 (f - -----)   ]
                [               1 - z                   1 - z   ]
                [ a - -----------------  d - ----------------- ]
                [                 9                       9     ]
                [     (- z) - ----- + 8       (- z) - ----- + 8 ]
                [               1 - z                   1 - z   ]
                [ --------------------  --------------------- ]
(%o7)           [         1 - z                   1 - z         ]
                [                                               ]
                [             3 a                     3 d       ]
                [       c - -----               f - -----       ]
                [           1 - z                   1 - z       ]
                [   -----------------       -----------------   ]
                [             9                       9         ]
                [   (- z) - ----- + 8       (- z) - ----- + 8   ]
                [           1 - z                   1 - z       ]
(%i8) ratsimp(A . x - B);
                                     [ 0  0 ]
(%o8)                                [      ]
                                     [ 0  0 ]
```

`lu_factor` (*M*, *field*) [Function]

Return a list of the form [*LU*, *perm*, *fld*], or [*LU*, *perm*, *fld*, *lower-cnd* *upper-cnd*], where

(1) The matrix LU contains the factorization of M in a packed form. Packed form means three things: First, the rows of LU are permuted according to the list *perm*. If, for example, *perm* is the list `[3,2,1]`, the actual first row of the LU factorization is the third row of the matrix LU. Second, the lower triangular factor of m is the lower triangular part of LU with the diagonal entries replaced by all ones. Third, the upper triangular factor of M is the upper triangular part of LU.

(2) When the field is either `floatfield` or `complexfield`, the numbers *lower-cnd* and *upper-cnd* are lower and upper bounds for the infinity norm condition number of M. For all fields, the condition number might not be estimated; for such fields, `lu_factor` returns a two item list. Both the lower and upper bounds can differ from their true values by arbitrarily large factors. (See also `mat_cond`.)

The argument M must be a square matrix.

The optional argument *fld* must be a symbol that determines a ring or field. The pre-defined fields and rings are:

(a) `generalring` – the ring of Maxima expressions,

(b) `floatfield` – the field of floating point numbers of the type double,

(c) `complexfield` – the field of complex floating point numbers of the type double,

(d) `crering` – the ring of Maxima CRE expressions,

(e) `rationalfield` – the field of rational numbers,

(f) `runningerror` – track the all floating point rounding errors,

(g) `noncommutingring` – the ring of Maxima expressions where multiplication is the non-commutative dot operator.

When the field is `floatfield`, `complexfield`, or `runningerror`, the algorithm uses partial pivoting; for all other fields, rows are switched only when needed to avoid a zero pivot.

Floating point addition arithmetic isn't associative, so the meaning of 'field' differs from the mathematical definition.

A member of the field `runningerror` is a two member Maxima list of the form `[x,n]`,where x is a floating point number and `n` is an integer. The relative difference between the 'true' value of `x` and `x` is approximately bounded by the machine epsilon times `n`. The running error bound drops some terms that of the order the square of the machine epsilon.

There is no user-interface for defining a new field. A user that is familiar with Common Lisp should be able to define a new field. To do this, a user must define functions for the arithmetic operations and functions for converting from the field representation to Maxima and back. Additionally, for ordered fields (where partial pivoting will be used), a user must define functions for the magnitude and for comparing field members. After that all that remains is to define a Common Lisp structure `mring`. The file `mring` has many examples.

To compute the factorization, the first task is to convert each matrix entry to a member of the indicated field. When conversion isn't possible, the factorization halts with an error message. Members of the field needn't be Maxima expressions. Members

of the `complexfield`, for example, are Common Lisp complex numbers. Thus after computing the factorization, the matrix entries must be converted to Maxima expressions.

See also `get_lu_factors`.

Examples:

```
(%i1) w[i,j] := random (1.0) + %i * random (1.0);
(%o1)          w     := random(1.) + %i random(1.)
                i, j
(%i2) showtime : true$
Evaluation took 0.00 seconds (0.00 elapsed)
(%i3) M : genmatrix (w, 100, 100)$
Evaluation took 7.40 seconds (8.23 elapsed)
(%i4) lu_factor (M, complexfield)$
Evaluation took 28.71 seconds (35.00 elapsed)
(%i5) lu_factor (M, generalring)$
Evaluation took 109.24 seconds (152.10 elapsed)
(%i6) showtime : false$

(%i7) M : matrix ([1 - z, 3], [3, 8 - z]);
                        [ 1 - z    3   ]
(%o7)                   [              ]
                        [   3    8 - z ]
(%i8) lu_factor (M, generalring);
          [ 1 - z         3        ]
          [                        ]
(%o8)    [[   3            9       ], [1, 2], generalring]
          [ -----  - z - ----- + 8 ]
          [ 1 - z        1 - z     ]
(%i9) get_lu_factors (%);
                   [   1    0 ]  [ 1 - z         3        ]
        [ 1  0 ]   [          ]  [                        ]
(%o9)  [[      ], [   3      ], [                9       ]]
        [ 0  1 ]   [ -----  1 ]  [   0    - z - ----- + 8 ]
                   [ 1 - z    ]  [             1 - z     ]
(%i10) %[1] . %[2] . %[3];
                        [ 1 - z    3   ]
(%o10)                  [              ]
                        [   3    8 - z ]
```

`mat_cond` [Function]

`mat_cond` (*M*, *1*)

`mat_cond` (*M*, *inf*)

Return the *p*-norm matrix condition number of the matrix *m*. The allowed values for *p* are 1 and *inf*. This function uses the LU factorization to invert the matrix *m*. Thus the running time for `mat_cond` is proportional to the cube of the matrix size; `lu_factor` determines lower and upper bounds for the infinity norm condition number in time proportional to the square of the matrix size.

`mat_norm` [Function]

`mat_norm` (*M*, *1*)
`mat_norm` (*M*, *inf*)
`mat_norm` (*M*, *frobenius*)

Return the matrix p-norm of the matrix M. The allowed values for p are 1, `inf`, and `frobenius` (the Frobenius matrix norm). The matrix M should be an unblocked matrix.

`matrixp` [Function]

`matrixp` (*e*, *p*)
`matrixp` (*e*)

Given an optional argument p, return `true` if e is a matrix and p evaluates to `true` for every matrix element. When `matrixp` is not given an optional argument, return `true` if `e` is a matrix. In all other cases, return `false`.

See also `blockmatrixp`

`matrix_size` (*M*) [Function]

Return a two member list that gives the number of rows and columns, respectively of the matrix M.

`mat_fullunblocker` (*M*) [Function]

If M is a block matrix, unblock the matrix to all levels. If M is a matrix, return M; otherwise, signal an error.

`mat_trace` (*M*) [Function]

Return the trace of the matrix M. If M isn't a matrix, return a noun form. When M is a block matrix, `mat_trace(M)` returns the same value as does `mat_trace(mat_unblocker(m))`.

`mat_unblocker` (*M*) [Function]

If M is a block matrix, unblock M one level. If M is a matrix, `mat_unblocker (M)` returns M; otherwise, signal an error.

Thus if each entry of M is matrix, `mat_unblocker (M)` returns an unblocked matrix, but if each entry of M is a block matrix, `mat_unblocker (M)` returns a block matrix with one less level of blocking.

If you use block matrices, most likely you'll want to set `matrix_element_mult` to `"."` and `matrix_element_transpose` to `'transpose`. See also `mat_fullunblocker`.

Example:

```
(%i1) A : matrix ([1, 2], [3, 4]);
                                [ 1  2 ]
(%o1)                           [      ]
                                [ 3  4 ]
(%i2) B : matrix ([7, 8], [9, 10]);
                                [ 7  8  ]
(%o2)                           [       ]
                                [ 9  10 ]
(%i3) matrix ([A, B]);
```

```
                          [ [ 1  2 ]  [ 7  8  ] ]
(%o3)                     [ [      ]  [       ] ]
                          [ [ 3  4 ]  [ 9  10 ] ]
(%i4) mat_unblocker (%);
                              [ 1  2  7  8  ]
(%o4)                         [              ]
                              [ 3  4  9  10 ]
```

`nullspace` (*M*) [Function]

If *M* is a matrix, return `span (v_1, ..., v_n)`, where the set `{v_1, ..., v_n}` is a basis for the nullspace of *M*. The span of the empty set is `{0}`. Thus, when the nullspace has only one member, return `span ()`.

`nullity` (*M*) [Function]

If *M* is a matrix, return the dimension of the nullspace of *M*.

`orthogonal_complement` (*v_1*, ..., *v_n*) [Function]

Return `span (u_1, ..., u_m)`, where the set `{u_1, ..., u_m}` is a basis for the orthogonal complement of the set `(v_1, ..., v_n)`.

Each vector *v_1* through *v_n* must be a column vector.

`polytocompanion` (*p*, *x*) [Function]

If *p* is a polynomial in *x*, return the companion matrix of *p*. For a monic polynomial *p* of degree *n*, we have *p* `= (-1)^n charpoly (polytocompanion (p, x))`.

When *p* isn't a polynomial in *x*, signal an error.

`ptriangularize` (*M*, *v*) [Function]

If *M* is a matrix with each entry a polynomial in *v*, return a matrix *M2* such that

(1) *M2* is upper triangular,

(2) *M2* `=` *E_n* `...` *E_1* *M*, where *E_1* through *E_n* are elementary matrices whose entries are polynomials in *v*,

(3) `|det (M)| = |det (M2)|`,

Note: This function doesn't check that every entry is a polynomial in *v*.

`rowop` (*M*, *i*, *j*, *theta*) [Function]

If *M* is a matrix, return the matrix that results from doing the row operation `R_i <- R_i - theta * R_j`. If *M* doesn't have a row *i* or *j*, signal an error.

`linalg_rank` (*M*) [Function]

Return the rank of the matrix *M*. This function is equivalent to function `rank`, but it uses a different algorithm: it finds the `columnspace` of the matrix and counts its elements, since the rank of a matrix is the dimension of its column space.

```
(%i1) linalg_rank(matrix([1,2],[2,4]));
(%o1)                          1
(%i2) linalg_rank(matrix([1,b],[c,d]));
(%o2)                          2
```

`rowswap` (*M*, *i*, *j*) [Function]

If *M* is a matrix, swap rows *i* and *j*. If *M* doesn't have a row *i* or *j*, signal an error.

toeplitz [Function]

toeplitz (*col*)
toeplitz (*col*, *row*)

Return a Toeplitz matrix T. The first first column of T is *col*; except for the first entry, the first row of T is *row*. The default for *row* is complex conjugate of *col*. Example:

```
(%i1)  toeplitz([1,2,3],[x,y,z]);
                                    [ 1  y  z ]
                                    [         ]
(%o1)                               [ 2  1  y ]
                                    [         ]
                                    [ 3  2  1 ]
(%i2)  toeplitz([1,1+%i]);

                                 [   1     1 - %I ]
(%o2)                            [                ]
                                 [ %I + 1    1    ]
```

vandermonde_matrix ([*x_1*, ..., *x_n*]) [Function]

Return a n by n matrix whose i-th row is `[1, x_i, x_i^2, ... x_i^(n-1)]`.

zerofor [Function]

zerofor (*M*)
zerofor (*M*, *fld*)

Return a zero matrix that has the same shape as the matrix M. Every entry of the zero matrix is the additive identity of the field *fld*; the default for *fld* is *generalring*.

The first argument M should be a square matrix or a non-matrix. When M is a matrix, each entry of M can be a square matrix – thus M can be a blocked Maxima matrix. The matrix can be blocked to any (finite) depth.

See also identfor

zeromatrixp (*M*) [Function]

If M is not a block matrix, return `true` if `is (equal (e, 0))` is true for each element e of the matrix M. If M is a block matrix, return `true` if `zeromatrixp` evaluates to `true` for each element of e.

77 Package lsquares

77.1 Introduction to lsquares

`lsquares` is a collection of functions to implement the method of least squares to estimate parameters for a model from numerical data.

77.2 Functions and Variables for lsquares

`lsquares_estimates` [Function]

`lsquares_estimates` (*D*, *x*, *e*, `a`)
`lsquares_estimates` (*D*, *x*, *e*, `a`, *initial* = *L*, *tol* = `t`)

Estimate parameters *a* to best fit the equation *e* in the variables *x* and *a* to the data *D*, as determined by the method of least squares. `lsquares_estimates` first seeks an exact solution, and if that fails, then seeks an approximate solution.

The return value is a list of lists of equations of the form `[a = ..., b = ..., c = ...]`. Each element of the list is a distinct, equivalent minimum of the mean square error.

The data *D* must be a matrix. Each row is one datum (which may be called a 'record' or 'case' in some contexts), and each column contains the values of one variable across all data. The list of variables *x* gives a name for each column of *D*, even the columns which do not enter the analysis. The list of parameters *a* gives the names of the parameters for which estimates are sought. The equation *e* is an expression or equation in the variables *x* and *a*; if *e* is not an equation, it is treated the same as `e = 0`.

Additional arguments to `lsquares_estimates` are specified as equations and passed on verbatim to the function `lbfgs` which is called to find estimates by a numerical method when an exact result is not found.

If some exact solution can be found (via `solve`), the data *D* may contain non-numeric values. However, if no exact solution is found, each element of *D* must have a numeric value. This includes numeric constants such as `%pi` and `%e` as well as literal numbers (integers, rationals, ordinary floats, and bigfloats). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.

If `lsquares_estimates` needs excessive amounts of time or runs out of memory `lsquares_estimates_approximate`, which skips the attempt to find an exact solution, might still succeed.

`load("lsquares")` loads this function.

See also `lsquares_estimates_exact`, `lsquares_estimates_approximate`, `lsquares_mse`, `lsquares_residuals`, and `lsquares_residual_mse`.

Examples:

A problem for which an exact solution is found.

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
```

```
        [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
                                [ 1  1  1 ]
                                [         ]
                                [ 3       ]
                                [ -  1  2 ]
                                [ 2       ]
                                [         ]
(%o2)                           [ 9       ]
                                [ -  2  1 ]
                                [ 4       ]
                                [         ]
                                [ 3  2  2 ]
                                [         ]
                                [ 2  2  1 ]
(%i3) lsquares_estimates (
        M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);
                  59        27      10921        107
(%o3)     [[A = - --, B = - --, C = -----, D = - ---]]
                  16        16      1024         32
```

A problem for which no exact solution is found, so `lsquares_estimates` resorts to numerical approximation.

```
(%i1) load ("lsquares")$
(%i2) M : matrix ([1, 1], [2, 7/4], [3, 11/4], [4, 13/4]);
                               [ 1  1  ]
                               [       ]
                               [    7  ]
                               [ 2  -  ]
                               [    4  ]
                               [       ]
(%o2)                          [    11 ]
                               [ 3  -- ]
                               [    4  ]
                               [       ]
                               [    13 ]
                               [ 4  -- ]
                               [    4  ]
(%i3) lsquares_estimates (
  M, [x,y], y=a*x^b+c, [a,b,c], initial=[3,3,3], iprint=[-1,0]);
(%o3) [[a = 1.375751433061394, b = 0.7148891534417651,
                                          c = - 0.4020908910062951]]
```

Exponential functions aren't well-conditioned for least min square fitting. In case that fitting to them fails it might be possible to get rid of the exponential function using an logarithm.

```
(%i1) load ("lsquares")$
(%i2) yvalues: [1,3,5,60,200,203,80]$
(%i3) time: [1,2,4,5,6,8,10]$
```

```
(%i4) f: y=a*exp(b*t);
                                      b t
(%o4)                          y = a %e
(%i5) yvalues_log: log(yvalues)$
(%i6) f_log: log(subst(y=exp(y),f));
                                       b t
(%o6)                      y = log(a %e   )
(%i7) lsquares_estimates (transpose(matrix(yvalues_log,time)),
                          [y,t], f_log, [a,b]);
************************************************
  N=    2   NUMBER OF CORRECTIONS=25
       INITIAL VALUES
 F=  6.802906290754687D+00   GNORM=  2.851243373781393D+01
************************************************

I NFN FUNC                    GNORM                  STEPLENGTH

1   3 1.141838765593467D+00 1.067358003667488D-01 1.390943719972406D-02
2   5 1.141118195694385D+00 1.237977833033414D-01 5.000000000000000D+00
3   6 1.136945723147959D+00 3.806696991691383D-01 1.000000000000000D+00
4   7 1.133958243220262D+00 3.865103550379243D-01 1.000000000000000D+00
5   8 1.131725773805499D+00 2.292258231154026D-02 1.000000000000000D+00
6   9 1.131625585698168D+00 2.664440547017370D-03 1.000000000000000D+00
7  10 1.131620564856599D+00 2.519366958715444D-04 1.000000000000000D+00

 THE MINIMIZATION TERMINATED WITHOUT DETECTING ERRORS.
 IFLAG = 0
(%o7)   [[a = 1.155904145765554, b = 0.5772666876959847]]
```

`lsquares_estimates_exact` (*MSE*, a) [Function]

Estimate parameters *a* to minimize the mean square error *MSE*, by constructing a system of equations and attempting to solve them symbolically via `solve`. The mean square error is an expression in the parameters *a*, such as that returned by `lsquares_mse`.

The return value is a list of lists of equations of the form `[a = ..., b = ..., c = ...]`. The return value may contain zero, one, or two or more elements. If two or more elements are returned, each represents a distinct, equivalent minimum of the mean square error.

See also `lsquares_estimates`, `lsquares_estimates_approximate`, `lsquares_mse`, `lsquares_residuals`, and `lsquares_residual_mse`.

Example:

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
         [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
                              [ 1  1  1 ]
                              [         ]
```

```
                                [ 3       ]
                                [ -  1  2 ]
                                [ 2       ]
                                [         ]
(%o2)                           [ 9       ]
                                [ -  2  1 ]
                                [ 4       ]
                                [         ]
                                [ 3  2  2 ]
                                [         ]
                                [ 2  2  1 ]
(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);
        5
       ====
       \                                          2     2
        >    ((- B M    ) - A M     + (M     + D)  - C)
       /           i, 3       i, 2     i, 1
       ====
       i = 1
(%o3)  ------------------------------------------------
                              5
(%i4) lsquares_estimates_exact (mse, [A, B, C, D]);
                  59        27      10921        107
(%o4)     [[A = - --, B = - --, C = -----, D = - ---]]
                  16        16      1024         32
```

`lsquares_estimates_approximate` (*MSE*, `a`, *initial* = *L*, *tol* = `t`) [Function]

Estimate parameters *a* to minimize the mean square error *MSE*, via the numerical minimization function `lbfgs`. The mean square error is an expression in the parameters *a*, such as that returned by `lsquares_mse`.

The solution returned by `lsquares_estimates_approximate` is a local (perhaps global) minimum of the mean square error. For consistency with `lsquares_estimates_exact`, the return value is a nested list which contains one element, namely a list of equations of the form `[a = ..., b = ..., c = ...]`.

Additional arguments to `lsquares_estimates_approximate` are specified as equations and passed on verbatim to the function `lbfgs`.

MSE must evaluate to a number when the parameters are assigned numeric values. This requires that the data from which *MSE* was constructed comprise only numeric constants such as `%pi` and `%e` and literal numbers (integers, rationals, ordinary floats, and bigfloats). Numerical calculations are carried out with ordinary floating-point arithmetic, so all other kinds of numbers are converted to ordinary floats for calculations.

`load("lsquares")` loads this function.

See also `lsquares_estimates`, `lsquares_estimates_exact`, `lsquares_mse`, `lsquares_residuals`, and `lsquares_residual_mse`.

Example:

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
         [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
                              [ 1  1  1 ]
                              [         ]
                              [ 3       ]
                              [ -  1  2 ]
                              [ 2       ]
                              [         ]
(%o2)                         [ 9       ]
                              [ -  2  1 ]
                              [ 4       ]
                              [         ]
                              [ 3  2  2 ]
                              [         ]
                              [ 2  2  1 ]
(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);
                5
               ====
               \                                          2     2
                >    ((- B M    ) - A M     + (M     + D)  - C)
               /            i, 3       i, 2     i, 1
               ====
               i = 1
(%o3)          ------------------------------------------------
                                    5
(%i4) lsquares_estimates_approximate (
         mse, [A, B, C, D], iprint = [-1, 0]);
(%o4) [[A = - 3.678504947401971, B = - 1.683070351177937,
                 C = 10.63469950148714, D = - 3.340357993175297]]
```

`lsquares_mse` (*D*, *x*, *e*) [Function]

Returns the mean square error (MSE), a summation expression, for the equation *e* in the variables *x*, with data *D*.

The MSE is defined as:

$$\frac{1}{n}\sum_{i=1}^{n}\left[\text{lhs}\left(e_i\right)-\text{rhs}\left(e_i\right)\right]^2,$$

where *n* is the number of data and `e[i]` is the equation *e* evaluated with the variables in *x* assigned values from the `i`-th datum, `D[i]`.

`load("lsquares")` loads this function.

Example:

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
         [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
```

```
                              [ 1  1  1 ]
                              [         ]
                              [ 3       ]
                              [ -  1  2 ]
                              [ 2       ]
                              [         ]
(%o2)                         [ 9       ]
                              [ -  2  1 ]
                              [ 4       ]
                              [         ]
                              [ 3  2  2 ]
                              [         ]
                              [ 2  2  1 ]
(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);
        5
       ====
       \                                          2     2
        >    ((- B M    ) - A M     + (M     + D)  - C)
       /           i, 3       i, 2     i, 1
       ====
       i = 1
(%o3)  ------------------------------------------------
                              5
(%i4) diff (mse, D);
(%o4)
     5
    ====
    \                                                    2
  4  >    (M     + D) ((- B M    ) - A M     + (M     + D)  - C)
    /       i, 1             i, 3       i, 2     i, 1
    ====
    i = 1
  ------------------------------------------------------------
                               5
(%i5) ''mse, nouns;
              2                2         9 2              2
(%o5) (((D + 3)  - C - 2 B - 2 A)  + ((D + -)  - C - B - 2 A)
                                       4
          2              2         3 2              2
 + ((D + 2)  - C - B - 2 A)  + ((D + -)  - C - 2 B - A)
                                    2
          2            2
 + ((D + 1)  - C - B - A) )/5
```

```
(%i3) mse : lsquares_mse (M, [z, x, y], (z + D)^2 = A*x + B*y + C);
           5
          ====
          \                 2                          2
           >    ((D + M    )  - C - M     B - M     A)
          /             i, 1           i, 3      i, 2
          ====
          i = 1
(%o3)     ------------------------------------------
                                5
(%i4) diff (mse, D);
          5
         ====
         \                                2
      4   >    (D + M    ) ((D + M    )  - C - M     B - M     A)
         /           i, 1          i, 1           i, 3      i, 2
         ====
         i = 1
(%o4) ------------------------------------------------------------
                                  5
(%i5) ''mse, nouns;
                2                 2          9 2                2
(%o5) (((D + 3)  - C - 2 B - 2 A)  + ((D + -)  - C - B - 2 A)
                                              4
           2                2          3 2                2
 + ((D + 2)  - C - B - 2 A)  + ((D + -)  - C - 2 B - A)
                                       2
           2             2
 + ((D + 1)  - C - B - A) )/5
```

`lsquares_residuals` (*D*, *x*, *e*, *a*) [Function]

Returns the residuals for the equation *e* with specified parameters *a* and data *D*.

D is a matrix, *x* is a list of variables, *e* is an equation or general expression; if not an equation, *e* is treated as if it were `e = 0`. *a* is a list of equations which specify values for any free parameters in *e* aside from *x*.

The residuals are defined as:

$$\mathrm{lhs}\left(e_i\right) - \mathrm{rhs}\left(e_i\right),$$

where `e[i]` is the equation *e* evaluated with the variables in *x* assigned values from the `i`-th datum, `D[i]`, and assigning any remaining free variables from *a*.

`load("lsquares")` loads this function.

Example:

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
         [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
```

```
                                [ 1  1  1 ]
                                [         ]
                                [ 3       ]
                                [ -  1  2 ]
                                [ 2       ]
                                [         ]
(%o2)                           [ 9       ]
                                [ -  2  1 ]
                                [ 4       ]
                                [         ]
                                [ 3  2  2 ]
                                [         ]
                                [ 2  2  1 ]
(%i3) a : lsquares_estimates (
          M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);
                   59        27      10921        107
(%o3)     [[A = - --, B = - --, C = -----, D = - ---]]
                   16        16      1024         32
(%i4) lsquares_residuals (
          M, [z,x,y], (z+D)^2 = A*x+B*y+C, first(a));
                      13    13    13  13  13
(%o4)                [--, - --, - --, --, --]
                      64    64    32  64  64
```

`lsquares_residual_mse` (*D*, *x*, *e*, `a`) [Function]

Returns the residual mean square error (MSE) for the equation *e* with specified parameters *a* and data *D*.

The residual MSE is defined as:

$$\frac{1}{n}\sum_{i=1}^{n}\left[\text{lhs}\left(e_i\right)-\text{rhs}\left(e_i\right)\right]^2,$$

where `e[i]` is the equation *e* evaluated with the variables in *x* assigned values from the `i`-th datum, `D[i]`, and assigning any remaining free variables from *a*.

`load("lsquares")` loads this function.

Example:

```
(%i1) load ("lsquares")$
(%i2) M : matrix (
          [1,1,1], [3/2,1,2], [9/4,2,1], [3,2,2], [2,2,1]);
                                [ 1  1  1 ]
                                [         ]
                                [ 3       ]
                                [ -  1  2 ]
                                [ 2       ]
                                [         ]
(%o2)                           [ 9       ]
```

```
                              [ -  2  1 ]
                              [ 4       ]
                              [         ]
                              [ 3  2  2 ]
                              [         ]
                              [ 2  2  1 ]
(%i3) a : lsquares_estimates (
       M, [z,x,y], (z+D)^2 = A*x+B*y+C, [A,B,C,D]);
                   59        27      10921        107
(%o3)     [[A = - --, B = - --, C = -----, D = - ---]]
                   16        16      1024         32
(%i4) lsquares_residual_mse (
       M, [z,x,y], (z + D)^2 = A*x + B*y + C, first (a));
                                169
(%o4)                           ----
                                2560
```

plsquares [Function]

plsquares (*Mat*,*VarList*,*depvars*)
plsquares (*Mat*,*VarList*,*depvars*,*maxexpon*)
plsquares (*Mat*,*VarList*,*depvars*,*maxexpon*,*maxdegree*)

Multivariable polynomial adjustment of a data table by the "least squares" method. *Mat* is a matrix containing the data, *VarList* is a list of variable names (one for each Mat column, but use "-" instead of varnames to ignore Mat columns), *depvars* is the name of a dependent variable or a list with one or more names of dependent variables (which names should be in *VarList*), *maxexpon* is the optional maximum exponent for each independent variable (1 by default), and *maxdegree* is the optional maximum polynomial degree (*maxexpon* by default); note that the sum of exponents of each term must be equal or smaller than *maxdegree*, and if `maxdgree = 0` then no limit is applied.

If *depvars* is the name of a dependent variable (not in a list), `plsquares` returns the adjusted polynomial. If *depvars* is a list of one or more dependent variables, `plsquares` returns a list with the adjusted polynomial(s). The Coefficients of Determination are displayed in order to inform about the goodness of fit, which ranges from 0 (no correlation) to 1 (exact correlation). These values are also stored in the global variable *DETCOEF* (a list if *depvars* is a list).

A simple example of multivariable linear adjustment:

```
(%i1) load("plsquares")$

(%i2) plsquares(matrix([1,2,0],[3,5,4],[4,7,9],[5,8,10]),
                [x,y,z],z);
     Determination Coefficient for z = .9897039897039897
                         11 y - 9 x - 14
(%o2)                z = ---------------
                                 3
```

The same example without degree restrictions:

```
(%i3) plsquares(matrix([1,2,0],[3,5,4],[4,7,9],[5,8,10]),
                [x,y,z],z,1,0);
     Determination Coefficient for z = 1.0
                     x y + 23 y - 29 x - 19
(%o3)           z = ----------------------
                                6
```

How many diagonals does a N-sides polygon have? What polynomial degree should be used?

```
(%i4) plsquares(matrix([3,0],[4,2],[5,5],[6,9],[7,14],[8,20]),
                [N,diagonals],diagonals,5);
     Determination Coefficient for diagonals = 1.0
                                 2
                                N  - 3 N
(%o4)              diagonals = --------
                                  2
(%i5) ev(%, N=9);   /* Testing for a 9 sides polygon */
(%o5)                 diagonals = 27
```

How many ways do we have to put two queens without they are threatened into a n x n chessboard?

```
(%i6) plsquares(matrix([0,0],[1,0],[2,0],[3,8],[4,44]),
                [n,positions],[positions],4);
     Determination Coefficient for [positions] = [1.0]
                        4       3      2
                     3 n  - 10 n  + 9 n  - 2 n
(%o6)    [positions = -------------------------]
                                6
(%i7) ev(%[1], n=8); /* Testing for a (8 x 8) chessboard */
(%o7)                 positions = 1288
```

An example with six dependent variables:

```
(%i8) mtrx:matrix([0,0,0,0,0,1,1,1],[0,1,0,1,1,1,0,0],
                  [1,0,0,1,1,1,0,0],[1,1,1,1,0,0,0,1])$
(%i8) plsquares(mtrx,[a,b,_And,_Or,_Xor,_Nand,_Nor,_Nxor],
                     [_And,_Or,_Xor,_Nand,_Nor,_Nxor],1,0);
      Determination Coefficient for
[_And, _Or, _Xor, _Nand, _Nor, _Nxor] =
[1.0, 1.0, 1.0, 1.0, 1.0, 1.0]
(%o2) [_And = a b, _Or = - a b + b + a,
_Xor = - 2 a b + b + a, _Nand = 1 - a b,
_Nor = a b - b - a + 1, _Nxor = 2 a b - b - a + 1]
```

To use this function write first `load("lsquares")`.

78 Package minpack

78.1 Introduction to minpack

`Minpack` is a Common Lisp translation (via `f2cl`) of the Fortran library MINPACK as obtained from Netlib.

78.2 Functions and Variables for minpack

`minpack_lsquares` (*flist*, *varlist*, *guess*, ['*tolerance* = *tolerance*, '*jacobian* = *jacobian*]) [Function]

Compute the point that minimizes the sum of the squares of the functions in the list *flist*. The variables are in the list *varlist*. An initial guess of the optimum point must be provided in *guess*.

Let *flist* be a list of m functions, $f_i(x_1, x_2, ..., x_n)$. Then this function can be used to find the values of $x_1, x_2, ..., x_n$ that solve the least squares problem

$$\sum_i^m f_i(x_1, x_2, ..., x_n)^2$$

The optional keyword arguments, *tolerance* and *jacobian* provide some control over the algorithm.

`tolerance`
: the estimated relative error desired in the sum of squares. The default value is approximately 1.0537×10^{-8}.

`jacobian`
: specifies the Jacobian. If *jacobian* is not given or is `true` (the default), the Jacobian is computed from *flist*. If *jacobian* is `false`, a numerical approximation is used. See [jacobian], page 1156.

`minpack_lsquares` returns a list of three items as follows:

1. The estimated solution
2. The sum of squares
3. The success of the algorithm. The possible values are
   0. improper input parameters.
   1. algorithm estimates that the relative error in the sum of squares is at most `tolerance`.
   2. algorithm estimates that the relative error between x and the solution is at most `tolerance`.
   3. conditions for info = 1 and info = 2 both hold.
   4. fvec is orthogonal to the columns of the jacobian to machine precision.
   5. number of calls to fcn with iflag = 1 has reached 100*(n+1).
   6. tol is too small. no further reduction in the sum of squares is possible.
   7. tol is too small. no further improvement in the approximate solution x is possible.

Here is an example using Powell's singular function.

```
(%i1) load("minpack")$
(%i2) powell(x1,x2,x3,x4) := [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2, sqrt(10)*(x
(%i3) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4], [3,-1,0,1]);
(%o3) [[1.6521175961683935e-17, - 1.6521175961683934e-18,
2.6433881538694683e-18, 2.6433881538694683e-18],
6.109327859207777e-34, 4]
```

Same problem but use numerical approximation to Jacobian.

```
(%i1) load("minpack")$
(%i2) powell(x1,x2,x3,x4) := [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2, sqrt(10)*(x
(%i3) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4], [3,-1,0,1], jacobian =
(%o3) [[5.060282149485331e-11, - 5.060282149491206e-12,
2.1794478435472183e-11, 2.1794478435472183e-11],
3.534491794847031e-21, 5]
```

`minpack_solve` (*flist*, *varlist*, *guess*, ['*tolerance* = *tolerance*, '*jacobian* = *jacobian*]) [Function]

Solve a system of `n` equations in `n` unknowns. The `n` equations are given in the list *flist*, and the unknowns are in *varlist*. An initial guess of the solution must be provided in *guess*.

Let *flist* be a list of m functions, $f_i(x_1, x_2, ..., x_n)$. Then this functions solves the system of m nonlinear equations in n variables:

$$f_i(x_1, x_2, ..., x_n) = 0$$

The optional keyword arguments, *tolerance* and *jacobian* provide some control over the algorithm.

`tolerance`
: the estimated relative error desired in the sum of squares. The default value is approximately 1.0537×10^{-8}.

`jacobian`
: specifies the Jacobian. If *jacobian* is not given or is `true` (the default), the Jacobian is computed from *flist*. If *jacobian* is `false`, a numerical approximation is used. See [jacobian], page 1156.

`minpack_solve` returns a list of three items as follows:

1. The estimated solution
2. The sum of squares
3. The success of the algorithm. The possible values are
   0. improper input parameters.
   1. algorithm estimates that the relative error in the solution is at most `tolerance`.
   2. number of calls to fcn with iflag = 1 has reached 100*(n+1).
   3. tol is too small. no further reduction in the sum of squares is possible.
   4. Iteration is not making good progress.

```
(%i1) load("minpack")$
(%i2) powell(x1,x2,x3,x4) := [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2, sqrt(10)*(x
(%i3) minpack_lsquares(powell(x1,x2,x3,x4), [x1,x2,x3,x4], [3,-1,0,1]);
(%o3) [[1.6521175961683935e-17, - 1.6521175961683934e-18,
2.6433881538694683e-18, 2.6433881538694683e-18],
6.109327859207777e-34, 4]
```

In this particular case, we can solve this analytically:

```
(%i1) powell(x1,x2,x3,x4) := [x1+10*x2, sqrt(5)*(x3-x4), (x2-2*x3)^2, sqrt(10)*(x
(%i2) solve(powell(x1,x2,x3,x4),[x1,x2,x3,x4]);
(%o2)          [[x1 = 0, x2 = 0, x3 = 0, x4 = 0]]
```

and we see that the numerical solution is quite close the analytical one.

79 Package makeOrders

79.1 Functions and Variables for makeOrders

`makeOrders` (*indvarlist*,*orderlist*) [Function]

Returns a list of all powers for a polynomial up to and including the arguments.

```
(%i1) load("makeOrders")$

(%i2) makeOrders([a,b],[2,3]);
(%o2) [[0, 0], [0, 1], [0, 2], [0, 3], [1, 0], [1, 1],
            [1, 2], [1, 3], [2, 0], [2, 1], [2, 2], [2, 3]]
(%i3) expand((1+a+a^2)*(1+b+b^2+b^3));
       2  3      3    3    2  2      2    2    2
(%o3) a  b  + a b  + b  + a  b  + a b  + b  + a  b + a b
                                                    2
                                                 + b + a  + a + 1
```

where `[0, 1]` is associated with the term b and `[2, 3]` with a^2b^3.

To use this function write first `load("makeOrders")`.

80 Package mnewton

80.1 Introduction to mnewton

`mnewton` is an implementation of Newton's method for solving nonlinear equations in one or more variables.

80.2 Functions and Variables for mnewton

`newtonepsilon` [Option variable]

Default value: `10.0^(-fpprec/2)`

Precision to determine when the `mnewton` function has converged towards the solution.

When `newtonepsilon` is a bigfloat, `mnewton` computations are done with bigfloats; otherwise, ordinary floats are used.

See also `mnewton`.

`newtonmaxiter` [Option variable]

Default value: `50`

Maximum number of iterations to stop the `mnewton` function if it does not converge or if it converges too slowly.

See also `mnewton`.

`newtondebug` [Option variable]

Default value: `false`

When `newtondebug` is `true`, `mnewton` prints out debugging information while solving a problem.

`mnewton` [Function]

`mnewton` (*FuncList*, *VarList*, *GuessList*)
`mnewton` (*FuncList*, *VarList*, *GuessList*, *DF*)

Approximate solution of multiple nonlinear equations by Newton's method.

FuncList is a list of functions to solve, *VarList* is a list of variable names, and *GuessList* is a list of initial approximations. The optional argument *DF* is the Jacobian matrix of the list of functions; if not supplied, it is calculated automatically from *FuncList*.

FuncList may be specified as a list of equations, in which case the function to be solved is the left-hand side of each equation minus the right-hand side.

If there is only a single function, variable, and initial point, they may be specified as a single expression, variable, and initial value; they need not be lists of one element.

A variable may be a simple symbol or a subscripted symbol.

The solution, if any, is returned as a list of one element, which is a list of equations, one for each variable, specifying an approximate solution; this is the same format as returned by `solve`. If the solution is not found, `[]` is returned.

Functions and initial points may contain complex numbers, and solutions likewise may contain complex numbers.

`mnewton` is governed by global variables `newtonepsilon` and `newtonmaxiter`, and the global flag `newtondebug`.

`load("mnewton")` loads this function.

See also `realroots`, `allroots`, `find_root` and `newton`.

Examples:

```
(%i1) load("mnewton")$

(%i2) mnewton([x1+3*log(x1)-x2^2, 2*x1^2-x1*x2-5*x1+1],
              [x1, x2], [5, 5]);
(%o2) [[x1 = 3.756834008012769, x2 = 2.779849592817897]]
(%i3) mnewton([2*a^a-5],[a],[1]);
(%o3)             [[a = 1.70927556786144]]
(%i4) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);
(%o4) [[u = 1.066618389595407, v = 1.552564766841786]]
```

The variable `newtonepsilon` controls the precision of the approximations. It also controls if computations are performed with floats or bigfloats.

```
(%i1) load("mnewton")$

(%i2) (fpprec : 25, newtonepsilon : bfloat(10^(-fpprec+5)))$

(%i3) mnewton([2*3^u-v/u-5, u+2^v-4], [u, v], [2, 2]);
(%o3) [[u = 1.066618389595406772591173b0,
                                  v = 1.552564766841786450100418b0]]
```

81 Package numericalio

81.1 Introduction to numericalio

`numericalio` is a collection of functions to read and write files and streams. Functions for plain-text input and output can read and write numbers (integer, float, or bigfloat), symbols, and strings. Functions for binary input and output can read and write only floating-point numbers.

If there already exists a list, matrix, or array object to store input data, `numericalio` input functions can write data into that object. Otherwise, `numericalio` can guess, to some degree, the structure of an object to store the data, and return that object.

81.1.1 Plain-text input and output

In plain-text input and output, it is assumed that each item to read or write is an atom: an integer, float, bigfloat, string, or symbol, and not a rational or complex number or any other kind of nonatomic expression. The `numericalio` functions may attempt to do something sensible faced with nonatomic expressions, but the results are not specified here and subject to change.

Atoms in both input and output files have the same format as in Maxima batch files or the interactive console. In particular, strings are enclosed in double quotes, backslash `\` prevents any special interpretation of the next character, and the question mark `?` is recognized at the beginning of a symbol to mean a Lisp symbol (as opposed to a Maxima symbol). No continuation character (to join broken lines) is recognized.

81.1.2 Separator flag values for input

The functions for plain-text input and output take an optional argument, *separator_flag*, that tells what character separates data.

For plain-text input, these values of *separator_flag* are recognized: `comma` for comma separated values, `pipe` for values separated by the vertical bar character `|`, `semicolon` for values separated by semicolon `;`, and `space` for values separated by space or tab characters. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab).

If the file name ends in `.csv` and *separator_flag* is not specified, `comma` is assumed. If the file name ends in something other than `.csv` and `separator_flag` is not specified, `space` is assumed.

In plain-text input, multiple successive space and tab characters count as a single separator. However, multiple comma, pipe, or semicolon characters are significant. Successive comma, pipe, or semicolon characters (with or without intervening spaces or tabs) are considered to have `false` between the separators. For example, `1234,,Foo` is treated the same as `1234,false,Foo`.

81.1.3 Separator flag values for output

For plain-text output, `tab`, for values separated by the tab character, is recognized as a value of *separator_flag*, as well as `comma`, `pipe`, `semicolon`, and `space`.

In plain-text output, `false` atoms are written as such; a list `[1234, false, Foo]` is written `1234,false,Foo`, and there is no attempt to collapse the output to `1234,,Foo`.

81.1.4 Binary floating-point input and output

`numericalio` functions can read and write 8-byte IEEE 754 floating-point numbers. These numbers can be stored either least significant byte first or most significant byte first, according to the global flag set by `assume_external_byte_order`. If not specified, `numericalio` assumes the external byte order is most-significant byte first.

Other kinds of numbers are coerced to 8-byte floats; `numericalio` cannot read or write binary non-numeric data.

Some Lisp implementations do not recognize IEEE 754 special values (positive and negative infinity, not-a-number values, denormalized values). The effect of reading such values with `numericalio` is undefined.

`numericalio` includes functions to open a stream for reading or writing a stream of bytes.

81.2 Functions and Variables for plain-text input and output

`read_matrix` [Function]

`read_matrix` (*S*)
`read_matrix` (*S*, *M*)
`read_matrix` (*S*, *separator_flag*)
`read_matrix` (*S*, *M*, *separator_flag*)

`read_matrix(`*S*`)` reads the source *S* and returns its entire content as a matrix. The size of the matrix is inferred from the input data; each line of the file becomes one row of the matrix. If some lines have different lengths, `read_matrix` complains.

`read_matrix(`*S*`, `*M*`)` read the source *S* into the matrix *M*, until *M* is full or the source is exhausted. Input data are read into the matrix in row-major order; the input need not have the same number of rows and columns as *M*.

The source *S* may be a file name or a stream which for example allows skipping the very first line of a file (that may be useful, if you read CSV data, where the first line often contains the description of the columns):

```
s : openr("data.txt");
readline(s);  /* skip the first line */
M : read_matrix(s, 'comma);  /* read the following (comma-separated) lines into m
close(s);
```

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, and `space`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openr`, `read_array`, `read_hashed_array`, `read_list`, `read_binary_matrix`, `write_data` and `read_nested_list`.

`read_array` [Function]

`read_array (S, A)`
`read_array (S, A, separator_flag)`

Reads the source S into the array A, until A is full or the source is exhausted. Input data are read into the array in row-major order; the input need not conform to the dimensions of A.

The source S may be a file name or a stream.

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, and `space`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openr`, `read_matrix`, `read_hashed_array`, `read_list`, `read_binary_array` and `read_nested_list`.

`read_hashed_array` [Function]

`read_hashed_array (S, A)`
`read_hashed_array (S, A, separator_flag)`

Reads the source S and returns its entire content as a `hashed array`. The source S may be a file name or a stream.

`read_hashed_array` treats the first item on each line as a hash key, and associates the remainder of the line (as a list) with the key. For example, the line `567 12 17 32 55` is equivalent to `A[567]: [12, 17, 32, 55]$`. Lines need not have the same numbers of elements.

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, and `space`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openr`, `read_matrix`, `read_array`, `read_list` and `read_nested_list`.

`read_nested_list` [Function]

`read_nested_list (S)`
`read_nested_list (S, separator_flag)`

Reads the source S and returns its entire content as a nested list. The source S may be a file name or a stream.

`read_nested_list` returns a list which has a sublist for each line of input. Lines need not have the same numbers of elements. Empty lines are *not* ignored: an empty line yields an empty sublist.

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, and `space`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openr`, `read_matrix`, `read_array`, `read_list` and `read_hashed_array`.

`read_list` [Function]

`read_list (`*S*`)`
`read_list (`*S*`, `*L*`)`
`read_list (`*S*`, `*separator_flag*`)`
`read_list (`*S*`, `*L*`, `*separator_flag*`)`

`read_list(`*S*`)` reads the source *S* and returns its entire content as a flat list.

`read_list(`*S*`, `*L*`)` reads the source *S* into the list *L*, until *L* is full or the source is exhausted.

The source *S* may be a file name or a stream.

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, and `space`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openr`, `read_matrix`, `read_array`, `read_nested_list`, `read_binary_list` and `read_hashed_array`.

`write_data` [Function]

`write_data (`*X*`, `*D*`)`
`write_data (`*X*`, `*D*`, `*separator_flag*`)`

Writes the object *X* to the destination *D*.

`write_data` writes a matrix in row-major order, with one line per row.

`write_data` writes an array created by `array` or `make_array` in row-major order, with a new line at the end of every slab. Higher-dimensional slabs are separated by additional new lines.

`write_data` writes a hashed array with each key followed by its associated list on one line.

`write_data` writes a nested list with each sublist on one line.

`write_data` writes a flat list all on one line.

The destination *D* may be a file name or a stream. When the destination is a file name, the global variable `file_output_append` governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by `write_data` after all the data are written; in particular, the stream remains open.

The recognized values of *separator_flag* are `comma`, `pipe`, `semicolon`, `space`, and `tab`. Equivalently, the separator may be specified as a string of one character: `","` (comma), `"|"` (pipe), `";"` (semicolon), `" "` (space), or `" "` (tab). If *separator_flag* is not specified, the file is assumed space-delimited.

See also `openw` and `read_matrix`.

81.3 Functions and Variables for binary input and output

`assume_external_byte_order (`*byte_order_flag*`)` [Function]

Tells `numericalio` the byte order for reading and writing binary data. Two values of *byte_order_flag* are recognized: `lsb` which indicates least-significant byte first, also called little-endian byte order; and `msb` which indicates most-significant byte first, also called big-endian byte order.

If not specified, `numericalio` assumes the external byte order is most-significant byte first.

`openr_binary (`*file_name*`)` [Function]
Returns an input stream of 8-bit unsigned bytes to read the file named by *file_name*.

See also `openw_binary` and `openr`.

`openw_binary (`*file_name*`)` [Function]
Returns an output stream of 8-bit unsigned bytes to write the file named by *file_name*.

See also `openr_binary`, `opena_binary` and `openw`.

`opena_binary (`*file_name*`)` [Function]
Returns an output stream of 8-bit unsigned bytes to append the file named by *file_name*.

`read_binary_matrix (`*S*, *M*`)` [Function]
Reads binary 8-byte floating point numbers from the source S into the matrix M until M is full, or the source is exhausted. Elements of M are read in row-major order.

The source S may be a file name or a stream.

The byte order in elements of the source is specified by `assume_external_byte_order`.

See also `read_matrix`.

`read_binary_array (`*S*, *A*`)` [Function]
Reads binary 8-byte floating point numbers from the source S into the array A until A is full, or the source is exhausted. A must be an array created by `array` or `make_array`. Elements of A are read in row-major order.

The source S may be a file name or a stream.

The byte order in elements of the source is specified by `assume_external_byte_order`.

See also `read_array`.

`read_binary_list` [Function]
`read_binary_list (`*S*`)`
`read_binary_list (`*S*, *L*`)`
`read_binary_list(`*S*`)` reads the entire content of the source S as a sequence of binary 8-byte floating point numbers, and returns it as a list. The source S may be a file name or a stream.

`read_binary_list(`*S*, *L*`)` reads 8-byte binary floating point numbers from the source S until the list L is full, or the source is exhausted.

The byte order in elements of the source is specified by `assume_external_byte_order`.

See also `read_list`.

`write_binary_data` (*X*, *D*) [Function]

Writes the object *X*, comprising binary 8-byte IEEE 754 floating-point numbers, to the destination *D*. Other kinds of numbers are coerced to 8-byte floats. `write_binary_data` cannot write non-numeric data.

The object *X* may be a list, a nested list, a matrix, or an array created by `array` or `make_array`; *X* cannot be a hashed array or any other type of object. `write_binary_data` writes nested lists, matrices, and arrays in row-major order.

The destination *D* may be a file name or a stream. When the destination is a file name, the global variable `file_output_append` governs whether the output file is appended or truncated. When the destination is a stream, no special action is taken by `write_binary_data` after all the data are written; in particular, the stream remains open.

The byte order in elements of the destination is specified by `assume_external_byte_order`.

See also `write_data`.

82 Package odepack

82.1 Introduction to ODEPACK

ODEPACK is a collection of Fortran solvers for the initial value problem for ordinary differential equation systems. It consists of nine solvers, namely a basic solver called LSODE and eight variants of it – LSODES, LSODA, LSODAR, LSODPK, LSODKR, LSODI, LSOIBT, and LSODIS. The collection is suitable for both stiff and nonstiff systems. It includes solvers for systems given in explicit form, dy/dt = f(t,y), and also solvers for systems given in linearly implicit form, A(t,y) dy/dt = g(t,y). Two of the solvers use general sparse matrix solvers for the linear systems that arise. Two others use iterative (preconditioned Krylov) methods instead of direct methods for these linear systems. The most recent addition is LSODIS, which solves implicit problems with general sparse treatment of all matrices involved.

[1]

References: [1] Fortran Code is from https://www.netlib.org/odepack/

82.1.1 Getting Started with ODEPACK

Of the eight variants of the solver, Maxima currently only has an interface to `dlsode`.

Let's say we have this system of equations to solve:

$$
\begin{aligned}
f_1 &= -0.04y_1 + 10^4 y_2 y_3 \\
f_3 &= 3 \times 10^7 y_2^2 \\
\frac{dy_1}{dt} &= f_1 \\
\frac{dy_2}{dt} &= -f_1 - f_3 \\
\frac{dy_3}{dt} &= f_3
\end{aligned}
$$

The independent variable is t; the dependent variables are y_1, y_2, and y_3.

To start the solution, set up the differential equations to solved:

```
(%i1) load("dlsode")$
(%i2) f1: -.04d0*y1 + 1d4*y2*y3$
(%i3) f3: 3d7*y2*y2$
(%i4) f2: -f1 - f3$
(%i5) fex: [f1, f2, f3];
(%o5) [10000.0 y2 y3 - 0.04 y1,
                                          2                    2
                - 10000.0 y2 y3 - 3.0e7 y2  + 0.04 y1, 3.0e7 y2 ]
```

Initialize the solver, where we have selected method 21:

```
(%i1) load("dlsode")$
```

[1] From https://www.netlib.org/odepack/opkd-sum

```
(%i2) f1: -.04d0*y1 + 1d4*y2*y3$
(%i3) f3: 3d7*y2*y2$
(%i4) f2: -f1 - f3$
(%i5) fex: [f1, f2, f3];
(%o5) [10000.0 y2 y3 - 0.04 y1,
                                                 2                    2
                - 10000.0 y2 y3 - 3.0e7 y2  + 0.04 y1, 3.0e7 y2 ]
(%i6) state : dlsode_init(fex, ['t,y1,y2,y3], 21);
(%o6) [[f, #<Function "LAMBDA ($T $Y1 $Y2 $Y3)" {6177C099}>],
[vars, [t, y1, y2, y3]], [mf, 21], [neq, 3], [lrw, 58],
[liw, 23], [rwork, {Lisp Array: #(0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.\
0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
              0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0\
.0 0.0 0.0 0.0
              0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0\
.0 0.0 0.0 0.0
              0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0)}],
[iwork,
{Lisp Array: #(0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)}],
[fjac, #<Function "LAMBDA ($T $Y1 $Y2 $Y3)" {616B2099}>]]
```

The arrays rwork and iwork carry state between calls to `dlsode_step`, so they should not be modified by the user. In fact, this *state* returned from `dlsode_init` should not be modified by the user at all.

Now that the algorithm has been initialized we can compute solutions to the differential equation, using the *state* returned above.

For this example, we want to compute the solution at times 0.4×10^k for k from 0 to 11, with the initial values of 1, 0, 0 for the dependent variables and with a relative tolerance of 10^{-4} and absolute tolerances of 10^{-6}, 10^{-10}, and 10^{-6} for the dependent variables.

Then, putting everything together

```
(%i1) load("dlsode")$
(%i2) f1: -.04d0*y1 + 1d4*y2*y3$
(%i3) f3: 3d7*y2*y2$
(%i4) f2: -f1 - f3$
(%i5) fex: [f1, f2, f3];
(%o5) [10000.0 y2 y3 - 0.04 y1,
                                                 2                    2
                - 10000.0 y2 y3 - 3.0e7 y2  + 0.04 y1, 3.0e7 y2 ]
(%i6) state : dlsode_init(fex, ['t,y1,y2,y3], 21)$
(%i7) y: [1d0, 0d0, 0d0]$
(%i8) t: 0d0$
(%i9) rtol : 1d-4$
(%i10) atol: [1d-6, 1d-10, 1d-6]$
(%i11) istate: 1$
(%i12) t:0d0$
(%i13) tout:.4d0$
```

```
(%i14) for k : 1 thru 12 do
  block([],
    result: dlsode_step(y, t, tout, rtol, atol, istate, state),
    printf(true, "At t = ~12,4,2e   y = ~{~14,6,2e~}~%", result[1], result[2]),
    istate : result[3],
    tout : tout * 10);
At t =   4.0000e-01   y =   9.851726e-01  3.386406e-05  1.479357e-02
At t =   4.0000e+00   y =   9.055142e-01  2.240418e-05  9.446344e-02
At t =   4.0000e+01   y =   7.158050e-01  9.184616e-06  2.841858e-01
At t =   4.0000e+02   y =   4.504846e-01  3.222434e-06  5.495122e-01
At t =   4.0000e+03   y =   1.831701e-01  8.940379e-07  8.168290e-01
At t =   4.0000e+04   y =   3.897016e-02  1.621193e-07  9.610297e-01
At t =   4.0000e+05   y =   4.935213e-03  1.983756e-08  9.950648e-01
At t =   4.0000e+06   y =   5.159269e-04  2.064759e-09  9.994841e-01
At t =   4.0000e+07   y =   5.306413e-05  2.122677e-10  9.999469e-01
At t =   4.0000e+08   y =   5.494530e-06  2.197824e-11  9.999945e-01
At t =   4.0000e+09   y =   5.129458e-07  2.051784e-12  9.999995e-01
At t =   4.0000e+10   y =  -7.170563e-08 -2.868225e-13  1.000000e+00
(%o14)                         done
(%i15) result;
(%o15) [4.0e10, [- 7.170562924983933e-8,
- 2.868224674710487e-13, 1.000000071705913], 2,
[[n_steps, 330], [n_f_eval, 405], [n_j_eval, 69],
[method_order, 1], [len_rwork, 58], [len_iwork, 23]]]
```

82.2 Functions and Variables for odepack

`dlsode_init` (*fex*, *vars*, *method*) [Function]

This must be called before running the solver. This function returns a state object for use in the solver. The user must not modify the state.

The ODE to be solved is given in *fex*, which is a list of the equations. *vars* is a list of independent variable and the dependent variables. The list of dependent variables must be in the same order as the equations if *fex*. Finally, *method* indicates the method to be used by the solver:

`10`	Nonstiff (Adams) method, no Jacobian used.
`21`	Stiff (BDF) method, user-supplied full Jacobian.
`22`	Stiff method, internally generated full Jacobian.

The returned state object is a list of lists. The sublist is a list of two elements:

`f`	The compiled function for the ODE.
`vars`	The list independent and dependent variables (*vars*).
`mf`	The method to be used (*method*).
`neq`	The number of equations.
`lrw`	Length of the work vector for real values.

`liw` Length of the work vector for integer values.

`rwork` Lisp array holding the real-valued work vector.

`iwork` Lisp array holding the integer-valued work vector.

`fjac` Compiled analytical Jacobian of the equations

See also `dlsode_step`. See Section 82.1.1 [Getting Started with ODEPACK], page 1189, for an example of usage.

`dlsode_step` (*inity*, *t*, *tout*, *rtol*, *atol*, *istate*, *state*) [Function]

Performs one step of the solver, returning the values of the independent and dependent variables, a success or error code.

The parameters for `dlsode_step` are:

`inity` For the first call (when istate = 1), the initial values

`t` Current value of the independent value

`tout` Next point where output is desired which must not be equal to *t*.

`rtol` relative tolerance parameter

`atol` Absolute tolerance parameter, scalar of vector. If scalar, it applies to all dependent variables. Otherwise it must be the tolerance for each dependent variable.

Use $rtol = 0$ for pure absolute error and use $atol = 0$ for pure relative error.

`istate` 1 for the first call to dlsode, 2 for subsequent calls.

`state` state returned by dlsode_init.

The output is a list of the following items:

`t` independent variable value

`y` list of values of the dependent variables at time t.

`istate` Integration status:

- `1` no work because tout = tt
- `2` successful result
- `-1` Excess work done on this call
- `-2` Excess accuracy requested
- `-3` Illegal input detected
- `-4` Repeated error test failures
- `-5` Repeated convergence failures (perhaps bad Jacobian or wrong choice of mf or tolerances)

`-6` Error weight because zero during problem (solution component is vanished and atol(i) = 0.

`info` association list of various bits of information:

`n_steps` total steps taken thus far

`n_f_eval` total number of function evals

`n_j_eval` total number of Jacobian evals

`method_order`
method order

`len_rwork`
Actual length used for real work array

`len_iwork`
Actual length used for integer work array

See also `dlsode_init`. See Section 82.1.1 [Getting Started with ODEPACK], page 1189, for an example of usage.

83 Package operatingsystem

83.1 Introduction to operatingsystem

Package `operatingsystem` contains functions for operatingsystem-tasks, like file system operations.

83.2 Directory operations

`chdir` (*dir*) [Function]
: Change to directory *dir*

`mkdir` (*dir*) [Function]
: Create directory *dir*

`rmdir` (*dir*) [Function]
: remove directory *dir*

`getcurrentdirectory` () [Function]
: returns the current working directory.

  See also `directory`.

Examples:

```
(%i1) load("operatingsystem")$
(%i2) mkdir("testdirectory")$
(%i3) chdir("testdirectory")$
(%i4) chdir("..")$
(%i5) rmdir("testdirectory")$
```

83.3 File operations

`copy_file` (*file1*, *file2*) [Function]
: copies file *file1* to *file2*

`rename_file` (*file1*, *file2*) [Function]
: renames file *file1* to *file2*

`delete_file` (*file1*) [Function]
: deletes file *file1*

83.4 Environment operations

`getenv` (*env*) [Function]
: Get the value of the environment variable *env*

  Example:

```
(%i1) load("operatingsystem")$
(%i2) getenv("PATH");
(%o2) /usr/local/sbin:/usr/local/bin:/usr/sbin:/usr/bin:/sbin:/bin
```

84 Package opsubst

84.1 Functions and Variables for opsubst

opsubst [Function]

opsubst (*f*,*g*,*e*)
opsubst (*g*=*f*,*e*)
opsubst ([*g1*=*f1*,*g2*=*f2*,..., *gn*=*fn*],*e*)

The function `opsubst` is similar to the function `subst`, except that `opsubst` only makes substitutions for the operators in an expression. In general, When f is an operator in the expression e, substitute g for f in the expression e.

To determine the operator, `opsubst` sets `inflag` to true. This means `opsubst` substitutes for the internal, not the displayed, operator in the expression.

Examples:

```
(%i1) load ("opsubst")$

(%i2) opsubst(f,g,g(g(x)));
(%o2)                      f(f(x))
(%i3) opsubst(f,g,g(g));
(%o3)                        f(g)
(%i4) opsubst(f,g[x],g[x](z));
(%o4)                        f(z)
(%i5) opsubst(g[x],f, f(z));
(%o5)                       g (z)
                             x
(%i6) opsubst(tan, sin, sin(sin));
(%o6)                      tan(sin)
(%i7) opsubst([f=g,g=h],f(x));
(%o7)                        h(x)
```

Internally, Maxima does not use the unary negation, division, or the subtraction operators; thus:

```
(%i8) opsubst("+","-",a-b);
(%o8)                      a - b
(%i9) opsubst("f","-",-a);
(%o9)                       - a
(%i10) opsubst("^^","/",a/b);
                             a
(%o10)                       -
                             b
```

The internal representation of -a*b is *(-1,a,b); thus

```
(%i11) opsubst("[","*", -a*b);
(%o11)                   [- 1, a, b]
```

When either operator isn't a Maxima symbol, generally some other function will signal an error:

```
(%i12) opsubst(a+b,f, f(x));
```

```
Improper name or value in functional position:
b + a
 -- an error.  Quitting.  To debug this try debugmode(true);
```

However, subscripted operators are allowed:

```
(%i13) opsubst(g[5],f, f(x));
(%o13)                      g (x)
                             5
```

To use this function write first `load("opsubst")`.

85 Package orthopoly

85.1 Introduction to orthogonal polynomials

`orthopoly` is a package for symbolic and numerical evaluation of several kinds of orthogonal polynomials, including Chebyshev, Laguerre, Hermite, Jacobi, Legendre, and ultraspherical (Gegenbauer) polynomials. Additionally, `orthopoly` includes support for the spherical Bessel, spherical Hankel, and spherical harmonic functions.

For the most part, `orthopoly` follows the conventions of Abramowitz and Stegun *Handbook of Mathematical Functions*, Chapter 22 (10th printing, December 1972); additionally, we use Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (1980 corrected and enlarged edition), and Eugen Merzbacher *Quantum Mechanics* (2nd edition, 1970).

Barton Willis of the University of Nebraska at Kearney (UNK) wrote the `orthopoly` package and its documentation. The package is released under the GNU General Public License (GPL).

85.1.1 Getting Started with orthopoly

`load ("orthopoly")` loads the `orthopoly` package.

To find the third-order Legendre polynomial,

```
(%i1) legendre_p (3, x);
                        3             2
             5 (1 - x)    15 (1 - x)
(%o1)      - ---------- + ----------- - 6 (1 - x) + 1
                  2             2
```

To express this as a sum of powers of x, apply *ratsimp* or *rat* to the result.

```
(%i2) [ratsimp (%), rat (%)];
                          3           3
                       5 x  - 3 x  5 x  - 3 x
(%o2)/R/              [----------, ----------]
                           2           2
```

Alternatively, make the second argument to `legendre_p` (its "main" variable) a canonical rational expression (CRE).

```
(%i1) legendre_p (3, rat (x));
                                3
                             5 x  - 3 x
(%o1)/R/                     ----------
                                 2
```

For floating point evaluation, `orthopoly` uses a running error analysis to estimate an upper bound for the error. For example,

```
(%i1) jacobi_p (150, 2, 3, 0.2);
(%o1) interval(- 0.062017037936715, 1.533267919277521E-11)
```

Intervals have the form `interval (c, r)`, where c is the center and r is the radius of the interval. Since Maxima does not support arithmetic on intervals, in some situations, such

as graphics, you want to suppress the error and output only the center of the interval. To do this, set the option variable `orthopoly_returns_intervals` to `false`.

```
(%i1) orthopoly_returns_intervals : false;
(%o1)                          false
(%i2) jacobi_p (150, 2, 3, 0.2);
(%o2)                  - 0.062017037936715
```

Refer to the section see [Floating point Evaluation], page 1203, for more information.

Most functions in `orthopoly` have a `gradef` property; thus

```
(%i1) diff (hermite (n, x), x);
(%o1)                       2 n H     (x)
                                n - 1
(%i2) diff (gen_laguerre (n, a, x), x);
                (a)                (a)
           n L    (x) - (n + a) L     (x) unit_step(n)
              n                  n - 1
(%o2)      ----------------------------------------
                              x
```

The unit step function in the second example prevents an error that would otherwise arise by evaluating with n equal to 0.

```
(%i3) ev (%, n = 0);
(%o3)                           0
```

The `gradef` property only applies to the "main" variable; derivatives with respect other arguments usually result in an error message; for example

```
(%i1) diff (hermite (n, x), x);
(%o1)                       2 n H     (x)
                                n - 1
(%i2) diff (hermite (n, x), n);

Maxima doesn't know the derivative of hermite with respect the first
argument
 -- an error.  Quitting.  To debug this try debugmode(true);
```

Generally, functions in `orthopoly` map over lists and matrices. For the mapping to fully evaluate, the option variables `doallmxops` and `listarith` must both be `true` (the defaults). To illustrate the mapping over matrices, consider

```
(%i1) hermite (2, x);
                                     2
(%o1)                     - 2 (1 - 2 x )
(%i2) m : matrix ([0, x], [y, 0]);
                             [ 0  x ]
(%o2)                        [      ]
                             [ y  0 ]
(%i3) hermite (2, m);
                [                               2  ]
                [     - 2         - 2 (1 - 2 x )  ]
(%o3)           [                                 ]
```

```
                    [             2                  ]
                    [ - 2 (1 - 2 y )       - 2       ]
```

In the second example, the `i, j` element of the value is `hermite (2, m[i,j])`; this is not the same as computing `-2 + 4 m . m`, as seen in the next example.

```
(%i4) -2 * matrix ([1, 0], [0, 1]) + 4 * m . m;
                     [ 4 x y - 2      0     ]
(%o4)                [                      ]
                     [     0      4 x y - 2 ]
```

If you evaluate a function at a point outside its domain, generally `orthopoly` returns the function unevaluated. For example,

```
(%i1) legendre_p (2/3, x);
(%o1)                          P   (x)
                                2/3
```

`orthopoly` supports translation into TeX; it also does two-dimensional output on a terminal.

```
(%i1) spherical_harmonic (l, m, theta, phi);
                             m
(%o1)                       Y (theta, phi)
                             l
(%i2) tex (%);
$$Y_{l}^{m}\left(\vartheta,\varphi\right)$$
(%o2)                           false
(%i3) jacobi_p (n, a, a - b, x/2);
                              (a, a - b) x
(%o3)                        P           (-)
                              n           2
(%i4) tex (%);
$$P_{n}^{\left(a,a-b\right)}\left({{x}\over{2}}\right)$$
(%o4)                           false
```

85.1.2 Limitations

When an expression involves several orthogonal polynomials with symbolic orders, it's possible that the expression actually vanishes, yet Maxima is unable to simplify it to zero. If you divide by such a quantity, you'll be in trouble. For example, the following expression vanishes for integers n greater than 1, yet Maxima is unable to simplify it to zero.

```
(%i1) (2*n - 1) * legendre_p (n - 1, x) * x - n * legendre_p (n, x)
      + (1 - n) * legendre_p (n - 2, x);
(%o1)  (2 n - 1) P     (x) x - n P (x) + (1 - n) P     (x)
                  n - 1           n               n - 2
```

For a specific n, we can reduce the expression to zero.

```
(%i2) ev (% ,n = 10, ratsimp);
(%o2)                            0
```

Generally, the polynomial form of an orthogonal polynomial is ill-suited for floating point evaluation. Here's an example.

```
(%i1) p : jacobi_p (100, 2, 3, x)$
```

```
(%i2) subst (0.2, x, p);
(%o2)                3.4442767023833592E+35
(%i3) jacobi_p (100, 2, 3, 0.2);
(%o3)  interval(0.18413609135169, 6.8990300925815987E-12)
(%i4) float(jacobi_p (100, 2, 3, 2/10));
(%o4)                   0.18413609135169
```

The true value is about 0.184; this calculation suffers from extreme subtractive cancellation error. Expanding the polynomial and then evaluating, gives a better result.

```
(%i5) p : expand(p)$
(%i6) subst (0.2, x, p);
(%o6) 0.18413609766122982
```

This isn't a general rule; expanding the polynomial does not always result in an expression that is better suited for numerical evaluation. By far, the best way to do numerical evaluation is to make one or more of the function arguments floating point numbers. By doing that, specialized floating point algorithms are used for evaluation.

Maxima's `float` function is somewhat indiscriminate; if you apply `float` to an expression involving an orthogonal polynomial with a symbolic degree or order parameter, these parameters may be converted into floats; after that, the expression will not evaluate fully. Consider

```
(%i1) assoc_legendre_p (n, 1, x);
                                1
(%o1)                          P (x)
                                n
(%i2) float (%);
                               1.0
(%o2)                         P   (x)
                               n
(%i3) ev (%, n=2, x=0.9);
                              1.0
(%o3)                        P   (0.9)
                              2
```

The expression in (%o3) will not evaluate to a float; `orthopoly` doesn't recognize floating point values where it requires an integer. Similarly, numerical evaluation of the `pochhammer` function for orders that exceed `pochhammer_max_index` can be troublesome; consider

```
(%i1) x :  pochhammer (1, 10), pochhammer_max_index : 5;
(%o1)                          (1)
                                  10
```

Applying `float` doesn't evaluate x to a float

```
(%i2) float (x);
(%o2)                       (1.0)
                                 10.0
```

To evaluate x to a float, you'll need to bind `pochhammer_max_index` to 11 or greater and apply `float` to x.

```
(%i3) float (x), pochhammer_max_index : 11;
```

```
(%o3)                          3628800.0
```

The default value of `pochhammer_max_index` is 100; change its value after loading `orthopoly`.

Finally, be aware that reference books vary on the definitions of the orthogonal polynomials; we've generally used the conventions of Abramowitz and Stegun.

Before you suspect a bug in orthopoly, check some special cases to determine if your definitions match those used by `orthopoly`. Definitions often differ by a normalization; occasionally, authors use "shifted" versions of the functions that makes the family orthogonal on an interval other than $(-1, 1)$. To define, for example, a Legendre polynomial that is orthogonal on $(0, 1)$, define

```
(%i1) shifted_legendre_p (n, x) := legendre_p (n, 2*x - 1)$

(%i2) shifted_legendre_p (2, rat (x));
                             2
(%o2)/R/                  6 x  - 6 x + 1
(%i3) legendre_p (2, rat (x));
                               2
                            3 x  - 1
(%o3)/R/                    --------
                               2
```

85.1.3 Floating point Evaluation

Most functions in `orthopoly` use a running error analysis to estimate the error in floating point evaluation; the exceptions are the spherical Bessel functions and the associated Legendre polynomials of the second kind. For numerical evaluation, the spherical Bessel functions call SLATEC functions. No specialized method is used for numerical evaluation of the associated Legendre polynomials of the second kind.

The running error analysis ignores errors that are second or higher order in the machine epsilon (also known as unit roundoff). It also ignores a few other errors. It's possible (although unlikely) that the actual error exceeds the estimate.

Intervals have the form `interval (c, r)`, where c is the center of the interval and r is its radius. The center of an interval can be a complex number, and the radius is always a positive real number.

Here is an example.

```
(%i1) fpprec : 50$

(%i2) y0 : jacobi_p (100, 2, 3, 0.2);
(%o2) interval(0.1841360913516871, 6.8990300925815987E-12)
(%i3) y1 : bfloat (jacobi_p (100, 2, 3, 1/5));
(%o3) 1.8413609135168563091370224958913493690868904463668b-1
```

Let's test that the actual error is smaller than the error estimate

```
(%i4) is (abs (part (y0, 1) - y1) < part (y0, 2));
(%o4)                          true
```

Indeed, for this example the error estimate is an upper bound for the true error.

Maxima does not support arithmetic on intervals.

```
(%i1) legendre_p (7, 0.1) + legendre_p (8, 0.1);
(%o1) interval(0.18032072148437508, 3.1477135311021797E-15)
        + interval(- 0.19949294375000004, 3.3769353084291579E-15)
```

A user could define arithmetic operators that do interval math. To define interval addition, we can define

```
(%i1) infix ("@+")$

(%i2) "@+"(x,y) := interval (part (x, 1) + part (y, 1), part (x, 2)
      + part (y, 2))$

(%i3) legendre_p (7, 0.1) @+ legendre_p (8, 0.1);
(%o3) interval(- 0.019172222265624955, 6.5246488395313372E-15)
```

The special floating point routines get called when the arguments are complex. For example,

```
(%i1) legendre_p (10, 2 + 3.0*%i);
(%o1) interval(- 3.876378825E+7 %i - 6.0787748E+7,
                                          1.2089173052721777E-6)
```

Let's compare this to the true value.

```
(%i1) float (expand (legendre_p (10, 2 + 3*%i)));
(%o1)          - 3.876378825E+7 %i - 6.0787748E+7
```

Additionally, when the arguments are big floats, the special floating point routines get called; however, the big floats are converted into double floats and the final result is a double.

```
(%i1) ultraspherical (150, 0.5b0, 0.9b0);
(%o1) interval(- 0.043009481257265, 3.3750051301228864E-14)
```

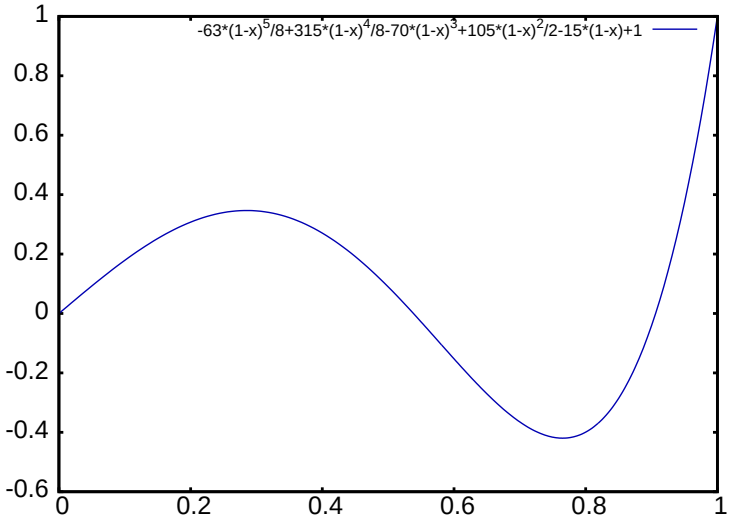
85.1.4 Graphics and `orthopoly`

To plot expressions that involve the orthogonal polynomials, you must do two things:

1. Set the option variable `orthopoly_returns_intervals` to `false`,
2. Quote any calls to `orthopoly` functions.

If function calls aren't quoted, Maxima evaluates them to polynomials before plotting; consequently, the specialized floating point code doesn't get called. Here is an example of how to plot an expression that involves a Legendre polynomial.

```
(%i1) plot2d ('(legendre_p (5, x)), [x, 0, 1]),
                        orthopoly_returns_intervals : false;
(%o1)
```



The *entire* expression `legendre_p (5, x)` is quoted; this is different than just quoting the function name using `'legendre_p (5, x)`.

85.1.5 Miscellaneous Functions

The `orthopoly` package defines the Pochhammer symbol and a unit step function. `orthopoly` uses the Kronecker delta function and the unit step function in `gradef` statements.

To convert Pochhammer symbols into quotients of gamma functions, use `makegamma`.

```
(%i1) makegamma (pochhammer (x, n));
                            gamma(x + n)
(%o1)                       ------------
                              gamma(x)
(%i2) makegamma (pochhammer (1/2, 1/2));
                                  1
(%o2)                           ---------
                                sqrt(%pi)
```

Derivatives of the Pochhammer symbol are given in terms of the `psi` function.

```
(%i1) diff (pochhammer (x, n), x);
(%o1)              (x)  (psi (x + n) - psi (x))
                     n     0             0
(%i2) diff (pochhammer (x, n), n);
(%o2)                     (x)  psi (x + n)
                            n    0
```

You need to be careful with the expression in (%o1); the difference of the `psi` functions has polynomials when *x* = `-1, -2, .., -`*n*. These polynomials cancel with factors in `pochhammer (`*x*`, `*n*`)` making the derivative a degree *n* `-` `1` polynomial when *n* is a positive integer.

The Pochhammer symbol is defined for negative orders through its representation as a quotient of gamma functions. Consider

```
(%i1) q : makegamma (pochhammer (x, n));
```

```
                              gamma(x + n)
(%o1)                         ------------
                                gamma(x)
(%i2) sublis ([x=11/3, n= -6], q);
                                   729
(%o2)                            - ----
                                   2240
```

Alternatively, we can get this result directly.

```
(%i1) pochhammer (11/3, -6);
                                   729
(%o1)                            - ----
                                   2240
```

The unit step function is left-continuous; thus

```
(%i1) [unit_step (-1/10), unit_step (0), unit_step (1/10)];
(%o1)                          [0, 0, 1]
```

If you need a unit step function that is neither left or right continuous at zero, define your own using `signum`; for example,

```
(%i1) xunit_step (x) := (1 + signum (x))/2$

(%i2) [xunit_step (-1/10), xunit_step (0), xunit_step (1/10)];
                                    1
(%o2)                          [0, -, 1]
                                    2
```

Do not redefine `unit_step` itself; some code in `orthopoly` requires that the unit step function be left-continuous.

85.1.6 Algorithms

Generally, `orthopoly` does symbolic evaluation by using a hypergeometic representation of the orthogonal polynomials. The hypergeometic functions are evaluated using the (undocumented) functions `hypergeo11` and `hypergeo21`. The exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. The half-integer Bessel functions are evaluated using an explicit representation, and the associated Legendre function of the second kind is evaluated using recursion.

For floating point evaluation, we again convert most functions into a hypergeometic form; we evaluate the hypergeometic functions using forward recursion. Again, the exceptions are the half-integer Bessel functions and the associated Legendre function of the second kind. Numerically, the half-integer Bessel functions are evaluated using the SLATEC code.

85.2 Functions and Variables for orthogonal polynomials

`assoc_legendre_p` (*n*, *m*, *x*) [Function]

The associated Legendre function of the first kind of degree n and order m, $P_n^m(z)$, is a solution of the differential equation:

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2}\right] w = 0$$

This is related to the Legendre polynomial, $P_n(x)$ via

$$P_n^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

Reference: A&S eqn 22.5.37, A&S eqn 8.6.6, and A&S eqn 8.2.5.

Some examples:

```
(%i1) assoc_legendre_p(2,0,x);
                                                 2
                                          3 (1 - x)
(%o1)                  (- 3 (1 - x)) + ---------- + 1
                                             2
(%i2) factor(%);
                                        2
                                     3 x  - 1
(%o2)                                --------
                                        2
(%i3) factor(assoc_legendre_p(2,1,x));
                                                2
(%o3)                          - 3 x sqrt(1 - x )

(%i4) (-1)^1*(1-x^2)^(1/2)*diff(legendre_p(2,x),x);
                                                    2
(%o4)                  - (3 - 3 (1 - x)) sqrt(1 - x )

(%i5) factor(%);
                                                2
(%o5)                          - 3 x sqrt(1 - x )
```

`assoc_legendre_q` (*n*, *m*, *x*) [Function]

The associated Legendre function of the second kind of degree n and order m, $Q_n^m(z)$, is a solution of the differential equation:

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2}\right] w = 0$$

Reference: Abramowitz and Stegun, equation 8.5.3 and 8.1.8.

Some examples:

```
(%i1) assoc_legendre_q(0,0,x);
                                       x + 1
                                  log(- -----)
                                       x - 1
(%o1)                             ------------
                                       2
(%i2) assoc_legendre_q(1,0,x);
                                     x + 1
                                log(- -----) x - 2
```

```
                                        x - 1
(%o2)/R/                          ------------------
                                          2
(%i3) assoc_legendre_q(1,1,x);
(%o3)/R/
          x + 1            2   2                2            x + 1            2
    log(- -----) sqrt(1 - x ) x  - 2 sqrt(1 - x ) x - log(- -----) sqrt(1 - x )
          x - 1                                             x - 1
  - ---------------------------------------------------------------------------
                                          2
                                        2 x  - 2
```

chebyshev_t (*n*, *x*) [Function]

The Chebyshev polynomial of the first kind of degree n, $T_n(x)$.

Reference: A&S eqn 22.5.47.

The polynomials $T_n(x)$ can be written in terms of a hypergeometric function:

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right)$$

The polynomials can also be defined in terms of the sum

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r}{n-r} \binom{n-r}{k} (2x)^{n-2r}$$

or the Rodrigues formula

$$T_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x)(1-x^2)^n\right)$$

where

$$w(x) = 1/\sqrt{1-x^2}$$
$$\kappa_n = (-2)^n \left(\frac{1}{2}\right)_n$$

Some examples:

```
(%i1) chebyshev_t(2,x);
                                                  2
(%o1)                    (- 4 (1 - x)) + 2 (1 - x)  + 1
(%i2) factor(%);
                                        2
(%o2)                                 2 x  - 1
(%i3) factor(chebyshev_t(3,x));
                                          2
(%o3)                               x (4 x  - 3)
(%i4) factor(hgfred([-3,3],[1/2],(1-x)/2));
                                          2
(%o4)                               x (4 x  - 3)
```

chebyshev_u (*n*, *x*) [Function]

The Chebyshev polynomial of the second kind of degree n, $U_n(x)$.

Reference: A&S eqn 22.5.48.

The polynomials $U_n(x)$ can be written in terms of a hypergeometric function:

$$U_n(x) = (n+1)\ {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right)$$

The polynomials can also be defined in terms of the sum

$$U_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r}$$

or the Rodrigues formula

$$U_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x)(1-x^2)^n\right)$$

where

$$w(x) = \sqrt{1-x^2}$$
$$\kappa_n = \frac{(-2)^n \left(\frac{3}{2}\right)_n}{n+1}$$

.

```
(%i1) chebyshev_u(2,x);
                                                        2
                                8 (1 - x)    4 (1 - x)
(%o1)                   3 ((- ---------) + ---------- + 1)
                                    3            3
(%i2) expand(%);
                                          2
(%o2)                                  4 x  - 1
(%i3) expand(chebyshev_u(3,x));
                                         3
(%o3)                                 8 x  - 4 x
(%i4) expand(4*hgfred([-3,5],[3/2],(1-x)/2));
                                         3
(%o4)                                 8 x  - 4 x
```

gen_laguerre (*n*, *a*, *x*) [Function]

The generalized Laguerre polynomial of degree n, $L_n^{(\alpha)}(x)$.

These can be defined by

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}\ {}_1F_1(-n; \alpha+1; x)$$

The polynomials can also be defined by the sum

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(\alpha+k+1)_{n-k}}{(n-k)!k!}(-x)^k$$

or the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n}\left(w(x)x^n\right)$$

where

$$w(x) = e^{-x}x^{\alpha}$$
$$\kappa_n = n!$$

Reference: A&S eqn 22.5.54.

Some examples:

```
(%i1) gen_laguerre(1,k,x);
                                                  x
(%o1)                              (k + 1) (1 - -----)
                                                k + 1
(%i2) gen_laguerre(2,k,x);
                                              2
                                             x            2 x
                   (k + 1) (k + 2) (--------------- - ----- + 1)
                                    (k + 1) (k + 2)   k + 1
(%o2)              ---------------------------------------------
                                            2
(%i3) binomial(2+k,2)*hgfred([-2],[1+k],x);
                                              2
                                             x            2 x
                   (k + 1) (k + 2) (--------------- - ----- + 1)
                                    (k + 1) (k + 2)   k + 1
(%o3)              ---------------------------------------------
                                            2
```

hermite (*n*, *x*) [Function]

The Hermite polynomial of degree n, $H_n(x)$.

These polynomials may be defined by a hypergeometric function

$$H_n(x) = (2x)^n \, {}_2F_0\left(-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}; ; -\frac{1}{x^2}\right)$$

or by the series

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}$$

or the Rodrigues formula

$$H_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x) \right)$$

where

$$w(x) = e^{-x^2/2}$$
$$\kappa_n = (-1)^n$$

Reference: A&S eqn 22.5.55.

Some examples:

```
(%i1) hermite(3,x);
                                                2
                                              2 x
(%o1)                          - 12 x (1 - ----)
                                               3
(%i2) expand(%);
                                     3
(%o2)                             8 x  - 12 x
(%i3) expand(hermite(4,x));
                                     4       2
(%o3)                           16 x  - 48 x  + 12
(%i4) expand((2*x)^4*hgfred([-2,-2+1/2],[],-1/x^2));
                                     4       2
(%o4)                           16 x  - 48 x  + 12
(%i5) expand(4!*sum((-1)^k*(2*x)^(4-2*k)/(k!*(4-2*k)!),k,0,floor(4/2)));
                                     4       2
(%o5)                           16 x  - 48 x  + 12
```

`intervalp (e)` [Function]

Return `true` if the input is an interval and return false if it isn't.

`jacobi_p (n, a, b, x)` [Function]

The Jacobi polynomial, $P_n^{(a,b)}(x)$.

The Jacobi polynomials are actually defined for all a and b; however, the Jacobi polynomial weight $(1-x)^a(1+x)^b$ isn't integrable for $a \leq -1$ or $b \leq -1$.

Reference: A&S eqn 22.5.42.

The polynomial may be defined in terms of hypergeometric functions:

$$P_n^{(a,b)}(x) = \binom{n+a}{n} {}_1F_2\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right)$$

or the Rodrigues formula

$$P_n^{(a,b)}(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x) \left(1-x^2\right)^n \right)$$

where

$$w(x) = (1-x)^a (1-x)^b$$
$$\kappa_n = (-2)^n n!$$

Some examples:

```
(%i1) jacobi_p(0,a,b,x);
(%o1)                                  1
(%i2) jacobi_p(1,a,b,x);
                                     (b + a + 2) (1 - x)
(%o2)                  (a + 1) (1 - ------------------)
                                          2 (a + 1)
```

`laguerre` (*n*, *x*) [Function]

The Laguerre polynomial, $L_n(x)$ of degree n.

Reference: A&S eqn 22.5.16 and A&S eqn 22.5.54.

These are related to the generalized Laguerre polynomial by

$$L_n(x) = L_n^{(0)}(x)$$

The polynomials are given by the sum

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k$$

Some examples:

```
(%i1) laguerre(1,x);
(%o1)                                 1 - x
(%i2) laguerre(2,x);
                                       2
                                      x
(%o2)                                 -- - 2 x + 1
                                      2
(%i3) gen_laguerre(2,0,x);
                                       2
                                      x
(%o3)                                 -- - 2 x + 1
                                      2
(%i4) sum((-1)^k/k!*binomial(2,k)*x^k,k,0,2);
                                       2
                                      x
(%o4)                                 -- - 2 x + 1
                                      2
```

`legendre_p` (*n*, *x*) [Function]

The Legendre polynomial of the first kind, $P_n(x)$, of degree n.

Reference: A&S eqn 22.5.50 and A&S eqn 22.5.51.

The Legendre polynomial is related to the Jacobi polynomials by

$$P_n(x) = P_n^{(0,0)}(x)$$

or the Rodrigues formula

$$P_n(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x) \left(1 - x^2\right)^n \right)$$

where

$$w(x) = 1$$
$$\kappa_n = (-2)^n n!$$

Some examples:

```
(%i1) legendre_p(1,x);
(%o1)                                  x
(%i2) legendre_p(2,x);
                                                  2
                                          3 (1 - x)
(%o2)                   (- 3 (1 - x)) + ---------- + 1
                                              2
(%i3) expand(%);
                                        2
                                     3 x    1
(%o3)                                ---- - -
                                      2     2
(%i4) expand(legendre_p(3,x));
                                       3
                                    5 x    3 x
(%o4)                               ---- - ---
                                     2      2
(%i5) expand(jacobi_p(3,0,0,x));
                                       3
                                    5 x    3 x
(%o5)                               ---- - ---
                                     2      2
```

legendre_q (*n*, *x*) [Function]

The Legendre function of the second kind, $Q_n(x)$ of degree n.

Reference: Abramowitz and Stegun, equations 8.5.3 and 8.1.8.

These are related to $Q_n^m(x)$ by

$$Q_n(x) = Q_n^0(x)$$

Some examples:

```
(%i1) legendre_q(0,x);
```

```
                                          x + 1
                                     log(- -----)
                                          x - 1
(%o1)                                ------------
                                          2

(%i2) legendre_q(1,x);
                                        x + 1
                                   log(- -----) x - 2
                                        x - 1
(%o2)/R/                           ------------------
                                          2

(%i3) assoc_legendre_q(1,0,x);
                                        x + 1
                                   log(- -----) x - 2
                                        x - 1
(%o3)/R/                           ------------------
                                          2
```

`orthopoly_recur` (*f*, *args*) [Function]

Returns a recursion relation for the orthogonal function family *f* with arguments *args*. The recursion is with respect to the polynomial degree.

```
(%i1) orthopoly_recur (legendre_p, [n, x]);
                     (2 n + 1) P (x) x - n P     (x)
                               n           n - 1
(%o1)   P     (x) = ------------------------------
         n + 1                  n + 1
```

The second argument to `orthopoly_recur` must be a list with the correct number of arguments for the function *f*; if it isn't, Maxima signals an error.

```
(%i1) orthopoly_recur (jacobi_p, [n, x]);

Function jacobi_p needs 4 arguments, instead it received 2
 -- an error.  Quitting.  To debug this try debugmode(true);
```

Additionally, when *f* isn't the name of one of the families of orthogonal polynomials, an error is signalled.

```
(%i1) orthopoly_recur (foo, [n, x]);

A recursion relation for foo isn't known to Maxima
 -- an error.  Quitting.  To debug this try debugmode(true);
```

`orthopoly_returns_intervals` [Variable]

Default value: `true`

When `orthopoly_returns_intervals` is `true`, floating point results are returned in the form `interval (c, r)`, where *c* is the center of an interval and *r* is its radius. The center can be a complex number; in that case, the interval is a disk in the complex plane.

`orthopoly_weight` (*f*, *args*) [Function]

Returns a three element list; the first element is the formula of the weight for the orthogonal polynomial family *f* with arguments given by the list *args*; the second and third elements give the lower and upper endpoints of the interval of orthogonality. For example,

```
(%i1) w : orthopoly_weight (hermite, [n, x]);
                              2
                           - x
(%o1)                  [%e    , - inf, inf]
(%i2) integrate(w[1]*hermite(3, x)*hermite(2, x), x, w[2], w[3]);
(%o2)                           0
```

The main variable of *f* must be a symbol; if it isn't, Maxima signals an error.

`pochhammer` (*x*, *n*) [Function]

The Pochhammer symbol, $(x)_n$. (See A&S eqn 6.1.22 and DLMF 5.2.iii).

For nonnegative integers n with *n* `<=` `pochhammer_max_index`, the expression $(x)_n$ evaluates to the product $x(x+1)(x+2)\cdots(x+n-1)$ when $n > 0$ and to 1 when $n = 0$. For negative n, $(x)_n$ is defined as $(-1)^n/(1-x)_{-n}$. Thus

```
(%i1) pochhammer (x, 3);
(%o1)                  x (x + 1) (x + 2)
(%i2) pochhammer (x, -3);
                                1
(%o2)             - ------------------------
                    (1 - x) (2 - x) (3 - x)
```

To convert a Pochhammer symbol into a quotient of gamma functions, (see A&S eqn 6.1.22) use `makegamma`; for example

```
(%i1) makegamma (pochhammer (x, n));
                           gamma(x + n)
(%o1)                      ------------
                             gamma(x)
```

When *n* exceeds `pochhammer_max_index` or when *n* is symbolic, `pochhammer` returns a noun form.

```
(%i1) pochhammer (x, n);
(%o1)                          (x)
                                 n
```

`pochhammer_max_index` [Variable]

Default value: 100

`pochhammer (n, x)` expands to a product if and only if *n* `<=` `pochhammer_max_index`.

Examples:

```
(%i1) pochhammer (x, 3), pochhammer_max_index : 3;
(%o1)                  x (x + 1) (x + 2)
(%i2) pochhammer (x, 4), pochhammer_max_index : 3;
(%o2)                          (x)
                                 4
```

Reference: A&S eqn 6.1.16.

`spherical_bessel_j (n, x)` [Function]

The spherical Bessel function of the first kind, $j_n(x)$.

Reference: A&S eqn 10.1.8 and A&S eqn 10.1.15.

It is related to the Bessel function by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$

Some examples:

```
(%i1) spherical_bessel_j(1,x);
                                  sin(x)
                                  ------ - cos(x)
                                    x
(%o1)                             ---------------
                                         x
(%i2) spherical_bessel_j(2,x);
                                  3             3 cos(x)
                          (- (1 - --) sin(x)) - --------
                                   2               x
                                  x
(%o2)                     -----------------------------
                                        x
(%i3) expand(%);
                             sin(x)    3 sin(x)   3 cos(x)
(%o3)                     (- ------) + -------- - --------
                               x           3          2
                                          x          x
(%i4) expand(sqrt(%pi/(2*x))*bessel_j(2+1/2,x)),besselexpand:true;
                             sin(x)    3 sin(x)   3 cos(x)
(%o4)                     (- ------) + -------- - --------
                               x           3          2
                                          x          x
```

`spherical_bessel_y (n, x)` [Function]

The spherical Bessel function of the second kind, $y_n(x)$.

Reference: A&S eqn 10.1.9 and A&S eqn 10.1.15.

It is related to the Bessel function by

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x)$$

```
(%i1) spherical_bessel_y(1,x);
                                          cos(x)
                                  (- sin(x)) - ------
                                                 x
(%o1)                             ------------------
                                         x
```

```
(%i2) spherical_bessel_y(2,x);
                              3 sin(x)        3
                              -------- - (1 - --) cos(x)
                                 x             2
                                              x
(%o2)                       - --------------------------
                                          x
(%i3) expand(%);
                             3 sin(x)    cos(x)   3 cos(x)
(%o3)                   (- --------) + ------ - --------
                               2          x          3
                              x                     x
(%i4) expand(sqrt(%pi/(2*x))*bessel_y(2+1/2,x)),besselexpand:true;
                             3 sin(x)    cos(x)   3 cos(x)
(%o4)                   (- --------) + ------ - --------
                               2          x          3
                              x                     x
```

`spherical_hankel1` (*n*, *x*) [Function]

The spherical Hankel function of the first kind, $h_n^{(1)}(x)$.

Reference: A&S eqn 10.1.36.

This is defined by

$$h_n^{(1)}(x) = j_n(x) + iy_n(x)$$

`spherical_hankel2` (*n*, *x*) [Function]

The spherical Hankel function of the second kind, $h_n^{(2)}(x)$.

Reference: A&S eqn 10.1.17.

This is defined by

$$h_n^{(2)}(x) = j_n(x) + iy_n(x)$$

`spherical_harmonic` (*n*, *m*, *theta*, *phi*) [Function]

The spherical harmonic function, $Y_n^m(\theta, \phi)$.

Spherical harmonics satisfy the angular part of Laplace's equation in spherical coordinates.

For integers n and m such that $n \geq |m|$ and for $\theta \in [0, \pi]$. Maxima's spherical harmonic function can be defined by

$$Y_n^m(\theta, \phi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}$$

Further, when $n < |m|$, the spherical harmonic function vanishes.

The factor $(-1)^m$, frequently used in Quantum mechanics, is called the Condon-Shortely phase. Some references, including *NIST Digital Library of Mathematical Functions* omit this factor; see `http://dlmf.nist.gov/14.30.E1`.

Reference: Merzbacher 9.64.

Some examples:

```
(%i1) spherical_harmonic(1,0,theta,phi);
                                sqrt(3) cos(theta)
(%o1)                           ------------------
                                   2 sqrt(%pi)
(%i2) spherical_harmonic(1,1,theta,phi);
                                      %i phi
                            sqrt(3) %e       sin(theta)
(%o2)                       ---------------------------
                                   3/2
                                 2    sqrt(%pi)
(%i3) spherical_harmonic(1,-1,theta,phi);
                                      - %i phi
                            sqrt(3) %e         sin(theta)
(%o3)                     - ----------------------------
                                    3/2
                                   2    sqrt(%pi)
(%i4) spherical_harmonic(2,0,theta,phi);
                                                                2
                                                3 (1 - cos(theta))
          sqrt(5) ((- 3 (1 - cos(theta))) + ------------------- + 1)
                                                        2
(%o4)     ----------------------------------------------------------
                                   2 sqrt(%pi)
(%i5) factor(%);
                                          2
                            sqrt(5) (3 cos (theta) - 1)
(%o5)                       ---------------------------
                                    4 sqrt(%pi)
```

unit_step (*x*) [Function]

The left-continuous unit step function; thus `unit_step (x)` vanishes for `x <= 0` and equals 1 for `x > 0`.

If you want a unit step function that takes on the value 1/2 at zero, use `hstep`.

ultraspherical (*n*, *a*, *x*) [Function]

The ultraspherical polynomial, $C_n^{(a)}(x)$ (also known as the Gegenbauer polynomial).

Reference: A&S eqn 22.5.46.

These polynomials can be given in terms of Jacobi polynomials:

$$C_n^{(\alpha)}(x) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)} \frac{\Gamma(n+2\alpha)}{\Gamma\left(n+\alpha+\frac{1}{2}\right)} P_n^{(\alpha-1/2,\alpha-1/2)}(x)$$

or the series

$$C_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\alpha)_{n-k}}{k!(n-2k)!} (2x)^{n-2k}$$

or the Rodrigues formula

$$C_n^{(\alpha)}(x) = \frac{1}{\kappa_n w(x)} \frac{d^n}{dx^n} \left(w(x) \left(1-x^2\right)^n \right)$$

where

$$w(x) = \left(1-x^2\right)^{\alpha - \frac{1}{2}}$$
$$\kappa_n = \frac{(-2)^n \left(\alpha + \frac{1}{2}\right)_n n!}{(2\alpha)_n}$$

Some examples:

```
(%i1) ultraspherical(1,a,x);
                                      (2 a + 1) (1 - x)
(%o1)                        2 a (1 - -----------------)
                                                1
                                           2 (a + -)
                                                2
(%i2) factor(%);
(%o2)                                    2 a x
(%i3) factor(ultraspherical(2,a,x));
                                        2      2
(%o3)                          a (2 a x  + 2 x  - 1)
```

86 Package pslq

86.1 Introduction to pslq

Package `pslq` contains two functions.

(1) `guess_exact_value` tries to find an exact equivalent for an inexact number (float or bigfloat).

(2) `pslq_integer_relation` tries to find integer coefficients such that a linear combination of inexact numbers is approximately zero.

86.2 Functions and Variables for pslq

`guess_exact_value` (*x*) [Function]

When *x* is a floating point number or bigfloat, `guess_exact_value` tries to find an exact expression (in terms of radicals, logarithms, exponentials, and the constant `%pi`) which is nearly equal to the given number. If `guess_exact_value` cannot find such an expression, *x* is returned unchanged.

When *x* is rational number or other mapatom (other than a float or bigfloat), *x* is returned unchanged.

Otherwise, *x* is a nonatomic expression, and `guess_exact_value` is applied to each of the arguments of *x*.

Example:

```
(%i1) load ("pslq.mac");
(%o1)                         pslq.mac
(%i2) root: float (sin (%pi/12));
(%o2)                    0.2588190451025207
(%i3) guess_exact_value (root);
                          sqrt(2 - sqrt(3))
(%o3)                     -----------------
                                  2
(%i4) L: makelist (root^i, i, 0, 4);
(%o4) [1.0, 0.2588190451025207, 0.06698729810778066,
                       0.01733758853025369, 0.004487298107780675]
(%i5) m: pslq_integer_relation(%);
(%o5)                  [- 1, 0, 16, 0, - 16]
(%i6) makelist (x^i, i, 0, 4) . m;
                               4        2
(%o6)                  (- 16 x ) + 16 x  - 1
(%i7) solve(%);
              sqrt(sqrt(3) + 2)      sqrt(sqrt(3) + 2)
(%o7) [x = - -----------------, x = -----------------,
                     2                      2
                        sqrt(2 - sqrt(3))      sqrt(2 - sqrt(3))
                   x = - -----------------, x = -----------------]
                                  2                      2
```

`pslq_integer_relation` (*L*) [Function]

Implements the PSLQ algorithm [1] to find integer relations between bigfloat numbers.

For a given list *L* of floating point numbers, `pslq_integer_relation` returns a list of integers *m* such that `m . L = 0` (with absolute residual error less than `pslq_threshold`).

[1] D.H.Bailey: Integer Relation Detection and Lattice Reduction.

Example:

```
(%i1) load ("pslq.mac");
(%o1)                         pslq.mac
(%i2) root: float (sin (%pi/12));
(%o2)                    0.2588190451025207
(%i3) L: makelist (root^i, i, 0, 4);
(%o3) [1.0, 0.2588190451025207, 0.06698729810778066,
                        0.01733758853025369, 0.004487298107780675]
(%i4) m: pslq_integer_relation(%);
(%o4)                  [- 1, 0, 16, 0, - 16]
(%i5) m . L;
(%o5)                - 2.359223927328458E-16
(%i6) float (10^(2 - fpprec));
(%o6)                           1.0E-14
(%i7) is (abs (m . L) < 10^(2 - fpprec));
(%o7)                            true
```

`pslq_precision` [Variable]

Default value: `10^(fpprec - 2)`

Maximum magnitude of some intermediate results in `pslq_integer_relation`. The search fails if one of the intermediate results has elements larger than `pslq_precision`.

`pslq_threshold` [Variable]

Default value: `10^(2 - fpprec)`

Threshold for absolute residual error of integer relation found by `pslq_integer_relation`.

`pslq_depth` [Variable]

Default value: `20 * n`

Number of iterations of the PSLQ algorithm.

The default value is 20 times *n*, where *n* is the length of the list of numbers supplied to `pslq_integer_relation`.

`pslq_status` [Variable]

Indicates success or failure for an integer relation search by `pslq_integer_relation`.

When `pslq_status` is 1, it indicates an integer relation was found, and the absolute residual error is less than `pslq_threshold`.

When `pslq_status` is 2, it indicates an integer relation was not found because some intermediate results are larger than `pslq_precision`.

When `pslq_status` is 3, it indicates an integer relation was not found because the number of iterations `pslq_depth` was reached.

87 Package pytranslate

87.1 Introduction to pytranslate

`pytranslate` package provides Maxima to Python translation functionality. The package is experimental, and the specifications of the functions in this package might change. It was written as a Google Summer of Code project by Lakshya A Agrawal (Undergraduate Student, IIIT-Delhi) in 2019. A detailed project report is available as a GitHub Gist.

The package needs to be loaded in a Maxima instance for use, by executing `load("pytranslate");`
The statements are converted to python3 syntax. The file pytranslate.py must be imported for all translations to run, as shown in example.

Example:

```
(%i1) load ("pytranslate")$
/* Define an example function to calculate factorial */
(%i2) pytranslate(my_factorial(x) := if (x = 1 or x = 0) then 1
                  else x * my_factorial(x - 1));
(%o2)
def my_factorial(x, v = v):
    v = Stack({}, v)
    v.ins({"x" : x})
    return((1 if ((v["x"] == 1) or (v["x"] == 0)) \
              else (v["x"] * my_factorial((v["x"] + (-1))))))
m["my_factorial"] = my_factorial
(%i3) my_factorial(5);
(%o3)                          120
>>> from pytranslate import *
>>> def my_factorial(x, v = v):
...     v = Stack({}, v)
...     v.ins({"x" : x})
...     return((1 if ((v["x"] == 1) or (v["x"] == 0)) \
...     else (v["x"] * my_factorial((v["x"] + (-1))))))
...
>>> my_factorial(5)
120
```

The Maxima to Python Translator works in two stages:
1. Conversion of the internal Maxima representation to a defined Intermediate Representation, henceforth referred as IR(mapping is present in `share/pytranslate/maxima-to-ir.html`)
2. The conversion of IR to Python.

Supported Maxima forms:
1. Section 7.1 Numbers(including complex numbers)
2. Section 9.6 Assignment operators
3. Section 9.2 Arithmetic operators(+, -, *, ^, /, !)

4. Section 9.4 Logical operators(and, or, not)
5. Section 9.3 Relational operators(>, <, >=, <=, !=, ==)
6. Section 7.4 Lists
7. Section 7.5 Arrays
8. block
9. Section 32.2 Function and function calls
10. if-else converted to Python conditionals
11. for loops
12. lambda form

87.1.1 Tests for pytranslate

The tests for pytranslate are present at share/pytranslate/rtest_pytranslate.mac and can be run by executing batch(rtest_pytranslate, test);

87.2 Functions in pytranslate

pytranslate (*expr*, [*print-ir*]) [Function]

Translates the expression *expr* to equivalent python3 statements. Output is printed in the stdout.

Example:

```
(%i1) load ("pytranslate")$
(%i2) pytranslate('(for i:8 step -1 unless i<3 do (print(i))));
(%o2)
v["i"] = 8
while not((v["i"] < 3)):
    m["print"](v["i"])
    v["i"] = (v["i"] + -1)
del v["i"]
```

expr is evaluated, and the return value is used for translation. Hence, for statements like assignment, it might be useful to quote the statement:

```
(%i1) load ("pytranslate")$
(%i2) pytranslate(x:20);
(%o2)
20
(%i3) pytranslate('(x:20));
(%o3)
v["x"] = 20
```

Passing the optional parameter (*print-ir*) to pytranslate as t, will print the internal IR representation of expr and return the translated python3 code.

```
(%i1) load("pytranslate");
(%o1) pytranslate
```

```
(%i2) pytranslate('(plot3d(lambda([x, y], x^2+y^(-1)), [x, 1, 10],
                    [y, 1, 10])), t);
(body
 (funcall (element-array "m" (string "plot3d"))
          (lambda
              ((symbol "x") (symbol "y")
               (op-no-bracket
                =
                (symbol "v")
                (funcall (symbol "stack") (dictionary) (symbol "v"))))
            (op +
                (funcall (element-array (symbol "m") (string "pow"))
                         (symbol "x") (num 2 0))
                (funcall (element-array (symbol "m") (string "pow"))
                         (symbol "y") (unary-op - (num 1 0)))))
          (struct-list (string "x") (num 1 0) (num 10 0))
          (struct-list (string "y") (num 1 0) (num 10 0))))
(%o2)
m["plot3d"](lambda x, y, v = Stack({}, v): (m["pow"](x, 2) + m["\
pow"](y, (-1))), ["x", 1, 10], ["y", 1, 10])
```

`show_form` (*expr*) [Function]

Displays the internal maxima form of `expr`

```
(%i4) show_form(a^b);
((mexpt) $a $b)
(%o4) a^b
```

87.3 Extending pytranslate

Working of pytranslate:

- The entry point for pytranslate is the function `$pytranslate` defined in `share/pytranslate/pytranslate.lisp`.
- `$pytranslate` calls the function `maxima-to-ir` with the Maxima expression as an argument(henceforth referred as `expr`).
- `maxima-to-ir` determines if `expr` is atomic or non-atomic(lisp cons form). If atomic, `atom-to-ir` is called with `expr` which returns the IR for the atomic expression.
  To define/modify translation for atomic expressions, make changes to the definition of `atom-to-ir` in accordance with the IR.
- If `expr` is non-atomic, the function `cons-to-ir` is called with `expr` as an argument.

  - `cons-to-ir` looks for `(caar expr)` which specifies the type of `expr`, in hash-table **maxima-direct-ir-map** and if the type is found, then appends the retrieved IR with the result of lisp call `(mapcar #'maxima-to-ir (cdr expr))`, which applies maxima-to-ir function to all the elements present in the list. Effectively, recursively generate IR for all the elements present in `expr` and append them to the IR map

for the type.
Example:

```
(%i9) show_form(a+b);
((MPLUS) $B $A)
(%i10) pytranslate(a+b, t);
(body (op + (element-array (symbol "v") (string "b")) \
(element-array (symbol "v") (string "a"))))
(%o10)
(v["b"] + v["a"])
```

Here, operator `+` with internal maxima representation, `(mplus)` is present in **maxima-direct-ir-map** and mapped to `(op +)` to which the result of generating IR for all other elements of the list (a b), i.e. `(ELEMENT-ARRAY (SYMBOL "v") (STRING "b")) (ELEMENT-ARRAY (SYMBOL "v") (STRING "a"))` is appended.

- If `(caar expr)` is not found in **maxima-direct-ir-map**, then `cons-to-ir` looks for the type in **maxima-special-ir-map** which returns the function to handle the translation of the type of `expr`. `cons-to-ir` then calls the returned function with argument `expr` as an argument.
  Example:

```
(%i11) show_form(g(x) := x^2);
((mdefine simp) (($g) $x) ((mexpt) $x 2))
(%i12) pytranslate(g(x):=x^2, t);
(body
 (body
  (func-def (symbol "g")
            ((symbol "x") (op-no-bracket = (symbol "v") (symbol "v")))
            (body-indented
                (op-no-bracket = (symbol "v") (funcall (symbol "stack") \
                (dictionary) (symbol "v")))
                (obj-funcall (symbol "v") (symbol "ins") (dictionary \
                ((string "x") (symbol "x"))))
                (funcall (symbol "return")
                    (funcall (element-array (symbol "f") (string "pow"))
                         (element-array (symbol "v") (string "x"))
                                  (num 2 0))))))
  (op-no-bracket = (element-array (symbol "f") (string "g")) \
  (symbol "g"))))
(%o12)
def g(x, v = v):
    v = Stack({}, v)
    v.ins({"x" : x})
    return(f["pow"](v["x"], 2))
f["g"] = g
```

Here, `mdefine`, which is the type of `expr` is present in **maxima-special-ir-map** which returns `func-def-to-ir` as handler function, which is then called with `expr`

to generate the IR.
To define/modify translation for a type, add an entry to **maxima-direct-ir-map** if only a part of the IR needs to be generated and the rest can be appended, otherwise, for complete handling of `expr`, add an entry to **maxima-special-ir-map** and define a function with the name defined in **maxima-special-ir-map** which returns the IR for the form. The function naming convention for ir generators is (type)-to-ir, where type is the `(caar expr)` for expression(`mdefine -> func-def-to-ir`). The function must return a valid IR for the specific type.

- After the generation of IR, the function `ir-to-python` is called with the generated `ir` as an argument, which performs the codegen in a recursive manner.
  - `ir-to-python` looks for lisp `(car ir)` in the hash-table **ir-python-direct-templates**, which maps IR type to function handlers and calls the function returned with `ir` as an argument.
- To extend the IR of pytranslate, define a function with the naming convention (type)-to-python and add the name to **ir-python-direct-templates**.

88 Package quantum_computing

88.1 Package quantum_computing

The `quantum_computing` package provides several functions to simulate quantum computing circuits. The state of a system of n qubits is represented by a list of 2^n complex numbers and an operator acting on m qubits is represented by a 2^m by 2^m matrix. A hash array *qmatrix* is defined with 6 common one-qubit matrices: the identity, the Pauli matrices, the Hadamard matrix and the phase matrix.

The major disadvantage compared to a real quantum computer is very slow computing times even with a few qubits. An advantage is that, unlike a quantum computer, in this simulator a quantum state can be cloned using `copylist`.

This is an additional package that must be loaded with `load("quantum_computing")` in order to use it.

88.2 Functions and Variables for Quantum_Computing

`binlist` [Function]

`binlist (k)`
`binlist (k, n)`

`binlist`(k), where k must be a natural number, returns a list of binary digits 0 or 1 corresponding to the digits of k in binary representation. `binlist`(k, n) does the same but returns a list of length n, with leading zeros as necessary. Notice that for the result to represent a possible state of m qubits, n should be equal to 2^m and k should be between 0 and 2^m-1.

`binlist2dec (lst)` [Function]

Given a list *lst* with n binary digits, it returns the decimal number it represents.

`CNOT (q, i, j)` [Function]

Changes the value of the j'th qubit, in a state q of m qubits, when the value of the i'th qubit equals 1. It modifies the list q and returns its modified value.

`controlled (U, q, c, i)` [Function]

Applies a matrix U, acting on m qubits, on qubits i through i+m-1 of the state q of n qubits (n > m), when the value of the c'th qubit in q equals 1. i should be an integer between 1 and n+1-m and c should be an integer between 1 and n, excluding the qubits to be modified (i through i+m-1).

U can be one of the indices of the array of common matrices *qmatrix* (see `qmatrix`). The state q is modified and shown in the output.

`gate` [Function]

`gate (U, q)`
`gate (U, q, i)`
`gate (U, q, i1, ..., im)`

U must be a matrix acting on states of m qubits; q a list corresponding to a state of n qubits (n >= m); i and the m numbers *i1*, ..., *im* must be different integers between 1 and n.

`gate`(U, q) applies matrix U to each qubit of q, when m equals 1, or to the first m qubits of q when m is bigger than 1.

`gate`(U, q, i) applies matrix U to the qubits i through i+m-1 of q.

`gate`(U, q, $i1$, ..., in) applies matrix U to the in the positions $i1$, ..., im.

U can be one of the indices of the array of common matrices *qmatrix* (see `qmatrix`). The state q is modified and shown in the output.

`gate_matrix` [Function]

`gate_matrix` (*U*, *n*)

`gate_matrix` (*U*, *n*, *i1*, ..., *im*)

U must be a 2 by 2 matrix or one of the indices of the array of common matrices *qmatrix* (see `qmatrix`). `gate_matrix`(U, n) returns the matrix corresponding to the action of U on each qubit in a state of n qubits.

`gate_matrix` (U, n, $i1$, ..., im) returns the matrix corresponding to the action of U on qubits $i1$, ..., im of a state of n qubits, where $i1$, ..., im are different integers between 1 and n.

`linsert` (*e*, *lst*, *p*) [Function]

Inserts the expression or list e into the list *lst* at position p. The list can be empty and p must be an integer between 1 and the length of *lst* plus 1.

`lreplace` (*e*, *lst*, *p*) [Function]

If e is a list of length n, the elements in the positions p, p+1, ..., p+n-1 of the list *lst* are replaced by e, or the first elements of e if the end of *lst* is reached. If e is an expression, the element in position p of list *lst* is replaced by that expression. p must be an integer between 1 and the length of *lst*.

`normalize` (*q*) [Function]

Returns the normalized version of a quantum state given as a list q.

`qdisplay` (*q*) [Function]

Represents the state q of a system of n qubits as a linear combination of the computational states with n binary digits. It returns an expression including strings and symbols.

`qmatrix` [System variable]

This variable is a predefined hash array of two by two matrices with the standard matrices: identity, Pauli matrices, Hadamard matrix and the phase matrix. The six possible indices are I, X, Y, Z, H, S. *qmatrix*[I] is the identity matrix, *qmatrix*[X] the Pauli x matrix, *qmatrix*[Y] the Pauli y matrix, *qmatrix*[Z] the Pauli z matrix, *qmatrix*[H] the Hadamard matrix and *qmatrix*[S] the phase matrix.

`qmeasure` [Function]

`qmeasure` (*q*)

`qmeasure` (*q*, *i1*, ..., *im*)

Measures the value of one or more qubits in a system of n qubits with state q. The m positive integers $i1$, ..., im are the positions of the qubits to be measured It requires 1 or more arguments. The first argument must be the state q. If the only argument given is q, all the n qubits will be measured.

It returns a list with the values of the qubits measured (either 0 or 1), in the same order they were requested or in ascending order if the only argument given was *q*. It modifies the list *q*, reflecting the collapse of the quantum state after the measurement.

`qubits` [Function]

`qubits (`*n*`)`
`qubits (`*i1*`, ..., `*in*`)`

`qubits`(*n*) returns a list representing the ground state of a system of *n* qubits.

`qubits`(*i1*, ..., *in*) returns a list with representing the state of *n* qubits with values *i1*, ..., *in*.

`qswap (`*q*`, `*i*`, `*j*`)` [Function]

Interchanges the states of qubits *i* and *j* in the state *q* of a system of several qubits. It modifies the list *q* and returns its modified value.

`Rx (a)` [Function]

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of *a* radians around the x axis.

`Ry (a)` [Function]

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of *a* radians around the y axis.

`Rz (a)` [Function]

Returns the 2 by two matrix (acting on one qubit) corresponding to a rotation of with an angle of *a* radians around the z axis.

`tprod (`*o1*`, ..., `*on*`)` [Function]

Returns the tensor product of the *n* matrices or lists *o1*, ..., *on*.

`toffoli (`*q*`, (`*i*`, (`*j*`, (`*k*`)` [Function]

Changes the value of the *k*'th qubit, in the state *q* of *n* qubits, if the values of the *i*'th anf *j*'th qubits are equal to 1. It modifies the list *q* and returns its new value.

89 Package ratpow

The package `ratpow` provides functions that return the coefficients of the numerator of a CRE polynomial in a given variable.

For example,

- `ratp_coeffs(5*x^7-3*x^2+4,x)` returns `[[7,5],[2,-3],[0,4]]`, which omits zero terms;
- `ratp_dense_coeffs(5*x^7-y*x^2+4,x)` returns `[5,0,0,0,0,-y,0,4]`, which includes zero terms;
- `ratp_dense_coeffs((x^4-y^4)/(x-y),x)` returns `[1,y,y^2,y^3]`, because CRE simplifies the expression to `x^3+y*x^2+y^2*x+y^3`;
- `ratp_dense_coeffs(x+sqrt(x),x)` returns `[1,sqrt(x)]` while `ratp_dense_coeffs(x+sqrt(x),sqrt(x))` returns `[1,x]`: in CRE form, `x` and `sqrt(x)` are treated as independent variables.

The returned coefficients are in CRE form except for numbers.

For the list of vars of a CRE polynomial, use `showratvars`. For the denominator of a CRE polynomial, use `ratdenom`.

For information about CREs see also `rat`, `ratdisrep` and `showratvars`.

89.1 Functions and Variables for ratpow

`ratp_hipow` (*expr*, *x*) [Function]

Returns the highest power of *x* in `ratnumer(expr)`

```
(%i1) load("ratpow")$
(%i2) ratp_hipow( x^(5/2) + x^2 , x);
(%o2)                          2
(%i3) ratp_hipow( x^(5/2) + x^2 , sqrt(x));
(%o3)                          5
```

`ratp_lopow` (*expr*, *x*) [Function]

Returns the lowest power of *x* in `ratnumer(expr)`

```
(%i1) load("ratpow")$
(%i2) ratp_lopow( x^5 + x^2 , x);
(%o2)                          2
```

The following example returns 0 since `1` equals `x^0`:

```
(%i1) load("ratpow")$
(%i2) ratp_lopow( x^5 + x^2 + 1, x);
(%o2)                          0
```

The CRE form of the following equation contains `sqrt(x)` and `x`. Since they are interpreted as independent variables, `ratp_lopow` returns `0`:

```
(%i1) load("ratpow")$
(%i2) g:sqrt(x)^5 + sqrt(x)^2;
                              5/2
(%o2)                        x    + x
```

```
(%i3) showratvars(g);
                                 1/2
(%o3)                          [x   , x]
(%i4) ratp_lopow( g, x);
(%o4)                            0
(%i5) ratp_lopow( g, sqrt(x));
(%o5)                            0
```

`ratp_coeffs` (*expr*, *x*) [Function]

Returns the powers and coefficients of *x* in `ratnumer(expr)` as a list of length-2 lists; returned coefficients are in CRE form except for numbers.

`ratnumer(expr)`.

```
(%i1) load("ratpow")$
(%i2) ratp_coeffs( 4*x^3 + x + sqrt(x), x);
(%o2)/R/         [[3, 4], [1, 1], [0, sqrt(x)]]
```

`ratp_dense_coeffs` (*expr*, *x*) [Function]

Returns the coefficients of powers of *x* in `ratnumer(expr)` from highest to lowest; returned coefficients are in CRE form except for numbers.

```
(%i1) load("ratpow")$
(%i2) ratp_dense_coeffs( 4*x^3 + x + sqrt(x), x);
(%o2)/R/               [4, 0, 1, sqrt(x)]
```

90 Package romberg

90.1 Functions and Variables for romberg

romberg [Function]

romberg (*expr*, *x*, a, *b*)
romberg (*F*, a, *b*)

Computes a numerical integration by Romberg's method.

romberg(*expr*, *x*, a, *b*) returns an estimate of the integral integrate(*expr*, *x*, a, *b*). *expr* must be an expression which evaluates to a floating point value when x is bound to a floating point value.

romberg(*F*, a, *b*) returns an estimate of the integral integrate(*F*(x), x, a, *b*) where x represents the unnamed, sole argument of F; the actual argument is not named x. F must be a Maxima or Lisp function which returns a floating point value when the argument is a floating point value. F may name a translated or compiled Maxima function.

The accuracy of romberg is governed by the global variables rombergabs and rombergtol. romberg terminates successfully when the absolute difference between successive approximations is less than rombergabs, or the relative difference in successive approximations is less than rombergtol. Thus when rombergabs is 0.0 (the default) only the relative error test has any effect on romberg.

romberg halves the stepsize at most rombergit times before it gives up; the maximum number of function evaluations is therefore 2^rombergit. If the error criterion established by rombergabs and rombergtol is not satisfied, romberg prints an error message. romberg always makes at least rombergmin iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

romberg repeatedly evaluates the integrand after binding the variable of integration to a specific value (and not before). This evaluation policy makes it possible to nest calls to romberg, to compute multidimensional integrals. However, the error calculations do not take the errors of nested integrations into account, so errors may be underestimated. Also, methods devised especially for multidimensional problems may yield the same accuracy with fewer function evaluations.

See also Section 21.3 Introduction to QUADPACK, a collection of numerical integration functions.

Examples:

A 1-dimensional integration.

```
(%i1) f(x) := 1/((x - 1)^2 + 1/100) + 1/((x - 2)^2 + 1/1000)
              + 1/((x - 3)^2 + 1/200);
                    1                 1                1
(%o1) f(x) := -------------- + --------------- + --------------
                       2    1           2    1            2    1
              (x - 1)  + ---   (x - 2)  + ----   (x - 3)  + ---
                         100             1000              200
```

```
(%i2) rombergtol : 1e-6;
(%o2)                 9.999999999999999e-7
(%i3) rombergit : 15;
(%o3)                           15
(%i4) estimate : romberg (f(x), x, -5, 5);
(%o4)                   173.6730736617464
(%i5) exact : integrate (f(x), x, -5, 5);
       3/2          3/2      3/2          3/2
(%o5) 10    atan(7 10   ) + 10    atan(3 10   )
      3/2         9/2       3/2         5/2
 + 5 2    atan(5 2   ) + 5 2    atan(5 2   ) + 10 atan(60)
 + 10 atan(40)
(%i6) abs (estimate - exact) / exact, numer;
(%o6)                 7.552722451569877e-11
```

A 2-dimensional integration, implemented by nested calls to `romberg`.

```
(%i1) g(x, y) := x*y / (x + y);
                                      x y
(%o1)                    g(x, y) := -----
                                     x + y
(%i2) rombergtol : 1e-6;
(%o2)                 9.999999999999999e-7
(%i3) estimate : romberg (romberg (g(x, y), y, 0, x/2), x, 1, 3);
(%o3)                  0.8193023962835647
(%i4) assume (x > 0);
(%o4)                         [x > 0]
(%i5) integrate (integrate (g(x, y), y, 0, x/2), x, 1, 3);
                                          3
                                    2 log(-) - 1
                   9                      2        9
(%o5)      (- 9 log(-)) + 9 log(3) + ------------ + -
                   2                     6         2
(%i6) exact : radcan (%);
                   26 log(3) - 26 log(2) - 13
(%o6)            - --------------------------
                                3
(%i7) abs (estimate - exact) / exact, numer;
(%o7)                 1.371197987185102e-10
```

`rombergabs` [Option variable]

Default value: 0.0

The accuracy of `romberg` is governed by the global variables `rombergabs` and `rombergtol`. `romberg` terminates successfully when the absolute difference between successive approximations is less than `rombergabs`, or the relative difference in successive approximations is less than `rombergtol`. Thus when `rombergabs` is 0.0 (the default) only the relative error test has any effect on `romberg`.

See also `rombergit` and `rombergmin`.

`rombergit` [Option variable]

Default value: 11

`romberg` halves the stepsize at most `rombergit` times before it gives up; the maximum number of function evaluations is therefore `2^rombergit`. `romberg` always makes at least `rombergmin` iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

See also `rombergabs` and `rombergtol`.

`rombergmin` [Option variable]

Default value: 0

`romberg` always makes at least `rombergmin` iterations; this is a heuristic intended to prevent spurious termination when the integrand is oscillatory.

See also `rombergit`, `rombergabs`, and `rombergtol`.

`rombergtol` [Option variable]

Default value: 1e-4

The accuracy of `romberg` is governed by the global variables `rombergabs` and `rombergtol`. `romberg` terminates successfully when the absolute difference between successive approximations is less than `rombergabs`, or the relative difference in successive approximations is less than `rombergtol`. Thus when `rombergabs` is 0.0 (the default) only the relative error test has any effect on `romberg`.

See also `rombergit` and `rombergmin`.

91 Package simplex

91.1 Introduction to simplex

`simplex` is a package for linear optimization using the simplex algorithm.

Example:

```
(%i1) load("simplex")$
(%i2) minimize_lp(x+y, [3*x+2*y>2, x+4*y>3]);
                    9        7       1
(%o2)              [--, [y = --, x = -]]
                    10       10      5
```

91.1.1 Tests for simplex

There are some tests in the directory `share/simplex/Tests`.

91.1.1.1 klee_minty

The function `klee_minty` produces input for `linear_program`, for which exponential time for solving is required without scaling.

Example:

```
load("klee_minty")$
apply(linear_program, klee_minty(6));
```

A better approach:

```
epsilon_sx : 0$
scale_sx : true$
apply(linear_program, klee_minty(10));
```

91.1.1.2 NETLIB

Some smaller problems from netlib (https://www.netlib.org/lp/data/) test suite are converted to a format, readable by Maxima. Problems are `adlittle`, `afiro`, `kb2`, `sc50a` and `share2b`. Each problem has three input files in CSV format for matrix A and vectors b and c.

Example:

```
A : read_matrix("adlittle_A.csv", 'csv)$
b : read_list("adlittle_b.csv", 'csv)$
c : read_list("adlittle_c.csv", 'csv)$
linear_program(A, b, c)$
%[2];
=> 225494.9631623802
```

Results:

```
PROBLEM         MINIMUM                  SCALING
adlittle        +2.2549496316E+05        false
afiro           -4.6475314286E+02        false
kb2             -1.7499001299E+03        true
sc50a           -6.4575077059E+01        false
```

```
share2b         -4.1573518187E+02        false
```

The Netlib website https://www.netlib.org/lp/data/readme lists the values as

```
PROBLEM         MINIMUM
adlittle        +2.2549496316E+05
afiro           -4.6475314286E+02
kb2             -1.7499001299E+03
sc50a           -6.4575077059E+01
share2b         -4.1573224074E+02
```

91.2 Functions and Variables for simplex

epsilon_lp [Option variable]

Default value: 10^-8

Epsilon used for numerical computations in linear_program; it is set to 0 in linear_program when all inputs are rational.

Example:

```
(%i1) load("simplex")$

(%i2) minimize_lp(-x, [1e-9*x + y <= 1], [x,y]);
Warning: linear_program(A,b,c): non-rat inputs found, epsilon_lp= 1.0e-8
Warning: Solution may be incorrect.
(%o2)                         Problem not bounded!
(%i3) minimize_lp(-x, [10^-9*x + y <= 1], [x,y]);
(%o3)                [- 1000000000, [y = 0, x = 1000000000]]
(%i4) minimize_lp(-x, [1e-9*x + y <= 1], [x,y]), epsilon_lp=0;
(%o4)     [- 9.999999999999999e+8, [y = 0, x = 9.999999999999999e+8]]
```

See also: linear_program, ratnump.

linear_program (*A*, *b*, *c*) [Function]

linear_program is an implementation of the simplex algorithm. linear_program(A, b, c) computes a vector *x* for which c.x is minimum possible among vectors for which A.x = b and x >= 0. Argument *A* is a matrix and arguments *b* and *c* are lists.

linear_program returns a list which contains the minimizing vector *x* and the minimum value c.x. If the problem is not bounded, it returns "Problem not bounded!" and if the problem is not feasible, it returns "Problem not feasible!".

To use this function first load the simplex package with load("simplex");.

Example:

```
(%i2) A: matrix([1,1,-1,0], [2,-3,0,-1], [4,-5,0,0])$
(%i3) b: [1,1,6]$
(%i4) c: [1,-2,0,0]$
(%i5) linear_program(A, b, c);
                     13     19        3
(%o5)             [[--, 4, --, 0], - -]
                     2      2         2
```

See also: minimize_lp, scale_lp, and epsilon_lp.

`maximize_lp` (*obj*, *cond*, [*pos*]) [Function]

Maximizes linear objective function *obj* subject to some linear constraints *cond*. See `minimize_lp` for detailed description of arguments and return value.

See also: `minimize_lp`.

`minimize_lp` (*obj*, *cond*, [*pos*]) [Function]

Minimizes a linear objective function *obj* subject to some linear constraints *cond*. *cond* a list of linear equations or inequalities. In strict inequalities `>` is replaced by `>=` and `<` by `<=`. The optional argument *pos* is a list of decision variables which are assumed to be positive.

If the minimum exists, `minimize_lp` returns a list which contains the minimum value of the objective function and a list of decision variable values for which the minimum is attained. If the problem is not bounded, `minimize_lp` returns `"Problem not bounded!"` and if the problem is not feasible, it returns `"Problem not feasible!"`.

The decision variables are not assumed to be non-negative by default. If all decision variables are non-negative, set `nonnegative_lp` to `true` or include `all` in the optional argument *pos*. If only some of decision variables are positive, list them in the optional argument *pos* (note that this is more efficient than adding constraints).

`minimize_lp` uses the simplex algorithm which is implemented in maxima `linear_program` function.

To use this function first load the `simplex` package with `load("simplex");`.

Examples:

```
(%i1) minimize_lp(x+y, [3*x+y=0, x+2*y>2]);
                       4       6        2
(%o1)                 [-, [y = -, x = - -]]
                       5       5        5
(%i2) minimize_lp(x+y, [3*x+y>0, x+2*y>2]), nonnegative_lp=true;
(%o2)                 [1, [y = 1, x = 0]]
(%i3) minimize_lp(x+y, [3*x+y>0, x+2*y>2], all);
(%o3)                          [1, [y = 1, x = 0]]
(%i4) minimize_lp(x+y, [3*x+y=0, x+2*y>2]), nonnegative_lp=true;
(%o4)                Problem not feasible!
(%i5) minimize_lp(x+y, [3*x+y>0]);
(%o5)                Problem not bounded!
```

There is also a limited ability to solve linear programs with symbolic constants.

```
(%i1) declare(c,constant)$
(%i2) maximize_lp(x+y, [y<=-x/c+3, y<=-x+4], [x, y]), epsilon_lp=0;
Is (c-1)*c positive, negative or zero?
p;
Is c*(2*c-1) positive, negative or zero?
p;
Is c positive or negative?
p;
Is c-1 positive, negative or zero?
```

```
p;
Is 2*c-1 positive, negative or zero?
p;
Is 3*c-4 positive, negative or zero?
p;
                                    1                 1
(%o2)                 [4, [x = -----, y = 3 - ---------]]
                                  1                 1
                                1 - -             (1 - -) c
                                  c                 c
(%i1) (assume(c>4/3), declare(c,constant))$
(%i2) maximize_lp(x+y, [y<=-x/c+3, y<=-x+4], [x, y]), epsilon_lp=0;
                                    1                 1
(%o2)                 [4, [x = -----, y = 3 - ---------]]
                                  1                 1
                                1 - -             (1 - -) c
                                  c                 c
```

See also: maximize_lp, nonnegative_lp, epsilon_lp.

nonnegative_lp [Option variable]
nonegative_lp [Option variable]

Default value: false

If nonnegative_lp is true all decision variables to minimize_lp and maximize_lp are assumed to be non-negative. nonegative_lp is a deprecated alias.

See also: minimize_lp.

scale_lp [Option variable]

Default value: false

When scale_lp is true, linear_program scales its input so that the maximum absolute value in each row or column is 1.

pivot_count_sx [Variable]

After linear_program returns, pivot_count_sx is the number of pivots in last computation.

pivot_max_sx [Variable]

pivot_max_sx is the maximum number of pivots allowed by linear_program.

92 Package simplification

92.1 Introduction to simplification

The directory `maxima/share/simplification` contains several scripts which implement simplification rules and functions, and also some functions not related to simplification.

92.2 Package absimp

The `absimp` package contains pattern-matching rules that extend the built-in simplification rules for the `abs` and `signum` functions. `absimp` respects relations established with the built-in `assume` function and by declarations such as `mode_declare (m, even, n, odd)` for even or odd integers.

`absimp` defines `unitramp` and `unitstep` functions in terms of `abs` and `signum`.

`load ("absimp")` loads this package. `demo ("absimp")` shows a demonstration of this package.

Examples:

```
(%i1) load ("absimp")$
(%i2) (abs (x))^2;
                                  2
(%o2)                            x
(%i3) diff (abs (x), x);
                                  x
(%o3)                           ------
                                abs(x)
(%i4) cosh (abs (x));
(%o4)                           cosh(x)
```

92.3 Package facexp

The `facexp` package contains several related functions that provide the user with the ability to structure expressions by controlled expansion. This capability is especially useful when the expression contains variables that have physical meaning, because it is often true that the most economical form of such an expression can be obtained by fully expanding the expression with respect to those variables, and then factoring their coefficients. While it is true that this procedure is not difficult to carry out using standard Maxima functions, additional fine-tuning may also be desirable, and these finishing touches can be more difficult to apply.

The function `facsum` and its related forms provide a convenient means for controlling the structure of expressions in this way. Another function, `collectterms`, can be used to add two or more expressions that have already been simplified to this form, without resimplifying the whole expression again. This function may be useful when the expressions are very large.

`load ("facexp")` loads this package. `demo ("facexp")` shows a demonstration of this package.

facsum (*expr*, *arg_1*, ..., *arg_n*) [Function]

Returns a form of *expr* which depends on the arguments *arg_1*, ..., *arg_n*. The arguments can be any form suitable for `ratvars`, or they can be lists of such forms. If the arguments are not lists, then the form returned is fully expanded with respect to the arguments, and the coefficients of the arguments are factored. These coefficients are free of the arguments, except perhaps in a non-rational sense.

If any of the arguments are lists, then all such lists are combined into a single list, and instead of calling `factor` on the coefficients of the arguments, `facsum` calls itself on these coefficients, using this newly constructed single list as the new argument list for this recursive call. This process can be repeated to arbitrary depth by nesting the desired elements in lists.

It is possible that one may wish to `facsum` with respect to more complicated subexpressions, such as `log (x + y)`. Such arguments are also permissible.

Occasionally the user may wish to obtain any of the above forms for expressions which are specified only by their leading operators. For example, one may wish to `facsum` with respect to all `log`'s. In this situation, one may include among the arguments either the specific `log`'s which are to be treated in this way, or alternatively, either the expression `operator (log)` or `'operator (log)`. If one wished to `facsum` the expression *expr* with respect to the operators *op_1*, ..., *op_n*, one would evaluate `facsum (expr, operator (op_1, ..., op_n))`. The `operator` form may also appear inside list arguments.

In addition, the setting of the switches `facsum_combine` and `nextlayerfactor` may affect the result of `facsum`.

nextlayerfactor [Global variable]

Default value: `false`

When `nextlayerfactor` is `true`, recursive calls of `facsum` are applied to the factors of the factored form of the coefficients of the arguments.

When `false`, `facsum` is applied to each coefficient as a whole whenever recursive calls to `facsum` occur.

Inclusion of the atom `nextlayerfactor` in the argument list of `facsum` has the effect of `nextlayerfactor: true`, but for the next level of the expression *only*. Since `nextlayerfactor` is always bound to either `true` or `false`, it must be presented single-quoted whenever it appears in the argument list of `facsum`.

facsum_combine [Global variable]

Default value: `true`

`facsum_combine` controls the form of the final result returned by `facsum` when its argument is a quotient of polynomials. If `facsum_combine` is `false` then the form will be returned as a fully expanded sum as described above, but if `true`, then the expression returned is a ratio of polynomials, with each polynomial in the form described above.

The `true` setting of this switch is useful when one wants to `facsum` both the numerator and denominator of a rational expression, but does not want the denominator to be multiplied through the terms of the numerator.

`factorfacsum` (*expr*, *arg_1*, ... *arg_n*) [Function]

Returns a form of *expr* which is obtained by calling `facsum` on the factors of *expr* with *arg_1*, ... *arg_n* as arguments. If any of the factors of *expr* is raised to a power, both the factor and the exponent will be processed in this way.

`collectterms` (*expr*, *arg_1*, ..., *arg_n*) [Function]

Collects all terms that contain *arg_1* ... *arg_n*. If several expressions have been simplified with the following functions `facsum`, `factorfacsum`, `factenexpand`, `facexpten` or `factorfacexpten`, and they are to be added together, it may be desirable to combine them using the function `collecterms`. `collecterms` can take as arguments all of the arguments that can be given to these other associated functions with the exception of `nextlayerfactor`, which has no effect on `collectterms`. The advantage of `collectterms` is that it returns a form similar to `facsum`, but since it is adding forms that have already been processed by `facsum`, it does not need to repeat that effort. This capability is especially useful when the expressions to be summed are very large.

See also `factor`.

Example:

```
(%i1) (exp(x)+2)*x+exp(x);
                                  x          x
(%o1)                     x (%e  + 2) + %e
(%i2) collectterms(expand(%),exp(x));
                                    x
(%o2)                     (x + 1) %e  + 2 x
```

92.4 Package functs

`rempart` (*expr*, *n*) [Function]

Removes part *n* from the expression *expr*.

If *n* is a list of the form `[l, m]` then parts *l* thru *m* are removed.

To use this function write first `load("functs")`.

`wronskian` ([*f_1*, ..., *f_n*], *x*) [Function]

Returns the Wronskian matrix of the list of expressions [*f_1*, ..., *f_n*] in the variable *x*. The determinant of the Wronskian matrix is the Wronskian determinant of the list of expressions.

To use `wronskian`, first `load("functs")`. Example:

```
(%i1) load ("functs")$
(%i2) wronskian([f(x), g(x)],x);
                      [   f(x)       g(x)    ]
                      [                      ]
(%o2)                 [ d          d         ]
                      [ -- (f(x))  -- (g(x)) ]
                      [ dx         dx        ]
```

`tracematrix` (*M*) [Function]

Returns the trace (sum of the diagonal elements) of matrix *M*.

To use this function write first `load("functs")`.

`rational` (*z*) [Function]

Multiplies numerator and denominator of *z* by the complex conjugate of denominator, thus rationalizing the denominator. Returns canonical rational expression (CRE) form if given one, else returns general form.

To use this function write first `load("functs")`.

`nonzeroandfreeof` (*x*, *expr*) [Function]

Returns `true` if *expr* is nonzero and `freeof (x, expr)` returns `true`. Returns `false` otherwise.

To use this function write first `load("functs")`.

`linear` (*expr*, *x*) [Function]

When *expr* is an expression of the form `a*x + b` where *a* is nonzero, and *a* and *b* are free of *x*, `linear` returns a list of three equations, one for each of the three formal variables *b*, *a*, and *x*. Otherwise, `linear` returns `false`.

`load("antid")` loads this function.

Example:

```
(%i1) load ("antid");
(%o1)  /maxima/share/integration/antid.mac
(%i2) linear ((1 - w)*(1 - x)*z, z);
(%o2) [bargumentb = 0, aargumenta = (w - 1) x - w + 1,
                                                 xargumentx = z]
(%i3) linear (cos(u - v) + cos(u + v), u);
(%o3)                          false
```

`gcdivide` (*p*, *q*) [Function]

When the option variable `takegcd` is `true` which is the default, `gcdivide` divides the polynomials *p* and *q* by their greatest common divisor and returns the ratio of the results. `gcdivde` calls the function `ezgcd` to divide the polynomials by the greatest common divisor.

When `takegcd` is `false`, `gcdivide` returns the ratio *p*/*q*.

To use this function write first `load("functs")`.

See also `ezgcd`, `gcd`, `gcdex`, and `poly_gcd`.

Example:

```
(%i1) load("functs")$

(%i2) p1:6*x^3+19*x^2+19*x+6;
                          3       2
(%o2)                  6 x  + 19 x  + 19 x + 6
(%i3) p2:6*x^5+13*x^4+12*x^3+13*x^2+6*x;
                    5       4       3       2
(%o3)          6 x  + 13 x  + 12 x  + 13 x  + 6 x
(%i4) gcdivide(p1, p2);
                               x + 1
```

```
(%o4)                          ------
                                3
                               x  + x
(%i5) takegcd:false;
(%o5)                           false
(%i6) gcdivide(p1, p2);
                          3       2
                     6 x  + 19 x  + 19 x + 6
(%o6)          ---------------------------------
                  5       4       3       2
                6 x  + 13 x  + 12 x  + 13 x  + 6 x
(%i7) ratsimp(%);
                               x + 1
(%o7)                          ------
                                3
                               x  + x
```

`arithmetic` (`a`, `d`, `n`) [Function]

Returns the *n*-th term of the arithmetic series `a`, `a + d`, `a + 2*d`, `...`, `a + (n - 1)*d`.

To use this function write first `load("functs")`.

`geometric` (`a`, `r`, `n`) [Function]

Returns the *n*-th term of the geometric series `a`, `a*r`, `a*r^2`, `...`, `a*r^(n - 1)`.

To use this function write first `load("functs")`.

`harmonic` (`a`, `b`, `c`, `n`) [Function]

Returns the *n*-th term of the harmonic series `a/b`, `a/(b + c)`, `a/(b + 2*c)`, `...`, `a/(b + (n - 1)*c)`.

To use this function write first `load("functs")`.

`arithsum` (`a`, `d`, `n`) [Function]

Returns the sum of the arithmetic series from 1 to *n*.

To use this function write first `load("functs")`.

`geosum` (`a`, `r`, `n`) [Function]

Returns the sum of the geometric series from 1 to *n*. If *n* is infinity (`inf`) then a sum is finite only if the absolute value of *r* is less than 1.

To use this function write first `load("functs")`.

`gaussprob` (`x`) [Function]

Returns the Gaussian probability function `%e^(-x^2/2) / sqrt(2*%pi)`.

To use this function write first `load("functs")`.

`gd` (`x`) [Function]

Returns the Gudermannian function `2*atan(%e^x)-%pi/2`.

To use this function write first `load("functs")`.

`agd (x)` [Function]
Returns the inverse Gudermannian function `log (tan (%pi/4 + x/2))`.
To use this function write first `load("functs")`.

`vers (x)` [Function]
Returns the versed sine `1 - cos (x)`.
To use this function write first `load("functs")`.

`covers (x)` [Function]
Returns the coversed sine `1 - sin (x)`.
To use this function write first `load("functs")`.

`exsec (x)` [Function]
Returns the exsecant `sec (x) - 1`.
To use this function write first `load("functs")`.

`hav (x)` [Function]
Returns the haversine `(1 - cos(x))/2`.
To use this function write first `load("functs")`.

`combination (n, r)` [Function]
Returns the number of combinations of n objects taken r at a time.
To use this function write first `load("functs")`.

`permutation (n, r)` [Function]
Returns the number of permutations of r objects selected from a set of n objects.
To use this function write first `load("functs")`.

92.5 Package ineq

The `ineq` package contains simplification rules for inequalities.

Example session:

```
(%i1) load("ineq")$
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
tellsimp: warning: rule will treat '+
                           ' as noncommutative and nonassociative.
```

```
tellsimp: warning: rule will treat '+
                        ' as noncommutative and nonassociative.
(%i2) a>=4;  /* a sample inequality */
(%o2)                         a >= 4
(%o3)                       b + a > c + 4
(%o4)                         7 x < 7 y
(%o5)                      - 2 x <= - 6 z
                                 2
(%o6)                        1 <= a  + 1
(%o8)                         2 x < 3 x
(%o9)                          a >= b
(%o10)                      a + 3 >= b + 3
(%o11)                         a >= b
(%o12)                       a >= c - b
(%o13)                       b + a >= c
(%o14)                   (- c) + b + a >= 0
(%o15)                     c - b - a <= 0
                               2
(%o16)                    (z - 1)  > - 2 z
                              2
(%o17)                      z  + 1 > 0
(%o18)                         true
(%i19) (b>c)+%; /* add a second, strict inequality */
```

Be careful about using parentheses around the inequalities: when the user types in `(A > B) + (C = 5)` the result is `A + C > B + 5`, but `A > B + C = 5` is a syntax error, and `(A > B + C) = 5` is something else entirely.

Do `disprule (all)` to see a complete listing of the rule definitions.

The user will be queried if Maxima is unable to decide the sign of a quantity multiplying an inequality.

The most common mis-feature is illustrated by:

```
(%i1) eq: a > b;
(%o1)                          a > b
(%i2) 2*eq;
(%o2)                        2 (a > b)
(%i3) % - eq;
(%o3)                          a > b
```

Another problem is 0 times an inequality; the default to have this turn into 0 has been left alone. However, if you type `X*some_inequality` and Maxima asks about the sign of `X` and you respond `zero` (or `z`), the program returns `X*some_inequality` and not use the information that `X` is 0. You should do `ev (%, x: 0)` in such a case, as the database will only be used for comparison purposes in decisions, and not for the purpose of evaluating `X`.

The user may note a slower response when this package is loaded, as the simplifier is forced to examine more rules than without the package, so you might wish to remove the rules after making use of them. Do `kill (rules)` to eliminate all of the rules (including

any that you might have defined); or you may be more selective by killing only some of them; or use `remrule` on a specific rule.

Note that if you load this package after defining your own rules you will clobber your rules that have the same name. The rules in this package are: `*rule1`, ..., `*rule8`, `+rule1`, ..., `+rule18`, and you must enclose the rulename in quotes to refer to it, as in `remrule ("+", "+rule1")` to specifically remove the first rule on `"+"` or `disprule ("*rule2")` to display the definition of the second multiplicative rule.

92.6 Package rducon

`reduce_consts` (*expr*) [Function]

Replaces constant subexpressions of *expr* with constructed constant atoms, saving the definition of all these constructed constants in the list of equations `const_eqns`, and returning the modified *expr*. Those parts of *expr* are constant which return `true` when operated on by the function `constantp`. Hence, before invoking `reduce_consts`, one should do

```
declare ([objects to be given the constant property], constant)$
```

to set up a database of the constant quantities occurring in your expressions.

If you are planning to generate Fortran output after these symbolic calculations, one of the first code sections should be the calculation of all constants. To generate this code segment, do

```
map ('fortran, const_eqns)$
```

Variables besides `const_eqns` which affect `reduce_consts` are:

`const_prefix` (default value: `xx`) is the string of characters used to prefix all symbols generated by `reduce_consts` to represent constant subexpressions.

`const_counter` (default value: 1) is the integer index used to generate unique symbols to represent each constant subexpression found by `reduce_consts`.

`load ("rducon")` loads this function. `demo ("rducon")` shows a demonstration of this function.

92.7 Package scifac

`gcfac` (*expr*) [Function]

`gcfac` is a factoring function that attempts to apply the same heuristics which scientists apply in trying to make expressions simpler. `gcfac` is limited to monomial-type factoring. For a sum, `gcfac` does the following:

1. Factors over the integers.
2. Factors out the largest powers of terms occurring as coefficients, regardless of the complexity of the terms.
3. Uses (1) and (2) in factoring adjacent pairs of terms.
4. Repeatedly and recursively applies these techniques until the expression no longer changes.

Item (3) does not necessarily do an optimal job of pairwise factoring because of the combinatorially-difficult nature of finding which of all possible rearrangements of the pairs yields the most compact pair-factored result.

`load ("scifac")` loads this function. `demo ("scifac")` shows a demonstration of this function.

93 Package solve_rec

93.1 Introduction to solve_rec

`solve_rec` is a package for solving linear recurrences with polynomial coefficients.

A demo is available with `demo("solve_rec");`.

Example:

```
(%i1) load("solve_rec")$
(%i2) solve_rec((n+4)*s[n+2] + s[n+1] - (n+1)*s[n], s[n]);
                                      n
                    %k  (2 n + 3) (- 1)          %k
                      1                            2
(%o2)         s  = -------------------- + ---------------
               n     (n + 1) (n + 2)      (n + 1) (n + 2)
```

93.2 Functions and Variables for solve_rec

`harmonic_number` (*x*) [Function]

When x is positive integer n, `harmonic_number` is the n'th harmonic number. More generally, `harmonic_number(x) = psi[0](x+1) + %gamma`. (See [polygamma], page 344).

```
(%i1) load("simplify_sum")$
(%i2) harmonic_number(5);
                                  137
(%o2)                             ---
                                  60
(%i3) sum(1/k, k, 1, 5);
                                  137
(%o3)                             ---
                                  60
(%i4) float(harmonic_number(sqrt(2)));
(%o4)              %gamma + 0.6601971549171388
(%i5) float(psi[0](1+sqrt(2)))+%gamma;
(%o5)              %gamma + 0.6601971549171388
```

`harmonic_to_psi` (*x*) [Function]

Converts expressions with `harmonic_number` to the equivalent expression involving `psi[0]` (see [polygamma], page 344).

```
(%i1) load("simplify_sum")$
(%i2) harmonic_to_psi(harmonic_number(sqrt(2)));
(%o2)              psi (sqrt(2) + 1) + %gamma
                      0
```

`reduce_order` (*rec*, *sol*, *var*) [Function]

Reduces the order of linear recurrence *rec* when a particular solution *sol* is known. The reduced recurrence can be used to get other solutions.

Example:

```
(%i3) rec: x[n+2] = x[n+1] + x[n]/n;
                                          x
                                           n
(%o3)               x      = x      + --
                     n + 2    n + 1   n
(%i4) solve_rec(rec, x[n]);
WARNING: found some hypergeometrical solutions!
(%o4)                     x  = %k  n
                           n     1
(%i5) reduce_order(rec, n, x[n]);
(%t5)                     x  = n %z
                           n       n

                             n - 1
                             ====
                             \
(%t6)                 %z  =  >     %u
                        n    /        %j
                             ====
                             %j = 0

(%o6)             (- n - 2) %u     - %u
                                n + 1     n
(%i6) solve_rec((n+2)*%u[n+1] + %u[n], %u[n]);
                                          n
                               %k  (- 1)
                                 1
(%o6)                   %u  = ----------
                          n    (n + 1)!
```

So the general solution is

$$\sum_{j=0}^{n-1} \frac{(-1)^j}{(j+1)!} + k_2 \, k_1^n$$

simplify_products [Option variable]

Default value: `true`

If `simplify_products` is `true`, `solve_rec` will try to simplify products in result.

See also: `solve_rec`.

simplify_sum (*expr*) [Function]

Tries to simplify all sums appearing in *expr* to a closed form.

To use this function first load the `simplify_sum` package with `load("simplify_sum")`.

Example:

```
(%i1) load("simplify_sum")$
(%i2) sum(binomial(n+k,k)/2^k, k, 1, n) + sum(binomial(2*n, 2*k), k, 1,n);
       n                          n
      ====                       ====
      \     binomial(n + k, k)   \
(%o2)  >    ------------------ +  >    binomial(2 n, 2 k)
      /              k           /
      ====          2            ====
      k = 1                      k = 1
(%i3) simplify_sum(%);
                         2 n - 1    n
(%o3)                   2        + 2  - 2
```

solve_rec (*eqn*, *var*, [*init*]) [Function]

Solves for hypergeometrical solutions to linear recurrence *eqn* with polynomials coefficient in variable *var*. Optional arguments *init* are initial conditions.

`solve_rec` can solve linear recurrences with constant coefficients, finds hypergeometrical solutions to homogeneous linear recurrences with polynomial coefficients, rational solutions to linear recurrences with polynomial coefficients and can solve Ricatti type recurrences.

Note that the running time of the algorithm used to find hypergeometrical solutions is exponential in the degree of the leading and trailing coefficient.

To use this function first load the `solve_rec` package with `load("solve_rec");`.

Example of linear recurrence with constant coefficients:

```
(%i2) solve_rec(a[n]=a[n-1]+a[n-2]+n/2^n, a[n]);
                        n          n
           (sqrt(5) - 1)  %k  (- 1)
                            1           n
(%o2) a  = ------------------------- - ----
       n               n                  n
                      2                5 2
                                                n
                                   (sqrt(5) + 1)  %k
                                                    2    2
                                 + ------------------ - ----
                                            n              n
                                           2            5 2
```

Example of linear recurrence with polynomial coefficients:

```
(%i7) 2*x*(x+1)*y[x] - (x^2+3*x-2)*y[x+1] + (x-1)*y[x+2];
                            2
(%o7) (x - 1) y      - (x  + 3 x - 2) y      + 2 x (x + 1) y
               x + 2                    x + 1                x
(%i8) solve_rec(%, y[x], y[1]=1, y[3]=3);
                                 x
                              3 2    x!
(%o9)                  y  = ---- - --
                        x    4     2
```

Example of Ricatti type recurrence:

```
(%i2) x*y[x+1]*y[x] - y[x+1]/(x+2) + y[x]/(x-1) = 0;
                               y         y
                                x + 1     x
(%o2)          x y  y      - ------ + ----- = 0
                  x  x + 1   x + 2    x - 1
(%i3) solve_rec(%, y[x], y[3]=5)$
(%i4) ratsimp(minfactorial(factcomb(%)));
                                    3
                                30 x  - 30 x
(%o4) y  = - ------------------------------------------------
       x        6      5       4       3       2
              5 x  - 3 x  - 25 x  + 15 x  + 20 x  - 12 x - 1584
```

See also: `solve_rec_rat`, `simplify_products` and `product_use_gamma`.

`solve_rec_rat` (*eqn*, *var*, [*init*]) [Function]

Solves for rational solutions to linear recurrences. See solve_rec for description of arguments.

To use this function first load the `solve_rec` package with `load("solve_rec");`.

Example:

```
(%i1) (x+4)*a[x+3] + (x+3)*a[x+2] - x*a[x+1] + (x^2-1)*a[x];
(%o1)  (x + 4) a      + (x + 3) a      - x a
                x + 3            x + 2      x + 1
                                                  2
                                             + (x  - 1) a
                                                        x
(%i2) solve_rec_rat(% = (x+2)/(x+1), a[x]);
                          1
(%o2)       a  = ---------------
             x   (x - 1) (x + 1)
```

See also: `solve_rec`.

`product_use_gamma` [Option variable]

Default value: `true`

When simplifying products, `solve_rec` introduces gamma function into the expression if `product_use_gamma` is `true`.

See also: `simplify_products`, `solve_rec`.

summand_to_rec [Function]

summand_to_rec (*summand*, *k*, *n*)
summand_to_rec (*summand*, [*k*, *lo*, *hi*], *n*)

Returns the recurrence satisfied by the sum

```
 hi
====
\
 >     summand
/
====
k = lo
```

where summand is hypergeometrical in *k* and *n*. If *lo* and *hi* are omitted, they are assumed to be `lo = -inf` and `hi = inf`.

To use this function first load the `simplify_sum` package with `load("simplify_sum")`.

Example:

```
(%i1) load("simplify_sum")$
(%i2) summand: binom(n,k);
(%o2)                            binomial(n, k)
(%i3) summand_to_rec(summand,k,n);
(%o3)                      2 sm  - sm      = 0
                              n     n + 1
(%i7) summand: binom(n, k)/(k+1);
                                 binomial(n, k)
(%o7)                            --------------
                                     k + 1
(%i8) summand_to_rec(summand, [k, 0, n], n);
(%o8)              2 (n + 1) sm  - (n + 2) sm      = - 1
                                 n             n + 1
```

94 Package stats

94.1 Introduction to stats

Package `stats` contains a set of classical statistical inference and hypothesis testing procedures.

All these functions return an `inference_result` Maxima object which contains the necessary results for population inferences and decision making.

Global variable `stats_numer` controls whether results are given in floating point or symbolic and rational format; its default value is `true` and results are returned in floating point format.

Package `descriptive` contains some utilities to manipulate data structures (lists and matrices); for example, to extract subsamples. It also contains some examples on how to use package `numericalio` to read data from plain text files. See `descriptive` and `numericalio` for more details.

Package `stats` loads packages `descriptive`, `distrib` and `inference_result`.

For comments, bugs or suggestions, please contact the author at

'mario AT edu DOT xunta DOT es'.

94.2 Functions and Variables for inference_result

`inference_result` (*title*, *values*, *numbers*) [Function]

Constructs an `inference_result` object of the type returned by the stats functions. Argument *title* is a string with the name of the procedure; *values* is a list with elements of the form `symbol = value` and *numbers* is a list with positive integer numbers ranging from one to `length(`*values*`)`, indicating which values will be shown by default.

Example:

This is a simple example showing results concerning a rectangle. The title of this object is the string `"Rectangle"`, it stores five results, named `'base`, `'height`, `'diagonal`, `'area`, and `'perimeter`, but only the first, second, fifth, and fourth will be displayed. The `'diagonal` is stored in this object, but it is not displayed; to access its value, make use of function `take_inference`.

```
(%i1) load("inference_result")$
(%i2) b: 3$ h: 2$
(%i3) inference_result("Rectangle",
                       ['base=b,
                        'height=h,
                        'diagonal=sqrt(b^2+h^2),
                        'area=b*h,
                        'perimeter=2*(b+h)],
                       [1,2,5,4] );
                       |   Rectangle
                       |
```

```
                              |    base = 3
                              |
(%o3)                         |   height = 2
                              |
                              | perimeter = 10
                              |
                              |    area = 6
(%i4) take_inference('diagonal,%);
(%o4)                          sqrt(13)
```

See also take_inference.

inferencep (*obj*) [Function]

Returns true or false, depending on whether *obj* is an inference_result object or not.

items_inference (*obj*) [Function]

Returns a list with the names of the items stored in *obj*, which must be an inference_result object.

Example:

The inference_result object stores two values, named 'pi and 'e, but only the second is displayed. The items_inference function returns the names of all items, no matter they are displayed or not.

```
(%i1) load("inference_result")$
(%i2) inference_result("Hi", ['pi=%pi,'e=%e],[2]);
                              |   Hi
(%o2)                         |
                              | e = %e
(%i3) items_inference(%);
(%o3)                          [pi, e]
```

take_inference [Function]

take_inference (*n*, *obj*)
take_inference (*name*, *obj*)
take_inference (*list*, *obj*)

Returns the *n*-th value stored in *obj* if *n* is a positive integer, or the item named *name* if this is the name of an item. If the first argument is a list of numbers and/or symbols, function take_inference returns a list with the corresponding results.

Example:

Given an inference_result object, function take_inference is called in order to extract some information stored in it.

```
(%i1) load("inference_result")$
(%i2) b: 3$ h: 2$
(%i3) sol: inference_result("Rectangle",
                            ['base=b,
                             'height=h,
                             'diagonal=sqrt(b^2+h^2),
```

```
                                    'area=b*h,
                                    'perimeter=2*(b+h)],
                                   [1,2,5,4] );
                              |   Rectangle
                              |
                              |    base = 3
                              |
(%o3)                         |   height = 2
                              |
                              | perimeter = 10
                              |
                              |    area = 6
(%i4) take_inference('base,sol);
(%o4)                                3
(%i5) take_inference(5,sol);
(%o5)                               10
(%i6) take_inference([1,'diagonal],sol);
(%o6)                         [3, sqrt(13)]
(%i7) take_inference(items_inference(sol),sol);
(%o7)                  [3, 2, sqrt(13), 6, 10]
```

See also `inference_result`, and `take_inference`.

94.3 Functions and Variables for stats

`stats_numer` [Option variable]

Default value: `true`

If `stats_numer` is `true`, inference statistical functions return their results in floating point numbers. If it is `false`, results are given in symbolic and rational format.

`test_mean` [Function]

`test_mean` (*x*)
`test_mean` (*x*, *options* ...)

This is the mean *t*-test. Argument *x* is a list or a column matrix containing an one dimensional sample. It also performs an asymptotic test based on the *Central Limit Theorem* if option `'asymptotic` is `true`.

Options:

- `'mean`, default `0`, is the mean value to be checked.
- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'dev`, default `'unknown`, this is the value of the standard deviation when it is known; valid values are: `'unknown` or a positive expression.
- `'conflevel`, default `95/100`, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- `'asymptotic`, default `false`, indicates whether it performs an exact *t*-test or an asymptotic one based on the *Central Limit Theorem*; valid values are `true` and `false`.

The output of function `test_mean` is an `inference_result` Maxima object showing the following results:

1. `'mean_estimate`: the sample mean.
2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: confidence interval for the population mean.
4. `'method`: inference procedure.
5. `'hypotheses`: null and alternative hypotheses to be tested.
6. `'statistic`: value of the sample statistic used for testing the null hypothesis.
7. `'distribution`: distribution of the sample statistic, together with its parameter(s).
8. `'p_value`: p-value of the test.

Examples:

Performs an exact t-test with unknown variance. The null hypothesis is $H_0 : mean = 50$ against the one sided alternative $H_1 : mean < 50$; according to the results, the p-value is too great, there are no evidence for rejecting H_0.

```
(%i1) load("stats")$
(%i2) data: [78,64,35,45,45,75,43,74,42,42]$
(%i3) test_mean(data,'conflevel=0.9,'alternative='less,'mean=50);
           |                  MEAN TEST
           |
           |            mean_estimate = 54.3
           |
           |              conf_level = 0.9
           |
           | conf_interval = [minf, 61.51314273502712]
           |
(%o3)      |  method = Exact t-test. Unknown variance.
           |
           | hypotheses = H0: mean = 50 , H1: mean < 50
           |
           |       statistic = .8244705235071678
           |
           |       distribution = [student_t, 9]
           |
           |        p_value = .7845100411786889
```

This time Maxima performs an asymptotic test, based on the *Central Limit Theorem*. The null hypothesis is $H_0 : equal(mean, 50)$ against the two sided alternative $H_1 : notequal(mean, 50)$; according to the results, the p-value is very small, H_0 should be rejected in favor of the alternative H_1. Note that, as indicated by the `Method` component, this procedure should be applied to large samples.

```
(%i1) load("stats")$
(%i2) test_mean([36,118,52,87,35,256,56,178,57,57,89,34,25,98,35,
              98,41,45,198,54,79,63,35,45,44,75,42,75,45,45,
              45,51,123,54,151],
```

```
                    'asymptotic=true,'mean=50);
                |                       MEAN TEST
                |
                |           mean_estimate = 74.88571428571429
                |
                |                   conf_level = 0.95
                |
                | conf_interval = [57.72848600856194, 92.04294256286663]
                |
        (%o2)   |    method = Large sample z-test. Unknown variance.
                |
                |       hypotheses = H0: mean = 50 , H1: mean # 50
                |
                |             statistic = 2.842831192874313
                |
                |             distribution = [normal, 0, 1]
                |
                |             p_value = .004471474652002261
```

`test_means_difference` [Function]

`test_means_difference` (*x1*, *x2*)

`test_means_difference` (*x1*, *x2*, *options ...*)

This is the difference of means *t*-test for two samples. Arguments *x1* and *x2* are lists or column matrices containing two independent samples. In case of different unknown variances (see options `'dev1`, `'dev2` and `'varequal` below), the degrees of freedom are computed by means of the Welch approximation. It also performs an asymptotic test based on the *Central Limit Theorem* if option `'asymptotic` is set to `true`.

Options:

-
- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'dev1`, default `'unknown`, this is the value of the standard deviation of the *x1* sample when it is known; valid values are: `'unknown` or a positive expression.
- `'dev2`, default `'unknown`, this is the value of the standard deviation of the *x2* sample when it is known; valid values are: `'unknown` or a positive expression.
- `'varequal`, default `false`, whether variances should be considered to be equal or not; this option takes effect only when `'dev1` and/or `'dev2` are `'unknown`.
- `'conflevel`, default `95/100`, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- `'asymptotic`, default `false`, indicates whether it performs an exact *t*-test or an asymptotic one based on the *Central Limit Theorem*; valid values are `true` and `false`.

The output of function `test_means_difference` is an `inference_result` Maxima object showing the following results:

1. `'diff_estimate`: the difference of means estimate.

2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: confidence interval for the difference of means.
4. `'method`: inference procedure.
5. `'hypotheses`: null and alternative hypotheses to be tested.
6. `'statistic`: value of the sample statistic used for testing the null hypothesis.
7. `'distribution`: distribution of the sample statistic, together with its parameter(s).
8. `'p_value`: p-value of the test.

Examples:

The equality of means is tested with two small samples x and y, against the alternative $H_1 : m_1 > m_2$, being m_1 and m_2 the populations means; variances are unknown and supposed to be different.

```
(%i1) load("stats")$
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]$
(%i3) y: [1.2,6.9,38.7,20.4,17.2]$
(%i4) test_means_difference(x,y,'alternative='greater);
                |              DIFFERENCE OF MEANS TEST
                |
                |         diff_estimate = 20.31999999999999
                |
                |                conf_level = 0.95
                |
                |    conf_interval = [- .04597417812882298, inf]
                |
(%o4)           |        method = Exact t-test. Welch approx.
                |
                | hypotheses = H0: mean1 = mean2 , H1: mean1 > mean2
                |
                |           statistic = 1.838004300728477
                |
                |    distribution = [student_t, 8.62758740184604]
                |
                |            p_value = .05032746527991905
```

The same test as before, but now variances are supposed to be equal.

```
(%i1) load("stats")$
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]$
(%i3) y: matrix([1.2],[6.9],[38.7],[20.4],[17.2])$
(%i4) test_means_difference(x,y,'alternative='greater,
                                                 'varequal=true);
                |              DIFFERENCE OF MEANS TEST
                |
                |         diff_estimate = 20.31999999999999
                |
                |                conf_level = 0.95
```

```
                    |
                    |     conf_interval = [- .7722627696897568, inf]
                    |
          (%o4)     |   method = Exact t-test. Unknown equal variances
                    |
                    | hypotheses = H0: mean1 = mean2 , H1: mean1 > mean2
                    |
                    |           statistic = 1.765996124515009
                    |
                    |           distribution = [student_t, 9]
                    |
                    |            p_value = .05560320992529344
```

`test_variance` [Function]

`test_variance` (*x*)

`test_variance` (*x*, *options*, ...)

This is the variance *chi^2*-test. Argument *x* is a list or a column matrix containing an one dimensional sample taken from a normal population.

Options:

- `'mean`, default `'unknown`, is the population's mean, when it is known.
- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'variance`, default `1`, this is the variance value (positive) to be checked.
- `'conflevel`, default `95/100`, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).

The output of function `test_variance` is an `inference_result` Maxima object showing the following results:

1. `'var_estimate`: the sample variance.
2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: confidence interval for the population variance.
4. `'method`: inference procedure.
5. `'hypotheses`: null and alternative hypotheses to be tested.
6. `'statistic`: value of the sample statistic used for testing the null hypothesis.
7. `'distribution`: distribution of the sample statistic, together with its parameter.
8. `'p_value`: *p*-value of the test.

Examples:

It is tested whether the variance of a population with unknown mean is equal to or greater than 200.

```
(%i1) load("stats")$
(%i2) x: [203,229,215,220,223,233,208,228,209]$
(%i3) test_variance(x,'alternative='greater,'variance=200);
                    |                  VARIANCE TEST
                    |
```

```
                      |              var_estimate = 110.75
                      |
                      |                conf_level = 0.95
                      |
                      |     conf_interval = [57.13433376937479, inf]
                      |
          (%o3)       | method = Variance Chi-square test. Unknown mean.
                      |
                      |    hypotheses = H0: var = 200 , H1: var > 200
                      |
                      |                 statistic = 4.43
                      |
                      |             distribution = [chi2, 8]
                      |
                      |           p_value = .8163948512777689
```

`test_variance_ratio` [Function]

`test_variance_ratio` (*x1*, *x2*)

`test_variance_ratio` (*x1*, *x2*, *options* ...)

This is the variance ratio F-test for two normal populations. Arguments *x1* and *x2* are lists or column matrices containing two independent samples.

Options:

- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'mean1`, default `'unknown`, when it is known, this is the mean of the population from which *x1* was taken.
- `'mean2`, default `'unknown`, when it is known, this is the mean of the population from which *x2* was taken.
- `'conflevel`, default `95/100`, confidence level for the confidence interval of the ratio; it must be an expression which takes a value in (0,1).

The output of function `test_variance_ratio` is an `inference_result` Maxima object showing the following results:

1. `'ratio_estimate`: the sample variance ratio.
2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: confidence interval for the variance ratio.
4. `'method`: inference procedure.
5. `'hypotheses`: null and alternative hypotheses to be tested.
6. `'statistic`: value of the sample statistic used for testing the null hypothesis.
7. `'distribution`: distribution of the sample statistic, together with its parameters.
8. `'p_value`: p-value of the test.

Examples:

The equality of the variances of two normal populations is checked against the alternative that the first is greater than the second.

```
(%i1) load("stats")$
(%i2) x: [20.4,62.5,61.3,44.2,11.1,23.7]$
(%i3) y: [1.2,6.9,38.7,20.4,17.2]$
(%i4) test_variance_ratio(x,y,'alternative='greater);
                |              VARIANCE RATIO TEST
                |
                |       ratio_estimate = 2.316933391522034
                |
                |               conf_level = 0.95
                |
                |    conf_interval = [.3703504689507268, inf]
                |
(%o4)           | method = Variance ratio F-test. Unknown means.
                |
                | hypotheses = H0: var1 = var2 , H1: var1 > var2
                |
                |         statistic = 2.316933391522034
                |
                |            distribution = [f, 5, 4]
                |
                |          p_value = .2179269692254457
```

`test_proportion` [Function]

`test_proportion` (*x*, *n*)
`test_proportion` (*x*, *n*, *options* ...)

Inferences on a proportion. Argument *x* is the number of successes in *n* trials in a Bernoulli experiment with unknown probability.

Options:

- `'proportion`, default `1/2`, is the value of the proportion to be checked.
- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'conflevel`, default `95/100`, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- `'asymptotic`, default `false`, indicates whether it performs an exact test based on the binomial distribution, or an asymptotic one based on the *Central Limit Theorem*; valid values are `true` and `false`.
- `'correct`, default `true`, indicates whether Yates correction is applied or not.

The output of function `test_proportion` is an `inference_result` Maxima object showing the following results:

1. `'sample_proportion`: the sample proportion.
2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: Wilson confidence interval for the proportion.

4. 'method: inference procedure.
5. 'hypotheses: null and alternative hypotheses to be tested.
6. 'statistic: value of the sample statistic used for testing the null hypothesis.
7. 'distribution: distribution of the sample statistic, together with its parameters.
8. 'p_value: p-value of the test.

Examples:

Performs an exact test. The null hypothesis is $H_0 : p = 1/2$ against the one sided alternative $H_1 : p < 1/2$.

```
(%i1) load("stats")$
(%i2) test_proportion(45, 103, alternative = less);
          |            PROPORTION TEST
          |
          | sample_proportion = .4368932038834951
          |
          |           conf_level = 0.95
          |
          | conf_interval = [0, 0.522714149150231]
          |
(%o2)     |     method = Exact binomial test.
          |
          | hypotheses = H0: p = 0.5 , H1: p < 0.5
          |
          |             statistic = 45
          |
          |  distribution = [binomial, 103, 0.5]
          |
          |      p_value = .1184509388901454
```

A two sided asymptotic test. Confidence level is 99/100.

```
(%i1) load("stats")$
(%i2) fpprintprec:7$
(%i3) test_proportion(45, 103,
                  conflevel = 99/100, asymptotic=true);
       |                PROPORTION TEST
       |
       |           sample_proportion = .43689
       |
       |                conf_level = 0.99
       |
       |        conf_interval = [.31422, .56749]
       |
(%o3)  | method = Asympthotic test with Yates correction.
       |
       |     hypotheses = H0: p = 0.5 , H1: p # 0.5
       |
```

```
                |               statistic = .43689
                |
                |      distribution = [normal, 0.5, .048872]
                |
                |                p_value = .19662
```

`test_proportions_difference` [Function]

`test_proportions_difference` (*x1*, *n1*, *x2*, *n2*)
`test_proportions_difference` (*x1*, *n1*, *x2*, *n2*, *options ...*)

Inferences on the difference of two proportions. Argument *x1* is the number of successes in *n1* trials in a Bernoulli experiment in the first population, and *x2* and *n2* are the corresponding values in the second population. Samples are independent and the test is asymptotic.

Options:

- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided` (`p1 # p2`), `'greater` (`p1 > p2`) and `'less` (`p1 < p2`).
- `'conflevel`, default `95/100`, confidence level for the confidence interval; it must be an expression which takes a value in (0,1).
- `'correct`, default `true`, indicates whether Yates correction is applied or not.

The output of function `test_proportions_difference` is an `inference_result` Maxima object showing the following results:

1. `'proportions`: list with the two sample proportions.
2. `'conf_level`: confidence level selected by the user.
3. `'conf_interval`: Confidence interval for the difference of proportions `p1 - p2`.
4. `'method`: inference procedure and warning message in case of any of the samples sizes is less than 10.
5. `'hypotheses`: null and alternative hypotheses to be tested.
6. `'statistic`: value of the sample statistic used for testing the null hypothesis.
7. `'distribution`: distribution of the sample statistic, together with its parameters.
8. `'p_value`: *p*-value of the test.

Examples:

A machine produced 10 defective articles in a batch of 250. After some maintenance work, it produces 4 defective in a batch of 150. In order to know if the machine has improved, we test the null hypothesis `H0:p1=p2`, against the alternative `H0:p1>p2`, where `p1` and `p2` are the probabilities for one produced article to be defective before and after maintenance. According to the p value, there is not enough evidence to accept the alternative.

```
(%i1) load("stats")$
(%i2) fpprintprec:7$
(%i3) test_proportions_difference(10, 250, 4, 150,
                                  alternative = greater);
      |       DIFFERENCE OF PROPORTIONS TEST
```

```
                |
                |       proportions = [0.04, .02666667]
                |
                |              conf_level = 0.95
                |
                |      conf_interval = [- .02172761, 1]
                |
          (%o3) | method = Asymptotic test. Yates correction.
                |
                |   hypotheses = H0: p1 = p2 , H1: p1 > p2
                |
                |            statistic = .01333333
                |
                |    distribution = [normal, 0, .01898069]
                |
                |             p_value = .2411936
```

Exact standard deviation of the asymptotic normal distribution when the data are unknown.

```
(%i1) load("stats")$
(%i2) stats_numer: false$
(%i3) sol: test_proportions_difference(x1,n1,x2,n2)$
(%i4) last(take_inference('distribution,sol));
                1    1                  x2 + x1
              (-- + --) (x2 + x1) (1 - -------)
                n2   n1                 n2 + n1
(%o4)    sqrt(---------------------------------)
                             n2 + n1
```

`test_sign` [Function]

`test_sign` (*x*)

`test_sign` (*x*, *options* ...)

This is the non parametric sign test for the median of a continuous population. Argument *x* is a list or a column matrix containing an one dimensional sample.

Options:

- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.
- `'median`, default `0`, is the median value to be checked.

The output of function `test_sign` is an `inference_result` Maxima object showing the following results:

1. `'med_estimate`: the sample median.
2. `'method`: inference procedure.
3. `'hypotheses`: null and alternative hypotheses to be tested.
4. `'statistic`: value of the sample statistic used for testing the null hypothesis.
5. `'distribution`: distribution of the sample statistic, together with its parameter(s).

6. 'p_value: p-value of the test.

Examples:

Checks whether the population from which the sample was taken has median 6, against the alternative $H_1 : median > 6$.

```
(%i1) load("stats")$
(%i2) x: [2,0.1,7,1.8,4,2.3,5.6,7.4,5.1,6.1,6]$
(%i3) test_sign(x,'median=6,'alternative='greater);
                |                   SIGN TEST
                |
                |               med_estimate = 5.1
                |
                |      method = Non parametric sign test.
                |
(%o3)           | hypotheses = H0: median = 6 , H1: median > 6
                |
                |                 statistic = 7
                |
                |      distribution = [binomial, 10, 0.5]
                |
                |         p_value = .05468749999999989
```

test_signed_rank [Function]

test_signed_rank (*x*)
test_signed_rank (*x*, *options* ...)

This is the Wilcoxon signed rank test to make inferences about the median of a continuous population. Argument x is a list or a column matrix containing an one dimensional sample. Performs normal approximation if the sample size is greater than 20, or if there are zeroes or ties.

See also pdf_rank_test and cdf_rank_test

Options:

- 'median, default 0, is the median value to be checked.
- 'alternative, default 'twosided, is the alternative hypothesis; valid values are: 'twosided, 'greater and 'less.

The output of function test_signed_rank is an inference_result Maxima object with the following results:

1. 'med_estimate: the sample median.
2. 'method: inference procedure.
3. 'hypotheses: null and alternative hypotheses to be tested.
4. 'statistic: value of the sample statistic used for testing the null hypothesis.
5. 'distribution: distribution of the sample statistic, together with its parameter(s).
6. 'p_value: p-value of the test.

Examples:

Checks the null hypothesis $H_0 : median = 15$ against the alternative $H_1 : median > 15$. This is an exact test, since there are no ties.

```
(%i1) load("stats")$
(%i2) x: [17.1,15.9,13.7,13.4,15.5,17.6]$
(%i3) test_signed_rank(x,median=15,alternative=greater);
                   |             SIGNED RANK TEST
                   |
                   |           med_estimate = 15.7
                   |
                   |           method = Exact test
                   |
(%o3)              | hypotheses = H0: med = 15 , H1: med > 15
                   |
                   |              statistic = 14
                   |
                   |     distribution = [signed_rank, 6]
                   |
                   |            p_value = 0.28125
```

Checks the null hypothesis H_0 : $equal(median, 2.5)$ against the alternative H_1 : $notequal(median, 2.5)$. This is an approximated test, since there are ties.

```
(%i1) load("stats")$
(%i2) y:[1.9,2.3,2.6,1.9,1.6,3.3,4.2,4,2.4,2.9,1.5,3,2.9,4.2,3.1]$
(%i3) test_signed_rank(y,median=2.5);
                |                 SIGNED RANK TEST
                |
                |                med_estimate = 2.9
                |
                |          method = Asymptotic test. Ties
                |
(%o3)           |    hypotheses = H0: med = 2.5 , H1: med # 2.5
                |
                |                 statistic = 76.5
                |
                | distribution = [normal, 60.5, 17.58195097251724]
                |
                |           p_value = .3628097734643669
```

`test_rank_sum` [Function]

`test_rank_sum` (*x1*, *x2*)
`test_rank_sum` (*x1*, *x2*, *option*)

This is the Wilcoxon-Mann-Whitney test for comparing the medians of two continuous populations. The first two arguments *x1* and *x2* are lists or column matrices with the data of two independent samples. Performs normal approximation if any of the sample sizes is greater than 10, or if there are ties.

Option:

- `'alternative`, default `'twosided`, is the alternative hypothesis; valid values are: `'twosided`, `'greater` and `'less`.

The output of function `test_rank_sum` is an `inference_result` Maxima object with the following results:

1. `'method`: inference procedure.
2. `'hypotheses`: null and alternative hypotheses to be tested.
3. `'statistic`: value of the sample statistic used for testing the null hypothesis.
4. `'distribution`: distribution of the sample statistic, together with its parameters.
5. `'p_value`: p-value of the test.

Examples:

Checks whether populations have similar medians. Samples sizes are small and an exact test is made.

```
(%i1) load("stats")$
(%i2) x:[12,15,17,38,42,10,23,35,28]$
(%i3) y:[21,18,25,14,52,65,40,43]$
(%i4) test_rank_sum(x,y);
                |                  RANK SUM TEST
                |
                |              method = Exact test
                |
                | hypotheses = H0: med1 = med2 , H1: med1 # med2
(%o4)           |
                |                  statistic = 22
                |
                |        distribution = [rank_sum, 9, 8]
                |
                |          p_value = .1995886466474702
```

Now, with greater samples and ties, the procedure makes normal approximation. The alternative hypothesis is $H_1 : median1 < median2$.

```
(%i1) load("stats")$
(%i2) x: [39,42,35,13,10,23,15,20,17,27]$
(%i3) y: [20,52,66,19,41,32,44,25,14,39,43,35,19,56,27,15]$
(%i4) test_rank_sum(x,y,'alternative='less);
              |                  RANK SUM TEST
              |
              |          method = Asymptotic test. Ties
              |
              |  hypotheses = H0: med1 = med2 , H1: med1 < med2
(%o4)         |
              |                 statistic = 48.5
              |
              | distribution = [normal, 79.5, 18.95419580097078]
              |
              |           p_value = .05096985666598441
```

`test_normality (x)` [Function]

Shapiro-Wilk test for normality. Argument x is a list of numbers, and sample size must be greater than 2 and less or equal than 5000, otherwise, function `test_normality` signals an error message.

Reference:

[1] Algorithm AS R94, Applied Statistics (1995), vol.44, no.4, 547-551

The output of function `test_normality` is an `inference_result` Maxima object with the following results:

1. `'statistic`: value of the W statistic.
2. `'p_value`: p-value under normal assumption.

Examples:

Checks for the normality of a population, based on a sample of size 9.

```
(%i1) load("stats")$
(%i2) x:[12,15,17,38,42,10,23,35,28]$
(%i3) test_normality(x);
                          |      SHAPIRO - WILK TEST
                          |
(%o3)                     | statistic = .9251055695162436
                          |
                          |  p_value = .4361763918860381
```

`linear_regression` [Function]

`linear_regression (x)`
`linear_regression (x option)`

Multivariate linear regression, $y_i = b0 + b1 * x_1 i + b2 * x_2 i + ... + bk * x_k i + u_i$, where u_i are $N(0, sigma)$ independent random variables. Argument x must be a matrix with more than one column. The last column is considered as the responses (y_i).

Option:

- `'conflevel`, default `95/100`, confidence level for the confidence intervals; it must be an expression which takes a value in (0,1).

The output of function `linear_regression` is an `inference_result` Maxima object with the following results:

1. `'b_estimation`: regression coefficients estimates.
2. `'b_covariances`: covariance matrix of the regression coefficients estimates.
3. `b_conf_int`: confidence intervals of the regression coefficients.
4. `b_statistics`: statistics for testing coefficient.
5. `b_p_values`: p-values for coefficient tests.
6. `b_distribution`: probability distribution for coefficient tests.
7. `v_estimation`: unbiased variance estimator.
8. `v_conf_int`: variance confidence interval.
9. `v_distribution`: probability distribution for variance test.
10. `residuals`: residuals.

11. `adc`: adjusted determination coefficient.
12. `aic`: Akaike's information criterion.
13. `bic`: Bayes's information criterion.

Only items 1, 4, 5, 6, 7, 8, 9 and 11 above, in this order, are shown by default. The rest remain hidden until the user makes use of functions `items_inference` and `take_inference`.

Example:

Fitting a linear model to a trivariate sample. The last column is considered as the responses (y_i).

```
(%i2) load("stats")$
(%i3) X:matrix(
    [58,111,64],[84,131,78],[78,158,83],
    [81,147,88],[82,121,89],[102,165,99],
    [85,174,101],[102,169,102])$
(%i4) fpprintprec: 4$
(%i5) res: linear_regression(X);
             |       LINEAR REGRESSION MODEL
             |
             | b_estimation = [9.054, .5203, .2397]
             |
             | b_statistics = [.6051, 2.246, 1.74]
             |
             | b_p_values = [.5715, .07466, .1423]
             |
(%o5)        |   b_distribution = [student_t, 5]
             |
             |         v_estimation = 35.27
             |
             |     v_conf_int = [13.74, 212.2]
             |
             |      v_distribution = [chi2, 5]
             |
             |             adc = .7922
(%i6) items_inference(res);
(%o6) [b_estimation, b_covariances, b_conf_int, b_statistics,
b_p_values, b_distribution, v_estimation, v_conf_int,
v_distribution, residuals, adc, aic, bic]
(%i7) take_inference('b_covariances, res);
                  [  223.9    - 1.12   - .8532  ]
                  [                             ]
(%o7)             [ - 1.12    .05367   - .02305 ]
                  [                             ]
                  [ - .8532  - .02305   .01898  ]
(%i8) take_inference('bic, res);
(%o8)                          30.98
```

```
(%i9) load("draw")$
(%i10) draw2d(
    points_joined = true,
    grid = true,
    points(take_inference('residuals, res)) )$
```

94.4 Functions and Variables for special distributions

`pdf_signed_rank` (*x*, *n*) [Function]

Probability density function of the exact distribution of the signed rank statistic. Argument *x* is a real number and *n* a positive integer.

See also `test_signed_rank`.

`cdf_signed_rank` (*x*, *n*) [Function]

Cumulative distribution function of the exact distribution of the signed rank statistic. Argument *x* is a real number and *n* a positive integer.

See also `test_signed_rank`.

`pdf_rank_sum` (*x*, *n*, *m*) [Function]

Probability density function of the exact distribution of the rank sum statistic. Argument *x* is a real number and *n* and *m* are both positive integers.

See also `test_rank_sum`.

`cdf_rank_sum` (*x*, *n*, *m*) [Function]

Cumulative distribution function of the exact distribution of the rank sum statistic. Argument *x* is a real number and *n* and *m* are both positive integers.

See also `test_rank_sum`.

95 Package stirling

95.1 Functions and Variables for stirling

`stirling` [Function]

`stirling` (*z*,*n*)
`stirling` (*z*,*n*,*pred*)

Replace `gamma(x)` with the $O(1/x^{2n-1})$ Stirling formula. when n isn't a nonnegative integer, signal an error. With the optional third argument `pred`, the Stirling formula is applied only when `pred` is true.

Reference: Abramowitz & Stegun, " Handbook of mathematical functions", 6.1.40.

Examples:

```
(%i1) load ("stirling")$

(%i2) stirling(gamma(%alpha+x)/gamma(x),1);
       1/2 - x             x + %alpha - 1/2
(%o2) x        (x + %alpha)
                                    1           1
                              --------------- - ---- - %alpha
                              12 (x + %alpha)   12 x
                            %e
(%i3) taylor(%,x,inf,1);
                     %alpha       2    %alpha
          %alpha   x       %alpha  - x       %alpha
(%o3)/T/ x       + -------------------------------- + . . .
                                 2 x
(%i4) map('factor,%);
                                          %alpha - 1
         %alpha   (%alpha - 1) %alpha x
(%o4)   x       + ------------------------------
                                2
```

The function `stirling` knows the difference between the variable 'gamma' and the function gamma:

```
(%i5) stirling(gamma + gamma(x),0);
                                     x - 1/2   - x
(%o5)    gamma + sqrt(2) sqrt(%pi) x        %e
(%i6) stirling(gamma(y) + gamma(x),0);
                           y - 1/2   - y
(%o6) sqrt(2) sqrt(%pi) y        %e
                                                 x - 1/2   - x
                          + sqrt(2) sqrt(%pi) x        %e
```

To apply the Stirling formula only to terms that involve the variable `k`, use an optional third argument; for example

```
(%i7) makegamma(pochhammer(a,k)/pochhammer(b,k));
```

```
(%o7) (gamma(b)*gamma(k+a))/(gamma(a)*gamma(k+b))
(%i8) stirling(%,1, lambda([s], not(freeof(k,s))));
(%o8) (%e^(b-a)*gamma(b)*(k+a)^(k+a-1/2)*(k+b)^(-k-b+1/2))/gamma(a)
```

The terms `gamma(a)` and `gamma(b)` are free of `k`, so the Stirling formula was not applied to these two terms.

To use this function write first `load("stirling")`.

96 Package stringproc

96.1 Introduction to String Processing

The package `stringproc` contains functions for processing strings and characters including formatting, encoding and data streams. This package is completed by some tools for cryptography, e.g. base64 and hash functions.

It can be directly loaded via `load("stringproc")` or automatically by using one of its functions.

For questions and bug reports please contact the author. The following command prints his e-mail-address.

```
printf(true, "~{~a~}@gmail.com", split(sdowncase("Volker van Nek")))$
```

A string is constructed by typing e.g. `"Text"`. When the option variable `stringdisp` is set to `false`, which is the default, the double quotes won't be printed. [stringp], page 1298, is a test, if an object is a string.

```
(%i1) str: "Text";
(%o1)                          Text
(%i2) stringp(str);
(%o2)                          true
```

Characters are represented by a string of length 1. [charp], page 1290, is the corresponding test.

```
(%i1) char: "e";
(%o1)                            e
(%i2) charp(char);
(%o2)                          true
```

In Maxima position indices in strings are like in list 1-indexed which results to the following consistency.

```
(%i1) is(charat("Lisp",1) = charlist("Lisp")[1]);
(%o1)                          true
```

A string may contain Maxima expressions. These can be parsed with [parse_string], page 1294.

```
(%i1) map(parse_string, ["42" ,"sqrt(2)", "%pi"]);
(%o1)                    [42, sqrt(2), %pi]
(%i2) map('float, %);
(%o2)        [42.0, 1.414213562373095, 3.141592653589793]
```

Strings can be processed as characters or in binary form as octets. Functions for conversions are [string_to_octets], page 1303, and [octets_to_string], page 1302. Usable encodings depend on the platform, the application and the underlying Lisp. (The following shows Maxima in GNU/Linux, compiled with SBCL.)

```
(%i1) obase: 16.$
(%i2) string_to_octets("$£€", "cp1252");
(%o2)                       [24, 0A3, 80]
(%i3) string_to_octets("$£€", "utf-8");
```

```
(%o3)                [24, 0C2, 0A3, 0E2, 82, 0AC]
```

Strings may be written to character streams or as octets to binary streams. The following example demonstrates file in and output of characters.

[openw], page 1284, returns an output stream to a file, [printf], page 1285, writes formatted to that file and by e.g. [close], page 1282, all characters contained in the stream are written to the file.

```
(%i1) s: openw("file.txt");
(%o1)                 #<output stream file.txt>
(%i2) printf(s, "~%~d ~f ~a ~a ~f ~e ~a~%",
42, 1.234, sqrt(2), %pi, 1.0e-2, 1.0e-2, 1.0b-2)$
(%i3) close(s)$
```

[openr], page 1284, then returns an input stream from the previously used file and [readline], page 1287, returns the line read as a string. The string may be tokenized by e.g. [split], page 1295, or [tokens], page 1298, and finally parsed by [parse_string], page 1294.

```
(%i4) s: openr("file.txt");
(%o4)                  #<input stream file.txt>
(%i5) readline(s);
(%o5)           42 1.234 sqrt(2) %pi 0.01 1.0E-2 1.0b-2
(%i6) map(parse_string, split(%));
(%o6)       [42, 1.234, sqrt(2), %pi, 0.01, 0.01, 1.0b-2]
(%i7) close(s)$
```

96.2 String Input and Output

Example: Formatted printing to a file.

```
(%i1) s: openw("file.txt");
(%o1)                        #<output stream file.txt>
(%i2) control:
"~2tAn atom: ~20t~a~%~2tand a list: ~20t~{~r ~}~%~2t\
and an integer: ~20t~d~%"$
(%i3) printf( s,control, 'true,[1,2,3],42 )$
(%o3)                                  false
(%i4) close(s);
(%o4)                                  true
(%i5) s: openr("file.txt");
(%o5)                        #<input stream file.txt>
(%i6) while stringp( tmp:readline(s) ) do print(tmp)$
  An atom:          true
  and a list:       one two three
  and an integer:   42
(%i7) close(s)$
```

`close` (*stream*) [Function]

Closes *stream* and returns `true` if *stream* had been open.

`flength` (*stream*) [Function]

stream has to be an open stream from or to a file. `flength` then returns the number of bytes which are currently present in this file.

Example: See [writebyte], page 1288, .

`flush_output` (*stream*) [Function]

Flushes *stream* where *stream* has to be an output stream to a file.

Example: See [writebyte], page 1288, .

`fposition` [Function]

`fposition` (*stream*)
`fposition` (*stream*, *pos*)

Returns the current position in *stream*, if *pos* is not used. If *pos* is used, `fposition` sets the position in *stream*. *stream* has to be a stream from or to a file and *pos* has to be a positive number.

Positions in data streams are like in strings or lists 1-indexed, i.e. the first element in *stream* is in position 1.

`freshline` [Function]

`freshline` ()
`freshline` (*stream*)

Writes a new line to the standard output stream if the position is not at the beginning of a line and returns `true`. Using the optional argument *stream* the new line is written to that stream. There are some cases, where `freshline()` does not work as expected.

See also [newline], page 1284.

`get_output_stream_string` (*stream*) [Function]

Returns a string containing all the characters currently present in *stream* which must be an open string-output stream. The returned characters are removed from *stream*.

Example: See [make_string_output_stream], page 1283, .

`make_string_input_stream` [Function]

`make_string_input_stream` (*string*)
`make_string_input_stream` (*string*, *start*)
`make_string_input_stream` (*string*, *start*, *end*)

Returns an input stream which contains parts of *string* and an end of file. Without optional arguments the stream contains the entire string and is positioned in front of the first character. *start* and *end* define the substring contained in the stream. The first character is available at position 1.

```
(%i1) istream : make_string_input_stream("text", 1, 4);
(%o1)              #<string-input stream from "text">
(%i2) (while (c : readchar(istream)) # false do sprint(c), newline())$
t e x
(%i3) close(istream)$
```

`make_string_output_stream` () [Function]

Returns an output stream that accepts characters. Characters currently present in this stream can be retrieved by [get_output_stream_string], page 1283.

```
(%i1) ostream : make_string_output_stream();
```

```
(%o1)               #<string-output stream 09622ea0>
(%i2) printf(ostream, "foo")$

(%i3) printf(ostream, "bar")$

(%i4) string : get_output_stream_string(ostream);
(%o4)                              foobar
(%i5) printf(ostream, "baz")$

(%i6) string : get_output_stream_string(ostream);
(%o6)                                baz
(%i7) close(ostream)$
```

`newline` [Function]

`newline ()`
`newline (`*`stream`*`)`

Writes a new line to the standard output stream. Using the optional argument *stream* the new line is written to that stream. There are some cases, where `newline()` does not work as expected.

See [sprint], page 1287, for an example of using `newline()`.

`opena (`*`file`*`)` [Function]

Returns a character output stream to *file*. If an existing file is opened, `opena` appends elements at the end of *file*.

For binary output see Section 81.3 [opena_binary], page 1186, .

`openr` [Function]

`openr (`*`file`*`)`
`openr (`*`file`*`, `*`encoding`*`)`

Returns a character input stream to *file*. `openr` assumes that *file* already exists. If reading the file results in a lisp error about its encoding passing the correct string as the argument *encoding* might help. The available encodings and their names depend on the lisp being used. For sbcl a list of suitable strings can be found at `http://www.sbcl.org/manual/#External-Formats`.

For binary input see Section 81.3 [openr_binary], page 1186, . See also `close` and `openw`.

```
(%i1) istream : openr("data.txt","EUC-JP");
(%o1)     #<FD-STREAM for "file /home/gunter/data.txt" {10099A3AE3}>
(%i2) close(istream);
(%o2)                                true
```

`openw (`*`file`*`)` [Function]

Returns a character output stream to *file*. If *file* does not exist, it will be created. If an existing file is opened, `openw` destructively modifies *file*.

For binary output see Section 81.3 [openw_binary], page 1186, .

See also `close` and `openr`.

printf [Function]

printf (*dest*, *string*)
printf (*dest*, *string*, *expr_1*, ..., *expr_n*)

Produces formatted output by outputting the characters of control-string *string* and observing that a tilde introduces a directive. The character after the tilde, possibly preceded by prefix parameters and modifiers, specifies what kind of formatting is desired. Most directives use one or more elements of the arguments *expr_1*, ..., *expr_n* to create their output.

If *dest* is a stream or `true`, then `printf` returns `false`. Otherwise, `printf` returns a string containing the output. By default the streams *stdin*, *stdout* and *stderr* are defined. If Maxima is running as a network client (which is the normal case if Maxima is communicating with a graphical user interface, which must be the server) `setup-client` will define *old_stdout* and *old_stderr*, too.

`printf` provides the Common Lisp function `format` in Maxima. The following example illustrates the general relation between these two functions.

```
(%i1) printf(true, "R~dD~d~%", 2, 2);
R2D2
(%o1)                                false
(%i2) :lisp (format t "R~dD~d~%" 2 2)
R2D2
NIL
```

The following description is limited to a rough sketch of the possibilities of `printf`. The Lisp function `format` is described in detail in many reference books. Of good help is e.g. the free available online-manual "Common Lisp the Language" by Guy L. Steele. See chapter 22.3.3 there.

In addition, `printf` recognizes two format directives which are not known to Lisp `format`. The format directive `~m` indicates Maxima pretty printer output. The format directive `~h` indicates a bigfloat number.

```
~%      new line
~&      fresh line
~t      tab
~$      monetary
~d      decimal integer
~b      binary integer
~o      octal integer
~x      hexadecimal integer
~br     base-b integer
~r      spell an integer
~p      plural
~f      floating point
~e      scientific notation
~g      ~f or ~e, depending upon magnitude
~h      bigfloat
~a      uses Maxima function string
~m      Maxima pretty printer output
```

```
~s       like ~a, but output enclosed in "double quotes"
~~       ~
~<       justification, ~> terminates
~(       case conversion, ~) terminates
~[       selection, ~] terminates
~{       iteration, ~} terminates
```

Note that the directive ~* is not supported.

If *dest* is a stream or `true`, then `printf` returns `false`. Otherwise, `printf` returns a string containing the output.

```
(%i1) printf( false, "~a ~a ~4f ~a ~@r",
              "String",sym,bound,sqrt(12),144), bound = 1.234;
(%o1)                 String sym 1.23 2*sqrt(3) CXLIV
(%i2) printf( false,"~{~a ~}",["one",2,"THREE"] );
(%o2)                           one 2 THREE
(%i3) printf(true,"~{~{~9,1f ~}~%~}",mat ),
          mat = args(matrix([1.1,2,3.33],[4,5,6],[7,8.88,9]))$
      1.1       2.0       3.3
      4.0       5.0       6.0
      7.0       8.9       9.0
(%i4) control: "~:(~r~) bird~p ~[is~;are~] singing."$
(%i5) printf( false,control, n,n,if n=1 then 1 else 2 ), n=2;
(%o5)                     Two birds are singing.
```

The directive ~h has been introduced to handle bigfloats.

```
~w,d,e,x,o,p@H
 w : width
 d : decimal digits behind floating point
 e : minimal exponent digits
 x : preferred exponent
 o : overflow character
 p : padding character
 @ : display sign for positive numbers
```

```
(%i1) fpprec : 1000$
(%i2) printf(true, "|~h|~%", 2.b0^-64)$
|0.00000000000000000005421010862427522170037264004349708557128906250|
(%i3) fpprec : 26$
(%i4) printf(true, "|~h|~%", sqrt(2))$
|1.4142135623730950488016887|
(%i5) fpprec : 24$
(%i6) printf(true, "|~h|~%", sqrt(2))$
|1.41421356237309504880169|
(%i7) printf(true, "|~28h|~%", sqrt(2))$
|   1.41421356237309504880169|
(%i8) printf(true, "|~28,,,,,'*h|~%", sqrt(2))$
|***1.41421356237309504880169|
(%i9) printf(true, "|~,18h|~%", sqrt(2))$
```

```
|1.414213562373095049|
(%i10) printf(true, "|~,,,-3h|~%", sqrt(2))$
|1414.213562373095048801688724209698078569671875376948073176679738-3|
(%i11) printf(true, "|~,,2,-3h|~%", sqrt(2))$
|1414.213562373095048801688724209698078569671875376948073176679738-03|
(%i12) printf(true, "|~20h|~%", sqrt(2))$
|1.41421356237309504880169|
(%i13) printf(true, "|~20,,,,'+h|~%", sqrt(2))$
|++++++++++++++++++++|
```

For conversion of objects to strings also see `concat`, `sconcat`, `string` and `simplode`.

`readbyte` (*stream*) [Function]

Removes and returns the first byte in *stream* which must be a binary input stream. If the end of file is encountered `readbyte` returns `false`.

Example: Read the first 16 bytes from a file encrypted with AES in OpenSSL.

```
(%i1) ibase: obase: 16.$

(%i2) in: openr_binary("msg.bin");
(%o2)                         #<input stream msg.bin>
(%i3) (L:[],  thru 16. do push(readbyte(in), L),  L:reverse(L));
(%o3) [53, 61, 6C, 74, 65, 64, 5F, 5F, 88, 56, 0DE, 8A, 74, 0FD,
       0AD, 0F0]
(%i4) close(in);
(%o4)                                    true
(%i5) map(ascii, rest(L,-8));
(%o5)                        [S, a, l, t, e, d, _, _]
(%i6) salt: octets_to_number(rest(L,8));
(%o6)                            8856de8a74fdadf0
```

`readchar` (*stream*) [Function]

Removes and returns the first character in *stream*. If the end of file is encountered `readchar` returns `false`.

Example: See [make_string_input_stream], page 1283.

`readline` (*stream*) [Function]

Returns a string containing all characters starting at the current position in *stream* up to the end of the line or `false` if the end of the file is encountered.

`sprint` (*expr_1*, ..., *expr_n*) [Function]

Evaluates and displays its arguments one after the other 'on a line' starting at the leftmost position. The expressions are printed with a space character right next to the number, and it disregards line length. `newline()` might be used for line breaking.

Example: Sequential printing with `sprint`. Creating a new line with `newline()`.

```
(%i1) for n:0 thru 19 do sprint(fib(n))$
0 1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181
(%i2) for n:0 thru 22 do (
        sprint(fib(n)),
```

```
         if mod(n,10) = 9 then newline() )$
0 1 1 2 3 5 8 13 21 34
55 89 144 233 377 610 987 1597 2584 4181
6765 10946 17711
```

`writebyte` (*byte*, *stream*) [Function]

Writes *byte* to *stream* which must be a binary output stream. `writebyte` returns `byte`.

Example: Write some bytes to a binary file output stream. In this example all bytes correspond to printable characters and are printed by `printfile`. The bytes remain in the stream until `flush_output` or `close` have been called.

```
(%i1) ibase: obase: 16.$

(%i2) bytes: map(cint, charlist("GNU/Linux"));
(%o2)                [47, 4E, 55, 2F, 4C, 69, 6E, 75, 78]
(%i3) out: openw_binary("test.bin");
(%o3)                       #<output stream test.bin>
(%i4) for i thru 3 do writebyte(bytes[i], out);
(%o4)                                  done
(%i5) printfile("test.bin")$

(%i6) flength(out);
(%o6)                                    0
(%i7) flush_output(out);
(%o7)                                  true
(%i8) flength(out);
(%o8)                                    3
(%i9) printfile("test.bin")$
GNU
(%i0A) for b in rest(bytes,3) do writebyte(b, out);
(%o0A)                                 done
(%i0B) close(out);
(%o0B)                                 true
(%i0C) printfile("test.bin")$
GNU/Linux
```

96.3 Characters

Characters are strings of length 1.

`adjust_external_format` () [Function]

Prints information about the current external format of the Lisp reader and in case the external format encoding differs from the encoding of the application which runs Maxima `adjust_external_format` tries to adjust the encoding or prints some help or instruction. `adjust_external_format` returns `true` when the external format has been changed and `false` otherwise.

Functions like [cint], page 1290, [unicode], page 1292, [octets_to_string], page 1302, and [string_to_octets], page 1303, need UTF-8 as the external format of the Lisp reader to work properly over the full range of Unicode characters.

Examples (Maxima on Windows, March 2016): Using `adjust_external_format` when the default external format is not equal to the encoding provided by the application.

1. Command line Maxima

In case a terminal session is preferred it is recommended to use Maxima compiled with SBCL. Here Unicode support is provided by default and calls to `adjust_external_format` are unnecessary.

If Maxima is compiled with CLISP or GCL it is recommended to change the terminal encoding from CP850 to CP1252. `adjust_external_format` prints some help.

CCL reads UTF-8 while the terminal input is CP850 by default. CP1252 is not supported by CCL. `adjust_external_format` prints instructions for changing the terminal encoding and external format both to iso-8859-1.

2. wxMaxima

In wxMaxima SBCL reads CP1252 by default but the input from the application is UTF-8 encoded. Adjustment is needed.

Calling `adjust_external_format` and restarting Maxima permanently changes the default external format to UTF-8.

```
(%i1)adjust_external_format();
The line
(setf sb-impl::*default-external-format* :utf-8)
has been appended to the init file
C:/Users/Username/.sbclrc
Please restart Maxima to set the external format to UTF-8.
(%i1) false
```

Restarting Maxima.

```
(%i1) adjust_external_format();
The external format is currently UTF-8
and has not been changed.
(%i1) false
```

`alphacharp` (*char*) [Function]

Returns `true` if *char* is an alphabetic character.

To identify a non-US-ASCII character as an alphabetic character the underlying Lisp must provide full Unicode support. E.g. a German umlaut is detected as an alphabetic character with SBCL in GNU/Linux but not with GCL. (In Windows Maxima, when compiled with SBCL, must be set to UTF-8. See [adjust_external_format], page 1288, for more.)

Example: Examination of non-US-ASCII characters.

The underlying Lisp (SBCL, GNU/Linux) is able to convert the typed character into a Lisp character and to examine.

```
(%i1) alphacharp("ü");
```

```
(%o1)                           true
```

In GCL this is not possible. An error break occurs.

```
(%i1) alphacharp("u");
(%o1)                           true
(%i2) alphacharp("ü");

package stringproc: ü cannot be converted into a Lisp character.
 -- an error.
```

`alphanumericp` (*char*) [Function]

Returns `true` if *char* is an alphabetic character or a digit (only corresponding US-ASCII characters are regarded as digits).

Note: See remarks on [alphacharp], page 1289.

`ascii` (*int*) [Function]

Returns the US-ASCII character corresponding to the integer *int* which has to be less than `128`.

See [unicode], page 1292, for converting code points larger than `127`.

Examples:

```
(%i1) for n from 0 thru 127 do (
        ch: ascii(n),
        if alphacharp(ch) then sprint(ch),
        if n = 96 then newline() )$
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
a b c d e f g h i j k l m n o p q r s t u v w x y z
```

`cequal` (*char_1*, *char_2*) [Function]

Returns `true` if *char_1* and *char_2* are the same character.

`cequalignore` (*char_1*, *char_2*) [Function]

Like `cequal` but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1289.

`cgreaterp` (*char_1*, *char_2*) [Function]

Returns `true` if the code point of *char_1* is greater than the code point of *char_2*.

`cgreaterpignore` (*char_1*, *char_2*) [Function]

Like `cgreaterp` but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1289.

`charp` (*obj*) [Function]

Returns `true` if *obj* is a Maxima-character. See introduction for example.

`cint` (*char*) [Function]

Returns the Unicode code point of *char* which must be a Maxima character, i.e. a string of length `1`.

Examples: The hexadecimal code point of some characters (Maxima with SBCL on GNU/Linux).

```
(%i1) obase: 16.$
(%i2) map(cint, ["$","£","€"]);
(%o2)                          [24, 0A3, 20AC]
```

Warning: It is not possible to enter characters corresponding to code points larger than 16 bit in wxMaxima with SBCL on Windows when the external format has not been set to UTF-8. See [adjust_external_format], page 1288.

CMUCL doesn't process these characters as one character. `cint` then returns `false`. Converting a character to a code point via UTF-8-octets may serve as a workaround:

```
utf8_to_unicode(string_to_octets(character));
```

See [utf8_to_unicode], page 1293, [string_to_octets], page 1303.

`clessp` (*char_1*, *char_2*) [Function]

Returns `true` if the code point of *char_1* is less than the code point of *char_2*.

`clesspignore` (*char_1*, *char_2*) [Function]

Like `clessp` but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1289.

`constituent` (*char*) [Function]

Returns `true` if *char* is a graphic character but not a space character. A graphic character is a character one can see, plus the space character. (`constituent` is defined by Paul Graham. See Paul Graham, ANSI Common Lisp, 1996, page 67.)

```
(%i1) for n from 0 thru 255 do (
tmp: ascii(n), if constituent(tmp) then sprint(tmp) )$
! " # % ' ( ) * + , - . / 0 1 2 3 4 5 6 7 8 9 : ; < = > ? @ A B
C D E F G H I J K L M N O P Q R S T U V W X Y Z [ \ ] ^ _ ‘ a b c
d e f g h i j k l m n o p q r s t u v w x y z { | } ~
```

`digitcharp` (*char*) [Function]

Returns `true` if *char* is a digit where only the corresponding US-ASCII-character is regarded as a digit.

`lowercasep` (*char*) [Function]

Returns `true` if *char* is a lowercase character.

Note: See remarks on [alphacharp], page 1289.

`newline` [Variable]

The newline character (ASCII-character 10).

`space` [Variable]

The space character.

`tab` [Variable]

The tab character.

`unicode` (*arg*) [Function]

Returns the character defined by *arg* which might be a Unicode code point or a name string if the underlying Lisp provides full Unicode support.

Example: Characters defined by hexadecimal code points (Maxima with SBCL on GNU/Linux).

```
(%i1) ibase: 16.$
(%i2) map(unicode, [24, 0A3, 20AC]);
(%o2)                              [$, £, €]
```

Warning: In wxMaxima with SBCL on Windows it is not possible to convert code points larger than 16 bit to characters when the external format has not been set to UTF-8. See [adjust_external_format], page 1288, for more information.

CMUCL doesn't process code points larger than 16 bit. In these cases `unicode` returns `false`. Converting a code point to a character via UTF-8 octets may serve as a workaround:

```
octets_to_string(unicode_to_utf8(code_point));
```

See [octets_to_string], page 1302, [unicode_to_utf8], page 1292.

In case the underlying Lisp provides full Unicode support the character might be specified by its name. The following is possible in ECL, CLISP and SBCL, where in SBCL on Windows the external format has to be set to UTF-8. `unicode(name)` is supported by CMUCL too but again limited to 16 bit characters.

The string argument to `unicode` is basically the same string returned by `printf` using the `"~@c"` specifier. But as shown below the prefix `"#\"` must be omitted. Underlines might be replaced by spaces and uppercase letters by lowercase ones.

Example (continued): Characters defined by names (Maxima with SBCL on GNU/Linux).

```
(%i3) printf(false, "~@c", unicode(0DF));
(%o3)                     #\LATIN_SMALL_LETTER_SHARP_S
(%i4) unicode("LATIN_SMALL_LETTER_SHARP_S");
(%o4)                                    ß
(%i5) unicode("Latin small letter sharp s");
(%o5)                                    ß
```

`unicode_to_utf8` (*code_point*) [Function]

Returns a list containing the UTF-8 code corresponding to the Unicode *code_point*.

Examples: Converting Unicode code points to UTF-8 and vice versa.

```
(%i1) ibase: obase: 16.$
(%i2) map(cint, ["$","£","€"]);
(%o2)                              [24, 0A3, 20AC]
(%i3) map(unicode_to_utf8, %);
(%o3)                  [[24], [0C2, 0A3], [0E2, 82, 0AC]]
(%i4) map(utf8_to_unicode, %);
(%o4)                              [24, 0A3, 20AC]
```

`uppercasep` (*char*) [Function]

Returns `true` if *char* is an uppercase character.

Note: See remarks on [alphacharp], page 1289.

`us_ascii_only` [Variable]

This option variable affects Maxima when the character encoding provided by the application which runs Maxima is UTF-8 but the external format of the Lisp reader is not equal to UTF-8.

On GNU/Linux this is true when Maxima is built with GCL and on Windows in wxMaxima with GCL- and SBCL-builds. With SBCL it is recommended to change the external format to UTF-8. Setting `us_ascii_only` is unnecessary then. See [adjust_external_format], page 1288, for details.

`us_ascii_only` is `false` by default. Maxima itself then (i.e. in the above described situation) parses the UTF-8 encoding.

When `us_ascii_only` is set to `true` it is assumed that all strings used as arguments to string processing functions do not contain Non-US-ASCII characters. Given that promise, Maxima avoids parsing UTF-8 and strings can be processed more efficiently.

`utf8_to_unicode` (*list*) [Function]

Returns a Unicode code point corresponding to the *list* which must contain the UTF-8 encoding of a single character.

Examples: See [unicode_to_utf8], page 1292.

96.4 String Processing

Position indices in strings are 1-indexed like in Maxima lists. See example in [charat], page 1293.

`charat` (*string*, *n*) [Function]

Returns the *n*-th character of *string*. The first character in *string* is returned with *n* = 1.

```
(%i1) charat("Lisp",1);
(%o1)                          L
(%i2) charlist("Lisp")[1];
(%o2)                          L
```

`charlist` (*string*) [Function]

Returns the list of all characters in *string*.

```
(%i1) charlist("Lisp");
(%o1)                    [L, i, s, p]
```

`eval_string` (*str*) [Function]

Parse the string *str* as a Maxima expression and evaluate it. The string *str* may or may not have a terminator (dollar sign `$` or semicolon `;`). Only the first expression is parsed and evaluated, if there is more than one.

Complain if *str* is not a string.

Examples:

```
(%i1) eval_string ("foo: 42; bar: foo^2 + baz");
(%o1)                        42
(%i2) eval_string ("(foo: 42, bar: foo^2 + baz)");
(%o2)                    baz + 1764
```

See also [parse_string], page 1294, and [eval_string_lisp], page 39.

`parse_string` (*str*) [Function]

Parse the string *str* as a Maxima expression (do not evaluate it). The string *str* may or may not have a terminator (dollar sign `$` or semicolon `;`). Only the first expression is parsed, if there is more than one.

Complain if *str* is not a string.

Examples:

```
(%i1) parse_string ("foo: 42; bar: foo^2 + baz");
(%o1)                     foo : 42
(%i2) parse_string ("(foo: 42, bar: foo^2 + baz)");
                                      2
(%o2)          (foo : 42, bar : foo  + baz)
```

See also [eval_string], page 1293.

`scopy` (*string*) [Function]

Returns a copy of *string* as a new string.

`sdowncase` [Function]

`sdowncase` (*string*)
`sdowncase` (*string*, *start*)
`sdowncase` (*string*, *start*, *end*)

Like [supcase], page 1298, but uppercase characters are converted to lowercase.

`sequal` (*string_1*, *string_2*) [Function]

Returns `true` if *string_1* and *string_2* contain the same sequence of characters.

`sequalignore` (*string_1*, *string_2*) [Function]

Like `sequal` but ignores case which is only possible for non-US-ASCII characters when the underlying Lisp is able to recognize a character as an alphabetic character. See remarks on [alphacharp], page 1289.

`sexplode` (*string*) [Function]

`sexplode` is an alias for function `charlist`.

`simplode` [Function]

`simplode` (*list*)
`simplode` (*list*, *delim*)

`simplode` takes a list of expressions and concatenates them into a string. If no delimiter *delim* is specified, `simplode` uses no delimiter. *delim* can be any string.

See also `concat`, `sconcat`, `string` and `printf`.

Examples:

```
(%i1) simplode(["xx[",3,"]:",expand((x+y)^3)]);
(%o1)             xx[3]:y^3+3*x*y^2+3*x^2*y+x^3
(%i2) simplode( sexplode("stars")," * " );
(%o2)                    s * t * a * r * s
(%i3) simplode( ["One","more","coffee."]," " );
(%o3)                    One more coffee.
```

sinsert (*seq*, *string*, *pos*) [Function]

Returns a string that is a concatenation of substring(*string*, 1, *pos*-1), the string *seq* and substring (*string*, *pos*). Note that the first character in *string* is in position 1.

Examples:

```
(%i1) s: "A submarine."$
(%i2) concat( substring(s,1,3),"yellow ",substring(s,3) );
(%o2)                  A yellow submarine.
(%i3) sinsert("hollow ",s,3);
(%o3)                  A hollow submarine.
```

sinvertcase [Function]

sinvertcase (*string*)
sinvertcase (*string*, *start*)
sinvertcase (*string*, *start*, *end*)

Returns *string* except that each character from position *start* to *end* is inverted. If *end* is not given, all characters from *start* to the end of *string* are replaced.

Examples:

```
(%i1) sinvertcase("sInvertCase");
(%o1)                     SiNVERTcASE
```

slength (*string*) [Function]

Returns the number of characters in *string*.

smake (*num*, *char*) [Function]

Returns a new string with a number of *num* characters *char*.

Example:

```
(%i1) smake(3,"w");
(%o1)                          www
```

smismatch [Function]

smismatch (*string_1*, *string_2*)
smismatch (*string_1*, *string_2*, *test*)

Returns the position of the first character of *string_1* at which *string_1* and *string_2* differ or false. Default test function for matching is sequal. If smismatch should ignore case, use sequalignore as test.

Example:

```
(%i1) smismatch("seven","seventh");
(%o1)                           6
```

split [Function]

split (*string*)
split (*string*, *delim*)
split (*string*, *delim*, *multiple*)

Returns the list of all tokens in *string*. Each token is an unparsed string. split uses *delim* as delimiter. If *delim* is not given, the space character is the default delimiter. *multiple* is a boolean variable with true by default. Multiple delimiters are read as

one. This is useful if tabs are saved as multiple space characters. If *multiple* is set to `false`, each delimiter is noted.

Examples:

```
(%i1) split("1.2   2.3   3.4   4.5");
(%o1)                 [1.2, 2.3, 3.4, 4.5]
(%i2) split("first;;third;fourth",";",false);
(%o2)               [first, , third, fourth]
```

`sposition` (*char*, *string*) [Function]

Returns the position of the first character in *string* which matches *char*. The first character in *string* is in position 1. For matching characters ignoring case see [ssearch], page 1296.

`sremove` [Function]

`sremove` (*seq*, *string*)
`sremove` (*seq*, *string*, *test*)
`sremove` (*seq*, *string*, *test*, *start*)
`sremove` (*seq*, *string*, *test*, *start*, *end*)

Returns a string like *string* but without all substrings matching *seq*. Default test function for matching is `sequal`. If `sremove` should ignore case while searching for *seq*, use `sequalignore` as test. Use *start* and *end* to limit searching. Note that the first character in *string* is in position 1.

Examples:

```
(%i1) sremove("n't","I don't like coffee.");
(%o1)                    I do like coffee.
(%i2) sremove ("DO ",%,'sequalignore);
(%o2)                     I like coffee.
```

`sremovefirst` [Function]

`sremovefirst` (*seq*, *string*)
`sremovefirst` (*seq*, *string*, *test*)
`sremovefirst` (*seq*, *string*, *test*, *start*)
`sremovefirst` (*seq*, *string*, *test*, *start*, *end*)

Like `sremove` except that only the first substring that matches *seq* is removed.

`sreverse` (*string*) [Function]

Returns a string with all the characters of *string* in reverse order.

See also `reverse`.

`ssearch` [Function]

`ssearch` (*seq*, *string*)
`ssearch` (*seq*, *string*, *test*)
`ssearch` (*seq*, *string*, *test*, *start*)
`ssearch` (*seq*, *string*, *test*, *start*, *end*)

Returns the position of the first substring of *string* that matches the string *seq*. Default test function for matching is `sequal`. If `ssearch` should ignore case, use `sequalignore` as test. Use *start* and *end* to limit searching. Note that the first character in *string* is in position 1.

Example:

```
(%i1) ssearch("~s","~{~S ~}~%",'sequalignore);
(%o1)                                4
```

ssort [Function]

ssort (*string*)
ssort (*string*, *test*)

Returns a string that contains all characters from *string* in an order such there are no two successive characters *c* and *d* such that `test (c, d)` is `false` and `test (d, c)` is `true`. Default test function for sorting is *clessp*. The set of test functions is `{clessp, clesspignore, cgreaterp, cgreaterpignore, cequal, cequalignore}`.

Examples:

```
(%i1) ssort("I don't like Mondays.");
(%o1)                    '.IMaddeiklnnoosty
(%i2) ssort("I don't like Mondays.",'cgreaterpignore);
(%o2)                 ytsoonnMlkIiedda.'
```

ssubst [Function]

ssubst (*new*, *old*, *string*)
ssubst (*new*, *old*, *string*, *test*)
ssubst (*new*, *old*, *string*, *test*, *start*)
ssubst (*new*, *old*, *string*, *test*, *start*, *end*)

Returns a string like *string* except that all substrings matching *old* are replaced by *new*. *old* and *new* need not to be of the same length. Default test function for matching is `sequal`. If `ssubst` should ignore case while searching for old, use `sequalignore` as test. Use *start* and *end* to limit searching. Note that the first character in *string* is in position 1.

Examples:

```
(%i1) ssubst("like","hate","I hate Thai food. I hate green tea.");
(%o1)          I like Thai food. I like green tea.
(%i2) ssubst("Indian","thai",%,'sequalignore,8,12);
(%o2)         I like Indian food. I like green tea.
```

ssubstfirst [Function]

ssubstfirst (*new*, *old*, *string*)
ssubstfirst (*new*, *old*, *string*, *test*)
ssubstfirst (*new*, *old*, *string*, *test*, *start*)
ssubstfirst (*new*, *old*, *string*, *test*, *start*, *end*)

Like `subst` except that only the first substring that matches *old* is replaced.

strim (*seq*,*string*) [Function]

Returns a string like *string*, but with all characters that appear in *seq* removed from both ends.

Examples:

```
(%i1) "/* comment */"$
(%i2) strim(" /*",%);
(%o2)                          comment
```

```
(%i3) slength(%);
(%o3)                          7
```

striml (*seq*, *string*) [Function]

Like `strim` except that only the left end of *string* is trimmed.

strimr (*seq*, *string*) [Function]

Like `strim` except that only the right end of *string* is trimmed.

stringp (*obj*) [Function]

Returns `true` if *obj* is a string. See introduction for example.

substring [Function]

`substring` (*string*, *start*)
`substring` (*string*, *start*, *end*)

Returns the substring of *string* beginning at position *start* and ending at position *end*. The character at position *end* is not included. If *end* is not given, the substring contains the rest of the string. Note that the first character in *string* is in position 1.

Examples:

```
(%i1) substring("substring",4);
(%o1)                        string
(%i2) substring(%,4,6);
(%o2)                          in
```

supcase [Function]

`supcase` (*string*)
`supcase` (*string*, *start*)
`supcase` (*string*, *start*, *end*)

Returns *string* except that lowercase characters from position *start* to *end* are replaced by the corresponding uppercase ones. If *end* is not given, all lowercase characters from *start* to the end of *string* are replaced.

Example:

```
(%i1) supcase("english",1,2);
(%o1)                        English
```

tokens [Function]

`tokens` (*string*)
`tokens` (*string*, *test*)

Returns a list of tokens, which have been extracted from *string*. The tokens are substrings whose characters satisfy a certain test function. If test is not given, *constituent* is used as the default test. `{constituent, alphacharp, digitcharp, lowercasep, uppercasep, charp, characterp, alphanumericp}` is the set of test functions. (The Lisp-version of `tokens` is written by Paul Graham. ANSI Common Lisp, 1996, page 67.)

Examples:

```
(%i1) tokens("24 October 2005");
(%o1)                  [24, October, 2005]
```

```
(%i2) tokens("05-10-24",'digitcharp);
(%o2)                      [05, 10, 24]
(%i3) map(parse_string,%);
(%o3)                       [5, 10, 24]
```

96.5 Octets and Utilities for Cryptography

base64 (*arg*) [Function]

Returns the base64-representation of *arg* as a string. The argument *arg* may be a string, a non-negative integer or a list of octets.

Examples:

```
(%i1) base64: base64("foo bar baz");
(%o1)                            Zm9vIGJhciBiYXo=
(%i2) string: base64_decode(base64);
(%o2)                               foo bar baz
(%i3) obase: 16.$
(%i4) integer: base64_decode(base64, 'number);
(%o4)                         666f6f206261722062617a
(%i5) octets: base64_decode(base64, 'list);
(%o5)            [66, 6F, 6F, 20, 62, 61, 72, 20, 62, 61, 7A]
(%i6) ibase: 16.$
(%i7) base64(octets);
(%o7)                            Zm9vIGJhciBiYXo=
```

Note that if *arg* contains umlauts (resp. octets larger than 127) the resulting base64-string is platform dependent. However the decoded string will be equal to the original.

base64_decode [Function]

base64_decode (*base64-string*)
base64_decode (*base64-string*, *return-type*)

By default `base64_decode` decodes the *base64-string* back to the original string.

The optional argument *return-type* allows `base64_decode` to alternatively return the corresponding number or list of octets. *return-type* may be `string`, `number` or `list`.

Example: See [base64], page 1299.

crc24sum [Function]

crc24sum (*octets*)
crc24sum (*octets*, *return-type*)

By default `crc24sum` returns the `CRC24` checksum of an octet-list as a string.

The optional argument *return-type* allows `crc24sum` to alternatively return the corresponding number or list of octets. *return-type* may be `string`, `number` or `list`.

Example:

```
-----BEGIN PGP SIGNATURE-----
Version: GnuPG v2.0.22 (GNU/Linux)

iQEcBAEBAgAGBQJVdCTzAAoJEG/1Mgf2DWAqCSYH/AhVFwhu1D89C3/QFcgVvZTM
wnOYzBUURJAL/cT+IngkLEpp3hEbREcugWp+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi
```

```
rBQu+I7+3ySK2hpryQ6Wx5J9uZSa4YmfsNteR8up0zGkaulJeWkS4pjiRM+auWVe
vajlKZCIK52P080DG7Q2dpshh4fgTeNwqCuCiBhQ73t8g1IaLdhDN6EzJVjGIzam
/spqT/sTo6sw8yDOJjvU+Qvn6/mSMjC/YxjhRMaQt9EMrR1AZ4ukBF5uG1S7mXOH
WdiwkSPZ3gnIBhM9SuC076gLWZUNs6NqTeE3UzMjDAFhH3jYk1T7mysCvdtIkms=
=WmeC
-----END PGP SIGNATURE-----
(%i1) ibase : obase : 16.$
(%i2) sig64 : sconcat(
 "iQEcBAEBAgAGBQJVdCTzAAoJEG/1Mgf2DWAqCSYH/AhVFwhu1D89C3/QFcgVvZTM",
 "wnOYzBUURJAL/cT+IngkLEpp3hEbREcugWp+Tm6aw3R4CdJ7G3FLxExBH/5KnDHi",
 "rBQu+I7+3ySK2hpryQ6Wx5J9uZSa4YmfsNteR8up0zGkaulJeWkS4pjiRM+auWVe",
 "vajlKZCIK52P080DG7Q2dpshh4fgTeNwqCuCiBhQ73t8g1IaLdhDN6EzJVjGIzam",
 "/spqT/sTo6sw8yDOJjvU+Qvn6/mSMjC/YxjhRMaQt9EMrR1AZ4ukBF5uG1S7mXOH",
 "WdiwkSPZ3gnIBhM9SuC076gLWZUNs6NqTeE3UzMjDAFhH3jYk1T7mysCvdtIkms=" )$
(%i3) octets: base64_decode(sig64, 'list)$
(%i4) crc24: crc24sum(octets, 'list);
(%o4)                          [5A, 67, 82]
(%i5) base64(crc24);
(%o5)                              WmeC
```

`md5sum` [Function]

`md5sum` (*arg*)
`md5sum` (*arg*, *return-type*)

Returns the `MD5` checksum of a string, non-negative integer, list of octets, or binary (not character) input stream. A file for which an input stream is opened may be an ordinary text file; it is the stream which needs to be binary, not the file itself.

When the argument is an input stream, `md5sum` reads the entire content of the stream, but does not close the stream.

The default return value is a string containing 32 hex characters. The optional argument *return-type* allows `md5sum` to alternatively return the corresponding number or list of octets. *return-type* may be `string`, `number` or `list`.

Note that in case *arg* contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the `MD5` checksum is platform dependent.

Examples:

```
(%i1) ibase: obase: 16.$
(%i2) msg: "foo bar baz"$
(%i3) string: md5sum(msg);
(%o3)                  ab07acbb1e496801937adfa772424bf7
(%i4) integer: md5sum(msg, 'number);
(%o4)                 0ab07acbb1e496801937adfa772424bf7
(%i5) octets: md5sum(msg, 'list);
(%o5)        [0AB,7,0AC,0BB,1E,49,68,1,93,7A,0DF,0A7,72,42,4B,0F7]
(%i6) sdowncase( printf(false, "~{~2,'0x~^:~}", octets) );
(%o6)           ab:07:ac:bb:1e:49:68:01:93:7a:df:a7:72:42:4b:f7
```

The argument may be a binary input stream.

```
(%i1) S: openr_binary (file_search ("md5.lisp"));
```

```
(%o1) #<INPUT BUFFERED FILE-STREAM (UNSIGNED-BYTE 8)
  /home/robert/maxima/maxima-code/share/stringproc/md5.lisp>
(%i2) md5sum (S);
(%o2)           31a512ed53daf5b99495c9d05559355f
(%i3) close (S);
(%o3)                          true
```

mgf1_sha1 [Function]

mgf1_sha1 (*seed*, *len*)
mgf1_sha1 (*seed*, *len*, *return-type*)

Returns a pseudo random number of variable length. By default the returned value is a number with a length of *len* octets.

The optional argument *return-type* allows `mgf1_sha1` to alternatively return the corresponding list of *len* octets. *return-type* may be `number` or `list`.

The computation of the returned value is described in `RFC 3447`, appendix `B.2.1 MGF1`. `SHA1` is used as hash function, i.e. the randomness of the computed number relies on the randomness of `SHA1` hashes.

Example:

```
(%i1) ibase: obase: 16.$
(%i2) number: mgf1_sha1(4711., 8);
(%o2)                         0e0252e5a2a42fea1
(%i3) octets: mgf1_sha1(4711., 8, 'list);
(%o3)                  [0E0,25,2E,5A,2A,42,0FE,0A1]
```

number_to_octets (*number*) [Function]

Returns an octet-representation of *number* as a list of octets. The *number* must be a non-negative integer.

Example:

```
(%i1) ibase : obase : 16.$
(%i2) octets: [0ca,0fe,0ba,0be]$
(%i3) number: octets_to_number(octets);
(%o3)                             0cafebabe
(%i4) number_to_octets(number);
(%o4)                      [0CA, 0FE, 0BA, 0BE]
```

octets_to_number (*octets*) [Function]

Returns a number by concatenating the octets in the list of *octets*.

Example: See [number_to_octets], page 1301.

octets_to_oid (*octets*) [Function]

Computes an object identifier (OID) from the list of *octets*.

Example: RSA encryption OID

```
(%i1) ibase : obase : 16.$
(%i2) oid: octets_to_oid([2A,86,48,86,0F7,0D,1,1,1]);
(%o2)                      1.2.840.113549.1.1.1
(%i3) oid_to_octets(oid);
(%o3)               [2A, 86, 48, 86, 0F7, 0D, 1, 1, 1]
```

octets_to_string [Function]

octets_to_string (*octets*)
octets_to_string (*octets*, *encoding*)

Decodes the list of *octets* into a string according to current system defaults. When decoding octets corresponding to Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

Example: Using system defaults (Maxima compiled with GCL, which uses no format definition and simply passes through the UTF-8-octets encoded by the GNU/Linux terminal).

```
(%i1) octets: string_to_octets("abc");
(%o1)                              [61, 62, 63]
(%i2) octets_to_string(octets);
(%o2)                                    abc
(%i3) ibase: obase: 16.$
(%i4) unicode(20AC);
(%o4)                                     €
(%i5) octets: string_to_octets(%);
(%o5)                             [0E2, 82, 0AC]
(%i6) octets_to_string(octets);
(%o6)                                     €
(%i7) utf8_to_unicode(octets);
(%o7)                                   20AC
```

In case the external format of the Lisp reader is equal to UTF-8 the optional argument *encoding* allows to set the encoding for the octet to string conversion. If necessary see [adjust_external_format], page 1288, for changing the external format.

Some names of supported encodings (see corresponding Lisp manual for more):
CCL, CLISP, SBCL: `utf-8`, `ucs-2be`, `ucs-4be`, `iso-8859-1`, `cp1252`, `cp850`
CMUCL: `utf-8`, `utf-16-be`, `utf-32-be`, `iso8859-1`, `cp1252`
ECL: `utf-8`, `ucs-2be`, `ucs-4be`, `iso-8859-1`, `windows-cp1252`, `dos-cp850`

Example (continued): Using the optional encoding argument (Maxima compiled with SBCL, GNU/Linux terminal).

```
(%i8) string_to_octets("€", "ucs-2be");
(%o8)                                 [20, 0AC]
```

oid_to_octets (*oid-string*) [Function]

Converts an object identifier (OID) to a list of *octets*.

Example: See [octets_to_oid], page 1301.

sha1sum [Function]

sha1sum (*arg*)
sha1sum (*arg*, *return-type*)

Returns the `SHA1` fingerprint of a string, a non-negative integer or a list of octets. The default return value is a string containing 40 hex characters.

The optional argument *return-type* allows `sha1sum` to alternatively return the corresponding number or list of octets. *return-type* may be `string`, `number` or `list`.

Example:

```
(%i1) ibase: obase: 16.$
(%i2) msg: "foo bar baz"$
(%i3) string: sha1sum(msg);
(%o3)              c7567e8b39e2428e38bf9c9226ac68de4c67dc39
(%i4) integer: sha1sum(msg, 'number);
(%o4)             0c7567e8b39e2428e38bf9c9226ac68de4c67dc39
(%i5) octets: sha1sum(msg, 'list);
(%o5)  [0C7,56,7E,8B,39,0E2,42,8E,38,0BF,9C,92,26,0AC,68,0DE,4C,67,0DC,39]
(%i6) sdowncase( printf(false, "~{~2,'0x~^:~}", octets) );
(%o6)     c7:56:7e:8b:39:e2:42:8e:38:bf:9c:92:26:ac:68:de:4c:67:dc:39
```

Note that in case *arg* contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the `SHA1` fingerprint is platform dependent.

`sha256sum` [Function]

`sha256sum` (*arg*)
`sha256sum` (*arg*, *return-type*)

Returns the `SHA256` fingerprint of a string, a non-negative integer or a list of octets. The default return value is a string containing 64 hex characters.

The optional argument *return-type* allows `sha256sum` to alternatively return the corresponding number or list of octets (see [sha1sum], page 1302).

Example:

```
(%i1) string: sha256sum("foo bar baz");
(%o1)  dbd318c1c462aee872f41109a4dfd3048871a03dedd0fe0e757ced57dad6f2d7
```

Note that in case *arg* contains German umlauts or other non-ASCII characters (resp. octets larger than 127) the `SHA256` fingerprint is platform dependent.

`string_to_octets` [Function]

`string_to_octets` (*string*)
`string_to_octets` (*string*, *encoding*)

Encodes a *string* into a list of octets according to current system defaults. When encoding strings containing Non-US-ASCII characters the result depends on the platform, application and underlying Lisp.

In case the external format of the Lisp reader is equal to UTF-8 the optional argument *encoding* allows to set the encoding for the string to octet conversion. If necessary see [adjust_external_format], page 1288, for changing the external format.

See [octets_to_string], page 1302, for examples and some more information.

96.6 Regular Expressions

96.6.1 Introduction to Regular Expressions

`sregex` is an interface to the portable regex engine by Dorai Sitaram. The syntax of the regular expressions is described in detail in the pregexp manual by Dorai Sitaram. See the manual for full details.

While `sregex` supports Unicode, the support for Unicode characters in strings is dependent on the support for Unicode characters in the Lisp used to run Maxima.

96.6.2 Functions and Variables

`regex_compile` (*pattern*) [Function]

Compile regex string in *pattern* to an internal form that is easier for the regex engine to process. This is not required, however. All the regex functions accept this compiled regex or a string. If the pattern is used many times, compiling the pattern will speed up matching.

```
(%i1) regex_compile("c.r");
(%o1)         Structure [COMPILED-REGEX for "c.r"]
```

`regex_match_pos` (*regex*, *str*) [Function]
`regex_match_pos` (*regex*, *str*, *start*) [Function]
`regex_match_pos` (*regex*, *str*, *start*, *end*) [Function]

Return a list consisting of a list of the start and end positions of *str* where the first match of *regex* occurred. If no match is found, returns `false`.

If a third argument, *start*, is supplied, it is the starting index of the text string *str*. The fourth argument, *end*, is the ending index of text string *str*.

```
(%i1) str : "his hay needle stack -- my hay needle stack -- her hay needle stack"
(%i2) regex : regex_compile("ne{2}dle")$
(%i3) regex_match_pos(regex, str);
(%o3)                       [[9, 15]]
(%i4) regex_match_pos("ne{2}dle", str);
(%o4)                       [[9, 15]]
(%i5) regex_match_pos("ne{2}dle", str, 25, 44);
(%o5)                      [[32, 38]]
```

Here is an example where `regex_match_pos` returns a list of more than one element:

```
(%i1) str : "jan 1, 1970";
(%o1)                      jan 1, 1970
(%i2) match: regex_match_pos("([a-z]+) ([0-9]+), ([0-9]+)", "jan 1, 1970");
(%o2)          [[1, 12], [1, 4], [5, 6], [8, 12]]
(%i3) map(lambda([posn], substring(str, posn[1], posn[2])), match);
(%o3)              [jan 1, 1970, jan, 1, 1970]
```

The first element is for the full match. Each subsequent element of the list is the substring that matches the *cluster* enclosed in parenthesis in the given regular expression.

`regex_match` (*regex*, *str*) [Function]
`regex_match` (*regex*, *str*, *start*) [Function]
`regex_match` (*regex*, *str*, *start*, *end*) [Function]

`regex_match` is very similar to `regex_match_pos` except that it returns the matching substrings instead of the indices of the match. If no match is found, returns `false`.

```
(%i1) regex_match("ne{2}dle", "hay needle stack");
(%o1)                       [needle]
(%i2) regex_match("ne{2}dle", "hay needle stack", 10);
(%o2)                         false
```

Here is examples using POSIX character classes. `[:alpha:]` matches any letter. The pattern matches any letter or underscore:

```
(%i1) regex_match("[[:alpha:]_]", "--x--");
(%o1)                          [x]
(%i2) regex_match("[[:alpha:]_]", "--_--");
(%o2)                          [_]
(%i3) regex_match("[[:alpha:]_]", "--:--");
(%o3)                        false
```

`sregex` supports *clusters* (see pregexp clusters) which are subpatterns denoted by being enclosed within parentheses. These cause the matcher to return the submatch along with the overall match.

Here we are looking for any number of letters followed by a space, any number of digits, a comma and space, then any number of digits.

```
(%i1) regex_match("([a-z]+) ([0-9]+), ([0-9]+)", "jan 1, 1970");
(%o1)              [jan 1, 1970, jan, 1, 1970]
```

The result is a list of strings. The first element is the full match. The second matches `"([a-z]+)"`, which is a cluster of any number of letters. Hence, `"jan"` matches this cluster. Likewise for the other clusters.

A more complicated example illustrates how a subpattern fails to match, but the overall pattern matches. In this case, `false` represents to failed match.

The regex pattern matches "month year" or "month day, year". The subpattern matches the day, if present.

```
(%i1) date_re : regex_compile("([a-z]+) +([0-9]+,)? *([0-9]+)");
(%o1)
  Structure [COMPILED-REGEX for "([a-z]+) +([0-9]+,)? *([0-9]+)"]
(%i2) regex_match(date_re, "jan 1, 1970");
(%o2)             [jan 1, 1970, jan, 1,, 1970]
(%i3) regex_match(date_re, "jan 1970");
(%o3)             [jan 1970, jan, false, 1970]
```

You can also do case-insensitive matches by using a *cloister* (see pregexp cloisters) with the `i` modifier:

```
(%i1) regex_match("hearth", "HeartH");
(%o1)                          false
(%i2) regex_match("(?i:hearth)", "HeartH");
(%o2)                       [HeartH]
```

Alternate subpatterns can be separated by `|`.

```
(%i1) regex_match("f(ee|i|o|um)", "a small, final fee");
(%o1)                         [fi, i]
```

The first element is the full match `"fi"`; the second shows that we matched `"i"` for the cluster.

`regex_split` (*regex*, *str*) [Function]

Returns a list of strings where *str* has been split into substrings where the *regex* identifies the delimiters to use for separating the substrings.

```
(%i1) regex_split("[,;]+", "split,pea;;;soup");
(%o1)                  [split, pea, soup]
```

regex_subst_first (*replacement*, *pattern*, *str*) [Function]

Returns a string where the first occurrence of *pattern* in *str* with *replacement*.

```
(%i1) regex_subst_first("ty", "t.", "liberte egalite fraternite");
(%o1)              liberty egalite fraternite
```

This example shows how to use back references. The replacement specifies that the first submatch is used as the replacement text.

```
(%i1) regex_match("_(.+?)_", "the _nina_, the _pinta_, and the _santa maria_");
(%o1)                     [_nina_, nina]
(%i2) regex_subst_first("*\\1*", "_(.+?)_", "the _nina_, the _pinta_, and the _sa
(%o2)    the *nina*, the _pinta_, and the _santa maria_
```

regex_subst (*replacement*, *pattern*, *str*) [Function]

Returns a string where every occurrence of *pattern* has been replaced by *replacement* in the string *str*.

```
(%i1) regex_subst("ty", "t.\\b", "liberte egalite fraternite");
(%o1)              liberty egality fraternity
```

string_to_regex (*str*) [Function]

Returns a regex string where any special reqex characters in *str* are quoted to remove the specialness of the character.

```
(%i1) re : string_to_regex(". :");
(%o1)                          \. :
(%i2) regex_match(re, "z :");
(%o2)                         false
(%i3) regex_match(re, ". :");
(%o3)                          [. :]
(%i4) regex_match(". :", "z :");
(%o4)                          [z :]
```

In this example, the regex will only match a substring consisting of a period, followed by a space and a colon. Without the quoting, the "." would match any single character.

97 Package sym

97.1 Introduction to Symmetries

`sym` is a package for working with symmetric groups of polynomials.

It was written for Macsyma-Symbolics by Annick Valibouze[1]. The algorithms are described in the following papers:

1. Fonctions symétriques et changements de bases[2]. Annick Valibouze. EUROCAL'87 (Leipzig, 1987), 323–332, Lecture Notes in Comput. Sci 378. Springer, Berlin, 1989.
2. Résolvantes et fonctions symétriques[3]. Annick Valibouze. Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation, ISSAC'89 (Portland, Oregon). ACM Press, 390-399, 1989.
3. Symbolic computation with symmetric polynomials, an extension to Macsyma[4]. Annick Valibouze. Computers and Mathematics (MIT, USA, June 13-17, 1989), Springer-Verlag, New York Berlin, 308-320, 1989.
4. Théorie de Galois Constructive. Annick Valibouze. Mémoire d'habilitation à diriger les recherches (HDR), Université P. et M. Curie (Paris VI), 1994.

97.2 Functions and Variables for Symmetries

97.2.1 Changing bases

`comp2pui` (*n*, *L*) [Function]

implements passing from the complete symmetric functions given in the list *L* to the elementary symmetric functions from 0 to *n*. If the list *L* contains fewer than *n+1* elements, it will be completed with formal values of the type *h1*, *h2*, etc. If the first element of the list *L* exists, it specifies the size of the alphabet, otherwise the size is set to *n*.

```
(%i1) comp2pui (3, [4, g]);
                            2                       2
(%o1)     [4, g, 2 h2 - g , 3 h3 - g h2 + g (g  - 2 h2)]
```

`ele2pui` (*m*, *L*) [Function]

goes from the elementary symmetric functions to the complete functions. Similar to `comp2ele` and `comp2pui`.

Other functions for changing bases: `comp2ele`.

`ele2comp` (*m*, *L*) [Function]

Goes from the elementary symmetric functions to the compete functions. Similar to `comp2ele` and `comp2pui`.

Other functions for changing bases: `comp2ele`.

---

[1] https://web.archive.org/web/20061125035035/http://www-calfor.lip6.fr/~avb/

[2] www.stix.polytechnique.fr/publications/1984-1994.html

[3] https://web.archive.org/web/20061125035035/http://www-calfor.lip6.fr/~avb/DonneesTelechargeables/MesArticles/issac89ACMValibouze.pdf

[4] www.stix.polytechnique.fr/publications/1984-1994.html

elem (*ele*, *sym*, *lvar*) [Function]

decomposes the symmetric polynomial *sym*, in the variables contained in the list *lvar*, in terms of the elementary symmetric functions given in the list *ele*. If the first element of *ele* is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial *sym*. If values are missing in the list *ele*, formal values of the type *e1*, *e2*, etc. will be added. The polynomial *sym* may be given in three different forms: contracted (`elem` should then be 1, its default value), partitioned (`elem` should be 3), or extended (i.e. the entire polynomial, and `elem` should then be 2). The function `pui` is used in the same way.

On an alphabet of size 3 with *e1*, the first elementary symmetric function, with value 7, the symmetric polynomial in 3 variables whose contracted form (which here depends on only two of its variables) is *x^4-2*x*y* decomposes as follows in elementary symmetric functions:

```
(%i1) elem ([3, 7], x^4 - 2*x*y, [x, y]);
(%o1) 7 (e3 - 7 e2 + 7 (49 - e2)) + 21 e3

                                              + (- 2 (49 - e2) - 2) e2
(%i2) ratsimp (%);
                                 2
(%o2)              28 e3 + 2 e2  - 198 e2 + 2401
```

Other functions for changing bases: `comp2ele`.

mon2schur (*L*) [Function]

The list L represents the Schur function S_L: we have $L = [i_1, i_2, \ldots, i_q]$, with $i_1 \leq i_2 \leq \ldots \leq i_q$. The Schur function $S_{i_1,i_2,\ldots,i_q}$ is the minor of the infinite matrix h_{i-j}, $i \geq 1, j \geq 1$, consisting of the q first rows and the columns $i_1 + 1, i_2 + 2, \ldots, i_q + q$.

This Schur function can be written in terms of monomials by using `treinat` and `kostka`. The form returned is a symmetric polynomial in a contracted representation in the variables $x_1, x_2, \ldots$

```
(%i1) mon2schur ([1, 1, 1]);
(%o1)                        x1 x2 x3
(%i2) mon2schur ([3]);
                                     2        3
(%o2)                 x1 x2 x3 + x1  x2 + x1
(%i3) mon2schur ([1, 2]);
                                       2
(%o3)                   2 x1 x2 x3 + x1  x2
```

which means that for 3 variables this gives:

```
2 x1 x2 x3 + x1^2 x2 + x2^2 x1 + x1^2 x3 + x3^2 x1
 + x2^2 x3 + x3^2 x2
```

Other functions for changing bases: `comp2ele`.

multi_elem (*l_elem*, *multi_pc*, *l_var*) [Function]

decomposes a multi-symmetric polynomial in the multi-contracted form *multi_pc* in the groups of variables contained in the list of lists *l_var* in terms of the elementary symmetric functions contained in *l_elem*.

```
(%i1) multi_elem ([[2, e1, e2], [2, f1, f2]], a*x + a^2 + x^3,
      [[x, y], [a, b]]);
                                                    3
(%o1)         - 2 f2 + f1 (f1 + e1) - 3 e1 e2 + e1
(%i2) ratsimp (%);
                          2                         3
(%o2)         - 2 f2 + f1  + e1 f1 - 3 e1 e2 + e1
```

Other functions for changing bases: `comp2ele`.

`multi_pui` [Function]

is to the function `pui` what the function `multi_elem` is to the function `elem`.

```
(%i1) multi_pui ([[2, p1, p2], [2, t1, t2]], a*x + a^2 + x^3,
      [[x, y], [a, b]]);
                                          3
                                  3 p1 p2   p1
(%o1)             t2 + p1 t1 + ------- - ---
                                  2       2
```

`pui` (*L*, *sym*, *lvar*) [Function]

decomposes the symmetric polynomial *sym*, in the variables in the list *lvar*, in terms of the power functions in the list *L*. If the first element of *L* is given, it will be the size of the alphabet, otherwise the size will be the degree of the polynomial *sym*. If values are missing in the list *L*, formal values of the type *p1*, *p2* , etc. will be added. The polynomial *sym* may be given in three different forms: contracted (`elem` should then be 1, its default value), partitioned (`elem` should be 3), or extended (i.e. the entire polynomial, and `elem` should then be 2). The function `pui` is used in the same way.

```
(%i1) pui;
(%o1)                          1
(%i2) pui ([3, a, b], u*x*y*z, [x, y, z]);
                        2
                   a (a  - b) u   (a b - p3) u
(%o2)              ------------ - ------------
                        6              3
(%i3) ratsimp (%);
                                       3
                      (2 p3 - 3 a b + a ) u
(%o3)                 ---------------------
                                6
```

Other functions for changing bases: `comp2ele`.

`pui2comp` (*n*, *lpui*) [Function]

renders the list of the first *n* complete functions (with the length first) in terms of the power functions given in the list *lpui*. If the list *lpui* is empty, the cardinal is *n*, otherwise it is its first element (as in `comp2ele` and `comp2pui`).

```
(%i1) pui2comp (2, []);
                                        2
                                  p2 + p1
(%o1)                    [2, p1, --------]
                                     2
(%i2) pui2comp (3, [2, a1]);
                                                2
                                   a1 (p2 + a1 )
                          2  p3 + ------------- + a1 p2
                   p2 + a1                2
(%o2)     [2, a1, --------, --------------------------]
                     2                    3
(%i3) ratsimp (%);
                              2                      3
                       p2 + a1   2 p3 + 3 a1 p2 + a1
(%o3)         [2, a1, --------, --------------------]
                          2                6
```

Other functions for changing bases: `comp2ele`.

`pui2ele` (*n*, *lpui*) [Function]

effects the passage from power functions to the elementary symmetric functions. If the flag `pui2ele` is `girard`, it will return the list of elementary symmetric functions from 1 to *n*, and if the flag is `close`, it will return the *n*-th elementary symmetric function.

Other functions for changing bases: `comp2ele`.

`puireduc` (*n*, *lpui*) [Function]

lpui is a list whose first element is an integer *m*. `puireduc` gives the first *n* power functions in terms of the first *m*.

```
(%i1) puireduc (3, [2]);
                                          2
                                    p1 (p1  - p2)
(%o1)           [2, p1, p2, p1 p2 - -------------]
                                          2
(%i2) ratsimp (%);
                                            3
                                  3 p1 p2 - p1
(%o2)               [2, p1, p2, -------------]
                                       2
```

`schur2comp` (*P*, *l_var*) [Function]

P is a polynomial in the variables of the list *l_var*. Each of these variables represents a complete symmetric function. In *l_var* the *i*-th complete symmetric function is represented by the concatenation of the letter `h` and the integer *i*: `hi`. This function expresses *P* in terms of Schur functions.

```
(%i1) schur2comp (h1*h2 - h3, [h1, h2, h3]);
(%o1)                          s
                                1, 2
```

```
(%i2) schur2comp (a*h3, [h3]);
(%o2)                          s  a
                                3
```

97.2.2 Changing representations

`cont2part` (*pc*, *lvar*) [Function]

returns the partitioned polynomial associated to the contracted form *pc* whose variables are in *lvar*.

```
(%i1) pc: 2*a^3*b*x^4*y + x^5;
                             3    4      5
(%o1)                     2 a  b x  y + x
(%i2) cont2part (pc, [x, y]);
                                      3
(%o2)              [[1, 5, 0], [2 a  b, 4, 1]]
```

`contract` (*psym*, *lvar*) [Function]

returns a contracted form (i.e. a monomial orbit under the action of the symmetric group) of the polynomial *psym* in the variables contained in the list *lvar*. The function `explose` performs the inverse operation. The function `tcontract` tests the symmetry of the polynomial.

```
(%i1) psym: explose (2*a^3*b*x^4*y, [x, y, z]);
        3     4     3     4     3    4       3    4
(%o1) 2 a  b y z  + 2 a  b x z  + 2 a  b y  z + 2 a  b x  z

                                          3     4     3    4
                                     + 2 a  b x y  + 2 a  b x  y
(%i2) contract (psym, [x, y, z]);
                               3    4
(%o2)                     2 a  b x  y
```

`explose` (*pc*, *lvar*) [Function]

returns the symmetric polynomial associated with the contracted form *pc*. The list *lvar* contains the variables.

```
(%i1) explose (a*x + 1, [x, y, z]);
(%o1)                  a z + a y + a x + 1
```

`part2cont` (*ppart*, *lvar*) [Function]

goes from the partitioned form to the contracted form of a symmetric polynomial. The contracted form is rendered with the variables in *lvar*.

```
(%i1) part2cont ([[2*a^3*b, 4, 1]], [x, y]);
                               3    4
(%o1)                     2 a  b x  y
```

`partpol` (*psym*, *lvar*) [Function]

psym is a symmetric polynomial in the variables of the list *lvar*. This function returns its partitioned representation.

```
(%i1) partpol (-a*(x + y) + 3*x*y, [x, y]);
(%o1)               [[3, 1, 1], [- a, 1, 0]]
```

`tcontract` (*pol*, *lvar*) [Function]

tests if the polynomial *pol* is symmetric in the variables of the list *lvar*. If so, it returns a contracted representation like the function `contract`.

`tpartpol` (*pol*, *lvar*) [Function]

tests if the polynomial *pol* is symmetric in the variables of the list *lvar*. If so, it returns its partitioned representation like the function `partpol`.

97.2.3 Groups and orbits

`direct` ([*p_1*, ..., *p_n*], *y*, *f*, [*lvar_1*, ..., *lvar_n*]) [Function]

calculates the direct image (see M. Giusti, D. Lazard et A. Valibouze, ISSAC 1988, Rome) associated to the function *f*, in the lists of variables *lvar_1*, ..., *lvar_n*, and in the polynomials *p_1*, ..., *p_n* in a variable *y*. The arity of the function *f* is important for the calculation. Thus, if the expression for *f* does not depend on some variable, it is useless to include this variable, and not including it will also considerably reduce the amount of computation.

```
(%i1) direct ([z^2  - e1* z + e2, z^2  - f1* z + f2],
              z, b*v + a*u, [[u, v], [a, b]]);
       2
(%o1) y  - e1 f1 y

                                  2            2             2   2
                  - 4 e2 f2 - (e1  - 2 e2) (f1  - 2 f2) + e1  f1
                + ----------------------------------------------
                                          2
(%i2) ratsimp (%);
              2                2                   2
(%o2)        y  - e1 f1 y + (e1  - 4 e2) f2 + e2 f1
```

```
(%i3) ratsimp (direct ([z^3-e1*z^2+e2*z-e3,z^2  - f1* z + f2],
             z, b*v + a*u, [[u, v], [a, b]]));
      6           5         2                        2    2   4
(%o3) y  - 2 e1 f1 y  + ((2 e1  - 6 e2) f2 + (2 e2 + e1 ) f1 ) y

                          3                                  3   3
 + ((9 e3 + 5 e1 e2 - 2 e1 ) f1 f2 + (- 2 e3 - 2 e1 e2) f1 ) y

        2       2        4    2
 + ((9 e2  - 6 e1  e2 + e1 ) f2

                   2       2       2                   2    4
 + (- 9 e1 e3 - 6 e2  + 3 e1  e2) f1  f2 + (2 e1 e3 + e2 ) f1 )

  2          2                       2     3          2
 y  + (((9 e1  - 27 e2) e3 + 3 e1 e2  - e1  e2) f1 f2

                2           2    3               5
 + ((15 e2 - 2 e1 ) e3 - e1 e2 ) f1  f2 - 2 e2 e3 f1 ) y

          2                   3           3     2   2    3
 + (- 27 e3  + (18 e1 e2 - 4 e1 ) e3 - 4 e2  + e1  e2 ) f2

        2      3                   3    2   2
 + (27 e3  + (e1  - 9 e1 e2) e3 + e2 ) f1  f2

                  2    4        2   6
 + (e1 e2 e3 - 9 e3 ) f1  f2 + e3  f1
```

Finding the polynomial whose roots are the sums $a+u$ where a is a root of $z^2-e_1z+e_2$ and u is a root of $z^2 - f_1z + f_2$.

```
(%i1) ratsimp (direct ([z^2 - e1* z + e2, z^2 - f1* z + f2],
                        z, a + u, [[u], [a]]));
      4                   3            2
(%o1) y  + (- 2 f1 - 2 e1) y  + (2 f2 + f1  + 3 e1 f1 + 2 e2

    2   2                              2              2
 + e1 ) y  + ((- 2 f1 - 2 e1) f2 - e1 f1  + (- 2 e2 - e1 ) f1

                 2                     2           2
 - 2 e1 e2) y + f2  + (e1 f1 - 2 e2 + e1 ) f2 + e2 f1  + e1 e2 f1

    2
 + e2
```

`direct` accepts two flags: `elementaires` and `puissances` (default) which allow decomposing the symmetric polynomials appearing in the calculation into elementary symmetric functions, or power functions, respectively.

Functions of `sym` used in this function:

`multi_orbit` (so `orbit`), `pui_direct`, `multi_elem` (so `elem`), `multi_pui` (so `pui`), `pui2ele`, `ele2pui` (if the flag `direct` is in `puissances`).

`multi_orbit` (*P*, [*lvar_1*, *lvar_2*,..., *lvar_p*]) [Function]

P is a polynomial in the set of variables contained in the lists *lvar_1*, *lvar_2*, ..., *lvar_p*. This function returns the orbit of the polynomial *P* under the action of the product of the symmetric groups of the sets of variables represented in these *p* lists.

```
(%i1) multi_orbit (a*x + b*y, [[x, y], [a, b]]);
(%o1)                [b y + a x, a y + b x]
(%i2) multi_orbit (x + y + 2*a, [[x, y], [a, b, c]]);
(%o2)        [y + x + 2 c, y + x + 2 b, y + x + 2 a]
```

Also see: `orbit` for the action of a single symmetric group.

`multsym` (*ppart_1*, *ppart_2*, *n*) [Function]

returns the product of the two symmetric polynomials in *n* variables by working only modulo the action of the symmetric group of order *n*. The polynomials are in their partitioned form.

Given the 2 symmetric polynomials in *x*, *y*: `3*(x + y) + 2*x*y` and `5*(x^2 + y^2)` whose partitioned forms are `[[3, 1], [2, 1, 1]]` and `[[5, 2]]`, their product will be

```
(%i1) multsym ([[3, 1], [2, 1, 1]], [[5, 2]], 2);
(%o1)         [[10, 3, 1], [15, 3, 0], [15, 2, 1]]
```

that is `10*(x^3*y + y^3*x) + 15*(x^2*y + y^2*x) + 15*(x^3 + y^3)`.

Functions for changing the representations of a symmetric polynomial:

`contract`, `cont2part`, `explose`, `part2cont`, `partpol`, `tcontract`, `tpartpol`.

`orbit` (*P*, *lvar*) [Function]

computes the orbit of the polynomial *P* in the variables in the list *lvar* under the action of the symmetric group of the set of variables in the list *lvar*.

```
(%i1) orbit (a*x + b*y, [x, y]);
(%o1)                [a y + b x, b y + a x]
(%i2) orbit (2*x + x^2, [x, y]);
                          2         2
(%o2)                [y  + 2 y, x  + 2 x]
```

See also `multi_orbit` for the action of a product of symmetric groups on a polynomial.

`pui_direct` (*orbite*, [*lvar_1*, ..., *lvar_n*], [*d_1*, *d_2*, ..., *d_n*]) [Function]

Let *f* be a polynomial in *n* blocks of variables *lvar_1*, ..., *lvar_n*. Let *c_i* be the number of variables in *lvar_i*, and *SC* be the product of *n* symmetric groups of degree *c_1*, ..., *c_n*. This group acts naturally on *f*. The list *orbite* is the orbit, denoted *SC(f)*, of the function *f* under the action of *SC*. (This list may be obtained by the function `multi_orbit`.) The *di* are integers s.t. $c_1 \le d_1, c_2 \le d_2, \ldots, c_n \le d_n$.

Let *SD* be the product of the symmetric groups $S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$. The function `pui_direct` returns the first *n* power functions of *SD(f)* deduced from the power functions of *SC(f)*, where *n* is the size of *SD(f)*.

The result is in multi-contracted form w.r.t. *SD*, i.e. only one element is kept per orbit, under the action of *SD*.

```
(%i1) l: [[x, y], [a, b]];
(%o1)                    [[x, y], [a, b]]
(%i2) pui_direct (multi_orbit (a*x + b*y, l), l, [2, 2]);
                                      2  2
(%o2)               [a x, 4 a b x y + a  x ]
(%i3) pui_direct (multi_orbit (a*x + b*y, l), l, [3, 2]);
                             2  2    2   2       3  3
(%o3) [2 a x, 4 a b x y + 2 a  x , 3 a  b x  y + 2 a  x ,

    2  2  2  2     3   3       4  4
12 a  b  x  y  + 4 a  b x  y + 2 a  x ,

    3  2  3  2     4   4       5  5
10 a  b  x  y  + 5 a  b x  y + 2 a  x ,

    3  3  3  3      4  2  4  2     5   5       6  6
40 a  b  x  y  + 15 a  b  x  y  + 6 a  b x  y + 2 a  x ]
(%i4) pui_direct ([y + x + 2*c, y + x + 2*b, y + x + 2*a],
      [[x, y], [a, b, c]], [2, 3]);
                               2            2
(%o4) [3 x + 2 a, 6 x y + 3 x  + 4 a x + 4 a ,

                 2                  3       2      2       3
               9 x  y + 12 a x y + 3 x  + 6 a x  + 12 a  x + 8 a ]
```

97.2.4 Partitions

`kostka` (*part_1*, *part_2*) [Function]

written by P. Esperet, calculates the Kostka number of the partition *part_1* and *part_2*.

```
(%i1) kostka ([3, 3, 3], [2, 2, 2, 1, 1, 1]);
(%o1)                           6
```

`lgtreillis` (*n*, *m*) [Function]

returns the list of partitions of weight *n* and length *m*.

```
(%i1) lgtreillis (4, 2);
(%o1)                   [[3, 1], [2, 2]]
```

Also see: `ltreillis`, `treillis` and `treinat`.

`ltreillis` (*n*, *m*) [Function]

returns the list of partitions of weight *n* and length less than or equal to *m*.

```
(%i1) ltreillis (4, 2);
(%o1)               [[4, 0], [3, 1], [2, 2]]
```

Also see: `lgtreillis`, `treillis` and `treinat`.

`treillis` (*n*) [Function]

returns all partitions of weight *n*.

```
(%i1) treillis (4);
(%o1)    [[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]]
```

See also: `lgtreillis`, `ltreillis` and `treinat`.

`treinat` (*part*) [Function]

returns the list of partitions inferior to the partition *part* w.r.t. the natural order.

```
(%i1) treinat ([5]);
(%o1)                          [[5]]
(%i2) treinat ([1, 1, 1, 1, 1]);
(%o2) [[5], [4, 1], [3, 2], [3, 1, 1], [2, 2, 1], [2, 1, 1, 1],

                                                    [1, 1, 1, 1, 1]]
(%i3) treinat ([3, 2]);
(%o3)                 [[5], [4, 1], [3, 2]]
```

See also: `lgtreillis`, `ltreillis` and `treillis`.

97.2.5 Polynomials and their roots

`ele2polynome` (*L*, *z*) [Function]

returns the polynomial in *z* s.t. the elementary symmetric functions of its roots are in the list `L = [n, e_1, ..., e_n]`, where *n* is the degree of the polynomial and *e_i* the *i*-th elementary symmetric function.

```
(%i1) ele2polynome ([2, e1, e2], z);
                            2
(%o1)                      z  - e1 z + e2
(%i2) polynome2ele (x^7 - 14*x^5 + 56*x^3  - 56*x + 22, x);
(%o2)          [7, 0, - 14, 0, 56, 0, - 56, - 22]
(%i3) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x);
                    7       5       3
(%o3)              x  - 14 x  + 56 x  - 56 x + 22
```

The inverse: `polynome2ele (P, z)`.

Also see: `polynome2ele`, `pui2polynome`.

`polynome2ele` (*P*, *x*) [Function]

gives the list `l = [n, e_1, ..., e_n]` where *n* is the degree of the polynomial *P* in the variable *x* and *e_i* is the *i*-the elementary symmetric function of the roots of *P*.

```
(%i1) polynome2ele (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x);
(%o1)          [7, 0, - 14, 0, 56, 0, - 56, - 22]
(%i2) ele2polynome ([7, 0, -14, 0, 56, 0, -56, -22], x);
                    7       5       3
(%o2)              x  - 14 x  + 56 x  - 56 x + 22
```

The inverse: `ele2polynome (l, x)`

`prodrac` (*L*, *k*) [Function]

L is a list containing the elementary symmetric functions on a set *A*. `prodrac` returns the polynomial whose roots are the *k* by *k* products of the elements of *A*.

Also see `somrac`.

`pui2polynome` (*x*, *lpui*) [Function]

calculates the polynomial in *x* whose power functions of the roots are given in the list *lpui*.

```
(%i1) pui;
(%o1)                           1
(%i2) kill(labels);
(%o0)                         done
(%i1) polynome2ele (x^3 - 4*x^2 + 5*x - 1, x);
(%o1)                      [3, 4, 5, 1]
(%i2) ele2pui (3, %);
(%o2)                      [3, 4, 6, 7]
(%i3) pui2polynome (x, %);
                         3      2
(%o3)                   x  - 4 x  + 5 x - 1
```

See also: `polynome2ele`, `ele2polynome`.

`somrac` (*L*, *k*) [Function]

The list *L* contains elementary symmetric functions of a polynomial *P* . The function computes the polynomial whose roots are the *k* by *k* distinct sums of the roots of *P*.

Also see `prodrac`.

97.2.6 Resolvents

`resolvante` (*P*, *x*, *f*, [*x_1*,..., *x_d*]) [Function]

calculates the resolvent of the polynomial *P* in *x* of degree `n >= d` by the function *f* expressed in the variables *x_1*, ..., *x_d*. For efficiency of computation it is important to not include in the list `[x_1, ..., x_d]` variables which do not appear in the transformation function *f*.

To increase the efficiency of the computation one may set flags in `resolvante` so as to use appropriate algorithms:

If the function *f* is unitary:

- A polynomial in a single variable,
- linear,
- alternating,
- a sum,
- symmetric,
- a product,
- the function of the Cayley resolvent (usable up to degree 5)

```
(x1*x2 + x2*x3 + x3*x4 + x4*x5 + x5*x1 -
     (x1*x3 + x3*x5 + x5*x2 + x2*x4 + x4*x1))^2
```

general,

the flag of `resolvante` may be, respectively:

- unitaire,
- lineaire,
- alternee,
- somme,
- produit,
- cayley,
- generale.

```
(%i1) resolvante: unitaire$
(%i2) resolvante (x^7 - 14*x^5 + 56*x^3 - 56*x + 22, x, x^3 - 1,
      [x]);

" resolvante unitaire " [7, 0, 28, 0, 168, 0, 1120, - 154, 7840,
                         - 2772, 56448, - 33880,

413952, - 352352, 3076668, - 3363360, 23114112, - 30494464,

175230832, - 267412992, 1338886528, - 2292126760]
  3       6      3       9      6      3
[x  - 1, x  - 2 x  + 1, x  - 3 x  + 3 x  - 1,

 12      9      6      3       15      12       9       6      3
x   - 4 x  + 6 x  - 4 x  + 1, x   - 5 x   + 10 x  - 10 x  + 5 x

       18      15       12       9       6      3
 - 1, x   - 6 x   + 15 x   - 20 x  + 15 x  - 6 x  + 1,

 21      18       15       12       9       6      3
x   - 7 x   + 21 x   - 35 x   + 35 x  - 21 x  + 7 x  - 1]
[- 7, 1127, - 6139, 431767, - 5472047, 201692519, - 3603982011]
       7      6        5         4          3           2
(%o2) y  + 7 y  - 539 y  - 1841 y  + 51443 y  + 315133 y

                                                 + 376999 y + 125253
(%i3) resolvante: lineaire$
(%i4) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);

" resolvante lineaire "
       24       20         16             12             8
(%o4) y   + 80 y   + 7520 y   + 1107200 y   + 49475840 y

                                                    4
                                          + 344489984 y  + 655360000
(%i5) resolvante: general$
```

```
(%i6) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3]);

" resolvante generale "
       24       20         16            12             8
(%o6) y   + 80 y   + 7520 y   + 1107200 y   + 49475840 y

                                                    4
                                           + 344489984 y  + 655360000
(%i7) resolvante (x^4 - 1, x, x1 + 2*x2 + 3*x3, [x1, x2, x3, x4]);

" resolvante generale "
       24       20         16            12             8
(%o7) y   + 80 y   + 7520 y   + 1107200 y   + 49475840 y

                                                    4
                                           + 344489984 y  + 655360000
(%i8) direct ([x^4 - 1], x, x1 + 2*x2 + 3*x3, [[x1, x2, x3]]);
       24       20         16            12             8
(%o8) y   + 80 y   + 7520 y   + 1107200 y   + 49475840 y

                                                    4
                                           + 344489984 y  + 655360000
(%i9) resolvante :lineaire$
(%i10) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);

" resolvante lineaire "
                              4
(%o10)                       y  - 1
(%i11) resolvante: symetrique$
(%i12) resolvante (x^4 - 1, x, x1 + x2 + x3, [x1, x2, x3]);

" resolvante symetrique "
                              4
(%o12)                       y  - 1
(%i13) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);

" resolvante symetrique "
                           6      2
(%o13)                    y  - 4 y  - 1
(%i14) resolvante: alternee$
(%i15) resolvante (x^4 + x + 1, x, x1 - x2, [x1, x2]);

" resolvante alternee "
           12      8       6        4        2
(%o15)    y   + 8 y  + 26 y  - 112 y  + 216 y  + 229
(%i16) resolvante: produit$
```

```
(%i17) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante produit "
       35      33         29        28         27        26
(%o17) y   - 7 y   - 1029 y   + 135 y   + 7203 y   - 756 y

        24           23          22            21           20
 + 1323 y   + 352947 y   - 46305 y   - 2463339 y   + 324135 y

         19           18             17              15
 - 30618 y   - 453789 y   - 40246444 y   + 282225202 y

            14              12             11            10
 - 44274492 y   + 155098503 y   + 12252303 y   + 2893401 y

             9            8            7             6
 - 171532242 y  + 6751269 y  + 2657205 y  - 94517766 y

           5             3
 - 3720087 y  + 26040609 y  + 14348907
(%i18) resolvante: symetrique$
(%i19) resolvante (x^7 - 7*x + 3, x, x1*x2*x3, [x1, x2, x3]);

" resolvante symetrique "
       35      33         29        28         27        26
(%o19) y   - 7 y   - 1029 y   + 135 y   + 7203 y   - 756 y

        24           23          22            21           20
 + 1323 y   + 352947 y   - 46305 y   - 2463339 y   + 324135 y

         19           18             17              15
 - 30618 y   - 453789 y   - 40246444 y   + 282225202 y

            14              12             11            10
 - 44274492 y   + 155098503 y   + 12252303 y   + 2893401 y

             9            8            7             6
 - 171532242 y  + 6751269 y  + 2657205 y  - 94517766 y

           5             3
 - 3720087 y  + 26040609 y  + 14348907
(%i20) resolvante: cayley$
```

```
(%i21) resolvante (x^5 - 4*x^2 + x + 1, x, a, []);

" resolvante de Cayley "
       6       5         4          3            2
(%o21) x  - 40 x  + 4080 x  - 92928 x  + 3772160 x  + 37880832 x

                                                          + 93392896
```

For the Cayley resolvent, the 2 last arguments are neutral and the input polynomial must necessarily be of degree 5.

See also:

⟨undefined⟩ resolvante_bipartite, ⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

`resolvante_alternee1` (*P*, *x*) [Function]

calculates the transformation `P(x)` of degree n by the function $\prod_{1\leq i<j\leq n-1}(x_i - x_j)$.

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale, ⟨undefined⟩ resolvante_bipartite.

`resolvante_bipartite` (*P*, *x*) [Function]

calculates the transformation of `P(x)` of even degree n by the function $x_1x_2\cdots x_{n/2} + x_{n/2+1}\cdots x_n$.

```
(%i1) resolvante_bipartite (x^6 + 108, x);
                10        8          6            4
(%o1)          y   - 972 y  + 314928 y  - 34012224 y
```

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale, ⟨undefined⟩ resolvante_alternee1.

`resolvante_diedrale` (*P*, *x*) [Function]

calculates the transformation of `P(x)` by the function *x_1 x_2 + x_3 x_4*.

```
(%i1) resolvante_diedrale (x^5 - 3*x^4 + 1, x);
       15       12       11       10        9         8         7
(%o1) x   - 21 x   - 81 x   - 21 x   + 207 x  + 1134 x  + 2331 x

        6         5          4          3          2
 - 945 x  - 4970 x  - 18333 x  - 29079 x  - 20745 x  - 25326 x

 - 697
```

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante.

`resolvante_klein` (*P*, *x*) [Function]

calculates the transformation of *P*(*x*) by the function *x_1 x_2 x_4* + *x_4*.

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

`resolvante_klein3` (*P*, *x*) [Function]

calculates the transformation of *P*(*x*) by the function *x_1 x_2 x_4* + *x_4*.

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

`resolvante_produit_sym` (*P*, *x*) [Function]

calculates the list of all product resolvents of the polynomial *P*(*x*).

```
(%i1) resolvante_produit_sym (x^5 + 3*x^4 + 2*x - 1, x);
       5     4            10     8      7      6      5
(%o1) [y  + 3 y  + 2 y - 1, y   - 2 y  - 21 y  - 31 y  - 14 y

    4      3     2       10     8      7    6      5      4
 - y  + 14 y  + 3 y  + 1, y   + 3 y  + 14 y  - y  - 14 y  - 31 y

       3      2      5      4
 - 21 y  - 2 y  + 1, y  - 2 y  - 3 y - 1, y - 1]
(%i2) resolvante: produit$
(%i3) resolvante (x^5 + 3*x^4 + 2*x - 1, x, a*b*c, [a, b, c]);

" resolvante produit "
      10     8      7    6       5      4      3     2
(%o3) y   + 3 y  + 14 y  - y  - 14 y  - 31 y  - 21 y  - 2 y  + 1
```

See also:

⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

`resolvante_unitaire` (*P*, *Q*, *x*) [Function]

computes the resolvent of the polynomial `P(x)` by the polynomial `Q(x)`.

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante_vierer, ⟨undefined⟩ resolvante_diedrale.

`resolvante_vierer` (*P*, *x*) [Function]

computes the transformation of `P(x)` by the function `x_1 x_2 - x_3 x_4`.

See also:

⟨undefined⟩ resolvante_produit_sym, ⟨undefined⟩ resolvante_unitaire, ⟨undefined⟩ resolvante_alternee1, ⟨undefined⟩ resolvante_klein, ⟨undefined⟩ resolvante_klein3, ⟨undefined⟩ resolvante, ⟨undefined⟩ resolvante_diedrale.

97.2.7 Miscellaneous

`multinomial` (*r*, *part*) [Function]

where *r* is the weight of the partition *part*. This function returns the associate multinomial coefficient: if the parts of *part* are *i_1*, *i_2*, ..., *i_k*, the result is `r!/(i_1! i_2! ... i_k!)`.

`permut` (*L*) [Function]

returns the list of permutations of the list *L*.

98 Package to_poly_solve

98.1 Functions and Variables for to_poly_solve

The packages `to_poly` and `to_poly_solve` are experimental; the specifications of the functions in these packages might change or the some of the functions in these packages might be merged into other Maxima functions.

Barton Willis (Professor of Mathematics, University of Nebraska at Kearney) wrote the `to_poly` and `to_poly_solve` packages and the English language user documentation for these packages.

`%and` [Operator]

The operator `%and` is a simplifying nonshort-circuited logical conjunction. Maxima simplifies an `%and` expression to either true, false, or a logically equivalent, but simplified, expression. The operator `%and` is associative, commutative, and idempotent. Thus when `%and` returns a noun form, the arguments of `%and` form a non-redundant sorted list; for example

```
(%i1) a %and (a %and b);
(%o1)                          a %and b
```

If one argument to a conjunction is the *explicit* the negation of another argument, `%and` returns false:

```
(%i2) a %and (not a);
(%o2)                           false
```

If any member of the conjunction is false, the conjunction simplifies to false even if other members are manifestly non-boolean; for example

```
(%i3) 42 %and false;
(%o3)                           false
```

Any argument of an `%and` expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination simplifier has a pre-processor that converts some, but not all, nonlinear inequations into linear inequations; for example the Fourier elimination code simplifies `abs(x) + 1 > 0` to true, so

```
(%i4) (x < 1) %and (abs(x) + 1 > 0);
(%o4)                            x < 1
```

Notes

- The option variable `prederror` does *not* alter the simplification `%and` expressions.
- To avoid operator precedence errors, compound expressions involving the operators `%and, %or`, and `not` should be fully parenthesized.
- The Maxima operators `and` and `or` are both short-circuited. Thus `and` isn't associative or commutative.

Limitations The conjunction `%and` simplifies inequations *locally, not globally.* This means that conjunctions such as

```
(%i5) (x < 1) %and (x > 1);
```

```
(%o5)                    (x > 1) %and (x < 1)
```

do *not* simplify to false. Also, the Fourier elimination code *ignores* the fact database;

```
(%i6) assume(x > 5);
(%o6)                            [x > 5]
(%i7) (x > 1) %and (x > 2);
(%o7)                    (x > 1) %and (x > 2)
```

Finally, nonlinear inequations that aren't easily converted into an equivalent linear inequation aren't simplified.

There is no support for distributing `%and` over `%or`; neither is there support for distributing a logical negation over `%and`.

To use `load("to_poly_solve")`

Related functions `%or, %if, and, or, not`

Status The operator `%and` is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

`%if` (*bool*, a, *b*) [Operator]

The operator `%if` is a simplifying conditional. The *conditional bool* should be boolean-valued. When the conditional is true, return the second argument; when the conditional is false, return the third; in all other cases, return a noun form.

Maxima inequations (either an inequality or an equality) are *not* boolean-valued; for example, Maxima does *not* simplify $5 < 6$ to true, and it does not simplify $5 = 6$ to false; however, in the context of a conditional to an `%if` statement, Maxima *automatically* attempts to determine the truth value of an inequation. Examples:

```
(%i1) f : %if(x # 1, 2, 8);
(%o1)                  %if(x - 1 # 0, 2, 8)
(%i2) [subst(x = -1,f), subst(x=1,f)];
(%o2)                            [2, 8]
```

If the conditional involves an inequation, Maxima simplifies it using the Fourier elimination package.

Notes

- If the conditional is manifestly non-boolean, Maxima returns a noun form:

```
(%i3) %if(42,1,2);
(%o3)                        %if(42, 1, 2)
```

- The Maxima operator `if` is nary, the operator `%if` *isn't* nary.

Limitations The Fourier elimination code only simplifies nonlinear inequations that are readily convertible to an equivalent linear inequation.

To use: `load("to_poly_solve")`

Status: The operator `%if` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`%or` [Operator]

The operator `%or` is a simplifying nonshort-circuited logical disjunction. Maxima simplifies an `%or` expression to either true, false, or a logically equivalent, but simplified,

expression. The operator %or is associative, commutative, and idempotent. Thus when %or returns a noun form, the arguments of %or form a non-redundant sorted list; for example

```
(%i1) a %or (a %or b);
(%o1)                          a %or b
```

If one member of the disjunction is the *explicit* the negation of another member, %or returns true:

```
(%i2) a %or (not a);
(%o2)                           true
```

If any member of the disjunction is true, the disjunction simplifies to true even if other members of the disjunction are manifestly non-boolean; for example

```
(%i3) 42 %or true;
(%o3)                           true
```

Any argument of an %or expression that is an inequation (that is, an inequality or equation), is simplified using the Fourier elimination package. The Fourier elimination code simplifies abs(x) + 1 > 0 to true, so we have

```
(%i4) (x < 1) %or (abs(x) + 1 > 0);
(%o4)                           true
```

Notes

- The option variable prederror does *not* alter the simplification of %or expressions.
- You should parenthesize compound expressions involving the operators %and, %or, and not; the binding powers of these operators might not match your expectations.
- The Maxima operators and and or are both short-circuited. Thus or isn't associative or commutative.

Limitations The conjunction %or simplifies inequations *locally, not globally.* This means that conjunctions such as

```
(%i1) (x < 1) %or (x >= 1);
(%o1) (x > 1) %or (x >= 1)
```

do *not* simplify to true. Further, the Fourier elimination code ignores the fact database;

```
(%i2) assume(x > 5);
(%o2)                          [x > 5]
(%i3) (x > 1) %and (x > 2);
(%o3)                  (x > 1) %and (x > 2)
```

Finally, nonlinear inequations that aren't easily converted into an equivalent linear inequation aren't simplified.

The algorithm that looks for terms that cannot both be false is weak; also there is no support for distributing %or over %and; neither is there support for distributing a logical negation over %or.

To use load("to_poly_solve")

Related functions `%or`, `%if`, `and`, `or`, `not`

Status The operator `%or` is experimental; the specifications of this function might change and its functionality might be merged into other Maxima functions.

`complex_number_p` (*x*) [Function]

The predicate `complex_number_p` returns true if its argument is either `a + %i * b`, `a`, `%i b`, or `%i`, where `a` and `b` are either rational or floating point numbers (including big floating point); for all other inputs, `complex_number_p` returns false; for example

```
(%i1) map('complex_number_p,[2/3, 2 + 1.5 * %i, %i]);
(%o1)                  [true, true, true]
(%i2) complex_number_p((2+%i)/(5-%i));
(%o2)                          false
(%i3) complex_number_p(cos(5 - 2 * %i));
(%o3)                          false
```

Related functions `isreal_p`

To use `load("to_poly_solve")`

Status The operator `complex_number_p` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`compose_functions` (*l*) [Function]

The function call `compose_functions(l)` returns a lambda form that is the *composition* of the functions in the list *l*. The functions are applied from *right* to *left*; for example

```
(%i1) compose_functions([cos, exp]);
                                              %g151
(%o1)             lambda([%g151], cos(%e     ))
(%i2) %(x);
                                     x
(%o2)                         cos(%e )
```

When the function list is empty, return the identity function:

```
(%i3) compose_functions([]);
(%o3)                lambda([%g152], %g152)
(%i4)  %(x);
(%o4)                            x
```

Notes

- When Maxima determines that a list member isn't a symbol or a lambda form, `funmake` (*not* `compose_functions`) signals an error:

```
(%i5) compose_functions([a < b]);

funmake: first argument must be a symbol, subscripted symbol,
string, or lambda expression; found: a < b
#0: compose_functions(l=[a < b])(to_poly_solve.mac line 40)
 -- an error. To debug this try: debugmode(true);
```

- To avoid name conflicts, the independent variable is determined by the function `new_variable`.

```
(%i6) compose_functions([%g0]);
```

```
(%o6)              lambda([%g154], %g0(%g154))
(%i7) compose_functions([%g0]);
(%o7)              lambda([%g155], %g0(%g155))
```

Although the independent variables are different, Maxima is able to to deduce that these lambda forms are semantically equal:

```
(%i8) is(equal(%o6,%o7));
(%o8)                          true
```

To use `load("to_poly_solve")`

Status The function `compose_functions` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`dfloat` (*x*) [Function]

The function `dfloat` is a similar to `float`, but the function `dfloat` applies `rectform` when `float` fails to evaluate to an IEEE double floating point number; thus

```
(%i1) float(4.5^(1 + %i));
                                  %i + 1
(%o1)                        4.5
(%i2) dfloat(4.5^(1 + %i));
(%o2)        4.48998802962884 %i + .3000124893895671
```

Notes

- The rectangular form of an expression might be poorly suited for numerical evaluation–for example, the rectangular form might needlessly involve the difference of floating point numbers (subtractive cancellation).
- The identifier `float` is both an option variable (default value false) and a function name.

Related functions `float, bfloat`

To use `load("to_poly_solve")`

Status The function `dfloat` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`elim` (*l*, *x*) [Function]

The function `elim` eliminates the variables in the set or list `x` from the equations in the set or list `l`. Each member of `x` must be a symbol; the members of `l` can either be equations, or expressions that are assumed to equal zero.

The function `elim` returns a list of two lists; the first is the list of expressions with the variables eliminated; the second is the list of pivots; thus, the second list is a list of expressions that `elim` used to eliminate the variables.

Here is an example of eliminating between linear equations:

```
(%i1) elim(set(x + y + z = 1, x - y  - z = 8, x - z = 1),
           set(x,y));
(%o1)             [[2 z - 7], [y + 7, z - x + 1]]
```

Eliminating `x` and `y` yields the single equation `2 z - 7 = 0`; the equations `y + 7 = 0` and `z - z + 1 = 1` were used as pivots. Eliminating all three variables from these equations, triangularizes the linear system:

```
(%i2) elim(set(x + y + z = 1, x - y  - z = 8, x - z = 1),
```

```
                set(x,y,z));
(%o2)                [[], [2 z - 7, y + 7, z - x + 1]]
```

Of course, the equations needn't be linear:

```
(%i3) elim(set(x^2 - 2 * y^3 = 1,  x - y = 5), [x,y]);
                          3    2
(%o3)         [[], [2 y  - y  - 10 y - 24, y - x + 5]]
```

The user doesn't control the order the variables are eliminated. Instead, the algorithm uses a heuristic to *attempt* to choose the best pivot and the best elimination order.

Notes

- Unlike the related function `eliminate`, the function `elim` does *not* invoke `solve` when the number of equations equals the number of variables.
- The function `elim` works by applying resultants; the option variable `resultant` determines which algorithm Maxima uses. Using `sqfr`, Maxima factors each resultant and suppresses multiple zeros.
- The `elim` will triangularize a nonlinear set of polynomial equations; the solution set of the triangularized set *can* be larger than that solution set of the untriangularized set. Thus, the triangularized equations can have *spurious* solutions.

Related functions *elim_allbut, eliminate_using, eliminate*

Option variables *resultant*

To use `load("to_poly")`

Status The function `elim` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`elim_allbut` (*l*, *x*) [Function]

This function is similar to `elim`, except that it eliminates all the variables in the list of equations `l` *except* for those variables that in in the list `x`

```
(%i1) elim_allbut([x+y = 1, x - 5*y = 1],[]);
(%o1)                  [[], [y, y + x - 1]]
(%i2) elim_allbut([x+y = 1, x - 5*y = 1],[x]);
(%o2)                [[x - 1], [y + x - 1]]
```

To use `load("to_poly")`

Option variables *resultant*

Related functions *elim, eliminate_using, eliminate*

Status The function `elim_allbut` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`eliminate_using` (*l*, *e*, *x*) [Function]

Using `e` as the pivot, eliminate the symbol `x` from the list or set of equations in `l`. The function `eliminate_using` returns a set.

```
(%i1) eq : [x^2 - y^2 - z^3 , x*y - z^2 - 5, x - y + z];
                3    2    2     2
(%o1)      [- z  - y  + x , - z  + x y - 5, z - y + x]
(%i2) eliminate_using(eq,first(eq),z);
```

```
        3              2      2      3    2
(%o2) {y  + (1 - 3 x) y  + 3 x  y - x  - x ,
                       4    3  3       2  2             4
                      y  - x  y  + 13 x  y  - 75 x y + x  + 125}
(%i3) eliminate_using(eq,second(eq),z);
       2            2       4    3  3       2  2            4
(%o3) {y  - 3 x y + x  + 5, y  - x  y  + 13 x  y  - 75 x y + x
                                                          + 125}
(%i4) eliminate_using(eq, third(eq),z);
       2            2       3             2      2      3    2
(%o4) {y  - 3 x y + x  + 5, y  + (1 - 3 x) y  + 3 x  y - x  - x }
```

Option variables *resultant*

Related functions *elim, eliminate, elim_allbut*

To use `load("to_poly")`

Status The function `eliminate_using` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`fourier_elim` ([*eq1*, *eq2*, ...], [*var1*, *var*, ...]) [Function]

Fourier elimination is the analog of Gauss elimination for linear inequations (equations or inequalities). The function call `fourier_elim([eq1, eq2, ...], [var1, var2, ...])` does Fourier elimination on a list of linear inequations `[eq1, eq2, ...]` with respect to the variables `[var1, var2, ...]`; for example

```
(%i1) fourier_elim([y-x < 5, x - y < 7, 10 < y],[x,y]);
(%o1)            [y - 5 < x, x < y + 7, 10 < y]
(%i2) fourier_elim([y-x < 5, x - y < 7, 10 < y],[y,x]);
(%o2)        [max(10, x - 7) < y, y < x + 5, 5 < x]
```

Eliminating first with respect to x and second with respect to y yields lower and upper bounds for x that depend on y, and lower and upper bounds for y that are numbers. Eliminating in the other order gives x dependent lower and upper bounds for y, and numerical lower and upper bounds for x.

When necessary, `fourier_elim` returns a *disjunction* of lists of inequations:

```
(%i3) fourier_elim([x # 6],[x]);
(%o3)                  [x < 6] or [6 < x]
```

When the solution set is empty, `fourier_elim` returns `emptyset`, and when the solution set is all reals, `fourier_elim` returns `universalset`; for example

```
(%i4) fourier_elim([x < 1, x > 1],[x]);
(%o4)                       emptyset
(%i5) fourier_elim([minf < x, x < inf],[x]);
(%o5)                     universalset
```

For nonlinear inequations, `fourier_elim` returns a (somewhat) simplified list of inequations:

```
(%i6) fourier_elim([x^3 - 1 > 0],[x]);
              2                                2
(%o6) [1 < x, x  + x + 1 > 0] or [x < 1, - (x  + x + 1) > 0]
(%i7) fourier_elim([cos(x) < 1/2],[x]);
```

```
(%o7)                    [1 - 2 cos(x) > 0]
```

Instead of a list of inequations, the first argument to `fourier_elim` may be a logical disjunction or conjunction:

```
(%i8) fourier_elim((x + y < 5) and (x - y >8),[x,y]);
                                                    3
(%o8)             [y + 8 < x, x < 5 - y, y < - -]
                                                    2
(%i9) fourier_elim(((x + y < 5) and x < 1) or  (x - y >8),[x,y]);
(%o9)           [y + 8 < x] or [x < min(1, 5 - y)]
```

The function `fourier_elim` supports the inequation operators `<`, `<=`, `>`, `>=`, `#`, and `=`.

The Fourier elimination code has a preprocessor that converts some nonlinear inequations that involve the absolute value, minimum, and maximum functions into linear in equations. Additionally, the preprocessor handles some expressions that are the product or quotient of linear terms:

```
(%i10) fourier_elim([max(x,y) > 6, x # 8, abs(y-1) > 12],[x,y]);
(%o10) [6 < x, x < 8, y < - 11] or [8 < x, y < - 11]
 or [x < 8, 13 < y] or [x = y, 13 < y] or [8 < x, x < y, 13 < y]
 or [y < x, 13 < y]
(%i11) fourier_elim([(x+6)/(x-9) <= 6],[x]);
(%o11)            [x = 12] or [12 < x] or [x < 9]
(%i12) fourier_elim([x^2 - 1 # 0],[x]);
(%o12)      [- 1 < x, x < 1] or [1 < x] or [x < - 1]
```

To use `load("fourier_elim")`

`isreal_p` (*e*) [Function]

The predicate `isreal_p` returns true when Maxima is able to determine that `e` is real-valued on the *entire* real line; it returns false when Maxima is able to determine that `e` *isn't* real-valued on some nonempty subset of the real line; and it returns a noun form for all other cases.

```
(%i1) map('isreal_p, [-1, 0, %i, %pi]);
(%o1)               [true, true, false, true]
```

Maxima variables are assumed to be real; thus

```
(%i2) isreal_p(x);
(%o2)                          true
```

The function `isreal_p` examines the fact database:

```
(%i3) declare(z,complex)$

(%i4) isreal_p(z);
(%o4)                       isreal_p(z)
```

Limitations Too often, `isreal_p` returns a noun form when it should be able to return false; a simple example: the logarithm function isn't real-valued on the entire real line, so `isreal_p(log(x))` should return false; however

```
(%i5) isreal_p(log(x));
```

```
(%o5)                    isreal_p(log(x))
```

To use `load("to_poly_solve")`

Related functions *complex_number_p*

Status The function `isreal_p` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`new_variable` (*type*) [Function]

Return a unique symbol of the form `%[z,n,r,c,g]k`, where `k` is an integer. The allowed values for *type* are *integer, natural_number, real, complex,* and *general*. (By natural number, we mean the *nonnegative integers*; thus zero is a natural number. Some, but not all, definitions of natural number *exclude* zero.)

When *type* isn't one of the allowed values, *type* defaults to *general*. For integers, natural numbers, and complex numbers, Maxima automatically appends this information to the fact database.

```
(%i1) map('new_variable,
          ['integer, 'natural_number, 'real, 'complex, 'general]);
(%o1)          [%z144, %n145, %r146, %c147, %g148]
(%i2) nicedummies(%);
(%o2)               [%z0, %n0, %r0, %c0, %g0]
(%i3) featurep(%z0, 'integer);
(%o3)                          true
(%i4) featurep(%n0, 'integer);
(%o4)                          true
(%i5) is(%n0 >= 0);
(%o5)                          true
(%i6) featurep(%c0, 'complex);
(%o6)                          true
```

Note Generally, the argument to `new_variable` should be quoted. The quote will protect against errors similar to

```
(%i7) integer : 12$

(%i8) new_variable(integer);
(%o8)                          %g149
(%i9) new_variable('integer);
(%o9)                          %z150
```

Related functions *nicedummies*

To use `load("to_poly_solve")`

Status The function `new_variable` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`nicedummies` [Function]

Starting with zero, the function `nicedummies` re-indexes the variables in an expression that were introduced by `new_variable`;

```
(%i1) new_variable('integer) + 52 * new_variable('integer);
(%o1)                   52 %z136 + %z135
```

```
(%i2) new_variable('integer) - new_variable('integer);
(%o2)                      %z137 - %z138
(%i3) nicedummies(%);
(%o3)                        %z0 - %z1
```

Related functions *new_variable*

To use `load("to_poly_solve")`

Status The function `nicedummies` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`parg` (*x*) [Function]

The function `parg` is a simplifying version of the complex argument function `carg`; thus

```
(%i1) map('parg,[1,1+%i,%i, -1 + %i, -1]);
                           %pi  %pi  3 %pi
(%o1)                [0, ---, ---, -----, %pi]
                          4    2     4
```

Generally, for a non-constant input, `parg` returns a noun form; thus

```
(%i2) parg(x + %i * sqrt(x));
(%o2)                   parg(x + %i sqrt(x))
```

When `sign` can determine that the input is a positive or negative real number, `parg` will return a non-noun form for a non-constant input. Here are two examples:

```
(%i3) parg(abs(x));
(%o3) 0
(%i4) parg(-x^2-1);
(%o4)                             %pi
```

Note The `sign` function mostly ignores the variables that are declared to be complex (`declare(x,complex)`); for variables that are declared to be complex, the `parg` can return incorrect values; for example

```
(%i1) declare(x,complex)$

(%i2) parg(x^2 + 1);
(%o2) 0
```

Related function *carg, isreal_p*

To use `load("to_poly_solve")`

Status The function `parg` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`real_imagpart_to_conjugate` (*e*) [Function]

The function `real_imagpart_to_conjugate` replaces all occurrences of `realpart` and `imagpart` to algebraically equivalent expressions involving the `conjugate`.

```
(%i1) declare(x, complex)$

(%i2) real_imagpart_to_conjugate(realpart(x) +  imagpart(x) = 3);
          conjugate(x) + x   %i (x - conjugate(x))
```

```
(%o2)     ---------------- - --------------------- = 3
                   2                     2
```

To use `load("to_poly_solve")`

Status The function `real_imagpart_to_conjugate` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`rectform_log_if_constant (e)` [Function]

The function `rectform_log_if_constant` converts all terms of the form `log(c)` to `rectform(log(c))`, where `c` is either a declared constant expression or explicitly declared constant

```
(%i1) rectform_log_if_constant(log(1-%i) - log(x - %i));
                                    log(2)   %i %pi
(%o1)            - log(x - %i) + ------ - ------
                                      2        4
(%i2) declare(a,constant, b,constant)$

(%i3) rectform_log_if_constant(log(a + %i*b));
                         2    2
                    log(b  + a )
(%o3)               ------------ + %i atan2(b, a)
                         2
```

To use `load("to_poly_solve")`

Status The function `rectform_log_if_constant` is experimental; the specifications of this function might change might change and its functionality might be merged into other Maxima functions.

`simp_inequality (e)` [Function]

The function `simp_inequality` applies basic simplifications to inequations, returning either a boolean value (true or false) or the original inequation.

The simplification rules used by `simp_inequality` include some facts about the ranges of the absolute value, power, and exponential functions along with some elementary algebra facts.

For conjunctions or disjunctions of inequations, `simp_inequality` is applied to each individual inequation, but no effort is made to simplify the entire logical expression.

Effectively, simp_inequality creates a new empty context, so database facts are not used to simplify inequations.

`load("to_poly_solve")` loads this function.

Examples:

```
(%i2) simp_inequality(1 # 0);
(%o2) true
(%i3) simp_inequality(1 < 0);
(%o3) false
(%i4) simp_inequality(a=a);
(%o4) true
(%i5) simp_inequality(a # a);
```

```
(%o5) false
(%i6) simp_inequality(a + 1 # a);
(%o6) true
(%i7) simp_inequality(a < a+1);
(%o7) true
(%i8) simp_inequality(abs(x) >= 0);
(%o8) true
(%i9) simp_inequality(exp(x)  > 0);
(%o9) true
(%i10) simp_inequality(x^2 >= 0);
(%o10) true
(%i11) simp_inequality(2^x  # 0);
(%o11) true
(%i12) simp_inequality(2^(x+1) > 2^x);
(%o12) true
```

The fact database is not consulted. For example:

```
(%i13) assume(xx > 0)$
(%i14) simp_inequality(xx > 0);
(%o14) xx>0
```

And finally, for conjunctions or disjunctions of inequations, each inequation is simplified, but no effort is made to simplify the entire logical expression; for example:

```
(%i15) simp_inequality((1 > 0) and (x < 0) and (x > 0));
(%o15) x<0 and x>0
```

`standardize_inverse_trig` (*e*) [Function]

This function applies the identities `cot(x) = atan(1/x)`, `acsc(x) = asin(1/x)`, and similarly for `asec`, `acoth`, `acsch` and `asech` to an expression. See Abramowitz and Stegun, Eqs. 4.4.6 through 4.4.8 and 4.6.4 through 4.6.6.

To use `load("to_poly_solve")`

Status The function `standardize_inverse_trig` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`subst_parallel` (*l*, *e*) [Function]

When `l` is a single equation or a list of equations, substitute the right hand side of each equation for the left hand side. The substitutions are made in parallel; for example

```
(%i1) load("to_poly_solve")$

(%i2) subst_parallel([x=y,y=x], [x,y]);
(%o2)                          [y, x]
```

Compare this to substitutions made serially:

```
(%i3) subst([x=y,y=x],[x,y]);
(%o3)                          [x, x]
```

The function `subst_parallel` is similar to `sublis` except that `subst_parallel` allows for substitution of nonatoms; for example

```
(%i4) subst_parallel([x^2 = a, y = b], x^2 * y);
(%o4)                          a b
(%i5) sublis([x^2 = a, y = b], x^2 * y);

                                                                  2
sublis: left-hand side of equation must be a symbol; found: x
 -- an error. To debug this try: debugmode(true);
```

The substitutions made by `subst_parallel` are literal, not semantic; thus `subst_parallel` *does not* recognize that $x * y$ is a subexpression of $x^2 * y$

```
(%i6) subst_parallel([x * y = a], x^2 * y);
                                2
(%o6)                          x  y
```

The function `subst_parallel` completes all substitutions *before* simplifications. This allows for substitutions into conditional expressions where errors might occur if the simplifications were made earlier:

```
(%i7) subst_parallel([x = 0], %if(x < 1, 5, log(x)));
(%o7)                            5
(%i8) subst([x = 0], %if(x < 1, 5, log(x)));

log: encountered log(0).
 -- an error. To debug this try: debugmode(true);
```

Related functions *subst, sublis, ratsubst*

To use `load("to_poly_solve_extra.lisp")`

Status The function `subst_parallel` is experimental; the specifications of this function might change might change and its functionality might be merged into other Maxima functions.

`to_poly` (*e*, *l*) [Function]

The function `to_poly` attempts to convert the equation `e` into a polynomial system along with inequality constraints; the solutions to the polynomial system that satisfy the constraints are solutions to the equation `e`. Informally, `to_poly` attempts to polynomialize the equation *e*; an example might clarify:

```
(%i1) load("to_poly_solve")$

(%i2) to_poly(sqrt(x) = 3, [x]);
                                2
(%o2) [[%g130 - 3, x = %g130 ],
                       %pi                                  %pi
                  [- --- < parg(%g130), parg(%g130) <= ---], []]
                       2                                    2
```

The conditions `-%pi/2<parg(%g130),parg(%g130)<=%pi/2` tell us that `%g130` is in the range of the square root function. When this is true, the solution set to `sqrt(x) = 3` is the same as the solution set to `%g130-3,x=%g130^2`.

To polynomialize trigonometric expressions, it is necessary to introduce a non algebraic substitution; these non algebraic substitutions are returned in the third list returned by `to_poly`; for example

```
(%i3) to_poly(cos(x),[x]);
                  2                                    %i x
(%o3)    [[%g131  + 1], [2 %g131 # 0], [%g131 = %e    ]]
```

Constant terms aren't polynomializied unless the number one is a member of the variable list; for example

```
(%i4) to_poly(x = sqrt(5),[x]);
(%o4)                [[x - sqrt(5)], [], []]
(%i5) to_poly(x = sqrt(5),[1,x]);
                              2
(%o5) [[x - %g132, 5 = %g132 ],
                       %pi                                  %pi
                   [- --- < parg(%g132), parg(%g132) <= ---], []]
                       2                                    2
```

To generate a polynomial with $sqrt(5)+sqrt(7)$ as one of its roots, use the commands

```
(%i6) first(elim_allbut(first(to_poly(x = sqrt(5) + sqrt(7),
                                      [1,x])), [x]));
                            4       2
(%o6)                      [x  - 24 x  + 4]
```

Related functions *to_poly_solve*

To use `load("to_poly")`

Status: The function `to_poly` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`to_poly_solve` (*e*, *l*, [*options*]) [Function]

The function `to_poly_solve` tries to solve the equations e for the variables l. The equation(s) e can either be a single expression or a set or list of expressions; similarly, l can either be a single symbol or a list of set of symbols. When a member of e isn't explicitly an equation, for example $x^2 - 1$, the solver assumes that the expression vanishes.

The basic strategy of `to_poly_solve` is to convert the input into a polynomial form and to call `algsys` on the polynomial system. Internally `to_poly_solve` defaults `algexact` to true. To change the default for `algexact`, append 'algexact=false to the `to_poly_solve` argument list.

When `to_poly_solve` is able to determine the solution set, each member of the solution set is a list in a `%union` object:

```
(%i1) load("to_poly_solve")$

(%i2) to_poly_solve(x*(x-1) = 0, x);
(%o2)                %union([x = 0], [x = 1])
```

When `to_poly_solve` is *unable* to determine the solution set, a `%solve` nounform is returned (in this case, a warning is printed)

```
(%i3) to_poly_solve(x^k + 2* x + 1 = 0, x);
```

```
Nonalgebraic argument given to 'to_poly'
unable to solve
                              k
(%o3)              %solve([x  + 2 x + 1 = 0], [x])
```

Substitution into a `%solve` nounform can sometimes result in the solution

```
(%i4) subst(k = 2, %);
(%o4)                     %union([x = - 1])
```

Especially for trigonometric equations, the solver sometimes needs to introduce an arbitrary integer. These arbitrary integers have the form `%zXXX`, where `XXX` is an integer; for example

```
(%i5) to_poly_solve(sin(x) = 0, x);
(%o5)   %union([x = 2 %pi %z33 + %pi], [x = 2 %pi %z35])
```

To re-index these variables to zero, use `nicedummies`:

```
(%i6) nicedummies(%);
(%o6)    %union([x = 2 %pi %z0 + %pi], [x = 2 %pi %z1])
```

Occasionally, the solver introduces an arbitrary complex number of the form `%cXXX` or an arbitrary real number of the form `%rXXX`. The function `nicedummies` will re-index these identifiers to zero.

The solution set sometimes involves simplifying versions of various of logical operators including `%and`, `%or`, or `%if` for conjunction, disjunction, and implication, respectively; for example

```
(%i7) sol : to_poly_solve(abs(x) = a, x);
(%o7) %union(%if(isnonnegative_p(a), [x = - a], %union()),
                       %if(isnonnegative_p(a), [x = a], %union()))
(%i8) subst(a = 42, sol);
(%o8)              %union([x = - 42], [x = 42])
(%i9) subst(a = -42, sol);
(%o9)                          %union()
```

The empty set is represented by `%union()`.

The function `to_poly_solve` is able to solve some, but not all, equations involving rational powers, some nonrational powers, absolute values, trigonometric functions, and minimum and maximum. Also, some it can solve some equations that are solvable in in terms of the Lambert W function; some examples:

```
(%i1) load("to_poly_solve")$

(%i2) to_poly_solve(set(max(x,y) = 5, x+y = 2), set(x,y));
(%o2)      %union([x = - 3, y = 5], [x = 5, y = - 3])
(%i3) to_poly_solve(abs(1-abs(1-x)) = 10,x);
(%o3)              %union([x = - 10], [x = 12])
(%i4) to_poly_solve(set(sqrt(x) + sqrt(y) = 5, x + y = 10),
                    set(x,y));
                     3/2               3/2
                    5    %i - 10      5    %i + 10
```

```
(%o4) %union([x = - ------------, y = ------------],
                         2                 2
                                3/2                 3/2
                               5    %i + 10        5    %i - 10
                          [x = ------------, y = - ------------])
                                    2                   2
(%i5) to_poly_solve(cos(x) * sin(x) = 1/2,x,
                    'simpfuncs = ['expand, 'nicedummies]);
                                          %pi
(%o5)              %union([x = %pi %z0 + ---])
                                           4
(%i6) to_poly_solve(x^(2*a) + x^a + 1,x);
                                         2 %i %pi %z81
                                         -------------
                                  1/a         a
                  (sqrt(3) %i - 1)    %e
(%o6) %union([x = -----------------------------------],
                                  1/a
                                 2
                                                   2 %i %pi %z83
                                                   -------------
                                             1/a         a
                          (- sqrt(3) %i - 1)    %e
                     [x = -------------------------------------])
                                             1/a
                                            2
(%i7) to_poly_solve(x * exp(x) = a, x);
(%o7)              %union([x = lambert_w(a)])
```

For *linear* inequalities, `to_poly_solve` automatically does Fourier elimination:

```
(%i8) to_poly_solve([x + y < 1, x - y >= 8], [x,y]);
                                7
(%o8) %union([x = y + 8, y < - -],
                                2
                                                                7
                                  [y + 8 < x, x < 1 - y, y < - -])
                                                                2
```

Each optional argument to `to_poly_solve` must be an equation; generally, the order of these options does not matter.

- `simpfuncs = l`, where `l` is a list of functions. Apply the composition of the members of l to each solution.

```
(%i1) to_poly_solve(x^2=%i,x);
                                  1/4              1/4
(%o1)       %union([x = - (- 1)   ], [x = (- 1)   ])
(%i2) to_poly_solve(x^2= %i,x, 'simpfuncs = ['rectform]);
                        %i         1              %i         1
(%o2) %union([x = - ------- - -------], [x = ------- + -------])
```

```
                          sqrt(2)   sqrt(2)        sqrt(2)   sqrt(2)
```

Sometimes additional simplification can revert a simplification; for example

```
(%i3) to_poly_solve(x^2=1,x);
(%o3)              %union([x = - 1], [x = 1])
(%i4) to_poly_solve(x^2= 1,x, 'simpfuncs = [polarform]);
                                              %i %pi
(%o4)            %union([x = 1], [x = %e      ])
```

Maxima doesn't try to check that each member of the function list `l` is purely a simplification; thus

```
(%i5) to_poly_solve(x^2 = %i,x, 'simpfuncs = [lambda([s],s^2)]);
(%o5)                    %union([x = %i])
```

To convert each solution to a double float, use `simpfunc = ['dfloat]`:

```
(%i6) to_poly_solve(x^3 +x + 1 = 0,x,
                    'simpfuncs = ['dfloat]), algexact : true;
(%o6) %union([x = - .6823278038280178],
[x = .3411639019140089 - 1.161541399997251 %i],
[x = 1.161541399997251 %i + .3411639019140089])
```

- `use_grobner = true` With this option, the function `poly_reduced_grobner` is applied to the equations before attempting their solution. Primarily, this option provides a workaround for weakness in the function `algsys`. Here is an example of such a workaround:

```
(%i7) to_poly_solve([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y],
                    'use_grobner = true);
                    sqrt(7) - 1      sqrt(7) + 1
(%o7) %union([x = - -----------, y = -----------],
                         2                2
                                  sqrt(7) + 1        sqrt(7) - 1
                              [x = -----------, y = - -----------])
                                       2                  2
(%i8) to_poly_solve([x^2+y^2=2^2,(x-1)^2+(y-1)^2=2^2],[x,y]);
(%o8)                       %union()
```

- `maxdepth = k`, where `k` is a positive integer. This function controls the maximum recursion depth for the solver. The default value for `maxdepth` is five. When the recursions depth is exceeded, the solver signals an error:

```
(%i9) to_poly_solve(cos(x) = x,x, 'maxdepth = 2);

Unable to solve
Unable to solve
(%o9)        %solve([cos(x) = x], [x], maxdepth = 2)
```

- `parameters = l`, where `l` is a list of symbols. The solver attempts to return a solution that is valid for all members of the list `l`; for example:

```
(%i10) to_poly_solve(a * x = x, x);
(%o10)                   %union([x = 0])
(%i11) to_poly_solve(a * x = x, x, 'parameters = [a]);
```

```
(%o11) %union(%if(a - 1 = 0, [x = %c111], %union()),
                                  %if(a - 1 # 0, [x = 0], %union()))
```

In (%o2), the solver introduced a dummy variable; to re-index the these dummy variables, use the function `nicedummies`:

```
(%i12) nicedummies(%);
(%o12) %union(%if(a - 1 = 0, [x = %c0], %union()),
                                  %if(a - 1 # 0, [x = 0], %union()))
```

The `to_poly_solve` uses data stored in the hashed array `one_to_one_reduce` to solve equations of the form $f(a) = f(b)$. The assignment `one_to_one_reduce['f,'f] : lambda([a,b], a=b)` tells `to_poly_solve` that the solution set of $f(a) = f(b)$ equals the solution set of $a = b$; for example

```
(%i13) one_to_one_reduce['f,'f] : lambda([a,b], a=b)$

(%i14) to_poly_solve(f(x^2-1) = f(0),x);
(%o14)             %union([x = - 1], [x = 1])
```

More generally, the assignment `one_to_one_reduce['f,'g] : lambda([a,b], w(a, b) = 0` tells `to_poly_solve` that the solution set of $f(a) = f(b)$ equals the solution set of $w(a, b) = 0$; for example

```
(%i15) one_to_one_reduce['f,'g] : lambda([a,b], a = 1 + b/2)$

(%i16) to_poly_solve(f(x) - g(x),x);
(%o16)                   %union([x = 2])
```

Additionally, the function `to_poly_solve` uses data stored in the hashed array `function_inverse` to solve equations of the form $f(a) = b$. The assignment `function_inverse['f] : lambda([s], g(s))` informs `to_poly_solve` that the solution set to `f(x) = b` equals the solution set to `x = g(b)`; two examples:

```
(%i17) function_inverse['Q] : lambda([s], P(s))$

(%i18) to_poly_solve(Q(x-1) = 2009,x);
(%o18)              %union([x = P(2009) + 1])
(%i19) function_inverse['G] : lambda([s], s+new_variable(integer));
(%o19)       lambda([s], s + new_variable(integer))
(%i20) to_poly_solve(G(x - a) = b,x);
(%o20)             %union([x = b + a + %z125])
```

Notes

- The solve variables needn't be symbols; when `fullratsubst` is able to appropriately make substitutions, the solve variables can be nonsymbols:

```
(%i1) to_poly_solve([x^2 + y^2 + x * y = 5, x * y = 8],
                    [x^2 + y^2, x * y]);
                                    2    2
(%o1)           %union([x y = 8, y  + x  = - 3])
```

- For equations that involve complex conjugates, the solver automatically appends the conjugate equations; for example

```
(%i1) declare(x,complex)$

(%i2) to_poly_solve(x + (5 + %i) * conjugate(x) = 1, x);
                                      %i + 21
(%o2)              %union([x = - -----------])
                                    25 %i - 125
(%i3) declare(y,complex)$

(%i4) to_poly_solve(set(conjugate(x) - y = 42 + %i,
                        x + conjugate(y) = 0), set(x,y));
                           %i - 42        %i + 42
(%o4)        %union([x = - -------, y = - -------])
                              2              2
```

- For an equation that involves the absolute value function, the `to_poly_solve` consults the fact database to decide if the argument to the absolute value is complex valued. When

```
(%i1) to_poly_solve(abs(x) = 6, x);
(%o1)              %union([x = - 6], [x = 6])
(%i2) declare(z,complex)$

(%i3) to_poly_solve(abs(z) = 6, z);
(%o3) %union(%if((%c11 # 0) %and (%c11 conjugate(%c11) - 36 =
                                                  0), [z = %c11], %union()))
```

  This is the only situation that the solver consults the fact database. If a solve variable is declared to be an integer, for example, `to_poly_solve` *ignores this declaration.*

Relevant option variables *algexact, resultant, algebraic*

Related functions *to_poly*

To use `load("to_poly_solve")`

Status: The function `to_poly_solve` is experimental; its specifications might change and its functionality might be merged into other Maxima functions.

`%union` (*soln_1*, *soln_2*, *soln_3*, ...) [Operator]
`%union` () [Operator]

`%union(`*soln_1*`, `*soln_2*`, `*soln_3*`, ...)` represents the union of its arguments, each of which represents a solution set, as determined by `to_poly_solve`. `%union()` represents the empty set.

In many cases, a solution is a list of equations `[`*x*` = ..., `*y*` = ..., `*z*` = ...]` where *x*, *y*, and *z* are one or more unknowns. In such cases, `to_poly_solve` returns a `%union` expression containing one or more such lists.

The solution set sometimes involves simplifying versions of various of logical operators including `%and`, `%or`, or `%if` for conjunction, disjunction, and implication, respectively.

Examples:

`%union(...)` represents the union of its arguments, each of which represents a solution set, as determined by `to_poly_solve`. In many cases, a solution is a list of equations.

```
(%i1) load ("to_poly_solve") $
(%i2) to_poly_solve ([sqrt(x^2 - y^2), x + y], [x, y]);
(%o2)    %union([x = 0, y = 0], [x = %c13, y = - %c13])
```

`%union()` represents the empty set.

```
(%i1) load ("to_poly_solve") $
(%i2) to_poly_solve (abs(x) = -1, x);
(%o2)                        %union()
```

The solution set sometimes involves simplifying versions of various of logical operators.

```
(%i1) load ("to_poly_solve") $
(%i2) sol : to_poly_solve (abs(x) = a, x);
(%o2) %union(%if(isnonnegative_p(a), [x = - a], %union()),
                       %if(isnonnegative_p(a), [x = a], %union()))
(%i3) subst (a = 42, sol);
(%o3)             %union([x = - 42], [x = 42])
(%i4) subst (a = -42, sol);
(%o4)                        %union()
```

99 Package trigtools

Trigtools Package

Aleksas Dormarkas
aleksasd873@gmail.com
aleksas.domarkas@mif.vu.lt
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99.1 Introduction to trigtools

We use open-source computer algebra system(CAS) maxima 5.31.2. The trigtools package[1] contains commands that help you work with trigonometric expressions. List of functions in trigtools package:

- c2sin
- c2cos
- c2trig
- c2hyp
- trigfactor
- trigsolve
- trigvalue
- trigeval
- atan_contract

99.2 Functions and Variables for trigtools

99.2.1 Convert to sin and cos

`c2sin (x)` [Function]
`c2cos (x)` [Function]

The function c2sin converts the expression $a\cos x + b\sin x$ to $r\sin(x+\phi)$.

The function c2cos converts the expression $a\cos x + b\sin x$ to $r\cos(x-\phi)$.

`load("trigtools")` loads these functions.

Examples:

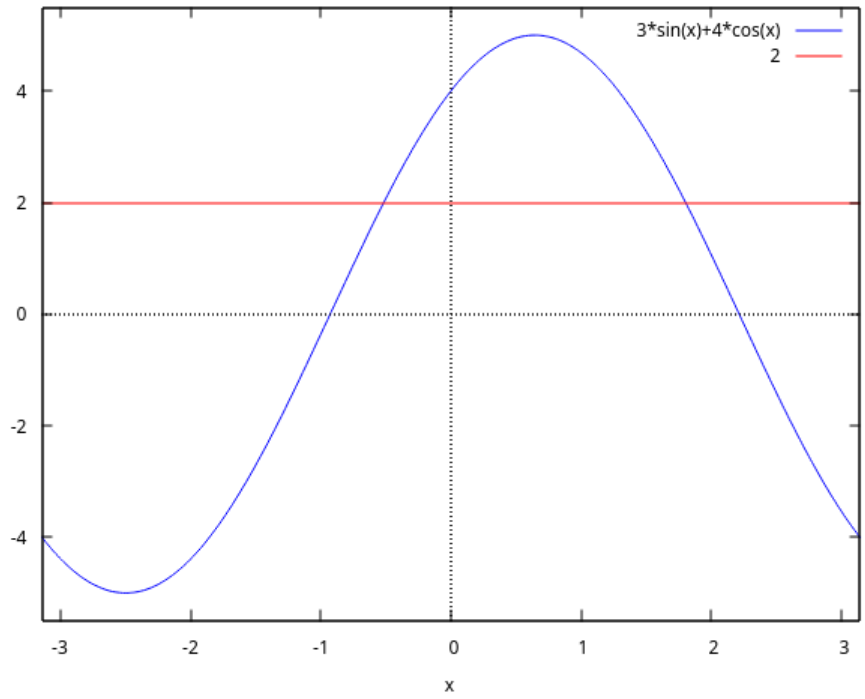
```
(%i1) load("trigtools")$
(%i2) c2sin(3*sin(x)+4*cos(x));
                                        4
(%o2)                  5 sin(x + atan(-))
                                        3
(%i3) trigexpand(%),expand;
(%o3)                  3 sin(x) + 4 cos(x)
```

[1] This is a conversion by hand of the original "trigtools-doc.pdf" file in "share/contrib/trigtools", by Raymond Toy. See the pdf for the definitive version.

```
(%i4) c2cos(3*sin(x)-4*cos(x));
                                        3
(%o4)                 - 5 cos(x + atan(-))
                                        4
(%i5) trigexpand(%),expand;
(%o5)                  3 sin(x) - 4 cos(x)
(%i6) c2sin(sin(x)+cos(x));
                                       %pi
(%o6)                 sqrt(2) sin(x + ---)
                                        4
(%i7) trigexpand(%),expand;
(%o7)                    sin(x) + cos(x)
(%i8) c2cos(sin(x)+cos(x));
                                       %pi
(%o8)                 sqrt(2) cos(x - ---)
                                        4
(%i9) trigexpand(%),expand;
(%o9)                    sin(x) + cos(x)
```

Example. Solve trigonometric equation

```
(%i1) eq:3*sin(x)+4*cos(x)=2;
(%o1)                3 sin(x) + 4 cos(x) = 2
(%i2) plot2d([3*sin(x)+4*cos(x),2],[x,-%pi,%pi]);
(%o2)                          false
```




```
(%i1) load("trigtools")$
```

```
(%i2) eq:3*sin(x)+4*cos(x)=2$
(%i3) eq1:c2sin(lhs(eq))=2;
                                      4
(%o3)                5 sin(x + atan(-)) = 2
                                      3
(%i4) solvetrigwarn:false$
(%i5) solve(eq1)[1]$ x1:rhs(%);
                               2         4
(%o6)                     asin(-) - atan(-)
                               5         3
(%i7) float(%), numer;
(%o7)                   - 0.5157783719341241
(%i8) eq2:c2cos(lhs(eq))=2;
                                      3
(%o8)                5 cos(x - atan(-)) = 2
                                      4
(%i9) solve(eq2,x)[1]$ x2:rhs(%);
                               3         2
(%o10)                    atan(-) + acos(-)
                               4         5
(%i11) float(%), numer;
(%o11)                    1.802780589520693
(%i12) sol:[x1,x2];
                    2         4        3         2
(%o12)       [asin(-) - atan(-), atan(-) + acos(-)]
                    5         3        4         5
```

Answ.: $x = x_1 + 2\pi k$, $x_1 = \sin^{-1}\frac{2}{5} - \tan^{-1}\frac{4}{3}$, or $x_1 = \tan^{-1}\frac{3}{4} + \cos^{-1}\frac{2}{5}$, for k any integer.

99.2.2 Convert to Trigonometric Functions

`c2trig (x)` [Function]

The function c2trig (convert to trigonometric) reduce expression with hyperbolic functions sinh, cosh, tanh, coth to trigonometric expression with sin, cos, tan, cot.

`load("trigtools")` loads these functions.

Examples:

1.

```
(%i1) load(trigtools)$
(%i2) sinh(x)=c2trig(sinh(x));
(%o2)                sinh(x) = - %i sin(%i x)
(%i3) cosh(x)=c2trig(cosh(x));
(%o3)                   cosh(x) = cos(%i x)
(%i4) tanh(x)=c2trig(tanh(x));
(%o4)                tanh(x) = - %i tan(%i x)
(%i5) coth(x)=c2trig(coth(x));
(%o5)                 coth(x) = %i cot(%i x)
```

2. see https://maxima.sourceforge.io/ext/list_archives/2013/msg03230.html

```
(%i1) load("trigtools")$
(%i2) cos(p+q*%i);
(%o2)                      cos(%i q + p)
(%i3) trigexpand(%);
(%o3)          cos(p) cosh(q) - %i sin(p) sinh(q)
(%i4) c2trig(%);
(%o4)                      cos(%i q + p)
```

3.

```
(%i1) load("trigtools")$
(%i2) sin(a+b*%i);
(%o2)                      sin(%i b + a)
(%i3) trigexpand(%);
(%o3)          %i cos(a) sinh(b) + sin(a) cosh(b)
(%i4) c2trig(%);
(%o4)                      sin(%i b + a)
```

4.

```
(%i1) load("trigtools")$
(%i2) cos(a*%i+b*%i);
(%o2)                    cos(%i b + %i a)
(%i3) trigexpand(%);
(%o3)           sinh(a) sinh(b) + cosh(a) cosh(b)
(%i4) c2trig(%);
(%o4)                    cos(%i b + %i a)
```

5.

```
(%i1) load("trigtools")$
(%i2) tan(a+%i*b);
(%o2)                      tan(%i b + a)
(%i3) trigexpand(%);
                        %i tanh(b) + tan(a)
(%o3)                  ---------------------
                       1 - %i tan(a) tanh(b)
(%i4) c2trig(%);
(%o4)                      tan(%i b + a)
```

6.

```
(%i1) load("trigtools")$
(%i2) cot(x+%i*y);
(%o2)                      cot(%i y + x)
(%i3) trigexpand(%);
                      - %i cot(x) coth(y) - 1
(%o3)                 -----------------------
                        cot(x) - %i coth(y)
(%i4) c2trig(%);
(%o4)                      cot(%i y + x)
```

99.2.3 Convert to Hyperbolic Functions

`c2hyp` (*x*) [Function]

The function c2hyp (convert to hyperbolic) convert expression with exp function to expression with hyperbolic functions sinh, cosh.

`load("trigtools")` loads this function.

Examples:

```
(%i1) load("trigtools")$
(%i2) c2hyp(exp(x));
(%o2)                    sinh(x) + cosh(x)
(%i3) c2hyp(exp(x)+exp(x^2)+1);
                  2          2
(%o3)      sinh(x ) + cosh(x ) + sinh(x) + cosh(x) + 1
(%i4) c2hyp(exp(x)/(2*exp(y)-3*exp(z)));
                         sinh(x) + cosh(x)
(%o4)     ------------------------------------------
          2 (sinh(y) + cosh(y)) - 3 (sinh(z) + cosh(z))
```

99.2.4 Factor Sums of sin and cos Functions

`trigfactor` (*x*) [Function]

The function trigfactor factors expressions of form $\pm \sin x \pm \cos y$.

`load("trigtools")` loads this function.

Examples:

1.

```
(%i1) load("trigtools")$
(%i2) trigfactor(sin(x)+cos(x));
                                        %pi
(%o2)                 sqrt(2) cos(x - ---)
                                         4
(%i3) trigrat(%);
(%o3)                    sin(x) + cos(x)
```

2.

```
(%i1) load("trigtools")$
(%i2) trigfactor(sin(x)+cos(y));
                       y   x   %pi      y   x   %pi
(%o2)          2 cos(- - - + ---) cos(- + - - ---)
                       2   2    4       2   2    4
(%i3) trigrat(%);
(%o3)                    cos(y) + sin(x)
```

3.

```
(%i1) load("trigtools")$
(%i2) trigfactor(sin(x)-cos(3*y));
                     3 y   x   %pi      3 y   x   %pi
(%o2)        2 sin(--- - - + ---) sin(--- + - - ---)
                      2    2    4        2    2    4
```

```
(%i3) trigrat(%);
(%o3)                    sin(x) - cos(3 y)
```

4.

```
(%i1) load("trigtools")$
(%i2) trigfactor(-sin(5*x)-cos(3*y));
                  3 y   5 x   %pi      3 y   5 x   %pi
(%o2)     - 2 cos(--- - --- + ---) cos(--- + --- - ---)
                   2     2     4        2     2     4
(%i3) trigrat(%);
(%o3)                 - cos(3 y) - sin(5 x)
```

5.

```
(%i1) load("trigtools")$
(%i2) sin(alpha)+sin(beta)=trigfactor(sin(alpha)+sin(beta));
                                      beta   alpha
(%o2) sin(beta) + sin(alpha) = 2 cos(---- - -----)
                                       2       2
                                                    beta   alpha
                                              sin(---- + -----)
                                                    2       2
(%i3) trigrat(%);
(%o3)    sin(beta) + sin(alpha) = sin(beta) + sin(alpha)
```

6.

```
(%i1) load("trigtools")$
(%i2) sin(alpha)-sin(beta)=trigfactor(sin(alpha)-sin(beta));
                                       beta   alpha
(%o2) sin(alpha) - sin(beta) = - 2 sin(---- - -----)
                                        2       2
                                                    beta   alpha
                                              cos(---- + -----)
                                                    2       2
```

7.

```
(%i1) load("trigtools")$
(%i2) cos(alpha)+cos(beta)=trigfactor(cos(alpha)+cos(beta));
                                      beta   alpha
(%o2) cos(beta) + cos(alpha) = 2 cos(---- - -----)
                                       2       2
                                                    beta   alpha
                                              cos(---- + -----)
                                                    2       2
```

8.

```
(%i1) load("trigtools")$
```

```
(%i2) cos(alpha)-cos(beta)=trigfactor(cos(alpha)-cos(beta));
                                        beta   alpha
(%o2) cos(alpha) - cos(beta) = 2 sin(---- - -----)
                                         2      2
                                                    beta   alpha
                                               sin(---- + -----)
                                                    2      2
```

9.

```
(%i1) load("trigtools")$
(%i2) trigfactor(3*sin(x)+7*cos(x));
(%o2)                  3 sin(x) + 7 cos(x)
(%i3) c2sin(%);
                                           7
(%o3)              sqrt(58) sin(x + atan(-))
                                           3
(%i4) trigexpand(%),expand;
(%o4)                  3 sin(x) + 7 cos(x)
```

10.

```
(%i1) load("trigtools")$
(%i2) trigfactor(sin(2*x));
(%o2)                       sin(2 x)
(%i3) trigexpand(%);
(%o3)                    2 cos(x) sin(x)
```

99.2.5 Solve Trigonometric Equations

`trigsolve` (*x*) [Function]

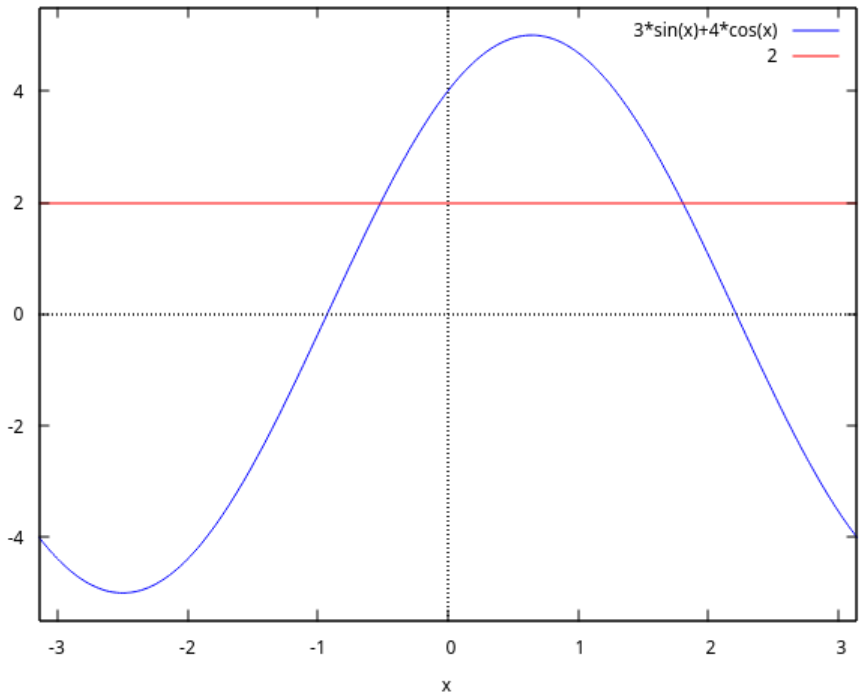
The function trigsolve find solutions of trigonometric equation from interval $[a, b)$.

`load("trigtools")` loads this function.

Examples:

1.

```
(%i1) eq:eq:3*sin(x)+4*cos(x)=2;
(%o1)                3 sin(x) + 4 cos(x) = 2
(%i2) plot2d([3*sin(x)+4*cos(x),2],[x,-%pi,%pi]);
(%o2)                         false
```

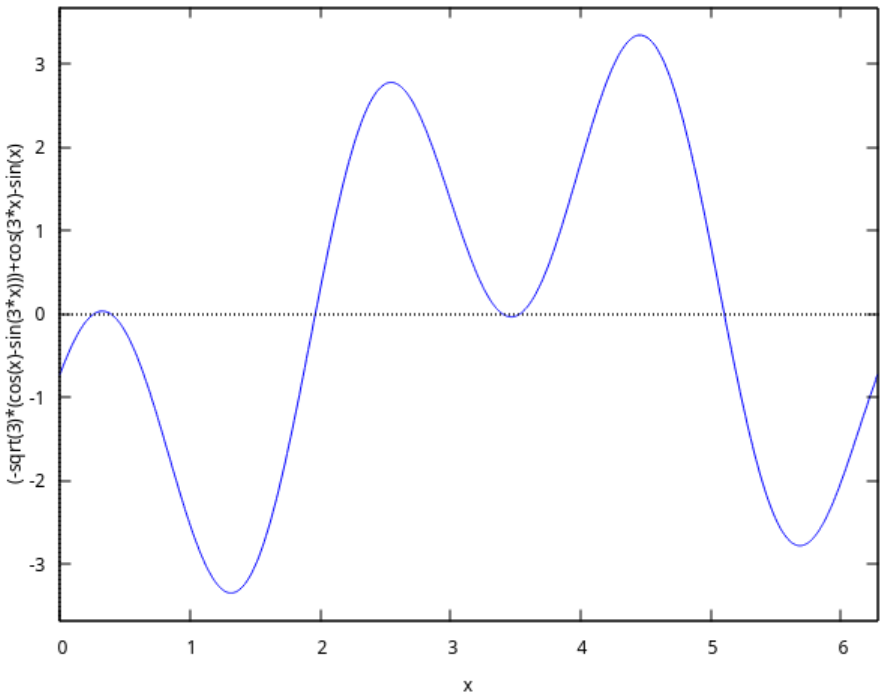



```
(%i1) load("trigtools")$
(%i2) eq:eq:3*sin(x)+4*cos(x)=2$
(%i3) sol:trigsolve(eq,-%pi,%pi);
            2 sqrt(21)   12              2 sqrt(21)   12
(%o3) {atan(---------- - --), %pi - atan(---------- + --)}
                5        5                   5        5
(%i4) float(%), numer;
(%o4)      {- 0.5157783719341241, 1.8027805895206928}
```

Answ. : $x = \tan^{-1}\left(\frac{2\sqrt{21}}{5} - \frac{12}{5}\right) + 2\pi k$; $x = \pi - \tan^{-1}\left(\frac{2\sqrt{21}}{5} + \frac{12}{5}\right) + 2\pi k$, k – any integer.

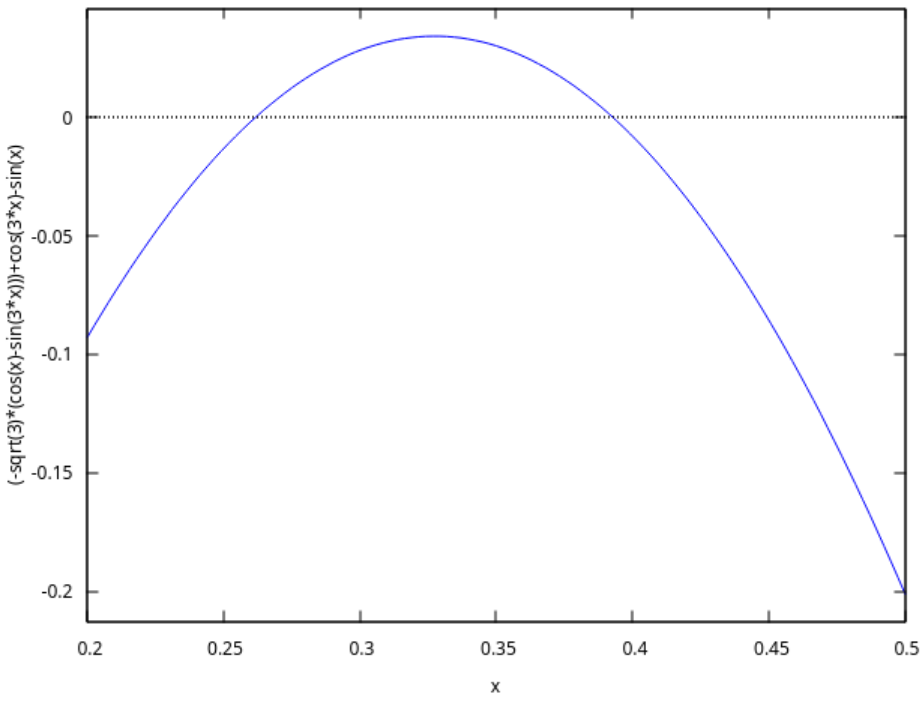
2.

```
(%i1) load("trigtools")$
(%i2) eq:cos(3*x)-sin(x)=sqrt(3)*(cos(x)-sin(3*x));
(%o2)    cos(3 x) - sin(x) = sqrt(3) (cos(x) - sin(3 x))
(%i3) plot2d([lhs(eq)-rhs(eq)], [x,0,2*%pi])$
```

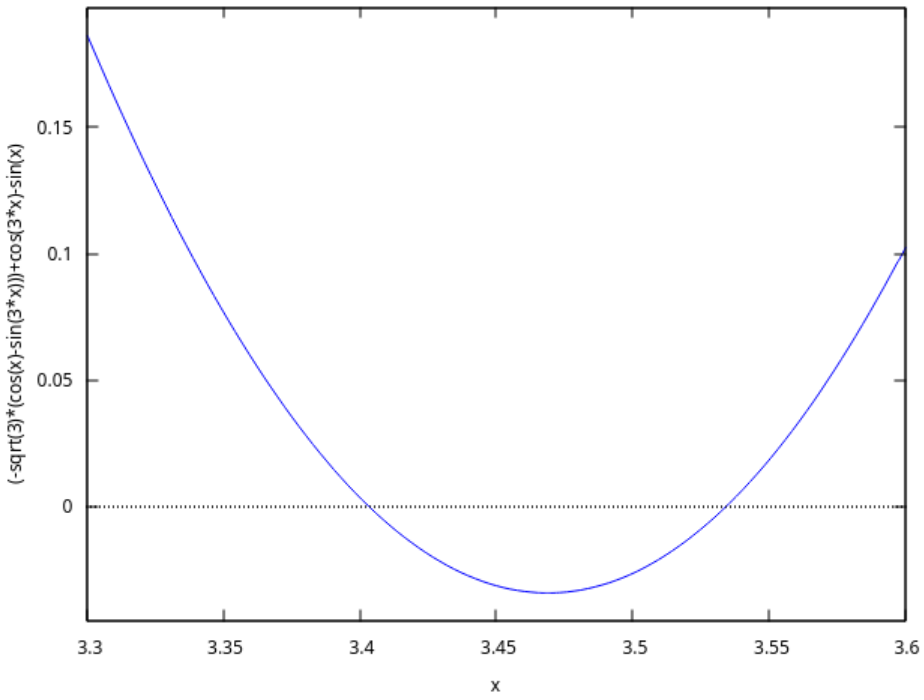


We have 6 solutions from [0, 2*pi].

```
(%i1) load("trigtools")$
(%i2) eq:cos(3*x)-sin(x)=sqrt(3)*(cos(x)-sin(3*x))$
(%i3) plot2d([lhs(eq)-rhs(eq)], [x,0.2,0.5])$
```



```
(%i1) load("trigtools")$
(%i2) load("trigtools")$
(%i3) eq:cos(3*x)-sin(x)=sqrt(3)*(cos(x)-sin(3*x))$
(%i4) plot2d([lhs(eq)-rhs(eq)], [x,3.3,3.6])$
```




```
(%i1) load("trigtools")$
(%i2) eq:cos(3*x)-sin(x)=sqrt(3)*(cos(x)-sin(3*x))$
(%i3) trigfactor(lhs(eq))=map(trigfactor,rhs(eq));
                   %pi             %pi
(%o3) - 2 sin(x + ---) sin(2 x - ---) =
                    4               4
                                              %pi             %pi
                            2 sqrt(3) sin(x - ---) sin(2 x - ---)
                                               4               4
(%i4) factor(lhs(%)-rhs(%));
                4 x + %pi                 4 x - %pi
(%o4) - 2 (sin(---------) + sqrt(3) sin(---------))
                    4                         4
                                                        8 x - %pi
                                                   sin(---------)
                                                          4
```

Equation is equivalent to

```
(%i1) load("trigtools")$
(%i2) eq:cos(3*x)-sin(x)=sqrt(3)*(cos(x)-sin(3*x))$
```

```
(%i3) trigfactor(lhs(eq))=map(trigfactor,rhs(eq));
                  %pi            %pi
(%o3) - 2 sin(x + ---) sin(2 x - ---) =
                   4              4
                                                  %pi            %pi
                             2 sqrt(3) sin(x - ---) sin(2 x - ---)
                                                   4              4
(%i4) L:factor(rhs(%)-lhs(%));
             4 x + %pi                4 x - %pi       8 x - %pi
(%o4) 2 (sin(---------) + sqrt(3) sin(---------)) sin(---------)
                 4                        4               4
(%i5) eq1:part(L,2)=0;
               4 x + %pi                4 x - %pi
(%o5)      sin(---------) + sqrt(3) sin(---------) = 0
                   4                        4
(%i6) eq2:part(L,3)=0;
                           8 x - %pi
(%o6)                  sin(---------) = 0
                               4
(%i7) S1:trigsolve(eq1,0,2*%pi);
                           %pi  13 %pi
(%o7)                     {---, ------}
                           12     12
(%i8) S2:trigsolve(eq2,0,2*%pi);
                    %pi  5 %pi  9 %pi  13 %pi
(%o8)               {---, -----, -----, ------}
                     8     8      8      8
(%i9) S:listify(union(S1,S2));
             %pi  %pi  5 %pi  13 %pi  9 %pi  13 %pi
(%o9)       [---, ---, -----, ------, -----, ------]
             12    8     8      12      8      8
(%i10) float(%), numer;
(%o10) [0.2617993877991494, 0.39269908169872414,
1.9634954084936207, 3.4033920413889422, 3.5342917352885173,
5.105088062083414]
```

Answer: $x = a + 2\pi k$, where a any from S, k any integer.

3.

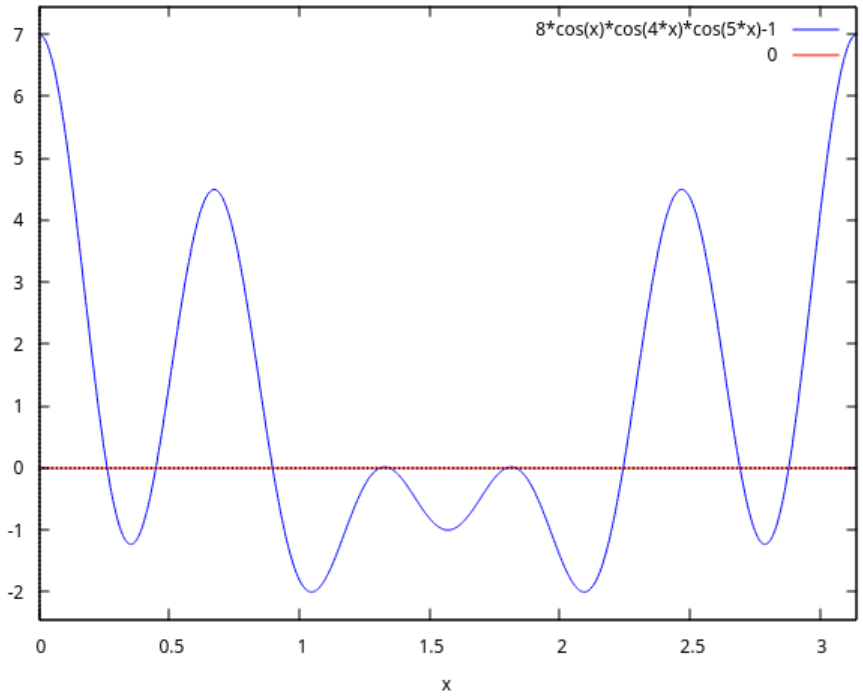
```
(%i1) load("trigtools")$
(%i2) eq:8*cos(x)*cos(4*x)*cos(5*x)-1=0;
(%o2)          8 cos(x) cos(4 x) cos(5 x) - 1 = 0
(%i3) trigrat(%);
(%o3)     2 cos(10 x) + 2 cos(8 x) + 2 cos(2 x) + 1 = 0
```

Left side is periodic with period $T = \pi$.

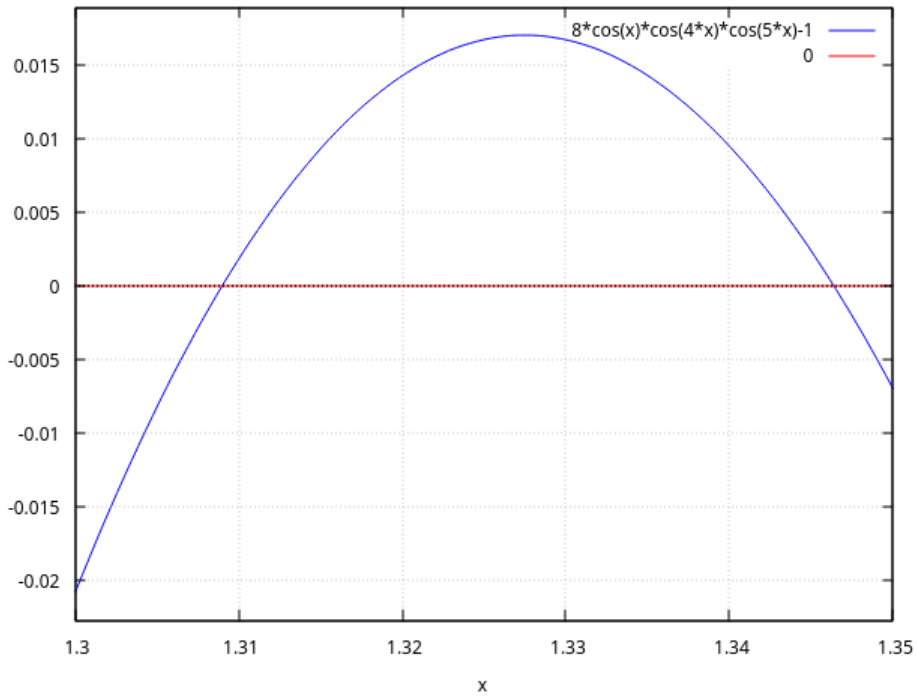
We have 10 solutions from [0, pi].

```
(%i1) load("trigtools")$
(%i2) eq:8*cos(x)*cos(4*x)*cos(5*x)-1=0$
```

```
(%i3) plot2d([lhs(eq),rhs(eq)],[x,0,%pi]);
(%o3)                          false
```




```
(%i1) load("trigtools")$
(%i2) eq:8*cos(x)*cos(4*x)*cos(5*x)-1=0$
(%i3) x4:find_root(eq, x, 1.3, 1.32);
(%o3)                  1.3089969389957472
(%i4) x5:find_root(eq, x, 1.32, 1.35);
(%o4)                  1.3463968515384828
(%i5) plot2d([lhs(eq),0], [x,1.3,1.35], [gnuplot_preamble, "set grid;"]);
(%o5)                          false
```



Equation we multiply by $2\sin x\cos 2x$:

```
(%i1) load("trigtools")$
(%i2) eq:8*cos(x)*cos(4*x)*cos(5*x)-1=0$
(%i3) eq*2*sin(x)*cos(2*x);
(%o3) 2 sin(x) cos(2 x) (8 cos(x) cos(4 x) cos(5 x) - 1) = 0
(%i4) eq1:trigreduce(%),expand;
(%o4)                 sin(13 x) + sin(x) = 0
(%i5) trigfactor(lhs(eq1))=0;
(%o5)                 2 cos(6 x) sin(7 x) = 0
(%i6) S1:trigsolve(cos(6*x),0,%pi);
              %pi  %pi  5 %pi  7 %pi  3 %pi  11 %pi
(%o6)        {---, ---, -----, -----, -----, ------}
              12    4    12     12      4      12
(%i7) S2:trigsolve(sin(7*x),0,%pi);
               %pi  2 %pi  3 %pi  4 %pi  5 %pi  6 %pi
(%o7)      {0, ---, -----, -----, -----, -----, -----}
                7     7      7      7      7      7
```

We remove solutions of $\sin x = 0$ and $\cos 2x = 0$.

```
(%i1) load("trigtools")$
(%i2) S1:trigsolve(cos(6*x),0,%pi)$
(%i3) S2:trigsolve(sin(7*x),0,%pi)$
(%i4) S3:trigsolve(sin(x),0,%pi);
(%o4)                          {0}
```

```
(%i5) S4:trigsolve(cos(2*x),0,%pi);
                              %pi  3 %pi
(%o5)                        {---, -----}
                               4     4
```

We find 10 solutions from $[0, \pi]$:

```
(%i1) load("trigtools")$
(%i2) S1:trigsolve(cos(6*x),0,%pi)$
(%i3) S2:trigsolve(sin(7*x),0,%pi)$
(%i4) S3:trigsolve(sin(x),0,%pi)$
(%i5) S4:trigsolve(cos(2*x),0,%pi)$
(%i6) union(S1,S2)$ setdifference(%,S3)$ setdifference(%,S4);
       %pi  %pi  2 %pi  5 %pi  3 %pi  4 %pi  7 %pi  5 %pi
(%o8) {---, ---, -----, -----, -----, -----, -----, -----,
       12    7     7     12      7      7     12      7
                                                   6 %pi  11 %pi
                                                   -----, ------}
                                                     7      12
(%i9) S:listify(%);
       %pi  %pi  2 %pi  5 %pi  3 %pi  4 %pi  7 %pi  5 %pi
(%o9) [---, ---, -----, -----, -----, -----, -----, -----,
       12    7     7     12      7      7     12      7
                                                   6 %pi  11 %pi
                                                   -----, ------]
                                                     7      12
(%i10) length(S);
(%o10)                          10
(%i11) float(S), numer;
(%o11) [0.2617993877991494, 0.4487989505128276,
0.8975979010256552, 1.3089969389957472, 1.3463968515384828,
1.7951958020513104, 1.8325957145940461, 2.243994752564138,
2.6927937030769655, 2.8797932657906435]
```

Answer: $x = a + 2\pi k$, where a any from S, k any integer.

99.2.6 Evaluation of Trigonometric Functions

`trigvalue` (*x*) [Function]

The function trigvalue compute values of $\sin \frac{m\pi}{n}$, $\cos \frac{m\pi}{n}$, $\tan \frac{m\pi}{n}$, and $\cot \frac{m\pi}{n}$ in radicals.

`trigvalue` is essentially an internal function. Use function `trigeval` in preference.

`load("trigtools")` loads this function.

`trigeval` (*x*) [Function]

The function trigeval compute values of expressions with $\sin \frac{m\pi}{n}$, $\cos \frac{m\pi}{n}$, $\tan \frac{m\pi}{n}$, and $\cot \frac{m\pi}{n}$ in radicals.

`load("trigtools")` loads this function.

Examples:

1. Values of trigonometric functions

```
(%i1) load("trigtools")$
(%i2) load(trigtools)$
(%i3) trigvalue(sin(%pi/10));
                           sqrt(5) - 1
(%o3)                      -----------
                                4
(%i4) trigvalue(cos(%pi/10));
                          sqrt(sqrt(5) + 5)
(%o4)                     -----------------
                                 3/2
                                2
(%i5) trigvalue(tan(%pi/10));
                         sqrt(5 - 2 sqrt(5))
(%o5)                    -------------------
                               sqrt(5)
(%i6) float(%), numer;
(%o6)                    0.3249196962329063
(%i7) float(tan(%pi/10)), numer;
(%o7)                    0.3249196962329063
(%i8) trigvalue(cot(%pi/10));
(%o8)                    sqrt(2 sqrt(5) + 5)
(%i9) float(%), numer;
(%o9)                    3.0776835371752536
(%i10) float(cot(%pi/10)), numer;
(%o10)                    3.077683537175254
(%i11) trigvalue(sin(%pi/32));
               sqrt(2 - sqrt(sqrt(sqrt(2) + 2) + 2))
(%o11)         -------------------------------------
                                2
(%i12) trigvalue(cos(%pi/32));
               sqrt(sqrt(sqrt(sqrt(2) + 2) + 2) + 2)
(%o12)         -------------------------------------
                                2
(%i13) trigvalue(cos(%pi/256));
(%o13) sqrt(sqrt(sqrt(sqrt(sqrt(sqrt(2) + 2) + 2) + 2) + 2)
                                                    + 2) + 2)/2
(%i14) trigvalue(cos(%pi/60));
(%o14)
  sqrt(sqrt(sqrt(2) sqrt(3) sqrt(sqrt(5) + 5) + sqrt(5) + 7) + 4)
  ---------------------------------------------------------------
                                3/2
                               2
```

```
(%i15) trigvalue(sin(%pi/60));
(%o15)
  sqrt(4 - sqrt(sqrt(2) sqrt(3) sqrt(sqrt(5) + 5) + sqrt(5) + 7))
  ---------------------------------------------------------------
                                3/2
                               2
(%i16) trigvalue(sin(%pi/18));
                                %pi
(%o16)                     sin(---)
                                18
(%i17) trigvalue(sin(%pi/20));
               sqrt(4 - sqrt(2) sqrt(sqrt(5) + 5))
(%o17)         -----------------------------------
                                3/2
                               2
```

2. ode example

```
(%i1) load("trigtools")$
(%i2) load(odes)$
(%i3) eq:'diff(y,x,5)+2*y=0;
                           5
                          d y
(%o3)                     --- + 2 y = 0
                            5
                          dx
```

```
(%i4) odeL(eq,y,x);
                1/5     4 %pi
             - 2    cos(-----) x
                           5               1/5     4 %pi
(%o4) y = %e                       C5 sin(2    sin(-----) x)
                                                     5

          1/5     4 %pi
      - 2    cos(-----) x
                   5               1/5     4 %pi
  + %e                      C4 cos(2    sin(-----) x)
                                           5

          1/5     2 %pi
      - 2    cos(-----) x
                   5               1/5     2 %pi
  + %e                      C3 sin(2    sin(-----) x)
                                           5

          1/5     2 %pi
      - 2    cos(-----) x
                   5               1/5     2 %pi
  + %e                      C2 cos(2    sin(-----) x)
                                           5

          1/5
      - 2    x
  + %e         C1
```

```
(%i5) sol:trigeval(%);
                (sqrt(5) - 1) x
              - ---------------
                      9/5
                     2                  sqrt(sqrt(5) + 5) x
(%o5) y = %e                   C3 sin(--------------------)
                                              13/10
                                             2

        (sqrt(5) - 1) x
      - ---------------
              9/5
             2                  sqrt(sqrt(5) + 5) x
 + %e                   C2 cos(--------------------)
                                       13/10
                                      2

     (sqrt(5) + 1) x
     ---------------
           9/5
          2                  sqrt(5 - sqrt(5)) x
 + %e                C5 sin(--------------------)
                                    13/10
                                   2

     (sqrt(5) + 1) x
     ---------------
           9/5                                              1/5
          2                  sqrt(5 - sqrt(5)) x      - 2    x
 + %e                C4 cos(--------------------) + %e         C1
                                    13/10
                                   2

(%i6) subst(sol,eq)$
(%i7) ev(%, nouns)$
(%i8) radcan(%);
(%o8)                          0 = 0
```

3. n-th root of complex number

   Example. Find the 4-th roots of %i

```
(%i1) load("trigtools")$
(%i2) solve(x^4=%i,x);
                 1/8                1/8              1/8
(%o2) [x = (- 1)    %i, x = - (- 1)   , x = - (- 1)    %i,
                                                            1/8
                                                  x = (- 1)   ]
```

```
(%i3) rectform(%);
                  %pi        %pi                %pi        %pi
(%o3) [x = %i cos(---) - sin(---), x = - %i sin(---) - cos(---),
                   8          8                  8          8
                  %pi           %pi             %pi        %pi
          x = sin(---) - %i cos(---), x = %i sin(---) + cos(---)]
                   8             8               8          8
(%i4) trigeval(%);
           sqrt(sqrt(2) + 2) %i   sqrt(2 - sqrt(2))
(%o4) [x = -------------------- - -----------------,
                     2                     2
      sqrt(2 - sqrt(2)) %i   sqrt(sqrt(2) + 2)
x = - -------------------- - -----------------,
                2                     2
    sqrt(2 - sqrt(2))   sqrt(sqrt(2) + 2) %i
x = ----------------- - --------------------,
           2                     2
    sqrt(2 - sqrt(2)) %i   sqrt(sqrt(2) + 2)
x = -------------------- + -----------------]
             2                     2
```

99.2.7 Contract atan Functions

`atan_contract` (*r*) [Function]

The function atan_contract(r) contracts atan functions. We assume: $|r| < \frac{\pi}{2}$.

`load("trigtools")` loads this function.

Examples:

1.

```
(%i1) load("trigtools")$
(%i2) atan_contract(atan(x)+atan(y));
(%o2)                    atan(y) + atan(x)
(%i3) assume(abs(atan(x)+atan(y))<%pi/2)$
(%i4) atan(x)+atan(y)=atan_contract(atan(x)+atan(y));
                                          y + x
(%o4)           atan(y) + atan(x) = atan(-------)
                                        1 - x y
```

2.

```
(%i1) load("trigtools")$
(%i2) atan(1/3)+atan(1/5)+atan(1/7)+atan(1/8)$ %=atan_contract(%);
                 1         1         1         1    %pi
(%o3)      atan(-) + atan(-) + atan(-) + atan(-) = ---
                 3         5         7         8     4
```

3. Machin's formulae

```
(%i1) load("trigtools")$
```

```
(%i2) 4*atan(1/5)-atan(1/239)=atan_contract(4*atan(1/5)-atan(1/239));
                            1          1     %pi
(%o2)              4 atan(-) - atan(---) = ---
                            5         239     4
```

4. see https://en.wikipedia.org/wiki/Machin-like_formula

```
(%i1) load("trigtools")$
(%i2) 12*atan(1/49)+32*atan(1/57)-5*atan(1/239)+12*atan(1/110443)$
(%i3) %=atan_contract(%);
                 1             1             1
(%o3) 12 atan(--) + 32 atan(--) - 5 atan(---)
                 49            57           239
                                                        1       %pi
                                                 + 12 atan(------) = ---
                                                          110443     4
```

99.3 References

1. https://maxima.sourceforge.io

100 Package unit

100.1 Introduction to Units

The *unit* package enables the user to convert between arbitrary units and work with dimensions in equations. The functioning of this package is radically different from the original Maxima units package - whereas the original was a basic list of definitions, this package uses rulesets to allow the user to chose, on a per dimension basis, what unit final answers should be rendered in. It will separate units instead of intermixing them in the display, allowing the user to readily identify the units associated with a particular answer. It will allow a user to simplify an expression to its fundamental Base Units, as well as providing fine control over simplifying to derived units. Dimensional analysis is possible, and a variety of tools are available to manage conversion and simplification options. In addition to customizable automatic conversion, *units* also provides a traditional manual conversion option.

Note - when unit conversions are inexact Maxima will make approximations resulting in fractions. This is a consequence of the techniques used to simplify units. The messages warning of this type of substitution are disabled by default in the case of units (normally they are on) since this situation occurs frequently and the warnings clutter the output. (The existing state of ratprint is restored after unit conversions, so user changes to that setting will be preserved otherwise.) If the user needs this information for units, they can set *unitverbose:on* to reactivate the printing of warnings from the unit conversion process.

unit is included in Maxima in the share/contrib/unit directory. It obeys normal Maxima package loading conventions:

```
(%i1) load("unit")$
*************************************************************
*                      Units version 0.50                   *
*          Definitions based on the NIST Reference on       *
*              Constants, Units, and Uncertainty            *
*       Conversion factors from various sources including   *
*                   NIST and the GNU units package          *
*************************************************************

Redefining necessary functions...
WARNING: DEFUN/DEFMACRO: redefining function TOPLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFMACRO: redefining function MSETCHK ...
WARNING: DEFUN/DEFMACRO: redefining function KILL1 ...
WARNING: DEFUN/DEFMACRO: redefining function NFORMAT ...
Initializing unit arrays...
Done.
```

The WARNING messages are expected and not a cause for concern - they indicate the *unit* package is redefining functions already defined in Maxima proper. This is necessary in order to properly handle units. The user should be aware that if other changes have been made to these functions by other packages those changes will be overwritten by this loading process.

The *unit.mac* file also loads a lisp file *unit-functions.lisp* which contains the lisp functions needed for the package.

Clifford Yapp is the primary author. He has received valuable assistance from Barton Willis of the University of Nebraska at Kearney (UNK), Robert Dodier, and other intrepid folk of the Maxima mailing list.

There are probably lots of bugs. Let me know. `float` and `numer` don't do what is expected.

TODO : dimension functionality, handling of temperature, showabbr and friends. Show examples with addition of quantities containing units.

100.2 Functions and Variables for Units

`setunits` (*list*) [Function]

By default, the *unit* package does not use any derived dimensions, but will convert all units to the seven fundamental dimensions using MKS units.

```
(%i2) N;
                                    kg m
(%o2)                               ----
                                      2
                                     s
(%i3) dyn;
                                  1      kg m
(%o3)                           (------) (----)
                                 100000     2
                                           s
(%i4) g;
                                   1
(%o4)                             (----) (kg)
                                   1000
(%i5) centigram*inch/minutes^2;
                                  127        kg m
(%o5)                       (-------------) (----)
                             1800000000000     2
                                              s
```

In some cases this is the desired behavior. If the user wishes to use other units, this is achieved with the `setunits` command:

```
(%i6) setunits([centigram,inch,minute]);
(%o6)                               done
(%i7) N;
                            1800000000000   %in cg
(%o7)                      (-------------) (------)
                                 127            2
                                              %min
(%i8) dyn;
                              18000000   %in cg
(%o8)                        (--------) (------)
                                 127          2
                                           %min
```

```
(%i9) g;
(%o9)                                (100) (cg)
(%i10) centigram*inch/minutes^2;
                                       %in cg
(%o10)                                 ------
                                           2
                                       %min
```

The setting of units is quite flexible. For example, if we want to get back to kilograms, meters, and seconds as defaults for those dimensions we can do:

```
(%i11) setunits([kg,m,s]);
(%o11)                                 done
(%i12) centigram*inch/minutes^2;
                                    127        kg m
(%o12)                       (-------------) (----)
                              1800000000000     2
                                                s
```

Derived units are also handled by this command:

```
(%i17) setunits(N);
(%o17)                                 done
(%i18) N;
(%o18)                                   N
(%i19) dyn;
                                       1
(%o19)                               (------) (N)
                                     100000
(%i20) kg*m/s^2;
(%o20)                                   N
(%i21) centigram*inch/minutes^2;
                                       127
(%o21)                           (-------------) (N)
                                 1800000000000
```

Notice that the *unit* package recognized the non MKS combination of mass, length, and inverse time squared as a force, and converted it to Newtons. This is how Maxima works in general. If, for example, we prefer dyne to Newtons, we simply do the following:

```
(%i22) setunits(dyn);
(%o22)                                 done
(%i23) kg*m/s^2;
(%o23)                             (100000) (dyn)
(%i24) centigram*inch/minutes^2;
                                       127
(%o24)                           (--------) (dyn)
                                 18000000
```

To discontinue simplifying to any force, we use the uforget command:

```
(%i26) uforget(dyn);
(%o26)                                 false
```

```
(%i27) kg*m/s^2;
                                        kg m
(%o27)                                  ----
                                          2
                                         s
(%i28) centigram*inch/minutes^2;
                                    127        kg m
(%o28)                         (-------------) (----)
                                1800000000000     2
                                                 s
```

This would have worked equally well with `uforget(N)` or `uforget(%force)`.

See also `uforget`. To use this function write first `load("unit")`.

`uforget` (*list*) [Function]

By default, the *unit* package converts all units to the seven fundamental dimensions using MKS units. This behavior can be changed with the `setunits` command. After that, the user can restore the default behavior for a particular dimension by means of the `uforget` command:

```
(%i13) setunits([centigram,inch,minute]);
(%o13)                                done
(%i14) centigram*inch/minutes^2;
                                      %in cg
(%o14)                                ------
                                          2
                                      %min
(%i15) uforget([cg,%in,%min]);
(%o15)                      [false, false, false]
(%i16) centigram*inch/minutes^2;
                                    127        kg m
(%o16)                         (-------------) (----)
                                1800000000000     2
                                                 s
```

`uforget` operates on dimensions, not units, so any unit of a particular dimension will work. The dimension itself is also a legal argument.

See also `setunits`. To use this function write first `load("unit")`.

`convert` (*expr*, *list*) [Function]

When resetting the global environment is overkill, there is the `convert` command, which allows one time conversions. It can accept either a single argument or a list of units to use in conversion. When a convert operation is done, the normal global evaluation system is bypassed, in order to avoid the desired result being converted again. As a consequence, for inexact calculations "rat" warnings will be visible if the global environment controlling this behavior (`ratprint`) is true. This is also useful for spot-checking the accuracy of a global conversion. Another feature is `convert` will allow a user to do Base Dimension conversions even if the global environment is set to simplify to a Derived Dimension.

```
(%i2) kg*m/s^2;
                                      kg m
(%o2)                                 ----
                                        2
                                       s
(%i3) convert(kg*m/s^2,[g,km,s]);
                                      g km
(%o3)                                 ----
                                        2
                                       s
(%i4) convert(kg*m/s^2,[g,inch,minute]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748
                              18000000000   %in g
(%o4)                        (-----------) (-----)
                                  127           2
                                             %min
(%i5) convert(kg*m/s^2,[N]);
(%o5)                                  N
(%i6) convert(kg*m^2/s^2,[N]);
(%o6)                                 m N
(%i7) setunits([N,J]);
(%o7)                                 done
(%i8) convert(kg*m^2/s^2,[N]);
(%o8)                                 m N
(%i9) convert(kg*m^2/s^2,[N,inch]);

'rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748
                                 5000
(%o9)                           (----) (%in N)
                                 127
(%i10) convert(kg*m^2/s^2,[J]);
(%o10)                                 J
(%i11) kg*m^2/s^2;
(%o11)                                 J
(%i12) setunits([g,inch,s]);
(%o12)                                done
(%i13) kg*m/s^2;
(%o13)                                 N
(%i14) uforget(N);
(%o14)                                false
(%i15) kg*m/s^2;
                                 5000000   %in g
(%o15)                          (-------) (-----)
                                   127       2
                                            s
```

```
(%i16) convert(kg*m/s^2,[g,inch,s]);

`rat' replaced 39.37007874015748 by 5000/127 = 39.37007874015748
                                  5000000   %in g
(%o16)                           (-------) (-----)
                                    127       2
                                             s
```

See also `setunits` and `uforget`. To use this function write first `load("unit")`.

`usersetunits` [Optional variable]

Default value: none

If a user wishes to have a default unit behavior other than that described, they can make use of *maxima-init.mac* and the *usersetunits* variable. The *unit* package will check on startup to see if this variable has been assigned a list. If it has, it will use setunits on that list and take the units from that list to be defaults. `uforget` will revert to the behavior defined by usersetunits over its own defaults. For example, if we have a *maxima-init.mac* file containing:

```
usersetunits : [N,J];
```

we would see the following behavior:

```
(%i1) load("unit")$
*************************************************************************
*                        Units version 0.50                             *
*          Definitions based on the NIST Reference on                   *
*              Constants, Units, and Uncertainty                        *
*       Conversion factors from various sources including               *
*                   NIST and the GNU units package                      *
*************************************************************************

Redefining necessary functions...
WARNING: DEFUN/DEFMACRO: redefining function
 TOPLEVEL-MACSYMA-EVAL ...
WARNING: DEFUN/DEFMACRO: redefining function MSETCHK ...
WARNING: DEFUN/DEFMACRO: redefining function KILL1 ...
WARNING: DEFUN/DEFMACRO: redefining function NFORMAT ...
Initializing unit arrays...
Done.
User defaults found...
User defaults initialized.
(%i2) kg*m/s^2;
(%o2)                                  N
(%i3) kg*m^2/s^2;
(%o3)                                  J
(%i4) kg*m^3/s^2;
(%o4)                                 J m
(%i5) kg*m*km/s^2;
(%o5)                            (1000) (J)
```

```
(%i6) setunits([dyn,eV]);
(%o6)                                done
(%i7) kg*m/s^2;
(%o7)                           (100000) (dyn)
(%i8) kg*m^2/s^2;
(%o8)                    (6241509596477042688) (eV)
(%i9) kg*m^3/s^2;
(%o9)                   (6241509596477042688) (eV m)
(%i10) kg*m*km/s^2;
(%o10)                 (624150959647704268800) (eV)
(%i11) uforget([dyn,eV]);
(%o11)                          [false, false]
(%i12) kg*m/s^2;
(%o12)                                N
(%i13) kg*m^2/s^2;
(%o13)                                J
(%i14) kg*m^3/s^2;
(%o14)                               J m
(%i15) kg*m*km/s^2;
(%o15)                           (1000) (J)
```

Without `usersetunits`, the initial inputs would have been converted to MKS, and uforget would have resulted in a return to MKS rules. Instead, the user preferences are respected in both cases. Notice these can still be overridden if desired. To completely eliminate this simplification - i.e. to have the user defaults reset to factory defaults - the `dontusedimension` command can be used. `uforget` can restore user settings again, but only if `usedimension` frees it for use. Alternately, `kill(usersetunits)` will completely remove all knowledge of the user defaults from the session. Here are some examples of how these various options work.

```
(%i2) kg*m/s^2;
(%o2)                                 N
(%i3) kg*m^2/s^2;
(%o3)                                 J
(%i4) setunits([dyn,eV]);
(%o4)                                done
(%i5) kg*m/s^2;
(%o5)                           (100000) (dyn)
(%i6) kg*m^2/s^2;
(%o6)                    (6241509596477042688) (eV)
(%i7) uforget([dyn,eV]);
(%o7)                          [false, false]
(%i8) kg*m/s^2;
(%o8)                                 N
(%i9) kg*m^2/s^2;
(%o9)                                 J
(%i10) dontusedimension(N);
(%o10)                             [%force]
```

```
(%i11) dontusedimension(J);
(%o11)                          [%energy, %force]
(%i12) kg*m/s^2;
                                      kg m
(%o12)                                ----
                                        2
                                       s
(%i13) kg*m^2/s^2;
                                         2
                                      kg m
(%o13)                                -----
                                        2
                                       s
(%i14) setunits([dyn,eV]);
(%o14)                                done
(%i15) kg*m/s^2;
                                      kg m
(%o15)                                ----
                                        2
                                       s
(%i16) kg*m^2/s^2;
                                         2
                                      kg m
(%o16)                                -----
                                        2
                                       s
(%i17) uforget([dyn,eV]);
(%o17)                          [false, false]
(%i18) kg*m/s^2;
                                      kg m
(%o18)                                ----
                                        2
                                       s
(%i19) kg*m^2/s^2;
                                         2
                                      kg m
(%o19)                                -----
                                        2
                                       s
(%i20) usedimension(N);
Done.  To have Maxima simplify to this dimension, use
setunits([unit]) to select a unit.
(%o20)                                true
(%i21) usedimension(J);
Done.  To have Maxima simplify to this dimension, use
setunits([unit]) to select a unit.
(%o21)                                true
```

```
(%i22) kg*m/s^2;
                                      kg m
(%o22)                                ----
                                        2
                                       s
(%i23) kg*m^2/s^2;
                                         2
                                      kg m
(%o23)                                -----
                                        2
                                       s
(%i24) setunits([dyn,eV]);
(%o24)                                done
(%i25) kg*m/s^2;
(%o25)                           (100000) (dyn)
(%i26) kg*m^2/s^2;
(%o26)                  (6241509596477042688) (eV)
(%i27) uforget([dyn,eV]);
(%o27)                            [false, false]
(%i28) kg*m/s^2;
(%o28)                                 N
(%i29) kg*m^2/s^2;
(%o29)                                 J
(%i30) kill(usersetunits);
(%o30)                                done
(%i31) uforget([dyn,eV]);
(%o31)                           [false, false]
(%i32) kg*m/s^2;
                                      kg m
(%o32)                                ----
                                        2
                                       s
(%i33) kg*m^2/s^2;
                                         2
                                      kg m
(%o33)                                -----
                                        2
                                       s
```

Unfortunately this wide variety of options is a little confusing at first, but once the user grows used to them they should find they have very full control over their working environment.

`metricexpandall` (*x*) [Function]

Rebuilds global unit lists automatically creating all desired metric units. *x* is a numerical argument which is used to specify how many metric prefixes the user wishes defined. The arguments are as follows, with each higher number defining all lower numbers' units:

```
            0 - none. Only base units
            1 - kilo, centi, milli
(default)   2 - giga, mega, kilo, hecto, deka, deci, centi, milli,
                micro, nano
            3 - peta, tera, giga, mega, kilo, hecto, deka, deci,
                centi, milli, micro, nano, pico, femto
            4 - all
```

Normally, Maxima will not define the full expansion since this results in a very large number of units, but `metricexpandall` can be used to rebuild the list in a more or less complete fashion. The relevant variable in the *unit.mac* file is *%unitexpand*.

`%unitexpand` [Variable]

Default value: `2`

This is the value supplied to `metricexpandall` during the initial loading of *unit*.

101 Package wrstcse

101.1 Introduction to wrstcse

`wrstcse` provides a few approaches to worst case calculations that are powerful enough for most engineering problems, but avoid the problems that make a true interval arithmetic unfeasible for non-trivial problems:

- Extreme worst case calculation by determining which combination of extreme input values will yield the most extreme result (`wc_mintypmax`, `wc_systematic`). Its approach isn't guaranteed to find local minima and maxima in the middle of the input parameter space. In most real-world problems, though, the derivation of the result to each parameter doesn't change its sign in the tolerance space, which moves the extreme cases to the edges of the tolerance space making them accessible to this method. Via `wc_ewc_simplify` this method can typically be reduced from a brute-force approach to being O(n)-complete.
- The monte carlo approach that assigns each parameter random values in the tolerance space and, when repeated n times will yield a view of the distribution n samples might have. The view provided by this method is pessimistic, as most real-wprld items are more probabible to lie in the middle of the tolerance space than at its end.
- The root mean square approach that knowing an approximation of the parameters of the gaussian distribution of the input values one can determine how big the tolerances will realistically be if one requires a given yield (see `wc_mintypmax_rss`). Note that even rectangles tend to quickly add up to approximation of gauss curve, if affecting the same tolerance which makes this method applicable to most real-world problems.

Tolerances are applied to parameters by providing each tolerance-laden parameter with a *tol[n]* that wrstcase will vary between -1 and 1 looking for extreme results, while *rol[n]*=0 should yield the typical value of a parameter.. Using the same *n* for two parameters will make both parameters have coupled tolerances. Therefore two 1% resistors that both will change over the temperature range by another 2% in the same way can be defined as follows:

```
(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01)*(1+tol[3]*.02),
   R_2= 2000.0*(1+tol[2]*.01)*(1+tol[3]*.02)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1) (0.02 tol  + 1),
                            1               3
                  R_2 = 2000.0 (0.01 tol  + 1) (0.02 tol  + 1)]
                                       2               3
(%i3) R_1+R_2=wc_mintypmax(subst(vals,R_1+R_2));
(%o3) R_2 + R_1 = [min = 2910.6, typ = 3000.0,
                                        max = 3090.6000000000004]
(%i4) R_1+R_2=wc_mintypmax_percent(subst(vals,R_1+R_2));
(%o4) R_2 + R_1 = [min = - 2.980000000000005 %, typ = 3000.0,
                                      max = 3.0200000000000227 %]
```

```
(%i5) R_1/R_2=wc_mintypmax(subst(vals,R_1/R_2));
      R_1
(%o5) --- = [min = 0.4900990099009901, typ = 0.5,
      R_2
                                          max = 0.5101010101010102]
(%i6) R_1/R_2=wc_mintypmax_percent(subst(vals,R_1/R_2));
      R_1
(%o6) --- = [min = - 1.980198019801982 %, typ = 0.5,
      R_2
                                        max = 2.020202020202033 %]
```

`load ("wrstcse")` loads this package.

101.2 Functions and Variables for wrstcse

`wc_typicalvalues` (*expression*) [Function]

Returns what happens if all *tol[n]* happen to be 0, which moves all parameter to the middle of their tolerance range.

Example:

```
(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1),
                             1
                                    R_2 = 2000.0 (0.01 tol  + 1)]
                                                          2
(%i3) divider:U_Out=U_In*R_1/(R_1+R_2);
                                  R_1 U_In
(%o3)                   U_Out = ---------
                                  R_2 + R_1
(%i4) wc_typicalvalues(vals);
(%o4)              [R_1 = 1000.0, R_2 = 2000.0]
(%i5) wc_typicalvalues(subst(vals,divider));
(%o5)            U_Out = 0.3333333333333333 U_In
```

`wc_inputvalueranges` (*expression*, [*show_tols*]) [Function]

Convenience function: Displays a list which parameter can vary between which values.

If *show_tols* is `true` then this function additionally displays which *tol[n]* each variable is affected by.

See also `wc_mintypmax2tol` and `wc_inputvalueassumptions`.

Example:

```
(%i1) load("wrstcse")$
```

```
(%i2) vals: [
   R_1= 1000.0*(1+tol["R_1"]*.01+tol["Temp"]*.001),
   R_2= 2000.0*(1+tol["R_2"]*.01+tol["Temp"]*.001),
   R_3= 2000.0*(1+tol["R_3"]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.001 tol     + 0.01 tol    + 1),
                              Temp           R_1
R_2 = 2000.0 (0.001 tol     + 0.01 tol    + 1),
                        Temp           R_2
R_3 = 2000.0 (0.01 tol    + 1)]
                      R_3
(%i3) wc_inputvalueranges(vals);
(%o3)
    [ R_1  min = 989.0   typ = 1000.0  max = 1010.9999999999999 ]
    [                                                           ]
    [ R_2  min = 1978.0  typ = 2000.0  max = 2021.9999999999998 ]
    [                                                           ]
    [ R_3  min = 1980.0  typ = 2000.0        max = 2020.0       ]
(%i4) wc_inputvalueranges(vals,true);
               [ R_1 ]         [ min = 989.0  ]
               [     ]         [              ]
(%o4)  Col 1 = [ R_2 ] Col 2 = [ min = 1978.0 ]
               [     ]         [              ]
               [ R_3 ]         [ min = 1980.0 ]
         [ typ = 1000.0 ]         [ max = 1010.9999999999999 ]
         [              ]         [                          ]
 Col 3 = [ typ = 2000.0 ] Col 4 = [ max = 2021.9999999999998 ]
         [              ]         [                          ]
         [ typ = 2000.0 ]         [       max = 2020.0       ]
         [ [tol    , tol   ] ]
         [     Temp     R_1  ]
         [                   ]
 Col 5 = [ [tol    , tol   ] ]
         [     Temp     R_2  ]
         [                   ]
         [     [tol   ]      ]
         [         R_3       ]
```

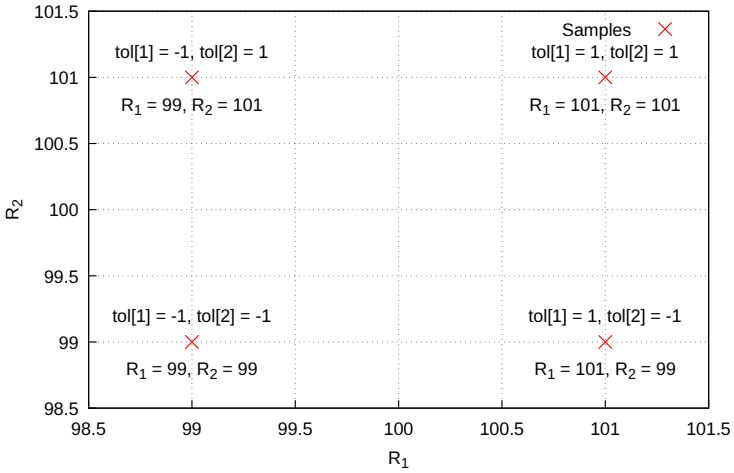
`wc_systematic` (*expression*, [*num*]) [Function]

Systematically introduces *num* values per parameter into *expression* and returns a list of the result. If no *num* is given, *num* defaults to *wc_defaultvaluespertol*. Negative values for *num* make `wc_systematic` use the monte carlo method with *num* samples, instead.

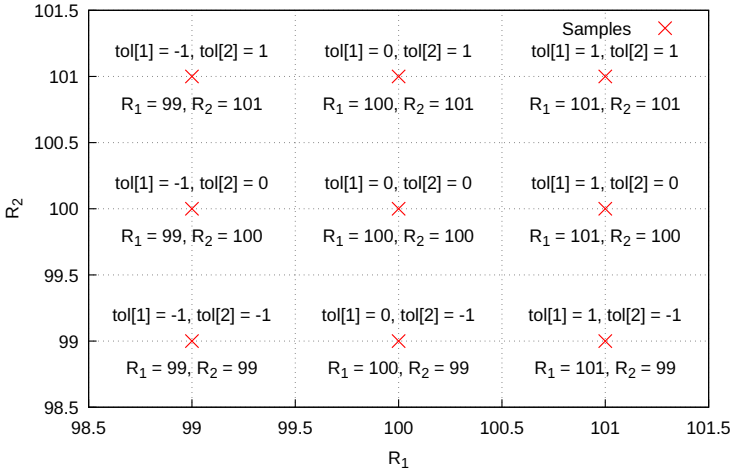
If an equation uses the following values with tolerances:

```
vals:[R_1=100+tol[1],R_2=100+tol[2]];
```

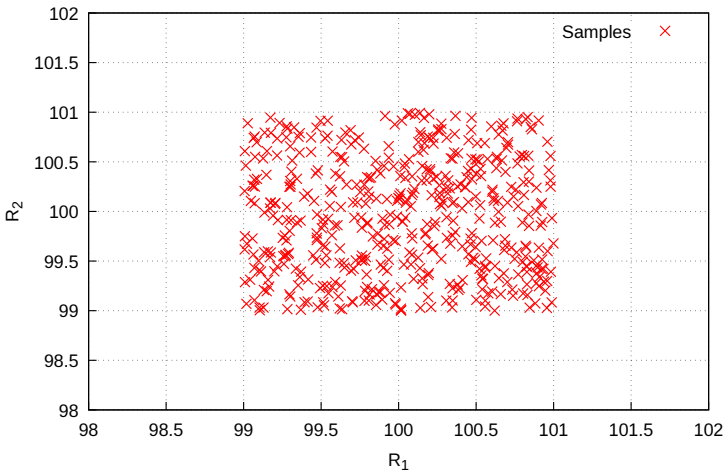
`num=2` will cause the following combinations of tolerances to be tested:



`num=3` will cause the following combinations of tolerances to be tested:



`num=-500` will cause the following combinations of tolerances to be tested, instead:



See also `wc_defaultvaluespertol`, `wc_mintypmax`, `wc_defaultvaluespertol`, `wc_ewc_simplify` and `wc_montecarlo`.

Example:

```
(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1),
                            1
                                  R_2 = 2000.0 (0.01 tol  + 1)]
                                                        2
(%i3) divider: U_Out=U_In*(R_1)/(R_1+R_2);
                                  R_1 U_In
(%o3)                    U_Out = ---------
                                  R_2 + R_1
(%i4) wc_systematic(subst(vals,rhs(divider)));
(%o4) [0.33333333333333337 U_In, 0.3311036789297659 U_In,
0.3289036544850498 U_In, 0.3355704697986577 U_In,
0.3333333333333333 U_In, 0.33112582781456956 U_In,
0.3377926421404682 U_In, 0.3355481727574751 U_In,
0.3333333333333333 U_In]
```

`wc_ewc_simplify` (*expression*, *definitions*...) [Function]

Brute-forcing through all combinations of *tol[n]* in order to find the worst-case combination is O(m^n)-complete and therefore computationally intensive for high numbers of tol[n].

`wc_ewc_simplify` uses the sign of the derivatives of *expression* to combine as many `tol[n]`, as possible. The result is an expression that might run much faster through `wc_mintypmax` and `wc_systematic`, but that, if the derivate of *expression* doesn't change sign in the *tol[n]* space, still yields the same results for the brute-force approaches to worst-case analysis.

Note that changing the number of *tol[n]* will change the statistical distribution of the results over the *tol[n]* space and therefore will change the statistical distribution of the montecarlo method and the results of the root sum square functions.

See also `wc_mintypmax` and `wc_montecarlo`.

definitions allows to temporarily asssign values to specific *tol[n]* during the optimizations (normally optimizagion is done with all *tol[n]* being zero) which has the side-effect of excempting these *tol[n]* from the optimization.

Example:

```
(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1),
                            1
                                   R_2 = 2000.0 (0.01 tol  + 1)]
                                                         2
(%i3) ratprint:false;
(%o3)                          false
(%i4) assume(U_In>0);
(%o4)                       [U_In > 0]
(%i5) divider: U_Out=U_In*(R_1)/(R_1+R_2);
                                 R_1 U_In
(%o5)                   U_Out = ---------
                                 R_2 + R_1
(%i6) divider_vals:subst(vals,divider);
                        1000.0 (0.01 tol  + 1) U_In
                                          1
(%o6) U_Out = ----------------------------------------------
              2000.0 (0.01 tol  + 1) + 1000.0 (0.01 tol  + 1)
                               2                          1
(%i7) divider_vals_simplified:lhs(divider_vals)=wc_ewc_simplify(rhs(divider_vals)
                        1000.0 (1 - 0.01 tol ) U_In
                                           2
(%o7) U_Out = ----------------------------------------------
              2000.0 (0.01 tol  + 1) + 1000.0 (1 - 0.01 tol )
                               2                            2
(%i8) wc_systematic(rhs(divider_vals));
(%o8) [0.33333333333333337 U_In, 0.3311036789297659 U_In,
0.3289036544850498 U_In, 0.3355704697986577 U_In,
0.3333333333333333 U_In, 0.33112582781456956 U_In,
0.3377926421404682 U_In, 0.3355481727574751 U_In,
0.3333333333333333 U_In]
(%i9) wc_systematic(rhs(divider_vals_simplified));
(%o9) [0.3377926421404682 U_In, 0.3333333333333333 U_In,
                                          0.3289036544850498 U_In]
```

```
(%i10) lhs(divider_vals)=wc_mintypmax(rhs(divider_vals));
(%o10) U_Out = [min = 0.3289036544850498 U_In,
    typ = 0.3333333333333333 U_In, max = 0.3377926421404682 U_In]
(%i11) lhs(divider_vals_simplified)=wc_mintypmax(rhs(divider_vals_simplified));
(%o11) U_Out = [min = 0.3289036544850498 U_In,
    typ = 0.3333333333333333 U_In, max = 0.3377926421404682 U_In]
```

Use case for adding definitions (*tol["E_Gain"]* cannot be optimized without knowing the sign of *tol["U_In"]*):

```
(%i1) load("wrstcse")$
(%i2) vals: [
   U_In=1.2*tol["U_In"], /* The input voltage range */
   n_Bits=16,            /* The ADC's number of bits */
   U_Ref=1.2*(1+.01*tol["U_Ref"]), /* The Adc's reference */
   E_Gain=1+1.2e-6*tol["E_Gain"], /* Gain error */
   E_Offset=1.5e-6               /* Offset error */
 ];
(%o2) [U_In = 1.2 tol    , n_Bits = 16,
                    U_In
U_Ref = 1.2 (0.01 tol      + 1), E_Gain = 1.2e-6 tol       + 1,
                    U_Ref                           E_Gain
E_Offset = 1.5e-6]
(%i3) ratprint:false;
(%o3)                         false
(%i4) wc_inputvalueranges(vals);
(%o4)
     [   U_In      min = - 1.2      typ = 0        max = 1.2    ]
     [                                                         ]
     [  n_Bits      min = 16        typ = 16       max = 16     ]
     [                                                         ]
     [  U_Ref      min = 1.188     typ = 1.2      max = 1.212   ]
     [                                                         ]
     [  E_Gain   min = 0.9999988    typ = 1     max = 1.0000012 ]
     [                                                         ]
     [ E_Offset   min = 1.5e-6    typ = 1.5e-6   max = 1.5e-6   ]
(%i5) ndsadc:n_DSADC=((U_In*E_Gain+E_Offset)/U_Ref+.5)*(2^(n_Bits)-1);
                  n_Bits       E_Gain U_In + E_Offset
(%o5) n_DSADC = (2       - 1) (---------------------- + 0.5)
                                      U_Ref
(%i6) lhs(ndsadc)=wc_ewc_simplify(subst(vals,rhs(ndsadc)));
(%o6) n_DSADC = 65535 ((0.8333333333333334
 (1.5e-6 - 1.2 (1.2e-6 tol       + 1) tol     ))
                          E_Gain         U_Ref
/(0.01 tol      + 1) + 0.5)
          U_Ref
```

```
(%i7) lhs(ndsadc)=wc_ewc_simplify(subst(vals,rhs(ndsadc)),tol["U_In"]=1);
(%o7) n_DSADC = 65535 ((0.8333333333333334
 (1.2 tol     (1 - 1.2e-6 tol     ) + 1.5e-6))
         U_In                U_Ref
/(0.01 tol      + 1) + 0.5)
          U_Ref
```

`wc_montecarlo` (*expression*, *num*) [Function]

Introduces *num* random values per parameter into *expression* and returns a list of the result.

See also `wc_systematic`.

Example:

```
(%i1) load("wrstcse")$
(%i2) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 2000.0*(1+tol[2]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1),
                             1
                                    R_2 = 2000.0 (0.01 tol  + 1)]
                                                          2
(%i3) divider: U_Out=U_In*(R_1)/(R_1+R_2);
                                 R_1 U_In
(%o3)                  U_Out = ---------
                                 R_2 + R_1
(%i4) wc_montecarlo(subst(vals,rhs(divider)),10);
(%o4) [0.3301505377099377 U_In, 0.33276843198354916 U_In,
0.33406203454153227 U_In, 0.33434833585190965 U_In,
0.3341006856369802 U_In, 0.3359647531928058 U_In,
0.32911646313213117 U_In, 0.333158315034511 U_In,
0.3325173140803672 U_In, 0.3331108406798784 U_In]
```

`wc_mintypmax` (*expr*, [*n*]) [Function]

Prints the minimum, maximum and typical value of *expr*. If *n* is positive, *n* values for each parameter will be tried systematically. If *n* is negative, -*n* random values are used instead. If no *n* is given, *wc_defaultvaluespertol* is assumed.

See also `wc_mintypmax_percent`, `wc_mintypmax_rss`, `wc_ewc_simplify` and `wc_systematic`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
```

```
(%i3) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o3) [R_1 = 1000.0 (0.01 tol  + 1),
                              1
                                     R_2 = 1000.0 (0.01 tol  + 1)]
                                                           2
(%i4) assume(U_In>0);
(%o4)                        [U_In > 0]
(%i5) divider:U_Out=U_In*R_1/(R_1+R_2);
                                  R_1 U_In
(%o5)                   U_Out = ---------
                                  R_2 + R_1
(%i6) lhs(divider)=wc_mintypmax(subst(vals,rhs(divider)));
(%o6) U_Out = [min = 0.495 U_In, typ = 0.5 U_In,
                                                 max = 0.505 U_In]
```

`wc_mintypmax_percent` (*expr*, *sigmas*) [Function]

Like `wc_mintypmax`, but outputs the tolerance range in percent.

See `wc_mintypmax`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) wc_defaultsigma:6$
(%i4) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o4) [R_1 = 1000.0 (0.01 tol  + 1),
                              1
                                     R_2 = 1000.0 (0.01 tol  + 1)]
                                                           2
(%i5) assume(U_In>0);
(%o5)                        [U_In > 0]
(%i6) divider:U_Out=U_In*R_1/(R_1+R_2);
                                  R_1 U_In
(%o6)                   U_Out = ---------
                                  R_2 + R_1
(%i7) lhs(divider)=wc_mintypmax(subst(vals,rhs(divider)));
(%o7) U_Out = [min = 0.495 U_In, typ = 0.5 U_In,
                                                 max = 0.505 U_In]
(%i8) lhs(divider)=wc_mintypmax_percent(subst(vals,rhs(divider)));
(%o8) U_Out = [min = - 1.0000000000000009 %, typ = 0.5 U_In,
                                      max = 1.0000000000000009 %]
```

`wc_mintypmax_rss` (*expr*, *sigmas*) [Function]

Prints the minimum and maximum of *expr*, as well as how high the probability is that *expr* will lie out of that range based on a root sum square calculation and assuming all input ranges will be gauss-shaped.

wc_defaultsigma defines how many sigma of an input value's tolerance distribution the range of *tol[n]*=[-1...1] corresponds to.

The RSS methods has a few advantages a brute-force worst-case calculation: It is fast even witput `wc_ewc_simplify`. It prevents overengineering resulting from assuming all tolerances to add up in a catastrophical way if this only happens in a neglectible number of cases. But it will yield only the correct results if the gaussian distribution of the input data is known.

Due to its nature this method doesn't support tolerances that aren't symmetrical. Therefore in that case the tolerances are extended in the direction that is less wide.

See also `wc_defaultsigma` `wc_mintypmax_rss_percent`, and `wc_mintypmax`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) wc_defaultsigma:6$
(%i4) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o4) [R_1 = 1000.0 (0.01 tol  + 1),
                            1
                                   R_2 = 1000.0 (0.01 tol  + 1)]
                                                         2
(%i5) assume(U_In>0);
(%o5)                       [U_In > 0]
(%i6) divider:U_Out=U_In*R_1/(R_1+R_2);
                                 R_1 U_In
(%o6)                  U_Out = ---------
                                 R_2 + R_1
(%i7) lhs(divider)=wc_mintypmax_rss(subst(vals,rhs(divider)),6);
(%o7) U_Out = [min = 0.49646446609406725 U_In, typ = 0.5 U_In,
 max = 0.5035355339059328 U_In, Fail = 0.0019731752898266564 ppm]
```

`wc_mintypmax_rss_percent` (*expr*, *sigmas*) [Function]

Like `wc_mintypmax_rss`, but outputs the tolerance range in percent.

See `wc_mintypmax_rss`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) wc_defaultsigma:6$
```

```
(%i4) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o4) [R_1 = 1000.0 (0.01 tol  + 1),
                              1
                                     R_2 = 1000.0 (0.01 tol  + 1)]
                                                           2
(%i5) assume(U_In>0);
(%o5)                        [U_In > 0]
(%i6) divider:U_Out=U_In*R_1/(R_1+R_2);
                                  R_1 U_In
(%o6)                    U_Out = ---------
                                  R_2 + R_1
(%i7) lhs(divider)=wc_mintypmax_rss(subst(vals,rhs(divider)),6);
(%o7) U_Out = [min = 0.49646446609406725 U_In, typ = 0.5 U_In,
 max = 0.5035355339059328 U_In, Fail = 0.0019731752898266564 ppm]
(%i8) lhs(divider)=float(wc_mintypmax_rss_percent(subst(vals,rhs(divider)),6));
(%o8) U_Out = [min = - 0.7071067811865506 %, typ = 0.5 U_In,
    max = 0.7071067811865506 %, Fail = 0.0019731752898266564 ppm]
```

`wc_defaultsigma` [Option variable]

Default value: `6`

Defines how many sigmas of the gauss distribution that represents the tolerances of a parameter correspond to a *tol[n]* value of -1...1.

See also `wc_mintypmax_rss`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) vals: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o3) [R_1 = 1000.0 (0.01 tol  + 1),
                              1
                                     R_2 = 1000.0 (0.01 tol  + 1)]
                                                           2
(%i4) assume(U_In>0);
(%o4)                        [U_In > 0]
(%i5) divider:U_Out=U_In*R_1/(R_1+R_2);
                                  R_1 U_In
(%o5)                    U_Out = ---------
                                  R_2 + R_1
(%i6) wc_defaultsigma:1$
(%i7) lhs(divider)=wc_mintypmax_rss(subst(vals,rhs(divider)),6);
(%o7) U_Out = [min = 0.4787867965644036 U_In, typ = 0.5 U_In,
 max = 0.5212132034355964 U_In, Fail = 0.0019731752898266564 ppm]
```

```
(%i8) wc_defaultsigma:3$
(%i9) lhs(divider)=wc_mintypmax_rss(subst(vals,rhs(divider)),6);
(%o9) U_Out = [min = 0.49292893218813455 U_In, typ = 0.5 U_In,
 max = 0.5070710678118655 U_In, Fail = 0.0019731752898266564 ppm]
(%i10) wc_defaultsigma:6$
(%i11) lhs(divider)=wc_mintypmax_rss(subst(vals,rhs(divider)),6);
(%o11) U_Out = [min = 0.49646446609406725 U_In, typ = 0.5 U_In,
 max = 0.5035355339059328 U_In, Fail = 0.0019731752898266564 ppm]
```

`wc_defaultvaluespertol` [Option variable]

Default value: `3`

Defines how many samples per *tol[n]* the EWC method of `wc_systematic` and `wc_mintypmax` shall use by default.

See also `wc_systematic` and `wc_mintypmax`.

Example:

```
(%i1) load("wrstcse")$
(%i2) ratprint:false$
(%i3) vals: [
   R_1= 100.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o3) [R_1 = 100.0 (0.01 tol  + 1), R_2 = 1000.0 (0.01 tol  + 1)]
                             1                               2
(%i4) wc_defaultvaluespertol:2$
(%i5) wc_systematic(vals);
(%o5) [[R_1 = 99.0, R_2 = 990.0], [R_1 = 99.0, R_2 = 1010.0],
         [R_1 = 101.0, R_2 = 990.0], [R_1 = 101.0, R_2 = 1010.0]]
(%i6) wc_defaultvaluespertol:3$
(%i7) wc_systematic(vals);
(%o7) [[R_1 = 99.0, R_2 = 990.0], [R_1 = 99.0, R_2 = 1000.0],
[R_1 = 99.0, R_2 = 1010.0], [R_1 = 100.0, R_2 = 990.0],
[R_1 = 100.0, R_2 = 1000.0], [R_1 = 100.0, R_2 = 1010.0],
[R_1 = 101.0, R_2 = 990.0], [R_1 = 101.0, R_2 = 1000.0],
[R_1 = 101.0, R_2 = 1010.0]]
(%i8) wc_defaultvaluespertol:5$
```

```
(%i9) wc_systematic(vals);
(%o9) [[R_1 = 99.0, R_2 = 990.0], [R_1 = 99.0, R_2 = 995.0],
[R_1 = 99.0, R_2 = 1000.0], [R_1 = 99.0,
R_2 = 1004.9999999999999], [R_1 = 99.0, R_2 = 1010.0],
[R_1 = 99.5, R_2 = 990.0], [R_1 = 99.5, R_2 = 995.0],
[R_1 = 99.5, R_2 = 1000.0], [R_1 = 99.5,
R_2 = 1004.9999999999999], [R_1 = 99.5, R_2 = 1010.0],
[R_1 = 100.0, R_2 = 990.0], [R_1 = 100.0, R_2 = 995.0],
[R_1 = 100.0, R_2 = 1000.0], [R_1 = 100.0,
R_2 = 1004.9999999999999], [R_1 = 100.0, R_2 = 1010.0],
[R_1 = 100.49999999999999, R_2 = 990.0],
[R_1 = 100.49999999999999, R_2 = 995.0],
[R_1 = 100.49999999999999, R_2 = 1000.0],
[R_1 = 100.49999999999999, R_2 = 1004.9999999999999],
[R_1 = 100.49999999999999, R_2 = 1010.0],
[R_1 = 101.0, R_2 = 990.0], [R_1 = 101.0, R_2 = 995.0],
[R_1 = 101.0, R_2 = 1000.0], [R_1 = 101.0,
R_2 = 1004.9999999999999], [R_1 = 101.0, R_2 = 1010.0]]
```

`wc_tolappend` (*list1*, *list2*,...) [Function]

Appends lists of parameters from independent sources making sure that tolerances of all elements in one list will stay independent from all elements in the others.

Works like append() in that it appends all values from all of its arguments to make one big list. But this command, if one list happens to contain a *tol[n]* with the same *n* as another list, changes one of these *n* to a new value that makes it independent from all tolerances from the other list.

See also `append`.

Example:

```
(%i1) load("wrstcse")$
(%i2) val_a: [
   R_1= 1000.0*(1+tol[1]*.01),
   R_2= 1000.0*(1+tol[2]*.01)
 ];
(%o2) [R_1 = 1000.0 (0.01 tol  + 1),
                            1
                                   R_2 = 1000.0 (0.01 tol  + 1)]
                                                         2
(%i3) val_b: [
   R_3= 1000.0*(1+tol[1]*.01),
   R_4= 1000.0*(1+tol[4]*.01)
 ];
(%o3) [R_3 = 1000.0 (0.01 tol  + 1),
                            1
                                   R_4 = 1000.0 (0.01 tol  + 1)]
                                                         4
```

```
(%i4) append(val_a,val_b);
(%o4) [R_1 = 1000.0 (0.01 tol  + 1),
                             1
R_2 = 1000.0 (0.01 tol  + 1), R_3 = 1000.0 (0.01 tol  + 1),
                        2                               1
R_4 = 1000.0 (0.01 tol  + 1)]
                        4
(%i5) wc_tolappend(val_a,val_b);
                            used = 1

(%o5) [R_1 = 1000.0 (0.01 tol  + 1),
                             1
R_2 = 1000.0 (0.01 tol  + 1), R_3 = 1000.0 (0.01 tol  + 1),
                        2                               5
R_4 = 1000.0 (0.01 tol  + 1)]
                        4
```

`wc_mintypmax2tol` (*tolname*, *minval*, *typval*, *maxval*) [Function]

Generates a parameter that uses the tolerance *tolname* and tolerates between the given values.

Example:

```
(%i1) load("wrstcse")$
(%i2) vals: [U_Diode=wc_mintypmax2tol(tol[1],.5,.75,.82),
       R=wc_mintypmax2tol(tol[2],1,1.1,1.3),
       U_In=wc_mintypmax2tol(tol[3],0,0,15)];
(%o2) [U_Diode = 0.034999999999999976 (|tol | + tol )
                                       |   1|      1
 + 0.125 (tol  - |tol |) + 0.75,
             1   |   1|
R = 0.09999999999999998 (|tol | + tol )
                         |   2|      2
 + 0.050000000000000044 (tol  - |tol |) + 1.1,
                            2   |   2|
       15 (|tol | + tol )
           |   3|      3
U_In = ------------------]
                2
(%i3) wc_inputvalueranges(vals);
         [ U_Diode  min = 0.5  typ = 0.75  max = 0.82 ]
         [                                            ]
(%o3)    [    R     min = 1.0  typ = 1.1   max = 1.3  ]
         [                                            ]
         [  U_In     min = 0    typ = 0     max = 15  ]
```

`wc_inputvalueassumptions` (*expr*) [Function]

Often it is good practice to keep all numeric values with tolerances in a list and to introduce them into the equations only when needed.

`wc_inputvalueassumptions` in this case can inform the `assume` database about the range each variable in that list will be in.

See also `wc_tolassumptions`, `wc_inputvalueranges` and `assume`.

Example:

```
(%i1) load("wrstcse");
(%o1) /home/gunter/src/maxima-code/share/contrib/wrstcse.mac
(%i2) vals:[
    R_1=100*(1+1/100*tol["R1"])*(1+1/100*tol["Temp"]),
    R_2=200*(1+1/100*tol["R2"])*(1+1/100*tol["Temp"])];
                  tol         tol
                     R1          Temp
(%o2) [R_1 = 100 (----- + 1) (------- + 1),
                   100          100
                                          tol         tol
                                             R2          Temp
                              R_2 = 200 (----- + 1) (------- + 1)]
                                           100          100
(%i3) float(wc_inputvalueassumptions(%));
(%o3) [R_2 >= 196.02, R_2 <= 204.02, R_1 >= 98.01, R_1 <= 102.01]
(%i4) is(R_1>R_2);
(%o4)                          false
(%i5) is(R_1>90);
(%o5)                          true
(%i6) is(R_1>200);
(%o6)                          false
(%i7) is(R_1<200);
(%o7)                          true
(%i8) is(R_1>100);
(%o8)                         unknown
```

`wc_tolassumptions` (*expr*) [Function]

Adds the range of the *tol[n]* contained in *expr* to the assume database.

See also `wc_inputvalueassumptions`, `wc_inputvalueranges` and `assume`.

Example:

```
(%i1) load("wrstcse");
(%o1) /home/gunter/src/maxima-code/share/contrib/wrstcse.mac
```

```
(%i2) vals:[
    R_1=100*(1+1/100*tol["R1"])*(1+1/100*tol["Temp"]),
    R_2=200*(1+1/100*tol["R2"])*(1+1/100*tol["Temp"])];
                  tol         tol
                     R1          Temp
(%o2) [R_1 = 100 (----- + 1) (------- + 1),
                   100          100
                                          tol         tol
                                             R2          Temp
                               R_2 = 200 (----- + 1) (------- + 1)]
                                           100          100
(%i3) float(wc_tolassumptions(%));
(%o3) [tol   >= - 1.0, tol   <= 1.0, tol     >= - 1.0,
          R1              R1            Temp
                     tol     <= 1.0, tol   >= - 1.0, tol   <= 1.0]
                        Temp            R2              R2
(%i4) is(tol[R_1]>tol[R_2]);
(%o4)                          unknown
(%i5) is(tol[R_1]>1);
(%o5)                          unknown
(%i6) is(tol[R_1]<-1);
(%o6)                          unknown
(%i7) is(tol[R_1]<0);
(%o7)                          unknown
(%i8) is(tol[R_2]>2);
(%o8)                          unknown
```

102 Package zeilberger

102.1 Introduction to zeilberger

`zeilberger` is an implementation of Zeilberger's algorithm for definite hypergeometric summation, and also Gosper's algorithm for indefinite hypergeometric summation.

`zeilberger` makes use of the "filtering" optimization method developed by Axel Riese.

`zeilberger` was developed by Fabrizio Caruso.

`load ("zeilberger")` loads this package.

102.1.1 The indefinite summation problem

`zeilberger` implements Gosper's algorithm for indefinite hypergeometric summation. Given a hypergeometric term F_k in k we want to find its hypergeometric anti-difference, that is, a hypergeometric term f_k such that

$$F_k = f_{k+1} - f_k.$$

102.1.2 The definite summation problem

`zeilberger` implements Zeilberger's algorithm for definite hypergeometric summation. Given a proper hypergeometric term (in n and k) $F_{n,k}$ and a positive integer d we want to find a d-th order linear recurrence with polynomial coefficients (in n) for $F_{n,k}$ and a rational function R in n and k such that

$$a_0\, F_{n,k} + \ldots + a_d\, F_{n+d,k} = \Delta_K \left(R\left(n,k\right) F_{n,k} \right),$$

where Δ_k is the k-forward difference operator, i.e., $\Delta_k \left(t_k\right) \equiv t_{k+1} - t_k$.

102.1.3 Verbosity levels

There are also verbose versions of the commands which are called by adding one of the following prefixes:

`Summary` Just a summary at the end is shown

`Verbose` Some information in the intermediate steps

`VeryVerbose`
More information

`Extra` Even more information including information on the linear system in Zeilberger's algorithm

For example:
`GosperVerbose`, `parGosperVeryVerbose`, `ZeilbergerExtra`, `AntiDifferenceSummary`.

102.2 Functions and Variables for zeilberger

`AntiDifference` (F_k, k) [Function]

Returns the hypergeometric anti-difference of F_k, if it exists.
Otherwise `AntiDifference` returns `no_hyp_antidifference`.

`Gosper` (F_k, k) [Function]

Returns the rational certificate $R(k)$ for F_k, that is, a rational function such that $F_k = R(k+1)\ F_{k+1} - R(k)\ F_k$, if it exists. Otherwise, `Gosper` returns `no_hyp_sol`.

`GosperSum` (F_k, k, `a`, b) [Function]

Returns the summation of F_k from $k = a$ to $k = b$ if F_k has a hypergeometric anti-difference. Otherwise, `GosperSum` returns `nongosper_summable`.

Examples:

```
(%i1) load ("zeilberger")$
(%i2) GosperSum ((-1)^k*k / (4*k^2 - 1), k, 1, n);
                            n + 1      3
                         (- 1)      (n + -)
                                         2    1
(%o2)                 - ------------------ - -
                                  2         4
                        2 (4 (n + 1)  - 1)
(%i3) GosperSum (1 / (4*k^2 - 1), k, 1, n);
                                  3
                              - n - -
                                    2       1
(%o3)                      -------------- + -
                                    2       2
                           4 (n + 1)  - 1
(%i4) GosperSum (x^k, k, 1, n);
                             n + 1
                            x          x
(%o4)                       ------ - -----
                            x - 1    x - 1
(%i5) GosperSum ((-1)^k*a! / (k!*(a - k)!), k, 1, n);
                         n + 1
                    (- 1)      a! (n + 1)          a!
(%o5)         - ------------------------ - ----------
                a (- n + a - 1)! (n + 1)!   (a - 1)! a
(%i6) GosperSum (k*k!, k, 1, n);
(%o6)                        (n + 1)! - 1
(%i7) GosperSum ((k + 1)*k! / (k + 1)!, k, 1, n);
                    (n + 1) (n + 2) (n + 1)!
(%o7)               ------------------------ - 1
                              (n + 2)!
(%i8) GosperSum (1 / ((a - k)!*k!), k, 1, n);
(%o8)                    NON_GOSPER_SUMMABLE
```

parGosper ($F_{n,k}$, k, n, d) [Function]

Attempts to find a d-th order recurrence for $F_{n,k}$.

The algorithm yields a sequence $[s_1, s_2, ..., s_m]$ of solutions. Each solution has the form

$$[R(n,k), [a_0, a_1, \ldots, a_d]].$$

parGosper returns [] if it fails to find a recurrence.

Zeilberger ($F_{n,k}$, k, n) [Function]

Attempts to compute the indefinite hypergeometric summation of $F_{n,k}$.

Zeilberger first invokes Gosper, and if that fails to find a solution, then invokes parGosper with order 1, 2, 3, ..., up to MAX_ORD. If Zeilberger finds a solution before reaching MAX_ORD, it stops and returns the solution.

The algorithms yields a sequence $[s_1, s_2, ..., s_m]$ of solutions. Each solution has the form

$$[R(n,k), [a_0, a_1, \ldots, a_d]].$$

Zeilberger returns [] if it fails to find a solution.

Zeilberger invokes Gosper only if Gosper_in_Zeilberger is true.

102.3 General global variables

MAX_ORD [Global variable]

Default value: 5

MAX_ORD is the maximum recurrence order attempted by Zeilberger.

simplified_output [Global variable]

Default value: false

When simplified_output is true, functions in the zeilberger package attempt further simplification of the solution.

linear_solver [Global variable]

Default value: linsolve

linear_solver names the solver which is used to solve the system of equations in Zeilberger's algorithm.

warnings [Global variable]

Default value: true

When warnings is true, functions in the zeilberger package print warning messages during execution.

Gosper_in_Zeilberger [Global variable]

Default value: true

When Gosper_in_Zeilberger is true, the Zeilberger function calls Gosper before calling parGosper. Otherwise, Zeilberger goes immediately to parGosper.

`trivial_solutions` [Global variable]

Default value: `true`

When `trivial_solutions` is `true`, `Zeilberger` returns solutions which have certificate equal to zero, or all coefficients equal to zero.

102.4 Variables related to the modular test

`mod_test` [Global variable]

Default value: `false`

When `mod_test` is `true`, `parGosper` executes a modular test for discarding systems with no solutions.

`modular_linear_solver` [Global variable]

Default value: `linsolve`

`modular_linear_solver` names the linear solver used by the modular test in `parGosper`.

`ev_point` [Global variable]

Default value: `big_primes[10]`

`ev_point` is the value at which the variable n is evaluated when executing the modular test in `parGosper`.

`mod_big_prime` [Global variable]

Default value: `big_primes[1]`

`mod_big_prime` is the modulus used by the modular test in `parGosper`.

`mod_threshold` [Global variable]

Default value: 4

`mod_threshold` is the greatest order for which the modular test in `parGosper` is attempted.

Appendix A Function and Variable Index

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